

Five-loop renormalisation of QCD in covariant gauges

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ABSTRACT: We present the complete set of vertex, wave function and charge renormalisation constants in QCD in a general simple gauge group and with the complete dependence on the covariant gauge parameter ξ in the minimal subtraction scheme of conventional dimensional regularisation. Our results confirm all already known results, which were obtained in the Feynman gauge, and allow the extraction of other useful gauges such as the Landau gauge. We use these results to extract the Landau gauge five-loop anomalous dimensions of the composite operator A^2 as well as the Landau gauge scheme independent gluon, ghost and fermion propagators at five loops.

KEYWORDS: QCD, Renormalization Group

Contents

1	Introduction	1
2	Setup	2
3	Calculation and results	5
4	Applications	10
4.1	Scheme independent propagators	10
4.2	Anomalous Dimension of the operator A^2	12
5	Summary	13

1 Introduction

By explaining both the confinement of hadrons as well as the asymptotic freedom of partons observed in highly energetic collisions, QCD is one of the phenomenologically richest, most predictive and well tested theories of particle and nuclear physics as a whole. Of fundamental importance for making predictions are the anomalous dimensions (ADs) and their associated renormalisation constants (RCs). These RCs are indispensable to arrive at finite predictions for observables, such as cross sections and decay rates which can be measured at present and future hadron and lepton collider experiments, where QCD effects play an important role. But the RCs, in particular in Landau gauge, also make up important ingredients when comparing perturbative with non-perturbative results from Lattice determinations.

The computation of QCD RCs in dimensional regularisation [1, 2] have a prominent history, beginning with the Nobel-prize winning discovery of asymptotic freedom in 1973 by Gross, Wilczek and Politzer at one loop [3, 4]. Since then enormous progress has pushed the current status of the art to five loops [5–15]. At five loops the QCD beta function was first published for the SU(3) gauge group in [16]. This result was confirmed and extended to an arbitrary simple gauge group; the highest power in the number of fermions is already known to all orders in perturbation theory [17], the subleading power in the number of fermions was presented in [18], while the full result was given in [19].

The fermion anomalous mass and field dimensions were similarly first computed for the SU(3) gauge group in [20] and then extended to an arbitrary gauge group in [21, 22]. The Feynman gauge ghost and fermion wave function- and vertex- RCs were presented first for an arbitrary simple gauge group in [23]. While in dimensional regularisation the beta function is entirely gauge independent, the same is not true for the vertex and wave function ADs. The Feynman gauge is often the simplest to work with in practice, nevertheless the

Landau gauge often plays a similarly important role, for instance in comparisons of the momentum dependence of propagators and vertex correlators between Lattice simulations and perturbative calculations [24–32].

An important 1-parameter family of gauges, which includes both the Feynman and the Landau gauge, are the covariant gauges. Until now not even the linear dependence on this gauge parameter of the RCs was known at five loops¹. In this article we address this issue and provide the full gauge parameter dependence in the covariant gauges, for an arbitrary simple gauge group at five loops. This calculation was possible by combining two extremely powerful approaches. Using the global R^* -method [34–38], see also [39] for a review of this technique, the required five-loop self energy and vertex integrals are written in terms of tadpole integrals, whose lines are, apart from one single massive line, all massless. These tadpoles integrals, which can always be factorised into products of four-loop massless propagator integrals times trivial one-loop tadpole integrals, are subsequently evaluated with the FORCER program [40–42]. This approach, which is at least an order of magnitude faster than any of the alternative approaches used so far, such as the local R^* -method recently developed in [43], enabled us to compute the full gauge parameter dependence of the ghost and fermion self energy and ghost-gluon vertex correlator. This has allowed us to reconstruct the full gauge parameter dependence of all QCD RCs at five loops and constitutes the main new result of this work. These results were previously known only at the four-loop level [23, 44]. For the special case of the Landau gauge to all orders in perturbation theory the gluon and ghost ADs are known at leading power in the number of fermions [45] while the quark ADs are known to subleading power [45, 46]; these results provide important cross checks for the parts of the results presented here.

The application of the global R^* -method to this problem was conceptually more challenging than in the case of the SU(3) gauge group [16]. We were able to solve these conceptual problems and in addition refined and simplified the global R^* method by making use of the QCD Ward Identities. We will present the details of this method in a future publication [47].

This paper is organised as follows. In Section 2 we will set up our notations and conventions. Explicit results will be given for the case of the Landau gauge in section 3. General gauge results can be found in an ancillary file. We apply these results to obtain the scheme independent gluon, ghost and fermion propagators as well as the anomalous dimension (AD) of the composite operator A^2 in section 4. We conclude in section 5.

2 Setup

In the covariant gauge the QCD Lagrangian with n_f quark flavors reads:

$$\mathcal{L} = \sum_{f=1}^{n_f} \bar{\psi}^f (i\not{D} - m_f)\psi^f - \frac{1}{4}G_{\mu\nu}^a G_{a\mu\nu} + \partial^\mu \bar{c}_a D_\mu^{ab} c_b - \frac{1}{2\xi_L}(\partial^\mu A_\mu^a)^2. \quad (2.1)$$

¹The linear terms in ξ were published in ref. [33], which appeared on the same day we submitted this paper. In the concluding section we add a few remarks on this point.

Here the gluon field strength is defined by

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g(A_\mu \times A_\nu)^a, \quad (A \times B)^a = f^{abc} A^b B^c, \quad (2.2)$$

the covariant derivatives are given by

$$D_\mu^{ab} = \delta^{ab} \partial_\mu - g f^{abc} A_\mu^c, \quad D_\mu = \partial_\mu - ig A_\mu^a T_R^a. \quad (2.3)$$

and are using the standard Feynman slash notation, e.g. $\not{p} = p_\mu \gamma^\mu$. The quark field ψ_i^f , with mass m_f , transforms in an arbitrary representation R of an arbitrary simple gauge group G . The gauge fields A_μ^a will be taken in the adjoint representation of the gauge group G . $(T_R)_i^a$ are the generators in the representation of the fermions and f^{abc} are the structure constants of the corresponding Lie algebra. Here c^a denote the ghost fields and ξ_L acts as the gauge parameter such that $\xi_L = 0$ corresponds to the Landau gauge, while $\xi_L = 1$ corresponds to the Feynman gauge.

Supplementing eq.(2.1) with all counterterms necessary to remove possible UV divergences from Green functions, one arrives at the *bare* QCD Lagrangian, written in terms of the renormalised quantities² :

$$\begin{aligned} \mathcal{L}_0 = & Z_2 \sum_{f=1}^{n_f} \bar{\psi}^f (i \not{\partial} + g Z_1^{\psi\psi g} Z_2^{-1} \not{A} - Z_m m_f) \psi^f + g Z_1^{ccg} \partial^\mu \bar{c} (A \times c) \\ & - \frac{1}{4} g^2 Z_1^{4g} (A_\mu \times A_\nu)^2 - Z_\xi^{2g} \frac{1}{2\xi_L} (\partial_\nu A_\mu)^2 + Z_3^c \partial_\nu \bar{c} (\partial_\nu c) \\ & - \frac{1}{4} Z_3 (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2} g Z_1^{3g} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) (A_\mu \times A_\nu)^a. \end{aligned} \quad (2.4)$$

The RC Z_ξ^{2g} is expressed through the RCs of the gauge fixing parameter ξ_L as follows

$$\xi_{L,0} = Z_\xi \xi_L, \quad Z_\xi^{2g} = Z_3 / Z_\xi.$$

The wave-function RCs, Z_3 , Z_2 and Z_3^c , arise in the relations between renormalised and bare gluon, quark and ghost fields, as follows

$$A_0^{a\mu} = \sqrt{Z_3} A^{a\mu}, \quad \psi_0^f = \sqrt{Z_2} \psi^f, \quad c_0^a = \sqrt{Z_3^c} c^a. \quad (2.5)$$

Similarly Z_m and Z_g arise in the relations between bare and renormalised charge and fermion mass parameters:

$$g_0 = \mu^\epsilon Z_g g, \quad m_0 = Z_m m. \quad (2.6)$$

The full set of vertex RCs

$$Z_1^V, \quad V \in \{3g, 4g, ccg, \psi\psi g\} \quad (2.7)$$

² For simplicity we set the t' Hooft mass $\mu = 1$ in eq. (2.4) below.

serve to renormalise the three-gluon, four-gluon, ghost-ghost-gluon and quark-quark-gluon vertex functions respectively. Gauge invariance leads to the Slavnov-Taylor identities among the RCs:

$$Z_\xi = Z_3, \quad Z_g = \frac{(Z_1^{4g})^{1/2}}{Z_3} = \frac{Z_1^{3g}}{Z_3^{3/2}} = \frac{Z_1^{cg}}{(Z_3)^{1/2} Z_3^c} = \frac{Z_1^{\psi\psi g}}{(Z_3)^{1/2} Z_2}. \quad (2.8)$$

To arrive at the complete set of RCs it therefore suffices to determine all RCs from the knowledge of only Z_m, Z_2, Z_3, Z_3^c and Z_1^{cg} . In the minimal subtraction scheme (MS), the RCs Z_i are pure poles in the dimensional regularization parameter ϵ

$$Z_i = 1 + \sum_{k=1}^{\infty} \frac{z_i^{(k)}}{\epsilon^k}, \quad (2.9)$$

and the coefficients $z_i^{(k)}$ do not depend explicitly on mass scales [48]. These features greatly simplify the calculation of the RCs, by imposing the finiteness condition on the propagators and vertex functions computed with the Lagrangian of eq. (2.4), that is

$$K_\epsilon \left[Z_3 (1 + \Pi_B(g_0, q^2)) \right] = 0, \quad (2.10)$$

$$K_\epsilon \left[Z_3^c (1 + \tilde{\Pi}_B(g_0, q^2)) \right] = 0, \quad (2.11)$$

$$K_\epsilon \left[Z_2 (1 + \Sigma_B(g_0, q^2)) \right] = 0, \quad (2.12)$$

$$K_\epsilon \left[Z_1^{cg} (1 + \delta\Gamma_B^{cg}(g_0, q^2)) \right] = 0, \quad (2.13)$$

where the operator K_ϵ extracts the pole part of the generic Laurent expansion

$$K_\epsilon \left[\sum_{k=-\infty}^{\infty} c_k \epsilon^k \right] = \sum_{k=-\infty}^{-1} c_k \epsilon^k, \quad (2.14)$$

while $\Pi_B, \tilde{\Pi}_B, \Sigma_B$ denote respectively the bare 1PI self energies of the gluon, of the ghost and of the fermion and $\delta\Gamma_B^{cg}$ is the bare ghost-gluon vertex correction. The constant Z_m is derived in a similar way from the finiteness of the renormalized operator $\bar{\Psi}\Psi$

$$Z_m = \frac{Z_{\bar{\Psi}\Psi}}{Z_2}, \quad K_\epsilon \left[Z_{\bar{\Psi}\Psi} (1 + \delta\Gamma_B^{\bar{\Psi}\Psi}(g_0)) \right] = 0, \quad (2.15)$$

where $\delta\Gamma_B^{\bar{\Psi}\Psi}(g_0)$ is the loop correction of the vertex of the operator $\bar{\Psi}\Psi$. Finally, the RCs give immediately the ADs that govern the renormalisation group equation of the theory: by using the independence of the RCs on mass scales, the evolution equations become

$$\frac{d}{d \log(\mu^2)} Z_i = \left[(-\epsilon + \beta) a \frac{\partial}{\partial a} + \gamma_3 \xi_L \frac{\partial}{\partial \xi_L} \right] Z_i = -\gamma_i Z_i, \quad (2.16)$$

where β and γ_3 are the ADs related to the renormalised coupling constant

$$a(\mu) \equiv \left(\frac{g(\mu)}{4\pi} \right)^2 = \frac{\alpha_s(\mu)}{4\pi} \quad (2.17)$$

and to the gluon renormalization, respectively

$$\beta = -\frac{d \log((Z_g)^2)}{d \log(\mu^2)} \equiv -\frac{d \log(Z_a)}{d \log(\mu^2)} = \frac{d \log(a)}{d \log(\mu^2)} + \epsilon, \quad \gamma_3 = -\frac{d \log(Z_3)}{d \log(\mu^2)}. \quad (2.18)$$

The expansion of eq. (2.16) in powers of ϵ by means of eq. (2.9) gives the relation between the poles of the RCs and ADs of the theory

$$\gamma_i = a \frac{\partial}{\partial a} z_i^{(1)}, \quad a \frac{\partial z_i^{(k+1)}}{\partial a} = \left[\beta a \frac{\partial}{\partial a} + \gamma_3 \xi_L \frac{\partial}{\partial \xi_L} + \gamma_i \right] z_i^{(k)}. \quad (2.19)$$

3 Calculation and results

For the computation of the RCs we employ the global R^* -method. This method, which has already been used in many important calculations in multi-loop QCD, see e.g. [20, 44, 49–51], was also used in the calculation of the 5-loop QCD RCs for the SU(3) gauge group in [16]. Generally the global R^* method relies on a global infrared rearrangement of a correlator by inserting a mass M into one or two propagators attached to one of the external vertices. Subsequently the correlator is expanded around the limit $M \rightarrow \infty$, or equivalently the limit of vanishing external momenta. In this way the 5-loop correlator becomes a “one-mass-tadpole”, which can be evaluated as a product of a 4-loop massless propagator integral times a trivial 1-loop tadpole. This makes it possible to use the FORCER program [40] in order to efficiently perform the reduction to Master integrals.

To extract the correct RCs from “one-mass-tadpole” correlators does however require a subtle derivation of global renormalisation counterterms. While the application of the global R^* method is comparably straightforward for fermion or ghost vertices and self energies, extra complications arise for the gluon propagator. These complications emerge from the fact that the type of the external gluon vertex can be any of the four QCD vertices in eq.(2.7), rather than just a single type in the case of external fermion or ghost vertices. This leads in particular to the phenomenon of operator mixing under renormalisation and further demands the introduction of new (renormalisable) operators into the QCD Lagrangian.

To extend the global R^* -method to a general gauge group required the introduction of yet new 4-gluon operators (whose color structures could not appear in the SU(3) case) for the construction of the global renormalisation counterterms. To obtain the results presented in this article, we have both extended and refined the global R^* method in order to deal with these extra complications. Since the details of the method are rather subtle, we will present them in a future publication [47].

All required Feynman diagrams were generated with QGRAF [52]. Further symbolic manipulations were performed with the computer algebra system FORM [53–56]. The

reduction to Master integrals is performed using the FORCER program [40–42]. Our computation of colour factors relied on the FORM package introduced in [57]. The total computation of all required components took approximately two months on a single computer with 32 cores.

Using the methods described above we have been able to directly compute Z_3^c, Z_1^{ccg} and Z_2 to all powers – and Z_3 to leading power – in the gauge-parameter up to five loops. Use of the Slavnov-Taylor identities then allows us to reconstruct the entire gauge parameter dependence of Z_3 , using the explicitly computed linear dependence as a cross check.

In the following we present analytic results for the 5-loop coefficients of the ADs γ_2, γ_3 and γ_{3c} , each of which have perturbative expansions of the form,

$$\gamma_i = - \sum_{k=0}^{\infty} a^{k+1} (\gamma_i)_k, \quad (3.1)$$

in the Landau gauge ($\xi_L = 0$):

$$(\gamma_2)_0 = 0, \quad (\gamma_2)_1 = -2 C_F n_f T_R - \frac{3}{2} C_F^2 + \frac{25}{4} C_A C_F, \quad (3.2)$$

$$\begin{aligned} (\gamma_2)_2 &= \frac{20}{9} C_F n_f^2 T_R^2 + 3 C_F^2 n_f T_R + \frac{3}{2} C_F^3 - \frac{287}{9} C_A C_F n_f T_R \\ &+ C_A C_F^2 \left[-\frac{143}{4} + 12 \zeta_3 \right] + C_A^2 C_F \left[\frac{9155}{144} - \frac{69}{8} \zeta_3 \right], \end{aligned} \quad (3.3)$$

$$\begin{aligned} (\gamma_2)_3 &= \frac{d_R^{abcd} d_A^{abcd}}{N_R} \left[-67 - \frac{855}{8} \zeta_5 + \frac{1113}{8} \zeta_3 \right] + 128 n_f \frac{d_R^{abcd} d_R^{abcd}}{N_R} \\ &+ \frac{280}{81} C_F n_f^3 T_R^3 + C_F^2 n_f^2 T_R^2 \left[\frac{304}{9} - 32 \zeta_3 \right] + C_F^3 n_f T_R \left[\frac{76}{3} - 64 \zeta_3 \right] \\ &+ C_F^4 \left[-\frac{1027}{8} + 640 \zeta_5 - 400 \zeta_3 \right] + C_A C_F n_f^2 T_R^2 \left[\frac{293}{9} + 32 \zeta_3 \right] \\ &+ C_A C_F^2 n_f T_R \left[-\frac{53}{18} + 24 \zeta_4 + 160 \zeta_5 \right] + C_A C_F^3 \left[\frac{5131}{12} - 1440 \zeta_5 + 848 \zeta_3 \right] \\ &+ C_A^2 C_F n_f T_R \left[-\frac{18371}{54} - \frac{69}{4} \zeta_4 - 80 \zeta_5 - \frac{3}{2} \zeta_3 \right] + C_A^2 C_F^2 \left[-\frac{23885}{36} - 66 \zeta_4 \right. \\ &\left. + 785 \zeta_5 - \frac{421}{2} \zeta_3 \right] + C_A^3 C_F \left[\frac{95261}{162} + \frac{759}{16} \zeta_4 - \frac{4145}{32} \zeta_5 - \frac{6209}{64} \zeta_3 \right], \end{aligned} \quad (3.4)$$

$$\begin{aligned} (\gamma_2)_4 &= C_F \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left[-\frac{1985}{24} + \frac{781753}{192} \zeta_7 - \frac{1458845}{384} \zeta_5 + \frac{135731}{192} \zeta_3 + \frac{3577}{64} \zeta_3^2 \right] \\ &+ T_R n_f \frac{d_R^{abcd} d_A^{abcd}}{N_R} \left[\frac{6200}{9} - \frac{1425}{4} \zeta_6 + \frac{27377}{6} \zeta_7 + \frac{1113}{4} \zeta_4 - \frac{9915}{2} \zeta_5 \right. \\ &\left. - \frac{2468}{3} \zeta_3 + \frac{91}{2} \zeta_3^2 \right] + T_R n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{N_R} \left[-\frac{7360}{9} + 640 \zeta_5 + \frac{704}{3} \zeta_3 \right] \\ &+ C_F n_f^4 T_R^4 \left[\frac{1328}{243} - \frac{256}{27} \zeta_3 \right] + C_F \frac{d_R^{abcd} d_A^{abcd}}{N_R} \left[\frac{113}{6} - \frac{125447}{8} \zeta_7 \right. \\ &\left. + 1015 \zeta_5 + 17554 \zeta_3 - 4884 \zeta_3^2 \right] + C_F n_f \frac{d_R^{abcd} d_R^{abcd}}{N_R} \left[-\frac{5984}{3} - 8680 \zeta_7 \right] \end{aligned}$$

$$\begin{aligned}
& +18080 \zeta_5 - 12096 \zeta_3 + 3648 \zeta_3^2 \Big] + C_F^2 n_f^3 T_R^3 \left[-\frac{2636}{243} - 64 \zeta_4 + \frac{832}{9} \zeta_3 \right] \\
& + C_F^3 n_f^2 T_R^2 \left[-\frac{2497}{27} - 128 \zeta_4 + 320 \zeta_5 + \frac{400}{9} \zeta_3 \right] + C_F^4 n_f T_R \left[\frac{29209}{36} \right. \\
& \left. + \frac{6400}{3} \zeta_6 - 800 \zeta_4 - \frac{46880}{9} \zeta_5 + \frac{22496}{9} \zeta_3 + \frac{1024}{3} \zeta_3^2 \right] + C_F^5 \left[\frac{4977}{8} \right. \\
& \left. - 47628 \zeta_7 + 22600 \zeta_5 + 16000 \zeta_3 + 2496 \zeta_3^2 \right] + C_A \frac{d_R^{abcd} d_A^{abcd}}{N_R} \left[-\frac{173959}{144} \right. \\
& \left. + \frac{15675}{16} \zeta_6 + \frac{3016307}{256} \zeta_7 - \frac{12243}{16} \zeta_4 + \frac{609425}{96} \zeta_5 - \frac{574393}{32} \zeta_3 + \frac{16935}{4} \zeta_3^2 \right] \\
& + C_A n_f \frac{d_R^{abcd} d_R^{abcd}}{N_R} \left[\frac{33464}{9} + \frac{23632}{3} \zeta_7 - \frac{48640}{3} \zeta_5 + 8992 \zeta_3 - 2320 \zeta_3^2 \right] \\
& + C_A C_F n_f^3 T_R^3 \left[-\frac{3566}{243} + 64 \zeta_4 - \frac{1984}{27} \zeta_3 \right] + C_A C_F^2 n_f^2 T_R^2 \left[\frac{101485}{162} \right. \\
& \left. + \frac{1600}{3} \zeta_6 + 176 \zeta_4 - \frac{3712}{3} \zeta_5 - \frac{6160}{9} \zeta_3 + \frac{256}{3} \zeta_3^2 \right] + C_A C_F^3 n_f T_R \left[-\frac{167263}{108} \right. \\
& \left. - 4800 \zeta_6 - 13944 \zeta_7 + 2120 \zeta_4 + \frac{58720}{3} \zeta_5 - \frac{25804}{9} \zeta_3 - 64 \zeta_3^2 \right] \\
& + C_A C_F^4 \left[-\frac{835739}{144} - \frac{17600}{3} \zeta_6 + 123977 \zeta_7 + 2200 \zeta_4 - \frac{248960}{9} \zeta_5 - \frac{530884}{9} \zeta_3 \right. \\
& \left. - \frac{24632}{3} \zeta_3^2 \right] + C_A^2 C_F n_f^2 T_R^2 \left[\frac{120037}{162} - \frac{800}{3} \zeta_6 - \frac{441}{2} \zeta_7 - 179 \zeta_4 + \frac{3584}{9} \zeta_5 \right. \\
& \left. + \frac{3140}{3} \zeta_3 - \frac{128}{3} \zeta_3^2 \right] + C_A^2 C_F^2 n_f T_R \left[\frac{717409}{432} + 1150 \zeta_6 + \frac{42203}{3} \zeta_7 - \frac{1411}{4} \zeta_4 \right. \\
& \left. - \frac{95792}{9} \zeta_5 - \frac{14287}{24} \zeta_3 - 1214 \zeta_3^2 \right] + C_A^2 C_F^3 \left[\frac{827215}{72} + 13200 \zeta_6 - \frac{1789067}{16} \zeta_7 \right. \\
& \left. - 4664 \zeta_4 - \frac{188795}{12} \zeta_5 + \frac{1365227}{18} \zeta_3 + \frac{18097}{2} \zeta_3^2 \right] + C_A^3 C_F n_f T_R \left[-\frac{31919039}{7776} \right. \\
& \left. + \frac{4825}{16} \zeta_6 - \frac{440419}{144} \zeta_7 - \frac{8705}{32} \zeta_4 + \frac{28721}{18} \zeta_5 - \frac{144377}{864} \zeta_3 + \frac{4067}{6} \zeta_3^2 \right] \\
& + C_A^3 C_F^2 \left[-\frac{42214139}{3888} - \frac{43175}{6} \zeta_6 + \frac{9074513}{192} \zeta_7 + \frac{3815}{4} \zeta_4 + \frac{5957573}{288} \zeta_5 \right. \\
& \left. - \frac{5503507}{144} \zeta_3 - \frac{78041}{24} \zeta_3^2 \right] + C_A^4 C_F \left[\frac{368712343}{62208} + \frac{227975}{192} \zeta_6 \right. \\
& \left. - \frac{312820991}{36864} \zeta_7 + \frac{87067}{128} \zeta_4 - \frac{16237513}{3072} \zeta_5 + \frac{46196783}{6912} \zeta_3 + \frac{23555}{128} \zeta_3^2 \right], \quad (3.5)
\end{aligned}$$

$$(\gamma_3)_0 = \frac{4}{3} n_f T_R - \frac{13}{6} C_A, \quad (3.6)$$

$$(\gamma_3)_1 = 4 C_F n_f T_R + 5 C_A n_f T_R - \frac{59}{8} C_A^2, \quad (3.7)$$

$$\begin{aligned}
(\gamma_3)_2 &= -\frac{44}{9} C_F n_f^2 T_R^2 - 2 C_F^2 n_f T_R - \frac{76}{9} C_A n_f^2 T_R^2 + C_A C_F n_f T_R \left[\frac{5}{18} \right. \\
&\left. + 24 \zeta_3 \right] + C_A^2 n_f T_R \left[\frac{911}{18} - 18 \zeta_3 \right] + C_A^3 \left[-\frac{9965}{288} + \frac{9}{16} \zeta_3 \right], \quad (3.8)
\end{aligned}$$

$$\begin{aligned}
(\gamma_3)_3 &= \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left[\frac{659}{144} - \frac{10185}{64} \zeta_5 - \frac{989}{12} \zeta_3 \right] + n_f \frac{d_R^{abcd} d_A^{abcd}}{N_A} \left[-\frac{512}{9} + 120 \zeta_5 \right. \\
&+ \left. \frac{1376}{3} \zeta_3 \right] + n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{N_A} \left[\frac{704}{9} - \frac{512}{3} \zeta_3 \right] + C_F n_f^3 T_R^3 \left[-\frac{1232}{243} \right] \\
&+ C_F^2 n_f^2 T_R^2 \left[-\frac{1352}{27} + \frac{704}{9} \zeta_3 \right] + C_F^3 n_f T_R \left[-46 \right] + C_A n_f^3 T_R^3 \left[-\frac{1420}{243} + \frac{64}{9} \zeta_3 \right] \\
&+ C_A C_F n_f^2 T_R^2 \left[-\frac{15082}{243} + 48 \zeta_4 - \frac{1168}{9} \zeta_3 \right] + C_A C_F^2 n_f T_R \left[\frac{10847}{54} - 240 \zeta_5 \right. \\
&+ \left. \frac{980}{9} \zeta_3 \right] + C_A^2 n_f^2 T_R^2 \left[-\frac{6674}{81} - 36 \zeta_4 + \frac{316}{9} \zeta_3 \right] + C_A^2 C_F n_f T_R \left[-\frac{146831}{972} \right. \\
&- 132 \zeta_4 + 120 \zeta_5 + \left. \frac{2240}{9} \zeta_3 \right] + C_A^3 n_f T_R \left[\frac{318907}{864} + \frac{801}{8} \zeta_4 + 110 \zeta_5 - \frac{3563}{12} \zeta_3 \right] \\
&+ C_A^4 \left[-\frac{10655437}{62208} - \frac{99}{32} \zeta_4 - \frac{47665}{512} \zeta_5 + \frac{50669}{768} \zeta_3 \right], \tag{3.9}
\end{aligned}$$

$$\begin{aligned}
(\gamma_3)_4 &= n_f T_R \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left[-\frac{3043}{36} - \frac{16975}{32} \zeta_6 + \frac{5285}{4} \zeta_7 - \frac{95}{2} \zeta_4 - \frac{232535}{288} \zeta_5 + \frac{505937}{288} \zeta_3 \right. \\
&- \left. \frac{2017}{16} \zeta_3^2 \right] + n_f^2 T_R \frac{d_R^{abcd} d_A^{abcd}}{N_A} \left[\frac{3680}{9} + 400 \zeta_6 + 640 \zeta_4 - \frac{4720}{9} \zeta_5 - \frac{37664}{9} \zeta_3 \right. \\
&+ \left. 96 \zeta_3^2 \right] + n_f^3 T_R \frac{d_R^{abcd} d_R^{abcd}}{N_A} \left[-\frac{3520}{9} - 256 \zeta_4 - \frac{1280}{3} \zeta_5 + \frac{2624}{3} \zeta_3 \right] \\
&+ C_F n_f \frac{d_R^{abcd} d_A^{abcd}}{N_A} \left[\frac{944}{3} - 476 \zeta_7 - \frac{3580}{3} \zeta_5 - \frac{1544}{3} \zeta_3 - 288 \zeta_3^2 \right] \\
&+ C_F n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{N_A} \left[-\frac{4160}{3} + \frac{12800}{3} \zeta_5 - \frac{5120}{3} \zeta_3 \right] + C_F n_f^4 T_R^4 \left[\frac{856}{243} + \frac{128}{27} \zeta_3 \right] \\
&+ C_F^2 n_f^3 T_R^3 \left[\frac{9922}{81} + \frac{352}{3} \zeta_4 - \frac{7616}{27} \zeta_3 \right] + C_F^3 n_f^2 T_R^2 \left[\frac{5018}{9} - \frac{4640}{3} \zeta_5 + \frac{2144}{3} \zeta_3 \right] \\
&+ C_F^4 n_f T_R \left[\frac{4157}{6} + 128 \zeta_3 \right] + C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left[\frac{204521}{1152} + \frac{186725}{128} \zeta_6 \right. \\
&+ \left. \frac{6235831}{4096} \zeta_7 + \frac{2101}{8} \zeta_4 + \frac{140675}{256} \zeta_5 - \frac{5513387}{1536} \zeta_3 - \frac{16887}{512} \zeta_3^2 \right] \\
&+ C_A n_f \frac{d_R^{abcd} d_A^{abcd}}{N_A} \left[-\frac{9752}{9} - 1100 \zeta_6 - 784 \zeta_7 - 2072 \zeta_4 - \frac{32885}{9} \zeta_5 + \frac{93551}{9} \zeta_3 \right. \\
&+ \left. 893 \zeta_3^2 \right] + C_A n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{N_A} \left[\frac{6328}{3} + 800 \zeta_4 - \frac{6560}{9} \zeta_5 - \frac{37720}{9} \zeta_3 + 256 \zeta_3^2 \right] \\
&+ C_A n_f^4 T_R^4 \left[-\frac{2476}{243} + \frac{128}{9} \zeta_4 \right] + C_A C_F n_f^3 T_R^3 \left[\frac{16973}{243} - \frac{704}{3} \zeta_4 + \frac{256}{3} \zeta_5 \right. \\
&+ \left. \frac{7072}{27} \zeta_3 \right] + C_A C_F^2 n_f^2 T_R^2 \left[-\frac{135571}{108} - 800 \zeta_6 + \frac{44}{3} \zeta_4 + 2800 \zeta_5 - \frac{16900}{27} \zeta_3 \right. \\
&+ \left. 448 \zeta_3^2 \right] + C_A C_F^3 n_f T_R \left[-3759 + 3360 \zeta_7 - \frac{1340}{3} \zeta_5 - \frac{6164}{3} \zeta_3 \right] \\
&+ C_A^2 n_f^3 T_R^3 \left[\frac{61223}{486} + \frac{392}{9} \zeta_4 - \frac{2176}{9} \zeta_5 + \frac{5792}{27} \zeta_3 \right] + C_A^2 C_F n_f^2 T_R^2 \left[\frac{92417}{486} \right. \\
&+ 400 \zeta_6 + \frac{3800}{3} \zeta_4 - \frac{5008}{3} \zeta_5 - \frac{20596}{27} \zeta_3 - 928 \zeta_3^2 \left. \right] + C_A^2 C_F^2 n_f T_R \left[\frac{1088107}{162} \right. \\
&+ 2200 \zeta_6 - 3360 \zeta_7 - \frac{1892}{3} \zeta_4 - 5650 \zeta_5 + \frac{183419}{54} \zeta_3 + 880 \zeta_3^2 \left. \right]
\end{aligned}$$

$$\begin{aligned}
& +C_A^3 n_f^2 T_R^2 \left[-\frac{575909}{324} + \frac{1100}{3} \zeta_6 - \frac{1929}{2} \zeta_4 + \frac{2881}{4} \zeta_5 + \frac{39451}{216} \zeta_3 + \frac{1328}{3} \zeta_3^2 \right] \\
& +C_A^3 C_F n_f T_R \left[-\frac{352669}{72} - 1100 \zeta_6 + \frac{16681}{12} \zeta_7 - \frac{41143}{24} \zeta_4 + \frac{15056}{3} \zeta_5 + \frac{16665}{16} \zeta_3 \right. \\
& \left. - 829 \zeta_3^2 \right] + C_A^4 n_f T_R \left[\frac{262835267}{62208} - \frac{337575}{256} \zeta_6 - \frac{224581}{96} \zeta_7 + \frac{2347015}{1152} \zeta_4 \right. \\
& \left. + \frac{5841025}{2304} \zeta_5 - \frac{16523531}{6912} \zeta_3 - \frac{887}{384} \zeta_3^2 \right] + C_A^5 \left[-\frac{630869659}{497664} + \frac{2621575}{3072} \zeta_6 \right. \\
& \left. + \frac{194793683}{98304} \zeta_7 - \frac{1697837}{4608} \zeta_4 - \frac{124549507}{36864} \zeta_5 + \frac{77047459}{110592} \zeta_3 + \frac{567483}{4096} \zeta_3^2 \right], \quad (3.10)
\end{aligned}$$

$$(\gamma_3^c)_0 = -\frac{3}{4} C_A, \quad (\gamma_3^c)_1 = \frac{5}{6} C_A n_f T_R - \frac{95}{48} C_A^2, \quad (3.11)$$

$$\begin{aligned}
(\gamma_3^c)_2 &= \frac{35}{27} C_A n_f^2 T_R^2 + C_A C_F n_f T_R \left[\frac{45}{4} - 12 \zeta_3 \right] \\
&+ C_A^2 n_f T_R \left[\frac{97}{108} + 9 \zeta_3 \right] + C_A^3 \left[-\frac{15817}{1728} - \frac{9}{32} \zeta_3 \right], \quad (3.12)
\end{aligned}$$

$$\begin{aligned}
(\gamma_3^c)_3 &= \frac{d^{abcd} d_A^{abcd}}{N_A} \left[\frac{69}{32} + \frac{10185}{128} \zeta_5 - \frac{609}{8} \zeta_3 \right] + n_f \frac{d_R^{abcd} d_A^{abcd}}{N_A} \left[-60 \zeta_5 + 48 \zeta_3 \right] \\
&+ C_A n_f^3 T_R^3 \left[\frac{166}{81} - \frac{32}{9} \zeta_3 \right] + C_A C_F n_f^2 T_R^2 \left[-\frac{115}{27} - 24 \zeta_4 + 40 \zeta_3 \right] \quad (3.13) \\
&+ C_A C_F^2 n_f T_R \left[-\frac{271}{12} + 120 \zeta_5 - 74 \zeta_3 \right] + C_A^2 n_f^2 T_R^2 \left[-\frac{628}{81} + 18 \zeta_4 - 30 \zeta_3 \right] \\
&+ C_A^2 C_F n_f T_R \left[\frac{13171}{216} + 66 \zeta_4 - 60 \zeta_5 - 88 \zeta_3 \right] + C_A^3 n_f T_R \left[\frac{295855}{5184} \right. \\
&\left. - \frac{801}{16} \zeta_4 - 55 \zeta_5 + \frac{3019}{24} \zeta_3 \right] + C_A^4 \left[-\frac{319561}{4608} + \frac{99}{64} \zeta_4 + \frac{47665}{1024} \zeta_5 - \frac{140743}{4608} \zeta_3 \right],
\end{aligned}$$

$$\begin{aligned}
(\gamma_3^c)_4 &= n_f T_R \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left[-\frac{191}{24} + \frac{16975}{64} \zeta_6 - \frac{5285}{8} \zeta_7 - \frac{609}{4} \zeta_4 + \frac{11615}{64} \zeta_5 + \frac{52709}{192} \zeta_3 \right. \\
&+ \left. \frac{2017}{32} \zeta_3^2 \right] + n_f^2 T_R \frac{d_R^{abcd} d_A^{abcd}}{N_A} \left[-200 \zeta_6 + 96 \zeta_4 + \frac{1000}{3} \zeta_5 - \frac{416}{3} \zeta_3 - 48 \zeta_3^2 \right] \\
&+ C_F n_f \frac{d_R^{abcd} d_A^{abcd}}{N_A} \left[\frac{8}{3} + 238 \zeta_7 - 470 \zeta_5 + 44 \zeta_3 + 144 \zeta_3^2 \right] \\
&+ C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left[-\frac{7145}{2304} - \frac{186725}{256} \zeta_6 - \frac{6235831}{8192} \zeta_7 + \frac{5643}{16} \zeta_4 + \frac{3520375}{1536} \zeta_5 \right. \\
&- \left. \frac{4069205}{3072} \zeta_3 + \frac{16887}{1024} \zeta_3^2 \right] + C_A n_f \frac{d_R^{abcd} d_A^{abcd}}{N_A} \left[-\frac{260}{3} + 550 \zeta_6 + 392 \zeta_7 - 108 \zeta_4 \right. \\
&- \left. \frac{11545}{6} \zeta_5 + \frac{11395}{6} \zeta_3 - \frac{893}{2} \zeta_3^2 \right] + C_A n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{N_A} \left[\frac{428}{3} - 48 \zeta_4 + 240 \zeta_5 \right. \\
&- \left. \frac{436}{3} \zeta_3 - 128 \zeta_3^2 \right] + C_A n_f^4 T_R^4 \left[\frac{260}{81} - \frac{64}{9} \zeta_4 + \frac{320}{81} \zeta_3 \right] \\
&+ C_A C_F n_f^3 T_R^3 \left[-\frac{14765}{486} + 80 \zeta_4 - \frac{128}{3} \zeta_5 - \frac{232}{9} \zeta_3 \right] + C_A C_F^2 n_f^2 T_R^2 \left[-\frac{53927}{216} \right.
\end{aligned}$$

$$\begin{aligned}
& +400\zeta_6 - 198\zeta_4 - \frac{2000}{3}\zeta_5 + \frac{7588}{9}\zeta_3 - 224\zeta_3^2 \Big] + C_A C_F^3 n_f T_R \left[21 - 1680\zeta_7 \right. \\
& + 1470\zeta_5 + 78\zeta_3 \Big] + C_A^2 n_f^3 T_R^3 \left[-\frac{23837}{972} - \frac{364}{9}\zeta_4 + \frac{928}{9}\zeta_5 - \frac{3820}{81}\zeta_3 \right] \\
& + C_A^2 C_F n_f^2 T_R^2 \left[-\frac{27380}{243} - 200\zeta_6 - 476\zeta_4 + 568\zeta_5 - \frac{976}{9}\zeta_3 + 464\zeta_3^2 \right] \\
& + C_A^2 C_F^2 n_f T_R \left[\frac{3119}{108} - 1100\zeta_6 + 1680\zeta_7 + 396\zeta_4 + \frac{2065}{3}\zeta_5 - \frac{27419}{36}\zeta_3 - 440\zeta_3^2 \right] \\
& + C_A^3 n_f^2 T_R^2 \left[\frac{41593}{1944} - \frac{550}{3}\zeta_6 + \frac{5995}{12}\zeta_4 + \frac{157}{24}\zeta_5 - \frac{187019}{432}\zeta_3 - \frac{664}{3}\zeta_3^2 \right] \\
& + C_A^3 C_F n_f T_R \left[\frac{345017}{972} + 550\zeta_6 - \frac{16681}{24}\zeta_7 + \frac{11309}{16}\zeta_4 - 1056\zeta_5 - \frac{19001}{32}\zeta_3 \right. \\
& + \left. \frac{829}{2}\zeta_3^2 \right] + C_A^4 n_f T_R \left[\frac{60298109}{124416} + \frac{337575}{512}\zeta_6 + \frac{224581}{192}\zeta_7 - \frac{2123143}{2304}\zeta_4 \right. \\
& - \frac{8674945}{4608}\zeta_5 + \frac{46881313}{41472}\zeta_3 + \frac{887}{768}\zeta_3^2 \Big] + C_A^5 \left[-\frac{47894269}{110592} - \frac{2621575}{6144}\zeta_6 \right. \\
& - \frac{194793683}{196608}\zeta_7 + \frac{1604909}{9216}\zeta_4 + \frac{128829827}{73728}\zeta_5 - \frac{224465897}{663552}\zeta_3 - \frac{567483}{8192}\zeta_3^2 \Big]. \quad (3.14)
\end{aligned}$$

These results are valid for any gauge group which is a simple Lie group and are expressed in terms of Casimirs. The notation for these Casimirs is explained in detail in [57]. For fermions transforming in the fundamental representation and the standard normalisation of the $SU(N)$ generators, the Casimirs evaluate to

$$\begin{aligned}
N_R &= N, \quad N_A = N^2 - 1, \quad T_R = \frac{1}{2}, \\
C_A &= N, \quad C_F = \frac{N_A}{2N} = \frac{N^2 - 1}{2N}, \quad \frac{d_A^{abcd} d_A^{abcd}}{N_A} = \frac{N^2(N^2 + 36)}{24}, \quad (3.15) \\
\frac{d_R^{abcd} d_A^{abcd}}{N_A} &= \frac{N(N^2 + 6)}{48}, \quad \frac{d_R^{abcd} d_R^{abcd}}{N_A} = \frac{N^4 - 6N^2 + 18}{96N^2}.
\end{aligned}$$

We have checked that in the Landau gauge the leading terms in n_f for quark, gluon, and ghost ADs agree with the results in [45] while the subleading n_f -term of the quark ADs are in agreement with the results in [46]. We further confirm that in the Landau gauge we have $Z_1^{\text{crg}} = 1$ [58, 59], constituting yet another non-trivial cross check of our calculation. Results of all the ADs and RCs valid for arbitrary gauges can be found in an ancillary file in computer-readable format with the `arxiv` submission of this paper.

4 Applications

4.1 Scheme independent propagators

In this section we apply our results to compute the $\overline{\text{MS}}$ -scheme independent gluon, ghost and fermion propagators to $N^4\text{LO}$ in perturbative QCD. We thereby extend the $N^3\text{LO}$ results which were, for gluon and ghost propagators, presented in [44] to $N^4\text{LO}$. Given the $\overline{\text{MS}}$ -renormalised propagator $G(\mu, a(\mu))$ with renormalisation group equation,

$$\frac{dG(\mu, a)}{d \log \mu^2} = \gamma(a)G(\mu, a), \quad (4.1)$$

one defines the scale invariant quantity $\hat{G}(\mu, a)$ such that

$$\hat{G}(\mu, a) = G(\mu, a)/f(a), \quad \frac{d\hat{G}(\mu, a)}{d \log \mu^2} = 0. \quad (4.2)$$

It follows that the function $f(a)$ satisfies the differential equation,

$$\gamma(a) f(a) - \beta(a) a \frac{d}{da} f(a) = 0, \quad (4.3)$$

which is formally solved by

$$f(a) = \exp \left(\int \frac{da}{a} \frac{\gamma(a)}{\beta(a)} \right). \quad (4.4)$$

Up to an overall exponential factor the function $f(a)$ can be expanded perturbatively as

$$\begin{aligned} f(a) &= a^{-\gamma_0/\beta_0} \left[1 + \bar{\gamma}_1 a + (\bar{\gamma}_2 - \bar{\gamma}_1 \bar{\beta}_1 + \bar{\gamma}_1^2) \frac{a^2}{2} \right. \\ &+ \left(2\bar{\beta}_1^2 \bar{\gamma}_1 - 3\bar{\beta}_1 \bar{\gamma}_1^2 + \bar{\gamma}_1^3 - 2\bar{\beta}_1 \bar{\gamma}_2 - 2\bar{\beta}_2 \bar{\gamma}_1 + 3\bar{\gamma}_1 \bar{\gamma}_2 + 2\bar{\gamma}_3 \right) \frac{a^3}{3!} \\ &+ \left(-6\bar{\beta}_1^3 \bar{\gamma}_1 + 11\bar{\beta}_1^2 \bar{\gamma}_1^2 - 6\bar{\beta}_1 \bar{\gamma}_1^3 + \bar{\gamma}_1^4 + 6\bar{\beta}_1^2 \bar{\gamma}_2 + 12\bar{\beta}_1 \bar{\beta}_2 \bar{\gamma}_1 - 14\bar{\beta}_1 \bar{\gamma}_1 \bar{\gamma}_2 \right. \\ &\quad \left. - 8\bar{\beta}_2 \bar{\gamma}_1^2 + 6\bar{\gamma}_1^2 \bar{\gamma}_2 - 6\bar{\beta}_1 \bar{\gamma}_3 - 6\bar{\beta}_2 \bar{\gamma}_2 - 6\bar{\beta}_3 \bar{\gamma}_1 + 8\bar{\gamma}_1 \bar{\gamma}_3 + 3\bar{\gamma}_2^2 + 6\bar{\gamma}_4 \right) \frac{a^4}{4!} + \mathcal{O}(a^5) \Big] \end{aligned} \quad (4.5)$$

where $\bar{\beta}_i = \beta_i/\beta_0$ and $\bar{\gamma}_i = (\gamma_i - \bar{\beta}_i \gamma_0)/\beta_0$.

We proceed by defining the scale invariant gluon, ghost and massless quark propagators:

$$\begin{aligned} \hat{D}^{-1}(q^2) &= f(a)(1 + \Pi(q^2)), & \hat{D}_{\mu\nu}^{ab} &= \frac{\delta^{ab}}{-q^2} \left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right] \hat{D}(q^2), \\ \hat{\Delta}^{-1}(q^2) &= f(a)(1 + \tilde{\Pi}(q^2)), & \hat{\Delta}^{ab} &= \frac{\delta^{ab}}{-q^2} \hat{\Delta}(q^2), \\ \hat{S}^{-1}(q^2) &= f(a)(1 + \Sigma(q^2)), & \hat{S}^{ij} &= \frac{\delta^{ij}}{-q^2} \hat{S}(q^2). \end{aligned} \quad (4.6)$$

where the scalar functions $\hat{D}(q^2)$, $\hat{\Delta}(q^2)$ and $\hat{S}(q^2)$ are related to the 1PI self energies. We provide the corresponding numeric values for the SU(3) gauge group by replacing $\gamma = \gamma_3, \gamma_3^c, \gamma_2$ respectively in eq. (4.4) and using the four-loop self energy published in [60]. The gluon propagator at different number of quark flavours is

$$\begin{aligned} a^{\frac{13}{22}} \hat{D}^{-1}(q^2)|_{n_F=0} &= 1 - 2.15952 a_s - 11.6762 a_s^2 - 88.8523 a_s^3 - 987.996 a_s^4, \\ a^{\frac{1}{2}} \hat{D}^{-1}(q^2)|_{n_F=3} &= 1 - 1.29514 a_s - 5.31783 a_s^2 - 34.0328 a_s^3 - 287.942 a_s^4, \\ a^{\frac{19}{46}} \hat{D}^{-1}(q^2)|_{n_F=5} &= 1 - 0.617944 a_s - 1.13804 a_s^2 - 7.32128 a_s^3 - 25.9219 a_s^4, \\ a^{\frac{5}{14}} \hat{D}^{-1}(q^2)|_{n_F=6} &= 1 - 0.214498 a_s + 1.03846 a_s^2 + 3.44460 a_s^3 + 51.6541 a_s^4, \end{aligned} \quad (4.7)$$

where $a_s = \frac{g^2}{4\pi^2} = 4a$ and we set for convenience $\mu^2 = -q^2$. Likewise, we get the ghost propagator

$$\begin{aligned}
a^{\frac{9}{44}} \hat{\Delta}^{-1}(q^2)|_{n_F=0} &= 1 - 0.680656 a_s - 4.52931 a_s^2 - 38.7849 a_s^3 - 466.453 a_s^4, \\
a^{\frac{1}{4}} \hat{\Delta}^{-1}(q^2)|_{n_F=3} &= 1 - 0.696181 a_s - 3.28694 a_s^2 - 21.2361 a_s^3 - 195.589 a_s^4, \\
a^{\frac{27}{92}} \hat{\Delta}^{-1}(q^2)|_{n_F=5} &= 1 - 0.757000 a_s - 2.41477 a_s^2 - 10.7375 a_s^3 - 70.6130 a_s^4, \\
a^{\frac{9}{28}} \hat{\Delta}^{-1}(q^2)|_{n_F=6} &= 1 - 0.819834 a_s - 1.97443 a_s^2 - 5.81361 a_s^3 - 23.5339 a_s^4, \quad (4.8)
\end{aligned}$$

and the fermion propagator

$$\begin{aligned}
\hat{S}^{-1}(q^2)|_{n_F=0} &= 1 + 0.507576 a_s + 2.63292 a_s^2 + 27.0067 a_s^3 + 330.042 a_s^4, \\
\hat{S}^{-1}(q^2)|_{n_F=3} &= 1 + 0.509259 a_s + 2.05204 a_s^2 + 14.8910 a_s^3 + 138.286 a_s^4, \\
\hat{S}^{-1}(q^2)|_{n_F=5} &= 1 + 0.510870 a_s + 1.65428 a_s^2 + 7.77199 a_s^3 + 51.1008 a_s^4, \\
\hat{S}^{-1}(q^2)|_{n_F=6} &= 1 + 0.511905 a_s + 1.45060 a_s^2 + 4.49450 a_s^3 + 18.7354 a_s^4. \quad (4.9)
\end{aligned}$$

The perturbative convergence of all three scaleless propagators appears within reason for low values of n_f and shows improvement as n_f is increased. This feature will be lost as n_f becomes closer to the value $n_f \sim 16.5$ where a singularity in $\frac{1}{\beta_0}$ forces a consequent loss of perturbativity in that region.

4.2 Anomalous Dimension of the operator A^2

In Landau gauge the AD of the composite operator A^2 is related to the beta function and the AD of the gluon wave function [61–64]:

$$-2\gamma_A|_{\xi_L=0} = \beta - \gamma_3. \quad (4.10)$$

This result was given at the 4-loop level in [44]. Here, specialising our result to QCD with gauge group SU(3), we extend it to 5 loop level:

$$\begin{aligned}
\gamma_A|_{\xi_L=0} &= a \left(-\frac{2}{3} n_f + \frac{35}{4} \right) + a^2 \left(\frac{1347}{16} - \frac{137}{12} n_f \right) \\
&+ a^3 \left(\frac{75607}{64} - \frac{243}{32} \zeta_3 + \left[\frac{33}{2} \zeta_3 - \frac{18221}{72} \right] n_f + \frac{755}{108} n_f^2 \right) \\
&+ a^4 \left(\frac{40905}{8} \zeta_5 - \frac{99639}{512} \zeta_3 + \frac{8019}{64} \zeta_4 + \frac{29764511}{1536} \right. \\
&\quad \left. + \left[-\frac{3355}{4} \zeta_5 - \frac{8955}{32} \zeta_4 + \frac{335585}{432} \zeta_3 - \frac{57858155}{10368} \right] n_f \right. \\
&\quad \left. + \left[\frac{8489}{162} \zeta_3 + \frac{33}{2} \zeta_4 + \frac{46549}{162} \right] n_f^2 + \left[-\frac{4}{3} \zeta_3 + \frac{6613}{2916} \right] n_f^3 \right) \\
&+ a^5 \left(\frac{1656617009}{4096} + \frac{662250297}{4096} \zeta_3 + \frac{16452855}{1024} \zeta_4 + \frac{2068873299}{8192} \zeta_5 \right. \\
&\quad - \frac{2249775}{16} \zeta_6 - \frac{9151899939}{32768} \zeta_7 - \frac{65529567}{4096} \zeta_3^2 + \left[\frac{440809}{128} \zeta_3^2 - \frac{66794279}{1728} \zeta_5 \right. \\
&\quad \left. + \frac{3662827}{96} \zeta_7 - \frac{26669221}{1536} \zeta_4 + \frac{63175}{2} \zeta_6 - \frac{929208143}{41472} \zeta_3 - \frac{36593462075}{248832} \right] n_f \\
&\quad \left. + \left[-\frac{16775}{12} \zeta_6 - \frac{2659}{6} \zeta_3^2 + \frac{33243821}{5184} \zeta_3 + \frac{78931}{144} \zeta_4 - \frac{9486899}{2592} \zeta_5 \right] \right) \quad (4.11)
\end{aligned}$$

$$\begin{aligned}
& + \frac{99923027}{7776} \Big] n_f^2 + \left[\frac{1312}{9} \zeta_5 + \frac{2939}{54} \zeta_4 - \frac{129493}{486} \zeta_3 - \frac{3520195}{23328} \right] n_f^3 \\
& + \left[-\frac{4}{3} \zeta_4 - \frac{92}{81} \zeta_3 + \frac{740}{729} \right] n_f^4 \Big) + \mathcal{O}(a^6)
\end{aligned} \tag{4.12}$$

5 Summary

In this work we complete the renormalisation of non-abelian gauge theory with n_f fermions in an arbitrary representation to five-loop order for a generic covariant gauge and an arbitrary simple gauge group. In particular we have extended the results of [16, 18, 20–23, 43], which were given in Feynman gauge only, by obtaining the complete dependence on the gauge-fixing parameter. All results are included in the `arXiv` submission of this paper as ancillary file in computer-readable format. Furthermore we applied our results to compute the five-loop Landau gauge scale-independent gluon, ghost and fermion propagators as well as the Landau gauge AD of the composite operator A^2 .

Remark: on the day on which this article was submitted, ref. [33] appeared where all the anomalous dimensions are computed in expansion around the Feynman gauge $\xi = 0$, including the linear terms. Compared to this work, ref. [33] follows an entirely different approach, involving the calculation of completely massive tadpoles at five loops. We verified that our results are in agreement with those reported in ref. [33], which is an important independent check on the ADs.

Remark 2: all scheme independent propagators considered in subsection 4.1 depend on only one even zeta, namely ζ_4 , which occurs *only* at order a^4 . All terms which contain ζ_6 in the 4-loop propagators and in the corresponding 5-loop anomalous dimensions neatly cancel each other in the scheme independent combinations considered in subsection 4.1. Analytic results for these scheme independent propagators in Landau gauge have been added in an ancillary file in computer-readable format with the `arXiv` submission of this paper. Even more, after transition to the so-called C-scheme [65] the results are fully free from any even zetas. The same is valid if one constructs scheme invariant versions of the vertex functions computed in [60] (again taken in the Landau gauge). The phenomenon was first discovered in [66] on examples of the scalar and gluon correlators that enter the hadronic decays of the Higgs boson. Many more examples of absence of π^2 terms in so-called physical anomalous dimensions in DIS have been very recently discussed in [67].

Note added: by the time this paper was published, we completed the calculation of the quark-gluon vertex to five loops with all the powers of ξ . This result was used to recompute the renormalisation constant $Z_1^{\psi\psi g}$, which was originally derived from Z_g , Z_3 and Z_2 by applying the Slavnov-Taylor identities in eq.(2.8). The results of the two procedures agree with each other, thus providing another check on our calculation.

Acknowledgements

We would like to thank T. Ueda and B. Ruijl for help with FORM and the FORCER package. We thank A. Vogt for comparing with us the result of an independent computation of the

linear gauge dependence of γ_2 which constitutes another cross check for our result. We are grateful for the help of J. Gracey for providing cross-checks for parts of the results presented here. This work is supported by the ERC Advanced Grant no. 320651, “HEPGAME”. The work by K. G. Chetyrkin was supported by the Deutsche Forschungsgemeinschaft through CH1479/1-1 and by the German Federal Ministry for Education and Research BMBF through Grant No. 05H15GUCC1.

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