The five-loop Beta function for a general gauge group*

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We present results for the gluon field anomalous dimension in perturbative QCD and derive the corresponding Beta function at five-loop order. All given results are valid for a general gauge group.

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1. Introduction

In modern high-energy physics experiments, in order to closely scrutinize (and eventually go beyond) our established particle physics models such as the Standard Model (SM), it is important to push the precision of theoretical predictions that follow from these models to the highest possible level. All parameters that appear in these quantum field theories such as the SM change as functions of the energy scale, in a well-defined way that is governed

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gauge), from which we extract the (gauge-invariant) Beta function. We refrain from presenting the details of the calculation but refer the reader to Ref. [18] for an extended description of the procedure. Throughout the paper, we work in dimensional regularization around $d = 4 - 2\varepsilon$ space-time dimensions and in the $\overline{\text{MS}}$ scheme.

2. Renormalization constants and anomalous dimensions

The fermion-, gauge- and ghost fields as well as fermion mass, gauge coupling and gauge-fixing parameter of the gauge theory are renormalized multiplicatively via

$$\psi_b = \sqrt{Z_2}\psi_r \,, \quad A_b = \sqrt{Z_3}A_r \,, \quad c_b = \sqrt{Z_3^c}c_r \,, \tag{1}$$

$$m_b = Z_m m_r \,, \quad g_b = \mu^\varepsilon Z_g g_r \,, \quad \xi_{L,b} = Z_\xi \xi_{L,r} \,. \tag{2}$$

We have used the subscript b and r for bare and renormalized quantities, respectively. All renormalization constants (RCs) have the form $Z_i = 1 + \mathcal{O}(g_r^2)$. There actually is no need to renormalize the gauge-fixing term $\sim (\partial A)^2/\xi_L$, such that setting $Z_\xi = Z_3$ leaves us with five independent renormalization constants only. A very economic way of recording the various renormalization constants Z_i is to merely list the corresponding anomalous dimensions, defined by

$$\gamma_i = -\partial_{\ln \mu^2} \ln Z_i \,. \tag{3}$$

Following usual conventions, instead of considering Z_g, one renormalizes the gauge coupling squared (which in our notation is

$$a \equiv \frac{C_\text{A}\, g_r^2(\mu)}{16\pi^2} \tag{4}$$

with C_A the quadratic Casimir operator of the adjoint representation of the gauge group) with the factor $Z_a \equiv Z_g^2$ and calls the corresponding anomalous dimension $\gamma_a = 2\gamma_g \equiv \beta$ the Beta function. Note that, due to the renormalization scale independence of the bare gauge coupling, using eqs. (2) and (3) this immediately implies

$$\beta = \varepsilon + \partial_{\ln \mu^2} \ln a \quad \Leftrightarrow \quad \partial_{\ln \mu^2} a = -a\Big[\varepsilon - \beta\Big] \,. \tag{5}$$

The Beta function is a gauge invariant object. The second gauge invariant anomalous dimension is γ_m, corresponding to the renormalization of the quark mass.

To complete the renormalization program, we are left with choosing (besides the gauge invariants β and γ_m) three further RCs. These three

$$d_3 = \frac{[\mathrm{sTr}(F^aF^bF^cF^d)]^2}{N_\mathrm{A}C_\mathrm{A}^4}\,. \tag{10}$$

Here, $\mathrm{sTr}(ABCD) = \frac{1}{6}\mathrm{Tr}(ABCD + ABDC + ACBD + ACDB + ADBC + ADCB)$ is a fully symmetrized trace.

Taking the gauge group to be SU(N) and setting $T_\mathrm{F} = \frac{1}{2}$ and $C_\mathrm{A} = N$, our set of normalized invariants then reads [29]

$$n_f = \frac{N_\mathrm{f}}{2N}\,,\quad c_f = \frac{N^2-1}{2N^2}\,, \tag{11}$$

$$d_1 = \frac{N^4-6N^2+18}{24N^4}\,,\quad d_2 = \frac{N^2+6}{24N^2}\,,\quad d_3 = \frac{N^2+36}{24N^2}\,. \tag{12}$$

From here, one can for example easily obtain the SU(3) coefficients, corresponding to physical QCD.

3. Results

In the following we present our results for the gauge field anomalous dimension γ_3 and the Beta function. The results for the remaining anomalous dimension have been presented in [16, 17].

In terms of the renormalized gauge coupling a as defined in eq. (4), we have obtained

$$\gamma_3 = -a\left[\frac{8n_f - (13-3\xi_L)}{6} + \gamma_{31}a + \gamma_{32}a^2 + \gamma_{33}a^3 + \gamma_{34}a^4 + \dots\right] \tag{13}$$

The coefficients γ_{3n} are functions of the group invariants and the gauge parameter. At five loops and in Feynman gauge $\xi_L = 1$, we have obtained [18]

$$2^{13}3^5\,\gamma_{34} = \gamma_{344}\,[16n_f]^4 + \gamma_{343}\,[16n_f]^3 + \gamma_{342}\,[16n_f]^2 + \gamma_{341}\,[16n_f] + \gamma_{340}\,,$$

$$\gamma_{344} = \{c_f, 1\}.\{107 + 144\zeta_3, -619/2 + 432\zeta_4\}\,, \tag{14}$$

$$\begin{aligned}\gamma_{343} = {} & \{c_f^2, c_f, d_1, 1\}.\{576(4961/48 - 238\zeta_3 + 99\zeta_4),\\ & 576(16973/288 + 221\zeta_3 - 198\zeta_4 + 72\zeta_5),\\ & -10368(55/3 - 41\zeta_3 + 12\zeta_4 + 20\zeta_5),\\ & 144(14843/36 + 722\zeta_3 + 165\zeta_4 - 816\zeta_5)\}\,,\end{aligned} \tag{15}$$

$$\begin{aligned}\gamma_{342} = {} & \{c_f^3, c_f^2, c_f d_1, c_f, d_2, d_1, 1\}.\{82944(2509/48 + 67\zeta_3 - 145\zeta_5),\\ & -1152(135571/16 + 4225\zeta_3 - 3024\zeta_3^2 - 99\zeta_4 - 18900\zeta_5 +\\ & 5400\zeta_6), -6635520(13/8 + 2\zeta_3 - 5\zeta_5), 288(476417/72\\ & -23035\zeta_3 - 25056\zeta_3^2 + 34929\zeta_4 - 44640\zeta_5 + 10800\zeta_6),\\ & 13824(230 - 2354\zeta_3 + 54\zeta_3^2 + 360\zeta_4 - 295\zeta_5 + 225\zeta_6),\end{aligned}$$

once the RCs Z_i are available, the corresponding anomalous dimensions can be extracted from the single poles, $\gamma_i = a\partial_a z_i^{(1)}$.

From the first of eq. (6), using the relation $\beta = 2(\gamma_1^{ccg} - \gamma_3^c) - \gamma_3$, this enables us to obtain the corresponding terms of the Beta function, whose coefficients we define as

$$\partial_{\ln\mu^2} a = -a\Big[\varepsilon - \beta\Big] = -a\Big[\varepsilon + b_0\, a + b_1\, a^2 + b_2\, a^3 + b_3\, a^4 + b_4\, a^5 + \dots\Big] \quad (20)$$

We refrain from listing the 2-4 loop results and only show the one-loop result for normalization and the result at five loops [18]

$$\begin{aligned}
3^1\, b_0 &= [-4]n_f + 11\,, && (21)\\
3^5\, b_4 &= b_{44}\, n_f^4 + b_{43}\, n_f^3 + b_{42}\, n_f^2 + b_{41}\, n_f + b_{40}\,, \\
b_{44} &= \Big\{c_f, 1\Big\}.\Big\{-8(107+144\zeta_3), 4(229-480\zeta_3)\Big\}\,, && (22)\\
b_{43} &= \Big\{c_f^2, c_f, d_1, 1\Big\}.\Big\{-6(4961-11424\zeta_3+4752\zeta_4), \\
&\quad -48(46+1065\zeta_3-378\zeta_4), 1728(55-123\zeta_3+36\zeta_4+60\zeta_5), \\
&\quad -3(6231+9736\zeta_3-3024\zeta_4-2880\zeta_5)\Big\}\,, && (23)\\
b_{42} &= \Big\{c_f^3, c_f^2, c_f d_1, c_f, d_2, d_1, 1\Big\}.\Big\{-54(2509+3216\zeta_3-6960\zeta_5), \\
&\quad 9(94749/2-28628\zeta_3+10296\zeta_4-39600\zeta_5), \\
&\quad 25920(13+16\zeta_3-40\zeta_5), 3(5701/2+79356\zeta_3-25488\zeta_4+43200\zeta_5), \\
&\quad -864(115-1255\zeta_3+234\zeta_4+40\zeta_5), \\
&\quad -432(1347-2521\zeta_3+396\zeta_4-140\zeta_5), \\
&\quad 843067/2+166014\zeta_3-8424\zeta_4-178200\zeta_5\Big\}\,, && (24)\\
b_{41} &= \Big\{c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_3, d_2, 1\Big\}.\Big\{-81(4157/2+384\zeta_3), \\
&\quad 81(11151+5696\zeta_3-7480\zeta_5), \\
&\quad -3(548732+151743\zeta_3+13068\zeta_4-346140\zeta_5), \\
&\quad -25920(3-4\zeta_3-20\zeta_5), \\
&\quad 8141995/8+35478\zeta_3+73062\zeta_4-706320\zeta_5, \\
&\quad 216(113-2594\zeta_3+396\zeta_4+500\zeta_5), \\
&\quad 216(1414-15967\zeta_3+2574\zeta_4+8440\zeta_5), \\
&\quad -5048959/4+31515\zeta_3-47223\zeta_4+298890\zeta_5\Big\}\,, && (25)\\
b_{40} &= \Big\{d_3, 1\Big\}.\Big\{-162(257-9358\zeta_3+1452\zeta_4+7700\zeta_5), \\
&\quad 8296235/16-4890\zeta_3+9801\zeta_4/2-28215\zeta_5\Big\}\,. && (26)
\end{aligned}$$

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