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The Three Loop Two-Mass Contribution to the Gluon Vacuum Polarization

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Abstract

We calculate the two-mass contribution to the 3-loop vacuum polarization of the gluon in Quantum Chromodynamics at virtuality $p^2 = 0$ for general masses and also present the analogous result for the photon in Quantum Electrodynamics.

The 3-loop heavy flavor corrections to deep-inelastic scattering at larger virtualities [1–7] form an important ingredient for the determination of the strong coupling constant $\alpha_s(M_Z^2)$, the parton distribution functions and the measurement of the mass of the charm quark m_c at high precision [8, 9]. Starting at 3-loop order the QCD corrections contain also 2-mass contributions in single Feynman diagrams [10–12]. The heavy flavor contributions to deep-inelastic structure functions in the region of larger virtualities $Q^2 \gg m_Q^2$, with m_Q the heavy quark mass, can be obtained in terms of massive on-shell operator matrix elements (OMEs) [1, 13]. These quantities also receive massive self-energy insertions, such as the on-shell vacuum polarization function $\hat{\hat{\Pi}}^{(3)}(0, m_1^2, m_2^2, \mu^2)$ and fermion self energy $\hat{\hat{\Sigma}}(0, m_1^2, m_2^2, \mu^2)$. Here $m_{1,2}$ denote the corresponding heavy quark masses and μ is the renormalization scale. These quantities are of more general interest, as they appear in various massive higher loop calculations. The expression for $\hat{\hat{\Sigma}}(0, m_1^2, m_2^2, \mu^2)$ for general ratios $\eta = m_1^2/m_2^2$ has been given in Ref. [10,14]. In the present note, we calculate the polarization function $\hat{\hat{\Pi}}^{(3)}(0, m_1^2, m_2^2, \mu^2)$, which is obtained as the 3rd term in the expansion

$$\hat{\Pi}^{\mu\nu}_{ab,H}(p^2, m_1^2, m_2^2, \mu^2, \varepsilon, \hat{a}_s) = i(-p^2 g^{\mu\nu} + p^\mu p^\nu)\delta_{ab} \sum_{k=1}^{\infty} \hat{a}_s^k \hat{\Pi}_H(p^2, m_1^2, m_2^2, \mu^2, \varepsilon) \quad (1)$$

in the limit $p^2 \to 0$. Here the index H labels the heavy quark part of the polarization function, $\hat{a}_s = g_s^2/(4\pi)^2$ denotes the unrenormalized strong coupling constant, $m_{1,2}$ are the bare heavy quark masses, and $\varepsilon = D - 4$ is the dimensional parameter. In the calculation we refer to the Feynman rules given in [15]. The corresponding (single mass) expressions $\hat{\Pi}_H^{(1,2)}(0, m_1^2, m_2^2, \mu^2)$ were calculated in [1, 16–20] for QED and/or QCD.

There are six physical topologies contributing to $\hat{\hat{\Pi}}^{(3)}(0, m_1^2, m_2^2, \mu^2)$, Eq. (2). The corresponding 3-loop Feynman diagrams have been generated using `QGRAF` [21] and the color-traces were calculated using `Color` [22]. Standard Feynman parameter integration has been applied, cf. [23], representing one of the integrals using the Mellin-Barnes contour integral [24–28], also using the package [29]. The sums over the residues have been performed analytically using the packages `Sigma` [30, 31], `EvaluateMultiSums` and `SumProduction` [32], applying procedures of `HarmonicSums` [33–37] also for the limiting processes in case of infinite sums.

The equal mass contributions have been calculated in [1], Eq. (4.7), before in case of a general R_ξ gauge using `MATAD` [38]. The 2-mass term is given by

$$\begin{aligned}
\hat{\hat{\Pi}}^{(3)}(0, m_1^2, m_2^2, \mu^2) &= \lim_{p^2 \to 0} \hat{\Pi}_H(p^2, m_1^2, m_2^2, \mu^2, \varepsilon) \\
&= -C_F T_F^2 \Biggl\{ \frac{256}{9\varepsilon^2} + \frac{64}{3\varepsilon}\left[\ln\left(\frac{m_1^2}{\mu^2}\right) + \ln\left(\frac{m_2^2}{\mu^2}\right) + \frac{5}{9}\right] - 5\eta - \frac{5}{\eta} \\
&\quad + \left(-\frac{5\eta}{8} - \frac{5}{8\eta} + \frac{51}{4}\right)\ln^2(\eta) + \left(\frac{5}{2\eta} - \frac{5\eta}{2}\right)\ln(\eta) + \frac{32\zeta_2}{3} \\
&\quad + 32\ln\left(\frac{m_1^2}{\mu^2}\right)\ln\left(\frac{m_2^2}{\mu^2}\right) + \frac{80}{9}\ln\left(\frac{m_1^2}{\mu^2}\right) + \frac{80}{9}\ln\left(\frac{m_2^2}{\mu^2}\right) + \frac{1246}{81} \\
&\quad + \left(\frac{5\eta^{3/2}}{2} + \frac{5}{2\eta^{3/2}} + \frac{3\sqrt{\eta}}{2} + \frac{3}{2\sqrt{\eta}}\right)\Biggl[\frac{1}{8}\ln\left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\right)\ln^2(\eta) \\
&\quad - \mathrm{Li}_3\left(-\sqrt{\eta}\right) + \mathrm{Li}_3\left(\sqrt{\eta}\right) - \frac{1}{2}\ln(\eta)\left(\mathrm{Li}_2\left(\sqrt{\eta}\right) - \mathrm{Li}_2\left(-\sqrt{\eta}\right)\right)\Biggr]\Biggr\}
\end{aligned}$$

$$
\begin{aligned}
&-C_A T_F^2 \Biggl\{ \frac{64}{9\varepsilon^3}(1+2\xi) + \frac{16}{3\varepsilon^2}\Biggl[(1+2\xi)\left(\ln\left(\frac{m_1^2}{\mu^2}\right)+\ln\left(\frac{m_2^2}{\mu^2}\right)\right) \\
&-\frac{35}{9}\Biggr] + \frac{4}{\varepsilon}\Biggl[\ln^2\left(\frac{m_1^2}{\mu^2}\right)+\ln^2\left(\frac{m_2^2}{\mu^2}\right)-\frac{35}{9}\ln\left(\frac{m_1^2}{\mu^2}\right)-\frac{35}{9}\ln\left(\frac{m_2^2}{\mu^2}\right) \\
&+\frac{2}{3}\zeta_2+\frac{37}{27}+\xi\left(\frac{4}{3}\ln^2(\eta)+4\ln\left(\frac{m_1^2}{\mu^2}\right)\ln\left(\frac{m_2^2}{\mu^2}\right)+\frac{4}{3}\zeta_2+\frac{292}{81}\right)\Biggr] \\
&+2(1+\xi)\left(\ln^3\left(\frac{m_1^2}{\mu^2}\right)+\ln^3\left(\frac{m_2^2}{\mu^2}\right)\right)-\frac{70}{3}\ln\left(\frac{m_1^2}{\mu^2}\right)\ln\left(\frac{m_2^2}{\mu^2}\right) \\
&+2\xi\ln^2\left(\frac{m_1^2}{\mu^2}\right)\ln\left(\frac{m_2^2}{\mu^2}\right)+2\xi\ln\left(\frac{m_1^2}{\mu^2}\right)\ln^2\left(\frac{m_2^2}{\mu^2}\right)-\frac{2}{9}(2+\xi)\ln^3(\eta) \\
&+\left(2(1+2\xi)\zeta_2+\frac{37}{9}+\frac{292}{27}\xi\right)\left(\ln\left(\frac{m_1^2}{\mu^2}\right)+\ln\left(\frac{m_2^2}{\mu^2}\right)\right) \\
&+\left[\frac{4}{3}(2+\xi)\ln(1-\eta)-\left(\eta+\frac{1}{\eta}\right)\left(\frac{2}{3}+\frac{5\xi}{24}\right)-\frac{179}{18}-\frac{43}{36}\xi\right]\ln^2(\eta) \\
&-\frac{1}{3}(16+5\xi)\left(\eta+\frac{1}{\eta}\right)-\frac{70}{9}\zeta_2-\frac{8}{9}\zeta_3(7+2\xi)-\frac{3769}{243}+\frac{262}{243}\xi \\
&+\left(\frac{1}{\eta}-\eta\right)\left(\frac{8}{3}+\frac{5}{6}\xi\right)\ln(\eta)+\frac{8}{3}(2+\xi)\left(\mathrm{Li}_2(\eta)\ln(\eta)-\mathrm{Li}_3(\eta)\right) \\
&+\left[\left(8+\frac{5}{2}\xi\right)\frac{1+\eta^3}{3\eta^{3/2}}+\left(10+\frac{9}{2}\xi\right)\frac{1+\eta}{\sqrt{\eta}}\right]\Biggl[\frac{1}{8}\ln\left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\right)\ln^2(\eta) \\
&-\mathrm{Li}_3(-\sqrt{\eta})+\mathrm{Li}_3(\sqrt{\eta})-\frac{1}{2}\ln(\eta)\left(\mathrm{Li}_2(\sqrt{\eta})-\mathrm{Li}_2(-\sqrt{\eta})\right)\Biggr]\Biggr\}. \qquad (2)
\end{aligned}
$$

Here $C_A = N_c, C_F = (N_c^2-1)/(2N_c), T_F = 1/2$ are the color factors and $N_c = 3$ in case of QCD.

Using Q2E/Exp [39,40] the first terms of the η-expansion of (2) have been obtained in Ref. [10] before. The corresponding expansion of (2) agrees with this expression. Eq. (2) is symmetric under the interchange $m_1 \Leftrightarrow m_2$ and depends on the gauge parameter ξ at most linearly. Although not explicitly visible in the representation given above, one can show that $\hat{\hat{\Pi}}^{(3)}(0, m_1^2, m_2^2, \mu^2)$ depends only on η and not on $\sqrt{\eta}$.

The corresponding expression in case of Quantum Electrodynamics is found by setting $C_F = 1, C_A = 0$ and $T_F = 1$. From [41] the 3-loop QED result can be inferred in principle.

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