Wigner functions for the pair angle and orbital angular momentum: Possible applications in quantum information theories

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The framework of Wigner functions for the canonical pair angle and orbital angular momentum, derived and analyzed in 2 recent papers [H. A. Kastrup, Phys. Rev. A **94**, 062113(2016) and Phys. Rev. A **95**, 052111(2017)], is applied to elementary concepts of quantum information like qubits and 2-qubits, e.g., entangled EPR/Bell states etc.. Properties of the associated Wigner functions are discussed and illustrated. The results may be useful for quantum information experiments with orbital angular momenta of light beams or electron beams.

In two recent papers [1, 2] basic properties of Wigner functions on cylindical phase spaces $\mathbb{S}^1 \times \mathbb{R}$ (angle and orbital angular momentum, denoted by "A-OAM" in the following) were derived and discussed. The possible usefulness of that concept has, of course, to be demonstrated by its applications to special systems. A few simple typical example were discussed in Ch. IV C of Ref. [1]. The present paper suggests possible applications to such elementary concepts as "qubits" and "2-qubits" of quantum information, see, e.g., Refs. [3, 4].

The quantized canonical system of the pair *angle and orbital angular momentum* is of special theoretical interest for *quantum information* because it provides as a basic framework an infinite dimensional Hilbert space $L^2(\mathbb{S}^1, d\varphi/2\pi)$, with orthonormal basis

$$e_m(\varphi) = e^{im\varphi}, \;\; m \in \mathbb{Z}, \tag{1}$$

scalar product

$$(\psi_2, \psi_1) = \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \psi_2^*(\varphi)\psi_1(\varphi), \;\; (e_m, e_n) = \delta_{mn}; \tag{2}$$

and expansions

$$\psi(\varphi) = \sum_{m\in\mathbb{Z}} c_m \, e_m(\varphi), \;\; c_m = (e_m, \psi). \tag{3}$$

One of the advantages of A-OAM systems for quantum information theories is that one can select finite dimensional subspaces of any dimension d: $d = 2$: qubits, $d = 3$: qutrits, ..., d: "qudits", like, e.g.,

$$(e_{m_0} + e_{m_1} + \ldots + e_{m_{d-1}})/\sqrt{d} \tag{4}$$

and associated tensor product spaces which then contain entangled states.

In those d-dimensional subspaces φ-independent ("global") unitary transformations $U(d)$ and other linear mappings may act.

Especially one can incorporate the usual elementary qubits from 2-dimensional spaces [5] , e.g.,

$$(|0\rangle \pm |1\rangle)/\sqrt{2}, \tag{5}$$

and associated entangled EPR/Bell product states [5]

$$(|00\rangle \pm |11\rangle)/\sqrt{2} \equiv (|0\rangle \otimes |0\rangle \pm |1\rangle \otimes |1\rangle)/\sqrt{2}, \tag{6}$$

$$(|01\rangle \pm |10\rangle)/\sqrt{2} \equiv (|0\rangle \otimes |1\rangle \pm |1\rangle \otimes |0\rangle)/\sqrt{2} \tag{7}$$

etc.

Experimentally A-OAM systems have been investigated particularly with laser light beams (see, e.g., the reviews [6–8]) and with electron beams [9]. The crucial property of certain such beams is that they carry OAM $\vec{p}$ along their directions, i.e., those beams "rotate" around their directions! For the experimental investigations of entangled OAM states see, e.g., the articles [10, 11].

Perhaps the use of associated A-OAM Wigner functions may be helpful for description and analysis of those experiments! Recall that, in principle, *all* statistical properties of a *quantum state of a system* can be derived from its Wigner function on the associated *classical phase space*!

In the following the A-OAM Wigner functions of the most general qubits and 2-qubits will be derived and some special cases discussed and illustrated in more detail. The discussion is restricted to pure states. The generalization to mixed states described by density matrices is straightforward [1, 2].

The most general qubit of a A-OAM system is given by

$$\chi^{(\alpha,\beta)}_{m_0,m_1}(\varphi) = \cos\beta \, e_{m_0}(\varphi) + \sin\beta \, e^{i\alpha} \, e_{m_1}(\varphi), \tag{8}$$

$$m_0, m_1 \in \mathbb{Z}, \; m_1 \neq m_0,$$

$$|\chi^{(\alpha,\beta)}_{m_0,m_1}(\varphi)|^2 = 1 + \sin 2\beta \cos[(m_0 - m_1)\varphi - \alpha]. \tag{9}$$

The states (8) are elements of the 2-dimensional subspace

$$Q^2_{m_0 m_1} = \{\chi = c_0 \, e_{m_0} + c_1 \, e_{m_1}, \; |c_0|^2 + |c_1|^2 = 1\} \tag{10}$$

of the overall Hilbert space $L^2(\mathbb{S}^1, d\varphi/2\pi)$.

The angular momentum operator ($\hbar = 1$ in the following) $L = (1/i)\partial_\varphi$ has the - obvious - expectation value

$$(\chi^{(\alpha,\beta)}_{m_0,m_1}, L\chi^{(\alpha,\beta)}_{m_0,m_1}) = m_0 \cos^2\beta + m_1 \sin^2\beta, \tag{11}$$

which vanishes for $m_1 = -m_0$ and $\cos^2\beta = \sin^2\beta = 1/2$. If

$$\chi^{(\hat{\alpha},\hat{\beta})}_{m_0,m_1}(\varphi) = \cos\hat{\beta} \, e_{m_0}(\varphi) + \sin\hat{\beta} \, e^{i\hat{\alpha}} \, e_{m_1}(\varphi) \tag{12}$$

is another qubit of the type (8) in the same 2-dimensional space, then the scalar product of both is given by

$$(\chi^{(\alpha,\beta)}_{m_0,m_1}, \chi^{(\hat\alpha,\hat\beta)}_{m_0,m_1}) = \cos\beta\,\cos\hat\beta + e^{-i(\alpha-\hat\alpha)}\sin\beta\sin\hat\beta, \quad (13)$$

with the associated transition probability

$$|(\chi^{(\alpha,\beta)}_{m_0,m_1}, \chi^{(\hat\alpha,\hat\beta)}_{m_0,m_1})|^2 = \cos^2\beta\cos^2\hat\beta + \sin^2\beta\sin^2\hat\beta + \frac{1}{2}\sin 2\beta\sin 2\hat\beta\cos(\alpha-\hat\alpha), \quad (14)$$

which equals $\cos^2(\beta-\hat\beta)$ for $\hat\alpha = \alpha$. Eq. (14) is of interest in a discussion below (see Eq. (24)).

According to Ch. IV of Ref. [1] the A-OAM Wigner function $V_\psi(\theta,p)$ – here θ describes (by means of the pair $(\cos\theta, \sin\theta)$) the points on the (configuration) circle $\mathbb{S}^1$ of the classical phase space and $p \in \mathbb{R}$ the classical canonically conjugate OAM, "V" stands for "Vortex" – for a wave function $\psi(\varphi)$ of Eq. (3) is given by

$$V_\psi(\theta,p) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\vartheta}{2\pi}e^{-ip\vartheta}\psi^*(\theta-\vartheta/2)\,\psi(\theta+\vartheta/2). \quad (15)$$

Taking for $\psi(\varphi)$ the wave function (8) gives

$$2\pi\,V^{(\alpha,\beta)}_{m_0,m_1}(\theta,p) \quad (16)$$
$$= \cos^2\beta\,\mathrm{sinc}\,\pi(p-m_0) + \sin^2\beta\,\mathrm{sinc}\,\pi(p-m_1)$$
$$+ \sin 2\beta\cos[(m_0-m_1)\theta-\alpha]\,\mathrm{sinc}\,\pi[p-(m_0+m_1)/2],$$

where

$$\mathrm{sinc}\,\pi x = \frac{\sin\pi x}{\pi x} = \frac{1}{2\pi}\int_{-\pi}^{\pi}d\vartheta\,e^{ix\vartheta}. \quad (17)$$

(The remarkable significance of the sinc-function in the context of the A-OAM Wigner function is elaborately discussed in Ref. [1].)

The third line in Eq. (16) represents the probability interference term in $(\chi^{(\alpha,\beta)}_{m_0,m_1}, \chi^{(\alpha,\beta)}_{m_0,m_1})$ – see Eq. (9) – and describes, therefore, essential *quantum mechanical* properties of the state!

Note that the interference term vanishes for $p-(m_0+m_1)/2 = k,\ k \in \{\mathbb{Z}-\{0\}\}$. It vanishes, of course, too, if α or/and θ are such that $\cos[(m_0-m_1)\theta-\alpha] = 0$. On the other hand, the last factor of the interference term is maximal ($=1$) for $p = (m_0+m_1)/2$!

The angles α and β may depend on other parameters, e.g., time t, space coordinates, external fields etc., and may, therefore, be manipulated from outside. Their values can be represented by points on the 2-dimensional surface of a "Bloch" sphere [12].

It is worthwhile to look at a few special examples:

$m_1 = -m_0$:

$$2\pi\,V^{(\alpha,\beta)}_{m_0,-m_0}(\theta,p) \quad (18)$$
$$= \cos^2\beta\,\mathrm{sinc}\,\pi(p-m_0) + \sin^2\beta\,\mathrm{sinc}\,\pi(p+m_0)$$
$$+ \sin 2\beta\cos[2m_0\theta-\alpha]\,\mathrm{sinc}\,\pi\,p.$$

The special case $\alpha = 0, \beta = \pi/4, m_0 = 1$ of Eq. (18) is illustrated in Fig. 1 (see also Fig. 2 in Ref. [1]):

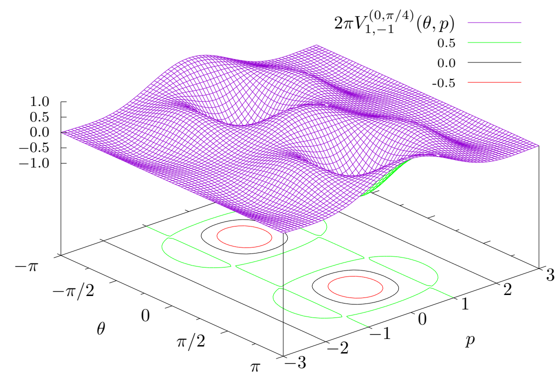

FIG. 1. A-OAM Wigner function $2\pi\,V^{(\alpha=0,\beta=\pi/4)}_{1,-1}(\theta,p) = \frac{1}{2}[\mathrm{sinc}\,\pi(p-1) + \mathrm{sinc}\,\pi(p+1)] + \cos 2\theta\,\mathrm{sinc}\,\pi p$ of the qubit $(e_{+1}+e_{-1})/\sqrt{2}$.

$m_1 = 0$ (ground state of $H \propto L^2$):

$$2\pi\,V^{(\alpha,\beta)}_{m_0,0}(\theta,p) \quad (19)$$
$$= \cos^2\beta\,\mathrm{sinc}\,\pi(p-m_0) + \sin^2\beta\,\mathrm{sinc}\,\pi\,p$$
$$+ \sin 2\beta\cos[m_0\theta-\alpha]\,\mathrm{sinc}\,\pi(p-m_0/2),$$

Here the interference term vanishes for $p = m_0/2+k,\ k \in \{\mathbb{Z}-\{0\}\}$. (Recall that $m_0 \neq 0$ because $m_1 = 0$.)

The Wigner function (19) for the special values $\alpha = 0, \beta = \pi/3, m_0 = 1$ is shown in Fig. 2:

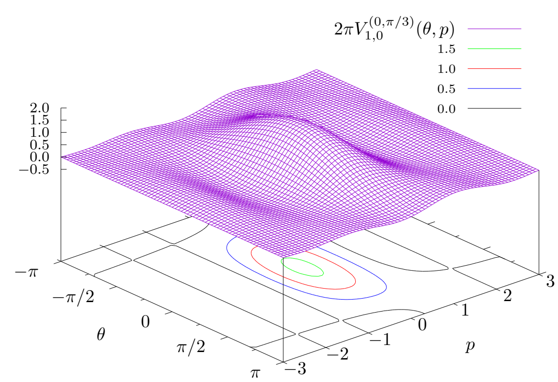

FIG. 2. A-OAM Wigner function $2\pi\,V^{(\alpha=0,\beta=\pi/3)}_{1,0}(\theta,p) = \frac{1}{4}[\mathrm{sinc}\,\pi(p-1) + 3\,\mathrm{sinc}\,\pi p] + \frac{\sqrt{3}}{2}\cos\theta\,\mathrm{sinc}\,\pi(p-1/2)$ for the qubit $(e_{+1}+e_0)/\sqrt{2}$.

The quantum mechanical *marginal* probability distributions $|\chi^{(\alpha,\beta)}_{m_0,m_1}(\theta)|^2$ (angular distribution density) and $\{\cos^2\beta, \sin^2\beta\}$ (OAM distribution) of the state (8) can be obtained, according to Ref. [1], from the A-OAM Wigner function (16) as follows:

$$\int_{-\infty}^{\infty} dp\, V_{m_0,m_1}^{(\alpha,\beta)}(\theta,p) \tag{20}$$
$$= \frac{1}{2\pi}\{1+\sin 2\beta\, \cos[(m_0-m_1)\theta-\alpha]\}$$
$$= \frac{1}{2\pi}|\chi_{m_0,m_1}^{(\alpha,\beta)}(\theta)|^2$$

– compare Eq. (9) –, where the relation

$$\int_{-\infty}^{\infty} dp\, \text{sinc}\,\pi(p+a) = 1,\; a\in\mathbb{R}, \tag{21}$$

has been used.

Integration over θ gives Whittaker's cardinal function [1]

$$\int_{-\pi}^{\pi} d\theta\, V_{m_0,m_1}^{(\alpha,\beta)}(\theta,p) = \omega_{m_0,m_1}^{(\alpha,\beta)}(p) \tag{22}$$
$$= \cos^2\beta\, \text{sinc}\,\pi(p-m_0) + \sin^2\beta\, \text{sinc}\,\pi(p-m_1),$$

from which the quantum mechanical OAM probabilities $\cos^2\beta$ and $\sin^2\beta$ can be extracted immediately with the help of the orthonormality relations [1]

$$\int_{-\infty}^{\infty} dp\, \text{sinc}\,\pi(p-m)\, \text{sinc}\,\pi(p-n) = \delta_{mn}. \tag{23}$$

If $V_{m_0,m_1}^{(\alpha,\beta)}(\theta,p)$ and $V_{m_0,m_1}^{(\hat\alpha,\hat\beta)}(\theta,p)$ are A-OAM Wigner functions of the qubits (8) and (12), then the transition probability (14) is now given [1] by the integral

$$2\pi\int_{-\infty}^{\infty} dp \int_{-\pi}^{\pi} d\theta V_{m_0,m_1}^{(\alpha,\beta)}(\theta,p) V_{m_0,m_1}^{(\hat\alpha,\hat\beta)}(\theta,p) = \tag{24}$$
$$\cos^2\beta\cos^2\hat\beta + \sin^2\beta\sin^2\hat\beta + \frac{1}{2}\sin 2\beta\sin 2\hat\beta\cos(\alpha-\hat\alpha),$$

where the relations (23), (21) and

$$\int_{-\infty}^{\infty} dp\, \text{sinc}^2\,\pi(p+a) = 1,\; a\in\mathbb{R}, \tag{25}$$

have been used [13].

For the discussion of the tensor product of the Hilbert space $L^2(\mathbb{S}^1, d\varphi/2\pi)$ from above – characterized by the Eqs. (1)–(3) – with itself we have to go slightly beyond the A-OAM framework discussed in Refs. [1, 2]:

There we had a phase space $\mathcal{P}^2(\theta,p) = \{(\theta,p)\in\mathbb{S}^1\times\mathbb{R}\}$ with the circle $\mathbb{S}^1$ as configuration space and the real line $\mathbb{R}$ as cotangent space. Coordinates for the former are provided by the pair $(\cos\theta,\sin\theta)$ and the angular momentum $p\in\mathbb{R}$ for the latter. By doubling the system we get the phase space

$$\mathcal{P}^4(\tilde\theta,\tilde p) = \{(\theta_1,\theta_2;p_1,p_2)\in\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{R}\times\mathbb{R})\}, \tag{26}$$
$$\tilde\theta \equiv (\theta_1,\theta_2),\quad \tilde p\equiv(p_1,p_2).$$

Configuration space is now the torus $\mathbb{S}^1\times\mathbb{S}^1$.

A crucial tool for the derivation of the Wigner function (15) in Ref. [1] are the unitary representations of the Euclidean group $E(2)$ of the plane [14]. In our case, $\mathcal{P}^4(\tilde\theta,\tilde p)$, we have to employ the direct product $E(2)\times E(2)$ and the associated unitary representations. The procedure for deriving the A-OAM Wigner function in question is then stricly analogue to that of Ch. II in Ref. [1] for the expression (15) and the result is as expected:

We have the Hilbert space

$$L^2(\mathbb{S}^1\times\mathbb{S}^1, d\varphi_1 d\varphi_2/(2\pi)^2), \tag{27}$$

with basis

$$e_{mn}(\tilde\varphi) = e_m(\varphi_1)\, e_n(\varphi_2) = e^{im\varphi_1+in\varphi_2},\; m,n\in\mathbb{Z}, \tag{28}$$

scalar product

$$(\psi_2,\psi_1) = \int_{-\pi}^{\pi}\frac{d^2\tilde\varphi}{(2\pi)^2}\psi_2^*(\tilde\varphi)\psi_1(\tilde\varphi), \tag{29}$$
$$(e_{km},e_{ln}) = \delta_{kl}\,\delta_{mn},$$

and expansions

$$\psi(\tilde\varphi) = \sum_{m,n\in\mathbb{Z}} c_{mn}\, e_{mn}(\tilde\varphi),\; c_{mn} = (e_{mn},\psi). \tag{30}$$

The functions (28) are eigenfunctions of the total OAM operator:

$$L = \frac{1}{i}L_{\varphi_1} + \frac{1}{i}L_{\varphi_2},\quad Le_{mn} = (m+n)e_{mn}. \tag{31}$$

Comparison of the basis (28) with Eqs. (6) and (7) suggests the following choice of correspondences:

$$e_{m_0n_0}\leftrightarrow|00\rangle,\; e_{m_1n_1}\leftrightarrow|11\rangle, \tag{32}$$
$$e_{m_0n_1}\leftrightarrow|01\rangle,\; e_{m_1n_0}\leftrightarrow|10\rangle.$$

Applying the same arguments of Ch. II in Ref. [1] – which lead to the Wigner function (15) – now to the products $\mathcal{P}^4(\tilde\theta,\tilde p)$ and $E(2)\times E(2)$ we then get, on the phase space $\mathcal{P}^4$ for the wave function ψ of Eq. (30) the A-OAM Wigner function

$$V_\psi(\tilde\theta,\tilde p) \equiv \hat V_\psi(\tilde\theta,\tilde p)/(2\pi)^2 = \tag{33}$$
$$\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\frac{d^2\tilde\vartheta}{(2\pi)^2}e^{-i(p_1\vartheta_1+p_2\vartheta_2)}\,\psi^*(\tilde\theta-\tilde\vartheta/2)\,\psi(\tilde\theta+\tilde\vartheta/2),$$

which is the obvious generalization of the expression (15). We now want to determine the A-OAM Wigner functions for the general elements

$$\psi_{2qb}(\tilde\varphi) = c_{00}\, e_{m_0n_0}(\tilde\varphi) + c_{10}\, e_{m_1n_0}(\tilde\varphi) \tag{34}$$
$$+ c_{01}\, e_{m_0n_1}(\tilde\varphi) + c_{11}\, e_{m_1n_1}(\tilde\varphi),$$
$$|c_{00}|^2+|c_{01}|^2+|c_{10}|^2+|c_{11}|^2 = 1.$$

of the 4-dimensional tensor product space $Q^4_{m_0 m_1, n_0 n_1}$ of the space (10) with itself. The four complex coefficients c_{jk} may be parametrized by real numbers as follows:

$$c_{00} = b_{00},\ c_{10} = e^{i\alpha_{10}} b_{10},\ c_{01} = e^{i\alpha_{01}} b_{01},\ c_{11} = e^{i\alpha_{11}} b_{11},$$
$$\alpha_{jk} \in [0, 2\pi),\ b_{jk} \in \mathbb{R},\ b_{00}^2 + b_{10}^2 + b_{01}^2 + b_{11}^2 = 1. \quad (35)$$

Inserting the wave function (34) into the expression (33) yields the most general 2-qubit A-OAM Wigner function

$$\begin{aligned}
&\hat{V}_{\psi_{2qb}}(\tilde{\theta}, \tilde{p}) \qquad (36)\\
&= b_{00}^2 \operatorname{sinc}\pi(p_1 - m_0) \operatorname{sinc}\pi(p_2 - n_0)\\
&+ b_{10}^2 \operatorname{sinc}\pi(p_1 - m_1) \operatorname{sinc}\pi(p_2 - n_0)\\
&+ b_{01}^2 \operatorname{sinc}\pi(p_1 - m_0) \operatorname{sinc}\pi(p_2 - n_1)\\
&+ b_{11}^2 \operatorname{sinc}\pi(p_1 - m_1) \operatorname{sinc}\pi(p_2 - n_1)\\
&+ 2\, b_{00} b_{10} \cos[(m_1 - m_0)\theta_1 + \alpha_{10}]\\
&\times \operatorname{sinc}\pi[p_1 - (m_0 + m_1)/2] \operatorname{sinc}\pi(p_2 - n_0)\\
&+ 2\, b_{00} b_{01} \cos[(n_1 - n_0)\theta_2 + \alpha_{01}]\\
&\times \operatorname{sinc}\pi(p_1 - m_0) \operatorname{sinc}\pi[p_2 - (n_0 + n_1)/2]\\
&+ 2\, b_{00} b_{11} \cos[(m_1 - m_0)\theta_1 + (n_1 - n_0)\theta_2 + \alpha_{11}]\\
&\times \operatorname{sinc}\pi[p_1 - (m_0 + m_1)/2] \operatorname{sinc}\pi[p_2 - (n_0 + n_1)/2]\\
&+ 2\, b_{01} b_{10} \cos[(m_1 - m_0)\theta_1 - (n_1 - n_0)\theta_2 + \alpha_{10} - \alpha_{01}]\\
&\times \operatorname{sinc}\pi[p_1 - (m_0 + m_1)/2] \operatorname{sinc}\pi[p_2 - (n_0 + n_1)/2]\\
&+ 2\, b_{01} b_{11} \cos[(m_1 - m_0)\theta_1 + \alpha_{11} - \alpha_{01}]\\
&\times \operatorname{sinc}\pi[p_1 - (m_0 + m_1)/2)] \operatorname{sinc}\pi(p_2 - n_1)\\
&+ 2\, b_{10} b_{11} \cos[(n_1 - n_0)\theta_2 + \alpha_{11} - \alpha_{10}]\\
&\times \operatorname{sinc}\pi(p_1 - m_1) \operatorname{sinc}\pi[p_2 - (n_1 + n_0)/2].
\end{aligned}$$

Here it is quite remarkable that in case of the four 2-dimensional specializations $(b_{00}, b_{10}, b_{01}, b_{11}) = (\cos\beta, \sin\beta, 0, 0)$, $(\cos\beta, 0, \sin\beta, 0)$, $(0, 0, \cos\beta, \sin\beta)$ or $(0, \cos\beta, 0, \sin\beta)$ the Wigner function (36) factorizes into a qubit Wigner function (16) with $\theta = \theta_1$ (or $\theta = \theta_2$) and $p = p_1$ (or $p = p_2$) times Wigner functions $V_{n_0}(\theta, p) = \operatorname{sinc}\pi(p - n_0)/(2\pi)$ etc. of the basis vectors $e_{n_0}, e_{n_1}, e_{m_0}$, or e_{m_1} [1]. We have, e.g.,

$$\begin{aligned}
&V^{(\alpha_{10},\beta)}_{00,10}(\tilde{\theta}, \tilde{p}) \qquad (37)\\
&= \frac{1}{(2\pi)^2}\{\cos^2\beta \operatorname{sinc}\pi(p_1 - m_0) + \sin^2\beta \operatorname{sinc}\pi(p_1 - m_1)\\
&+ \sin 2\beta \cos[(m_1 - m_0)\theta_1 + \alpha_{10}] \operatorname{sinc}\pi[p_1 - (m_0 + m_1)/2]\}\\
&\times \operatorname{sinc}\pi(p_2 - n_0) = V^{(\alpha_{10},\beta)}_{m_0,m_1}(\theta_1, p_1)\, V_{n_0}(\theta_2, p_2).
\end{aligned}$$

No such factorization occurs for the "entangled" cases – compare Eqs. (6) and (7) – $(\cos\beta, 0, 0, \sin\beta)$ and $(0, \cos\beta, \sin\beta, 0)$, where we get for the former

$$\begin{aligned}
&V^{(\alpha_{11},\beta)}_{00,11}(\tilde{\theta}, \tilde{p}) \qquad (38)\\
&= \frac{1}{(2\pi)^2}\{\cos^2\beta \operatorname{sinc}\pi(p_1 - m_0) \operatorname{sinc}\pi(p_2 - n_0)\\
&+ \sin^2\beta \operatorname{sinc}\pi(p_1 - m_1) \operatorname{sinc}\pi(p_2 - n_1)\\
&+ \sin 2\beta \cos[(m_1 - m_0)\theta_1 + (n_1 - n_0)\theta_2 + \alpha_{11}]\\
&\times \operatorname{sinc}\pi[p_1 - (m_0 + m_1)/2] \operatorname{sinc}\pi[p_2 - (n_0 + n_1)/2]\}.
\end{aligned}$$

It follows that the two basic entangled EPR/Bell states (6) have the Wigner functions $V^{(\alpha_{11}=0,\beta=\pm\pi/4)}_{00.11}(\tilde{\theta}, \tilde{p})$ and the states (7) a corresponding $V^{(\alpha_{10}-\alpha_{01}=0,\beta=\pm\pi/4)}_{10,01}(\tilde{\theta}, \tilde{p})$.

What is remarkable here is that the four entangled EPR/Bell states (6) and (7) do have *irreducible* A-OAM Wigner functions, i.e. they cannot be written as products of "lower" ones like in Eq. (37)! We see that this irreducibility of their Wigner functions is characteristic for EPR/Bell states!

Considerable simplifications are obtained for the terms in expression (36) if $m_0 + m_1 = 0$, $n_0 + n_1 = 0$, $m_0 \neq 0 \neq n_0$, and additionally $n_0 = m_0$ and $\alpha_{jk} = 0$ as well. This corresponds to a physical situation where a system with total angular momentum zero is decomposed or decays into two subsystems which move in opposite directions, one with angular momentum m_0 and the other with $m_1 = -m_0$. Example is a particle of spin zero which - in its rest frame - decays into 2 photons with opposite spins one.

Examples with the simplifications mentioned:

The A-OAM Wigner function for the basis vector $e_{m_0 m_0}(\tilde{\varphi})$ is given by

$$V_{m_0 m_0}(\tilde{\theta}, \tilde{p}) = \frac{1}{(2\pi)^2}[\operatorname{sinc}\pi(p_1 - m_0) \operatorname{sinc}\pi(p_2 - m_0)]. \qquad (39)$$

This follows from Eq. (36) with $b_{00} = 1, b_{01} = b_{10} = b_{11} = 0$.

According to the correspondences (32) we get for the EPR/Bell states

$$\begin{aligned}
&\psi_{00,11;\pm}(\tilde{\varphi}) \qquad (40)\\
&= \frac{1}{\sqrt{2}}[e_{m_0}(\varphi_1) e_{m_0}(\varphi_2) \pm e_{-m_0}(\varphi_1) e_{-m_0}(\varphi_2)],
\end{aligned}$$
$$L\, \psi_{00,11;\pm}(\tilde{\varphi}) = (2m_0 \mp 2m_0)\, \psi_{00,11;\pm}(\tilde{\varphi}), \qquad (41)$$

from Eq. (38) (with $\alpha_{11} = 0, \beta = \pm\pi/4$) the A-OAM Wigner functions

$$\begin{aligned}
&\hat{V}_{00,11;\pm}(\tilde{\theta}, \tilde{p}) \qquad (42)\\
&= \frac{1}{2}[\operatorname{sinc}\pi(p_1 - m_0) \operatorname{sinc}\pi(p_2 - m_0)]\\
&+ \frac{1}{2}[\operatorname{sinc}\pi(p_1 + m_0) \operatorname{sinc}\pi(p_2 + m_0)]\\
&\pm \cos[2m_0(\theta_1 + \theta_2)] \operatorname{sinc}\pi p_1 \operatorname{sinc}\pi p_2.
\end{aligned}$$

For the states

$$\psi_{01,10;\pm}(\tilde{\varphi}) \tag{43}$$
$$= \frac{1}{\sqrt{2}}[e_{m_0}(\varphi_1)e_{-m_0}(\varphi_2) \pm e_{-m_0}(\varphi_1)e_{m_0}(\varphi_2)],$$
$$L\,\psi_{01,10;\pm}(\tilde{\varphi}) = 0, \tag{44}$$

we get accordingly

$$\hat{V}_{01,10;\pm}(\tilde{\theta},\tilde{p}) \tag{45}$$
$$= \frac{1}{2}[\operatorname{sinc}\pi(p_1 - m_0)\operatorname{sinc}\pi(p_2 + m_0)]$$
$$+ \frac{1}{2}[\operatorname{sinc}\pi(p_1 + m_0)\operatorname{sinc}\pi(p_2 - m_0)]$$
$$\pm \cos[2m_0(\theta_1 - \theta_2)]\operatorname{sinc}\pi p_1 \operatorname{sinc}\pi p_2.$$

A special example of the function (45) with $m_0 = 1$ at $p_2 = 1/2$ is shown in Fig. 3.

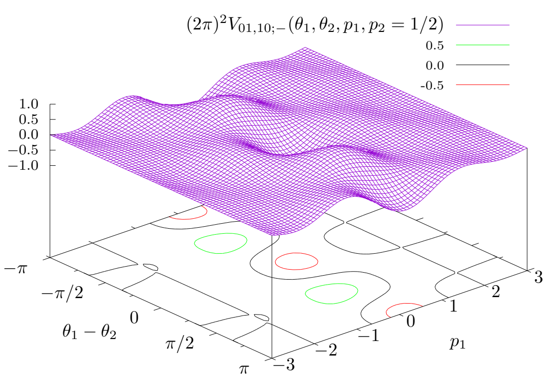

FIG. 3. Wigner function $(2\pi)^2V_{01,10;-}(\theta_1,\theta_2,p_1,p_2=1/2) = \frac{1}{\pi}\{-\frac{1}{3}\operatorname{sinc}\pi(p_1-1)+\operatorname{sinc}\pi(p_1+1)-2\cos[2(\theta_1-\theta_2)]\operatorname{sinc}\pi p_1\}$ of the EPR/Bell state $\psi_{01,10;-}(\tilde{\varphi})$ from Eq. (43).

Note again the θ-dependences in Eqs. (42) and (45): The Wigner functions of the entangled EPR/Bell states depend on both angles θ_1 and θ_2, either only on the sum $\theta_1 + \theta_2$ or the difference $\theta_1 - \theta_2$, respectively. Consider the curve

$$\mathcal{C}_- = \{\theta_- = \theta_1 - \theta_2 \in \mathbb{R};\ \ \theta_1 + \theta_2 = \mu = \text{const.},\ \ \tilde{p} = \text{const.}\} \subset \mathcal{P}^4(\theta_1,\theta_2,p_1,p_2), \tag{46}$$

so that

$$\theta_1 = (\theta_- + \mu)/2,\ \ \theta_2 = (-\theta_- + \mu)/2. \tag{47}$$

That means, if θ_- increases then θ_1 increases and θ_2 decreases at the same rate, i.e. the curve (46) spirals around a torus.

We see that non-classical properties of the EPR/Bell states (6) and (7) amazingly have their correspondence in the irreducibility of their A-OAM Wigner functions, which itself is related to the topology of the toroidal configuration subspace of the phase space $\mathcal{P}^4(\tilde{\theta},\tilde{p})$.

Let us look at some details of the expression (45), with the minus sign in Eq. (43) and the related one in Eq. (45) as well:

Integrating $V_{01,10;-}$ in Eq. (45) over p_1 and p_2 gives, with the help of relation (21):

$$\int_{-\infty}^{\infty} dp_1 dp_2 V_{01,10;-}(\tilde{\theta},\tilde{p}) = \tag{48}$$
$$\frac{1}{4\pi^2}\{1 - \cos[2m_0(\theta_1 - \theta_2)]\} = \frac{1}{4\pi^2}\{2\sin^2[m_0(\theta_1 - \theta_2)]\}$$
$$= \frac{1}{4\pi^2}|\psi_{01,10;-}(\tilde{\theta})|^2,$$

that is, the integral (48) gives the marginal angle probability density, as required by the general theory [1].

On the other hand, the angle integral

$$\int_{-\pi}^{\pi} d\theta_1 d\theta_2 V_{01,10;-}(\tilde{\theta},\tilde{p}) \tag{49}$$
$$= \frac{1}{2}[\operatorname{sinc}\pi(p_1 - m_0)\operatorname{sinc}\pi(p_2 + m_0)]$$
$$+ \frac{1}{2}[\operatorname{sinc}\pi(p_1 + m_0)\operatorname{sinc}\pi(p_2 - m_0)]$$
$$= \omega_{01,10;-}(\tilde{p})$$

gives Whittaker's cardinal function [1] for two variables p_1 and p_2. Again, using the orthonormality relations (23), the marginal OAM probabilities

$$|c_{01}|^2 = \frac{1}{2},\ \ |c_{10}|^2 = \frac{1}{2} \tag{50}$$

can be extracted from $\omega_{01,10;-}(\tilde{p})$.

Special additional properties are:

$$V_{01,10;-}(\tilde{\theta},p_1,p_2) = 0 \text{ for } p_1,p_2 \in \{\mathbb{Z} - \{0\}\}, \tag{51}$$

and

$$\hat{V}_{01,10;-}(\tilde{\theta},p_1,p_2 = 0) \tag{52}$$
$$= -\cos[2m_0(\theta_1 - \theta_2)]\operatorname{sinc}\pi p_1,$$

with the corresponding relation for $p_1 = 0$. This shows that the "classical" probabilities (50) as derived from Whittaker's cardinal function (49) do not get any contributions from the phase space subsets $p_1 = 0$ and $p_2 = 0$ respectively, only the interference term does!

A graphical illustration of the function (52) with $m_0 = 1$ is given by Fig. 4.

Furthermore,

$$V_{01,10;-}(\theta_1,\theta_2,p_1 = 0,p_2 = 0) = -\frac{1}{4\pi^2}\cos[2m_0(\theta_1 - \theta_2)], \tag{53}$$

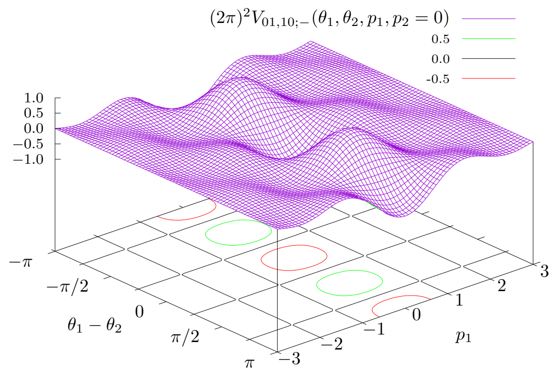

FIG. 4. Wigner function $(2\pi)^2 V_{01,10;-}(\theta_1,\theta_2,p_1,p_2=0) = -\cos[2(\theta_1-\theta_2)]\,\mathrm{sinc}\,\pi p_1$, according to Eq. (52) with $m_0 = 1$, of the EPR/Bell state $\psi_{01,10;-}(\tilde{\varphi})$ from Eq. (43).

showing explicitely that the Wigner function is negative on certain subsets of the phase space.

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[13] For Eq. (25) see the literature on the sinc function as quoted in Ref. [1].
[14] For details see the discussion in Ch. II of Ref. [1].