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# Texture zeros and hierarchical masses from flavour (mis)alignment


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**Abstract**

We introduce an unconventional interpretation of the fermion mass matrix elements. As the full rotational freedom of the gauge-kinetic terms renders a set of infinite bases called weak bases, basis-dependent structures as mass matrices are unphysical. Matrix invariants, on the other hand, provide a set of basis-independent objects which are of more relevance. We employ one of these invariants to give a new parametrisation of the mass matrices. By virtue of it, one gains control over its implicit implications on several mass matrix structures. The key element is the trace invariant which resembles the equation of a hypersphere with a radius equal to the Frobenius norm of the mass matrix. With the concepts of alignment or misalignment we can identify texture zeros with certain alignments whereas Froggatt–Nielsen structures in the matrix elements are governed by misalignment. This method allows further insights of traditional approaches to the underlying flavour geometry.



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# 1 Introduction

After different trials to understand the various unsolved aspects of fermion masses and mixing, the so called *flavour puzzle* still lacks for a satisfactory explanation. In spite of this, some hints could already be pointing out for a theory of flavour, see for example Refs. [1–5]. The common approaches have mainly concerned on introducing zeros (texture zeros) in the mass matrices in order to reduce the number of parameters [6–13], the use of flavour symmetries which at the same time can justify some of the aforementioned zeros [14], the use of hierarchical fermion masses to unveil the structure in fermion mixing [1, 15], the Froggatt–Nielsen mechanism [16] or extra dimensions to produce hierarchical fermion masses and mixing angles [17], among others.

The main puzzle arises from the *complete arbitrariness* in which the mass matrices appear in the Standard Model (SM), proportional to the Yukawa couplings of fermions to the Higgs field, such that after electroweak symmetry breaking, a generic fermion mass matrix is given by

$$\boldsymbol{M} = \frac{v}{\sqrt{2}}\begin{pmatrix} |y_{11}|e^{i\delta_{11}} & |y_{12}|e^{i\delta_{12}} & |y_{13}|e^{i\delta_{13}} \\ |y_{21}|e^{i\delta_{21}} & |y_{22}|e^{i\delta_{22}} & |y_{23}|e^{i\delta_{23}} \\ |y_{31}|e^{i\delta_{31}} & |y_{32}|e^{i\delta_{32}} & |y_{33}|e^{i\delta_{33}} \end{pmatrix}, \tag{1}$$

with $v = 246$ GeV the Higgs vacuum expectation value. There are in general much more parameters allowed than physical. Moreover, the question why there are three generations, so why are they $3 \times 3$ matrices stays unclear. We do not intend to resolve this open question here but rather like to scrutinise the underlying arbitrariness. A new level of understanding may be gained by a study of the generic properties of these mass matrices and identification which or how many of the available parameters can be physical at the end. Later, one may find a fundamental reason behind its construction. Regarding this two-level approach, in this letter, we provide a way to dissolve the initial arbitrariness and understand some of the phenomenological observations that have already been made. The second part lies beyond the scope of our present work.

In the limit of massless fermions, e. g. vanishing Yukawa couplings, the matter sector of the Standard Model reveals a very large accidental symmetry. This symmetry allows for some arbitrariness in the choice of a weak basis.[1] The largest flavour symmetry is given by

[1] A weak basis is a particular choice of U(3) transformations which leave the neutral and charged current interactions invariant.

the following global symmetries on the fermion fields:

$$\mathcal{G}_F \supset \mathrm{U}(3)_L^F \times \mathrm{U}(3)_R^a \times \mathrm{U}(3)_R^b, \tag{2}$$

which holds for both quarks and leptons, where $F = Q, \ell$ stands for the left-handed doublet fields and $a = u, \nu$ and $b = d, e$ for the right-handed singlets if we add 3 right-handed neutrinos to the Standard Model to be symmetric in the quark and lepton sector.[2] The mass matrices $\boldsymbol{M}_a$ and $\boldsymbol{M}_b$ are modified by these weak basis transformations,

$$\boldsymbol{M}_a' = \boldsymbol{L}_Q \boldsymbol{M}_a \boldsymbol{R}_a^\dagger \qquad \text{and} \qquad \boldsymbol{M}_b' = \boldsymbol{L}_Q \boldsymbol{M}_b \boldsymbol{R}_b^\dagger. \tag{3}$$

where left- and right-handed fields are transformed independently

$$\psi_L^F \to \boldsymbol{L}_F, \tag{4a}$$
$$\psi_R^a \to \boldsymbol{R}_a, \tag{4b}$$
$$\psi_R^b \to \boldsymbol{R}_b, \tag{4c}$$

with $\boldsymbol{X}_y \in \mathrm{U}(3)_X^y$ unitary transformations, meaning $\boldsymbol{X}_y^\dagger \boldsymbol{X}_y = \boldsymbol{X}_y \boldsymbol{X}_y^\dagger = \mathbf{1}$.

Basically, this ambiguity reveals $(3 \times 9) = 27$ free parameters which have to be balanced with $(9 \times 2 \times 2) = 36$ arbitrary parameters in the mass matrices like Eq. (1). In addition, there is a freedom of a global rephasing in each fermion sector, known as global baryon or lepton number which remains after introducing the masses. Thus, the number of physical parameters[3] apparently is given by $36 - 27 + 1 = 10$ which decomposes to the six masses, three mixing angles and one complex phase. In the case of e. g. light Majorana neutrinos, their mass matrix is constrained to be complex, but symmetric, so the counting is slightly different, especially because no $\mathrm{U}(3)_R^\nu$ freedom exists. We then have $2\times 9$ arbitrary parameters from the complex $3 \times 3$ charged lepton masses and $2 \times 6 = 12$ parameters from the complex symmetric neutrino Majorana mass, see also Section 6. In total, we are left with $30-18 = 12$ physical parameters: compared to the pure Dirac case there are two more complex phases, the well-known Majorana phases.

In the course of this letter, we present a novel route on how to relate the initially free parameters of the mass matrices with the weak basis transformations and define a new *inter-*

[2]In general, models for neutrino masses involve a much broader range of possibilities. For our study, the explicit UV complete theory of neutrino masses does not play a role and we can even work with the field content of the pure SM only (no right-handed neutrinos and only an effective mass operators for the light neutrinos).

[3]Unphysical is the full rotational freedom of the gauge-kinetic terms.

*pretation* for the individual mass matrix elements on a geometrical argument. By geometrical reasoning (as e. g. alignment/misalignment), we can dissolve the arbitrariness within a weak basis and give a way to study underlying flavour patterns through a systematical procedure. While there exists already an exhaustive literature on the problem how weak basis transformations affects flavour structures and texture zeros in a general way, see e. g. Refs. [8,10–13], our geometrical approach differs from them in its easiness and originality.

This letter is organised as follows: in Section 2, we propose a new spherical parametrisation for the magnitude of the mass matrix elements following from the matrix invariants. In Section 3, we relate the angles of the spherical mass matrix to the physical angles and discuss an explicit two-family description in Section 4. We examine the nature of texture zeros in Section 5 and in Section 6 we explore similar considerations for the case of Majorana neutrinos. The description of large fermion mass hierarchies by small angles can be found in Section 7 relating to Froggatt–Nielsen-like models. Finally, in Section 8 we conclude.

## 2 The spherical mass matrix interpretation

Let $\boldsymbol{M}$ be a generic $3\times 3$ complex mass matrix,

$$\boldsymbol{M} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}. \tag{5}$$

Its Singular Value Decomposition (SVD) is given as

$$\boldsymbol{M} = \sum_{j=1}^{3} \ell_j m_j r_j^\dagger \tag{6}$$

where $\ell_j$ and $r_j$ are the singular vectors corresponding to the $j$-th singular value (mass) $m_j$. They set up the left and right unitary transformations $\boldsymbol{L}$ and $\boldsymbol{R}$ of Eq. (3), which diagonalise the two hermitian products of $\boldsymbol{M}$: $\boldsymbol{L}^\dagger \boldsymbol{M}\boldsymbol{M}^\dagger \boldsymbol{L} = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$ and $\boldsymbol{R}^\dagger \boldsymbol{M}^\dagger \boldsymbol{M}\boldsymbol{R} = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$, respectively.

A complex $3\times 3$ matrix has three invariants that do not change under the left and right unitary transformations:

$$\xi = \frac{1}{2}\left[\mathrm{Tr}\left[\boldsymbol{M}\boldsymbol{M}^\dagger\right]^2 - \mathrm{Tr}\left[\left(\boldsymbol{M}\boldsymbol{M}^\dagger\right)^2\right]\right] = m_1^2 m_2^2 + m_2^2 m_3^2 + m_1^2 m_3^2, \tag{7}$$

$$D = \det\left[\boldsymbol{M}\boldsymbol{M}^\dagger\right] = m_1^2 m_2^2 m_3^2, \tag{8}$$

$$R^2 = \mathrm{Tr}\left[\boldsymbol{M}\boldsymbol{M}^\dagger\right] = m_1^2 + m_2^2 + m_3^2, \tag{9}$$

which can be expressed in terms of the singular values or masses. Conversely, this same set can be written using the mass matrix elements,

$$\xi = x_1 x_2 + x_1 x_3 + x_2 x_3 - (|y_1|^2 + |y_2|^2 + |y_3|^2), \tag{10}$$

$$D = x_1 x_2 x_3 - x_1 |y_3|^2 - x_2 |y_2|^2 - x_3 |y_1|^2 + 2\,\mathrm{Re}(y_1 y_2^* y_3), \tag{11}$$

$$R^2 = x_1 + x_2 + x_3, \tag{12}$$

where we have abbreviated

$$x_1 = |m_{11}|^2 + |m_{12}|^2 + |m_{13}|^2, \tag{13a}$$

$$x_2 = |m_{21}|^2 + |m_{22}|^2 + |m_{23}|^2, \tag{13b}$$

$$x_3 = |m_{31}|^2 + |m_{32}|^2 + |m_{33}|^2, \tag{13c}$$

$$y_1 = m_{11} m_{21}^* + m_{12} m_{22}^* + m_{13} m_{23}^*, \tag{13d}$$

$$y_2 = m_{11} m_{31}^* + m_{12} m_{32}^* + m_{13} m_{33}^*, \tag{13e}$$

$$y_3 = m_{21} m_{31}^* + m_{22} m_{32}^* + m_{23} m_{33}^*. \tag{13f}$$

Of course, all these equations are well-known facts and these relations already have been exploited in the flavour physics context, see e. g. Ref. [18, 19]. Nevertheless, we want to state a very pictorial *interpretation*, which can be shown to be a powerful parametrisation of the mass matrix arbitrariness. In this interpretation the trace invariant suggests a parametrisation of the matrix elements describing the surface of a hypersphere. As can be easily seen, the trace of the hermitian product is given by the sum of squared matrix elements which also defines the Frobenius norm $||\boldsymbol{M}||_F$. Thus, we have the relation

$$R^2 = \mathrm{Tr}\left[\boldsymbol{M}\boldsymbol{M}^\dagger\right] = ||\boldsymbol{M}||_F^2 = \sum_{i,j} |m_{ij}|^2. \tag{14}$$

This is the equation of a hypersphere in $n^2$ dimensions, for $i, j = 1, \ldots n$ and $n = 2, 3$ for most of our purposes. It suggests a very elegant way of parametrising the individual matrix elements in terms of spherical coordinates.

In the following, we define a slightly different notion of flavour space than what is usually understood. Mass terms are usually written in terms of Lorentz-invariants and are explicitly flavour dependent. If we wished to introduce flavour invariance we would find that it requires a more careful treatment. The notion of a flavour symmetry or a democratic approach as the one proposed in Ref. [2] are part of some of the trials to extend the flavour invariance of the kinetic terms to the Yukawa sector.

Let us already put our personal bias in the choice of coordinate system. The final values, however, do not depend explicitly on that choice as always a certain transformation can be found that redefines the axes.[4] For the hypersphere equation (14), the complex nature of the matrix elements plays no role, so for the following we consider a *real* $3\times 3$ matrix

$$\widetilde{\boldsymbol{M}} = \begin{pmatrix} \widetilde{m}_{11} & \widetilde{m}_{12} & \widetilde{m}_{13} \\ \widetilde{m}_{21} & \widetilde{m}_{22} & \widetilde{m}_{23} \\ \widetilde{m}_{31} & \widetilde{m}_{32} & \widetilde{m}_{33} \end{pmatrix}, \tag{15}$$

with

$$\widetilde{m}_{11} = R\sin\chi\,\sin\phi_1\sin\phi_2\sin\phi_3\sin\phi_4\sin\phi_5\sin\phi_6\sin\phi_7, \tag{16a}$$
$$\widetilde{m}_{12} = R\sin\chi\,\sin\phi_1\sin\phi_2\sin\phi_3\sin\phi_4\sin\phi_5\sin\phi_6\cos\phi_7, \tag{16b}$$
$$\widetilde{m}_{13} = R\sin\chi\,\sin\phi_1\sin\phi_2\sin\phi_3\sin\phi_4\sin\phi_5\cos\phi_6, \tag{16c}$$
$$\widetilde{m}_{21} = R\sin\chi\,\sin\phi_1\sin\phi_2\sin\phi_3\sin\phi_4\cos\phi_5, \tag{16d}$$
$$\widetilde{m}_{22} = R\sin\chi\,\sin\phi_1\sin\phi_2\sin\phi_3\cos\phi_4, \tag{16e}$$
$$\widetilde{m}_{23} = R\sin\chi\,\sin\phi_1\sin\phi_2\cos\phi_3, \tag{16f}$$
$$\widetilde{m}_{31} = R\sin\chi\,\sin\phi_1\cos\phi_2, \tag{16g}$$
$$\widetilde{m}_{32} = R\sin\chi\,\cos\phi_1, \tag{16h}$$
$$\widetilde{m}_{33} = R\cos\chi\,. \tag{16i}$$

[4] This freedom can be characterised by the independent permutation of columns and rows $S_{3L}\times S_{3R}$, where $S_3$ is the group of permutations of three identical objects.

The angles are $\phi_i \in [0, 2\pi)$, $i = 1, \ldots, 7$, and $\chi \in [0, \pi]$. The mass matrix is then written as,

$$\widetilde{\boldsymbol{M}} = R \begin{pmatrix} \sin\chi \left(\prod_{i=1}^{6} \sin\phi_i\right) \sin\phi_7 & \sin\chi \left(\prod_{i=1}^{6} \sin\phi_i\right) \cos\phi_7 & \sin\chi \left(\prod_{i=1}^{5} \sin\phi_i\right) \cos\phi_6 \\ \sin\chi \left(\prod_{i=1}^{4} \sin\phi_i\right) \cos\phi_5 & \sin\chi \left(\prod_{i=1}^{3} \sin\phi_i\right) \cos\phi_4 & \sin\chi \left(\prod_{i=1}^{2} \sin\phi_i\right) \cos\phi_3 \\ \sin\chi \sin\phi_1 \cos\phi_2 & \sin\chi \cos\phi_1 & \cos\chi \end{pmatrix}. \tag{17}$$

Although it does not look very advantageous to express the mass matrix elements like this, we can immediately draw some useful applications out. First, we see directly how the matrix elements can be interrelated: an adjustment in one element also affects the others unless it means exact alignment in one angle or only a small misalignment. Second, we can with a certain choice of angles immediately produce "texture zeros": null mass matrix elements at distinct positions. For example, a vanishing $m_{11}$ then could be obtained by setting $\phi_7 = 0$ without severely influencing any other matrix element (notice that $\cos\phi_7 = 1$ in $m_{12}$ and the angle appears nowhere else). Similarly, for $m_{13} = 0$ one chooses $\phi_6 = \frac{\pi}{2}$, and so on. Third, we discover that Froggatt–Nielsen-like patterns can easily be produced for small angles, see Section 7: misalignment instead of alignment. We are going to give a more physical connection to the observable and well-known flavour angles in Section 3.

It is easy to relate the mass matrix entries in this interpretation as a 9-dimensional vector

$$\overrightarrow{\mathfrak{m}} = (\widetilde{m}_{11}, \widetilde{m}_{12}, \widetilde{m}_{13}, \widetilde{m}_{21}, \widetilde{m}_{22}, \widetilde{m}_{23}, \widetilde{m}_{31}, \widetilde{m}_{32}, \widetilde{m}_{33})^T$$

to some flavour space, where we define the axes accordingly:

$$-\mathcal{L} = \sum_{i,j=1}^{3} \overline{\psi}_{L,i} \widetilde{m}_{ij} \psi_{R,j} \equiv \sum_{i,j=1}^{3} \widetilde{m}_{ij} \hat{x}_{ij}, \tag{18}$$

with $\hat{x}_{ij}$ a unit vector in the $i$-$j$ direction, where the first index refers to the left-handed fermions and the second one to the right-handed. Surely, the individual $\hat{x}_{ij}$-directions cannot be treated independently as they are the outer product of some flavour vectors and calculus rules for outer products apply. Nevertheless, we consider the vectors $\hat{x}_{ij}$ as basis of the 9 dimensional vector space spanned by the mass matrix elements describing the surface of a hypersphere. The apparent redundancy gets reduced later on.

In this interpretation, it can be easily seen that the angle $\chi$ represents the deviation of the mass vector $\overrightarrow{\mathfrak{m}}$ from the 3-3 axis ($\chi = 0$ means full alignment with the third generation

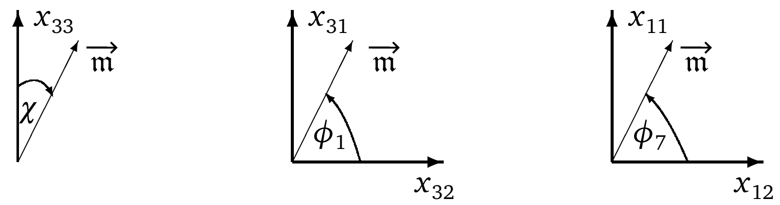


Figure 1: Visualization of the angles $\chi$, $\phi_1$, and $\phi_7$. The other $\phi_i$ follow analogously; the coordinate $x_{ij}$ represents the axis relating $i$-th and $j$-th generation $\sim \bar{\psi}_{L,i}\psi_{R,j}$.

of left- and right-handed fields[5]). The other angles represent the relative orientation with respect to two axes, so $\phi_1$ interpolates between the 3-2 and the 3-1 axis and $\phi_2$ between 3-1 and 2-3 and so forth, see Fig. 1. Notice that in our specific parametrisation from above, the last angle $\phi_7$ has the axis flipped with respect to the usual convention (i. e. in three dimensions) and $\phi_7 = 0$ means alignment with the 1-2 axis rather than 1-1, which is very useful for the application in flavour physics.

# 3 Relating mass matrix elements to physical angles

We show briefly in the following how the eight angles in the spherical mass matrix interpretation can be related to the physical angles in the mixing matrices and masses. The Frobenius norm of a general complex and rectangular $m \times n$ matrix $\boldsymbol{A}$ is given by the square root of the sum of its matrix elements $a_{ij}$ squared,

$$||\boldsymbol{A}||_F = \sqrt{\mathrm{Tr}\left(\boldsymbol{A}\boldsymbol{A}^\dagger\right)} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}. \tag{19}$$

In return, this relation may be seen as an hypersphere equation in $m \times n$ dimensions with the Frobenius norm as radius of the sphere. The corresponding spherical coordinates require $(m \times n - 1)$ angles and one radius.

On the other hand, this complex matrix has a number of $q$ non-zero and positive singular values, $\sigma_i > 0$. This defines its rank to be $q$. The Frobenius norm can also be expressed in terms of the singular values as

$$||\boldsymbol{A}||_F = \sqrt{\sum_{i=1}^{q}\sigma_i^2}, \tag{20}$$

and similarly, this characterises the surface of a $q$-dimensional hypersphere.

[5] It is interesting to notice how this is approximately true for the known values of the charged fermion masses.

For the following, we restrict ourselves to the flavour-physical case of square matrices, in particular with dimension three. We work on the surface of unit sphere, where the radius is an overall scaling factor and can be factored out by normalizing the matrix to its Frobenius norm

$$\bar{\boldsymbol{A}} = \frac{\boldsymbol{A}}{||\boldsymbol{A}||_F}. \tag{21}$$

For the normalised singular values, we define

$$\bar{\sigma}_1 = \sin\alpha \sin\beta, \qquad \bar{\sigma}_2 = \sin\alpha\cos\beta, \qquad \bar{\sigma}_3 = \cos\alpha, \tag{22}$$

with $\alpha, \beta \in [0, \frac{\pi}{2}]$ for all $\bar{\sigma}_i > 0$. The three matrix invariants expressed through Eqs. (22) are then

$$\bar{R}^2 = \mathrm{Tr}\left(\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\dagger\right) = 1, \tag{23}$$

$$\bar{D} = \det\left(\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\dagger\right) = \sin^4\alpha \sin^2\beta \cos^2\alpha \cos^2\beta, \tag{24}$$

$$\bar{\xi} = \tfrac{1}{2}\left[\mathrm{Tr}\left[\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\dagger\right]^2 - \mathrm{Tr}\left[\left(\bar{\boldsymbol{A}}\bar{\boldsymbol{A}}^\dagger\right)^2\right]\right] = \sin^2\alpha\left(\sin^2\alpha\sin^2\beta\cos^2\beta + \cos^2\alpha\right). \tag{25}$$

Eq. (22) shows that two angles are enough to describe the normalised singular values spectra, which is equivalent to the fact that only two independent mass ratios are relevant. This can be trivially extended to the $n$ family case.

The next step is to reconsider the hypersphere made out of the matrix elements which carries more information than the singular value spectrum. In this case, a nine dimensional hypersphere requires a set of eight angles as written in Eq. (16). These eight angles are to be related with the two "angles" describing the span of the singular values and furthermore $2\times3$ from the left and right unitary rotations. The two angles most tightly related to the singular values can be read from comparison with Eq. (16) and we find $\chi$ and $\phi_3$ to be important here. The other six angles, however, have to be related via the usual SVD

$$\boldsymbol{M}_f = \boldsymbol{L}_f^\dagger \boldsymbol{\Sigma}_f \boldsymbol{R}_f, \tag{26}$$

where we have three mixing angles in $\boldsymbol{L}_f$ and $\boldsymbol{R}_f$ each. Furthermore, the unitary transformations acting on the left-handed fields are physical in the sense that their combined product

$$\boldsymbol{V} = \boldsymbol{L}_a \boldsymbol{L}_b^\dagger, \tag{27}$$

describes the mixing matrix of the charged current interaction and thus the angles of $\boldsymbol{V}$ are the observable quantities. The right-handed rotations disappear from phenomenology.

The SVD is independent of the normalisation factor and is given in an explicit form with the singular values of Eq. (22)

$$\bar{\boldsymbol{M}} \equiv \frac{\boldsymbol{M}}{||\boldsymbol{M}||_F} = \boldsymbol{L}^\dagger \begin{pmatrix} \sin\alpha\sin\beta & 0 & 0 \\ 0 & \sin\alpha\cos\beta & 0 \\ 0 & 0 & \cos\alpha \end{pmatrix} \boldsymbol{R}. \tag{28}$$

The unitary transformations $\boldsymbol{L}$ and $\boldsymbol{R}$ can be parametrised by three angles and six complex phases each. Some of the phases are redundant and can be absorbed in the fermion fields, so let us for simplicity first study the rotation matrices as real matrices. The right hand side of Eq. (28) embraces eight independent angles: two from the singular values and three coming from each unitary transformation, the same amount as in $\bar{\boldsymbol{M}}$.[6] The right transformations $\boldsymbol{R}$, however, are unphysical in the sense that they drop out from physical observables and only the left rotations $\boldsymbol{L}$ play a role. Furthermore, whenever the same left transformation $\boldsymbol{L}$ is used in both mass matrices, the charged current remains invariant, so this adds three more unobservable angles. Hence, from the right transformations, there are three unphysical angles for up- and down-type fermions each, whereas from the left ones, three more are included to the sum, reaching a total of nine unphysical angles; this freedom can be used e. g. to remove mass matrix elements, i. e. introduce "texture zeros". The singular values in the reduced form lack one more parameter each, which is the Frobenius norm and sets the scale of the largest mass.

It is then a simple task to determine the "angles" $\alpha$ and $\beta$ as functions of the (normalised) singular values. With the definition of Eq. (22), the $\bar{\sigma}_i$ are the singular values of the matrix $\bar{\boldsymbol{M}}$ and one easily finds $\tan\beta = \bar{\sigma}_1/\bar{\sigma}_2$ and correspondingly $\sin\beta\tan\alpha = \bar{\sigma}_1/\bar{\sigma}_3$ for the ratios of first to second and third generation masses. So we have the identities,[7]

$$\sin\beta = \sqrt{\frac{\bar{\sigma}_1^2}{\bar{\sigma}_1^2 + \bar{\sigma}_2^2}}, \tag{29a}$$

$$\sin\alpha = \sqrt{\frac{\bar{\sigma}_1^2 + \bar{\sigma}_2^2}{\bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \bar{\sigma}_3^2}}. \tag{29b}$$

[6] This observation is rather trivial, since the number of independent parameters has to be balanced on the two sides, and for the SVD an overall factor plays no role.

[7] We employ $\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$.

# 4 The two-flavour philosophy

Although two-flavour scenarios mostly lack the complexity of the "true" three-family construction, it is very helpful to see what is going on and provide a gateway to further complications.

Let us consider an arbitrary $2 \times 2$ mass matrix,

$$\boldsymbol{m} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \tag{30}$$

with real matrix elements $m_{ij}$. A singular value decomposition of this matrix is given by $\boldsymbol{m} = \boldsymbol{L}^\dagger \boldsymbol{\Sigma} \boldsymbol{R}$ with U(2)-matrices $\boldsymbol{L}$ and $\boldsymbol{R}$ and the diagonal matrix of singular values $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \sigma_2)$ with $\sigma_2 \geqslant \sigma_1 > 0$. The matrix invariants relate the (somewhat arbitrary) entries of $\boldsymbol{m}$ with the singular values, so from the trace,

$$\mathrm{Tr}\big[\boldsymbol{m}\boldsymbol{m}^\dagger\big] = |m_{11}|^2 + |m_{12}|^2 + |m_{21}|^2 + |m_{22}|^2 = \sigma_1^2 + \sigma_2^2 = \mathrm{Tr}\big[\boldsymbol{\Sigma}^2\big] \equiv r^2. \tag{31}$$

This equation constrains the matrix elements to the surface of a four-dimensional sphere and also correlates the two singular values with a circle, $\sigma_1 = r\sin\zeta$ and $\sigma_2 = r\cos\zeta$ with $\zeta \in [0, \frac{\pi}{2}]$ to avoid any negative $\sigma_k$. Consequently, we can write

$$\begin{aligned} \boldsymbol{m} &= r \begin{pmatrix} \cos\theta_L & -\sin\theta_L \\ \sin\theta_L & \cos\theta_L \end{pmatrix} \begin{pmatrix} \sin\zeta & 0 \\ 0 & \cos\zeta \end{pmatrix} \begin{pmatrix} \cos\theta_R & \sin\theta_R \\ -\sin\theta_R & \cos\theta_R \end{pmatrix} \\ &= r \begin{pmatrix} \sin\zeta\cos\theta_L\cos\theta_R + \cos\zeta\sin\theta_L\sin\theta_R & \sin\zeta\cos\theta_L\cos\theta_R - \cos\zeta\sin\theta_L\cos\theta_R \\ \sin\zeta\sin\theta_L\cos\theta_R - \cos\zeta\cos\theta_L\sin\theta_R & \cos\zeta\cos\theta_L\cos\theta_R + \sin\zeta\sin\theta_L\sin\theta_R \end{pmatrix}. \end{aligned} \tag{32}$$

It is very intriguing to also look at the left-symmetric product in this way and discuss its relation to the choice of a weak basis. We have

$$\boldsymbol{m}\boldsymbol{m}^\dagger = \frac{r^2}{2} \begin{pmatrix} 1 - \cos(2\zeta)\cos(2\theta_L) & -\cos(2\zeta)\sin(2\theta_L) \\ -\cos(2\zeta)\sin(2\theta_L) & 1 + \cos(2\zeta)\cos(2\theta_L) \end{pmatrix}, \tag{33}$$

what trivially tells us, that $\theta_L = 0$ is the basis in which $\boldsymbol{m}\boldsymbol{m}^\dagger$ is diagonal and $\Delta\sigma^2 = \sigma_2^2 - \sigma_1^2 = \cos(2\zeta)$.

On the other hand, we can also use the spherical mass matrix interpretation to find

$$\boldsymbol{m} = r\begin{pmatrix} \sin\chi\sin\phi_1\sin\phi_2 & \sin\chi\sin\phi_1\cos\phi_2 \\ \sin\chi\cos\phi_1 & \cos\chi \end{pmatrix}. \tag{34}$$

It is not a straightforward task to build a direct connection between these angles and those appearing in Eq. (32).[8] However, the usefulness of this approach does not lie in a functional relation between matrix elements and mixing angles but rather in the minimalistic picture it offers to generate zero matrix elements or hierarchical elements and a complementary understanding of both of them.

For small angles $\rho \equiv \chi \sim \phi_1 \sim \phi_2 \ll 0$ we can perform a Taylor expansion and find

$$\boldsymbol{m} \sim \begin{pmatrix} \rho^3 & \rho^2 \\ \rho - \frac{2}{3}\rho^3 & 1 - \frac{\rho^2}{2} \end{pmatrix} + \mathcal{O}(\rho^4) \quad \text{and} \quad \boldsymbol{m}\boldsymbol{m}^\dagger \sim \begin{pmatrix} 0 & \rho^2 \\ \rho^2 & 1 \end{pmatrix} + \mathcal{O}(\rho^4), \tag{35}$$

which also justifies the discussion about hierarchical matrix elements and a vanishing 1-1 entry in the Appendix of Ref. [1]. Similarly, by setting $\phi_2 \to 0$, we insert one texture zero. Therefore, we see that there is a basis where,

$$\boldsymbol{m} = r\begin{pmatrix} 0 & \sin\chi\sin\phi_1 \\ \sin\chi\cos\phi_1 & \cos\chi \end{pmatrix}, \tag{36}$$

and one reaches the same conclusion up to $\mathcal{O}(\rho^3)$ as under the small angle approximation from Eq. (35). Furthermore, we can put Eq. (36) into the form of a Cheng–Sher ansatz $|\boldsymbol{m}_{ij}| \sim \sqrt{m_i m_j}$ [20], exploiting $\sin(\chi) = \sqrt{1-\cos^2\chi} = \sqrt{(1-\cos\chi)(1+\cos\chi)}$ (which works for $\chi \in [0, \frac{\pi}{2}]$). Defining

$$m_1 = \tfrac{r}{\sqrt{2}}(1-\cos\chi) \qquad \text{and} \qquad m_2 = \tfrac{r}{\sqrt{2}}(1+\cos\chi), \tag{37}$$

[8] A similar structure, however, can appear if instead of rotating flavour space one *shears* it. So, e. g. one finds

$$\begin{aligned} \boldsymbol{m} &= r\begin{pmatrix} 1 & \tan\zeta\sin\phi_1\cos\phi_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sin\zeta & 0 \\ 0 & \cos\zeta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \tan\zeta\cos\phi_1 & 1 \end{pmatrix} \\ &= r\begin{pmatrix} \tan\zeta\sin\zeta\sin\phi_1\cos\phi_1\cos\phi_2 & \sin\zeta\sin\phi_1\cos\phi_2 \\ \sin\zeta\cos\phi_1 & \cos\zeta \end{pmatrix} + r\begin{pmatrix} \sin\zeta & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

we have together with $\phi_1 = \frac{\pi}{4}$

$$\boldsymbol{m} = \begin{pmatrix} 0 & \sqrt{m_1 m_2} \\ \sqrt{m_1 m_2} & \frac{1}{\sqrt{2}}(m_2 - m_1) \end{pmatrix}. \tag{38}$$

# 5 Physical and unphysical zeros

It has become common use to introduce null mass matrix elements defined as a certain "ansatz" and (or) put initially complex mass matrices into hermitian form, arguing that weak basis transformations allow them [6–8, 10, 11, 13]. In this section, we shall give a direct explanation of their origin in our interpretation of mass matrices and comment on which texture zeros can be called unphysical and which other can only be due to a physical origin (e.g. a symmetries of the Lagrangian), reproducing the conclusions already reached in the literature, see Refs. [8, 10, 11, 13].

Consider the $n$ family case. As no right-handed charged currents have been observed, right-handed transformations in family space are unphysical; thus, giving a total of $n(n-1)$ arbitrary unphysical angles per fermion sector. On the other hand, unitary transformations preserving flavour invariance in the charged current interactions (weak basis transformations) will contribute to this number with $n(n-1)/2$. This set of angles, $3n(n-1)/2$ in total, is the one responsible for producing unphysical zeros in a mass matrix or equal mass matrix elements. The key difference from our approach with others is that in a very simple manner one can track the consequences of making a null element on the other matrix elements. By introducing these zeros, the vector on the surface of the hypersphere gets aligned along certain axes in flavour space as can be seen from the following subsection.

## 5.1 Nearest-Neighbour-Interaction form

For $n = 3$, we have 9 arbitrary and unphysical angles to which we can assign any value. From Eq. (16), we see that under the choice

$$\phi_{2,4,6} = \frac{\pi}{2} \quad \text{and} \quad \phi_7 = 0, \tag{39}$$

we easily generate the following well-known mass matrix, so called Nearest-Neighbour-Interaction form [8]

$$|\boldsymbol{M}| = \begin{pmatrix} 0 & A & 0 \\ A' & 0 & B \\ 0 & B' & C \end{pmatrix}, \tag{40}$$

with

$$\begin{aligned} A &= R \sin\chi \sin\phi_1 \sin\phi_3 \sin\phi_5, && (41a)\\ A' &= R \sin\chi \sin\phi_1 \sin\phi_3 \cos\phi_5, && (41b)\\ B &= R \sin\chi \sin\phi_1 \cos\phi_3, && (41c)\\ B' &= R \sin\chi \cos\phi_1, && (41d)\\ C &= R \cos\chi\,. && (41e) \end{aligned}$$

We can then reexpress the spherical coordinates by the mass matrix elements as

$$\begin{aligned} \tan\phi_5 &= \frac{A}{A'}, && (42a)\\ \tan\phi_3 &= \sqrt{1 + \left(\frac{A}{A'}\right)^2}, && (42b)\\ \tan\phi_1 &= \sqrt{1 + \left(1 + \left(\frac{A}{A'}\right)^2\right)\left(\frac{A^2}{A'B}\right)^2 \frac{B}{B'}}. && (42c) \end{aligned}$$

Moreover, we still have one more free angle by which we could choose $A' = A$, that is $\phi_5 = \pi/4$. Although this would only hold for one of the two mass matrices per fermion sector.

## 5.2 Inclusion of complex phases

The three unitary matrices giving rise to the weak basis transformations imply a total of $[3n(n+1)-2]/2$ arbitrary (unphysical) complex phases. For $n = 3$ we have 17 free phases. In order to correctly introduce them in the spherical mass matrix interpretation, we need to subtract the number of phases gone when producing null mass matrix elements. Take for example our previous case, this implies having $17 - 8 = 9$ unphysical phases left. The matrices have in total 10 complex phases. Through an appropriate choice of phases, we are allowed to keep one independent phase; which could also have been anticipated if after introducing the textures zeros, one realises that only one linear combination of phases remains in Eqs.(13),

$\gamma = \delta_{21} + \delta_{33} - \delta_{31} - \delta_{23}$. Therefore, by redefining them in such a way that only one $\delta_{21}$ survives we get

$$\boldsymbol{M}_a = \begin{pmatrix} 0 & A_a & 0 \\ A'_a & 0 & B_a \\ 0 & B'_a & C_a \end{pmatrix}, \qquad \boldsymbol{M}_b = \begin{pmatrix} 0 & A_b e^{i\gamma} & 0 \\ A_b e^{-i\gamma} & 0 & B_b \\ 0 & B'_b & C_b \end{pmatrix}, \tag{43}$$

giving a total of ten independent parameters in accordance with the ones appearing in the mass basis. So we see that by relating the weak basis transformations to the spherical mass matrix interpretation allows us to directly write the matrix forms with all their redundancy now ripped off.

## 5.3 Hermiticity and texture zeros

By demanding hermitian matrices, there is a cost one should pay, which is on one hand 6 of the 9 angles have been employed while on the other, 12 of the 17 available phases have also been used. Therefore, the introduction of further constraints as null matrix elements should be limited to only 3 free angles and 5 complex phases. So equally distributing three null mass matrix elements between two matrices is impossible. That is, within the traditional approach, no parallel structures with zero elements can be obtained via weak basis transformations when hermiticity has been first invoked.[9] Within our approach this can also be done. However, taking a look at Eqs. (16), an alternative scenario appears in which parallel structures *seem* to be allowed. In the following we will discuss the former scenario (no-parallel structures) and then we will clarify the issue of the alternative one (parallel structures).

Let us show it. For the former point, first apply the hermiticity condition and thereafter the spherical mass matrix interpretation. The space of the hypersphere now gets reduced

[9] Parallel structures are such matrix structures, where both matrices in the same fermion sector (quark or lepton) shares their matrix form.

from nine dimensions to only six with the matrix elements given by

$$\widetilde{m}_{11} = R\sin\chi\,\sin\phi_1\sin\phi_2\sin\phi_3\sin\phi_4, \tag{44a}$$
$$\widetilde{m}_{12} = R\sin\chi\,\sin\phi_1\sin\phi_2\sin\phi_3\cos\phi_4, \tag{44b}$$
$$\widetilde{m}_{13} = R\sin\chi\,\sin\phi_1\sin\phi_2\cos\phi_3, \tag{44c}$$
$$\widetilde{m}_{22} = R\sin\chi\,\sin\phi_1\cos\phi_2, \tag{44d}$$
$$\widetilde{m}_{23} = R\sin\chi\,\cos\phi_1, \tag{44e}$$
$$\widetilde{m}_{33} = R\cos\chi\,. \tag{44f}$$

Note, that for a hermitian matrix, one has an overcounting for the Frobenius norm from the off-diagonal elements, so we define the mass matrix as

$$\boldsymbol{M} = \begin{pmatrix} \widetilde{m}_{11} & \frac{1}{\sqrt{2}}\widetilde{m}_{12}e^{i\delta_{12}} & \frac{1}{\sqrt{2}}\widetilde{m}_{13}e^{i\delta_{13}} \\ \frac{1}{\sqrt{2}}\widetilde{m}_{12}e^{-i\delta_{12}} & \widetilde{m}_{22} & \frac{1}{\sqrt{2}}\widetilde{m}_{23}e^{i\delta_{23}} \\ \frac{1}{\sqrt{2}}\widetilde{m}_{13}e^{-i\delta_{13}} & \frac{1}{\sqrt{2}}\widetilde{m}_{23}e^{-i\delta_{23}} & \widetilde{m}_{33} \end{pmatrix}. \tag{45}$$

In this sense, we have now five angles from which two may correspond to the singular values and the other three allow to introduce texture zeros. Nevertheless, in total we have no more than three free angles for both matrices. So following this, we can produce the next kind of no-parallel weak basis matrices,

$$\boldsymbol{M}_a = \begin{pmatrix} 0 & A_a & 0 \\ A_a & B_a & C_a \\ 0 & C_a & D_a \end{pmatrix}, \qquad \boldsymbol{M}_b = \begin{pmatrix} A_b & B_b e^{i\beta} & 0 \\ B_b e^{-i\beta} & C_b & D_b \\ 0 & D_b & E_b \end{pmatrix}, \tag{46}$$

thus reaching the traditional conclusions [11]. Apparently, we have chosen $\phi_4^a = 0$, $\phi_3^{a(b)} = \frac{\pi}{2}$. First of all, there is no *physical meaning* attached to any of those zeros in a certain weak basis like the one we have singled out here. We have to reduce the number of free parameters to ten—how this is achieved should have no influence on the observable physics. Second, there can be no parallel structures for hermitian matrices with only one complex phase. However, with $\phi_4^b = 0$, one either has to introduce an additional phase or one can construct a *prediction* of one of the SM parameters in terms of the others. This is only valid by *adhoc* assumptions or proposing a kind of flavour symmetry. In the latter case, there is, of course, a physical meaning associated with it; see for example [21].

One remark about alternative scenarios and possible loopholes in our interpretation: Notice that if we had considered $\phi_2^b = \frac{\pi}{2}$ in the second matrix we could have found a parallel structure. And moreover, for $\phi_3^b = 0$ and $\phi_3^a = 0$ plus an adequate initial reordering of the matrix entries, we could have found another parallel structure. Therefore, it *seems* that we can indeed build parallel structures with more than three independent texture zeros together with hermiticity. What seems to be wrong? Both alternative scenarios reach a weak basis with less than ten arbitrary parameters. But this contradicts our interpretation on the angles which corresponded to the freedom in the weak basis transformations (one cannot have a *weak basis* with less than ten arbitrary parameters). Hence, the alternative scenarios are not valid within the approach.

## 5.4 Deviations from hermiticity in the Nearest-Neighbour-Interaction form

From the point of view of our approach and the traditional ones, producing the Nearest-Neighbour-Interaction form together with an hermitian matrix, is impossible. However, from Eq.(43), we could work out the deviations from hermiticity if we work in the small angle approximation (further results about small angles in the next section). With the assignment

$$\phi_7^{a(b)} = 0, \phi_{2,4,6}^{a(b)} = \frac{\pi}{2}, \phi_5^a = \frac{\pi}{4}, \phi_1^a = \frac{\pi}{4} + \varepsilon_1^a, \text{ and } \phi_{1,5}^b = \frac{\pi}{4} + \varepsilon_{1,5}^b,$$

where $\varepsilon_j \ll 1$, we get the following:

$$\boldsymbol{M}_a \simeq \begin{pmatrix} 0 & A_a & 0 \\ A_a & 0 & B_a \\ 0 & B_a & C_a \end{pmatrix} + \varepsilon_1^a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B_a \\ 0 & -B_a & 0 \end{pmatrix} + \mathcal{O}\left((\varepsilon_5^a)^2\right), \quad (47)$$

$$\boldsymbol{M}_b \simeq \begin{pmatrix} 0 & A_b e^{i\beta} & 0 \\ A_b e^{-i\beta} & 0 & B_b \\ 0 & B_b & C_b \end{pmatrix} + \varepsilon_1^b \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B_b \\ 0 & -B_b & 0 \end{pmatrix} + \varepsilon_5^b \begin{pmatrix} 0 & A_b e^{i\beta} & 0 \\ -A_b e^{-i\beta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{O}\left((\varepsilon_{1,5}^b)^2\right). \quad (48)$$

It can be readily seen how the presence of the small deviations helps to the counting of ten free parameters within the weak basis. This approach reduces from four to three parameters, as previously used [12,22], to measure the deviations from hermiticity. It is a straightforward task to determine that this set of parameters reproduce both the masses and the mixing in the quark sector.

# 6 Majorana neutrinos

Massive neutrinos are not part of the renormalisable Standard Model. There is, however, one single operator at dimension five that can generate very small neutrino masses for the left-handed neutrinos only [23], without introducing right-handed neutrinos. The UV-completion of this operator will reveal some new physics at the scale $\Lambda_{\text{NP}}$. This operator requests the resulting mass matrix to be of the Majorana type, meaning complex but symmetric. It is a gauge- and Lorentz-invariant construction:

$$\mathcal{L}_5 = \frac{1}{2}\frac{c_{\alpha\beta}}{\Lambda_{\text{NP}}}\left(\bar{L}^c_{L\alpha}\widetilde{H}^*\right)\left(\widetilde{H}^\dagger L_{L\beta}\right) + \text{h. c.}\,, \tag{49}$$

where $L_L = (\nu_L, e_L)^T$ and $H = (H^+, H^0)^T$ are the left-handed lepton and the Higgs doublet of the SM, respectively; we follow the usual notation for the charged conjugated Higgs field as $\widetilde{H} = i\sigma_2 H^*$. The coefficients $c_{\alpha\beta}$ are arbitrary numbers, but supposed to be $\mathcal{O}(1)$ numbers or show some rather mild hierarchy which is imprinted in the neutrino mass spectrum. The whole operator is suppressed by the new physics scale, $\sim 1/\Lambda_{NP}$, which can be $\mathcal{O}(10^{10\ldots14}\,\text{GeV})$.

We want to study the different zero elements that could arise from weak basis transformations. The flavour group for the lepton sector is

$$\mathcal{G}_F \supset U^\ell_L(3) \times U^e_R(3). \tag{50}$$

The above group of transformations can be used to diagonalise the charged lepton mass matrix. In this weak basis, which we could call the charged lepton basis, the symmetrical mass matrix of neutrinos gets diagonalised by the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix. So we immediately reach the conclusion that as no freedom is left to still make weak basis transformations, any texture zero in the neutrino mass matrix will be physical as long as we are in the charged lepton basis.

## 6.1 Weak bases

Let us consider those weak bases where the charged lepton mass matrices are still non diagonal. This discussion not only reproduces some known facts, as of Ref. [24], but, if extended, may provide further observations. From the six free angles, we can choose four of them as

$\phi^e_7 = 0$ and $\phi^e_{2,4,6} = \frac{\pi}{2}$ in the spherical mass matrix interpretation, to get e. g.

$$\boldsymbol{M}_e = \begin{pmatrix} 0 & A_a e^{-i\delta} & 0 \\ A'_e e^{i\delta} & 0 & B_e \\ 0 & B'_e & C_e \end{pmatrix}. \tag{51}$$

The neutrino mass matrix, however, has to be symmetric. We change the notation slightly and perform a renaming $\phi \to \omega$ in the angles to show the difference. Hence, we have the following entries

$$\widetilde{m}^{\nu}_{11} = R^{\nu} \sin\chi^{\nu} \sin\omega^{\nu}_1 \sin\omega^{\nu}_2 \sin\omega^{\nu}_3 \sin\omega^{\nu}_4, \tag{52a}$$
$$\widetilde{m}^{\nu}_{12} = R^{\nu} \sin\chi^{\nu} \sin\omega^{\nu}_1 \sin\omega^{\nu}_2 \sin\omega^{\nu}_3 \cos\omega^{\nu}_4, \tag{52b}$$
$$\widetilde{m}^{\nu}_{13} = R^{\nu} \sin\chi^{\nu} \sin\omega^{\nu}_1 \sin\omega^{\nu}_2 \cos\omega^{\nu}_3, \tag{52c}$$
$$\widetilde{m}^{\nu}_{22} = R^{\nu} \sin\chi^{\nu} \sin\omega^{\nu}_1 \cos\omega^{\nu}_2, \tag{52d}$$
$$\widetilde{m}^{\nu}_{23} = R^{\nu} \sin\chi^{\nu} \cos\omega^{\nu}_1, \tag{52e}$$
$$\widetilde{m}^{\nu}_{33} = R^{\nu} \cos\chi^{\nu}, \tag{52f}$$

of the complex symmetric matrix

$$\boldsymbol{M}^{\nu} = \begin{pmatrix} \widetilde{m}^{\nu}_{11} e^{i\varphi^{\nu}_{11}} & \frac{1}{\sqrt{2}}\widetilde{m}^{\nu}_{12} e^{i\varphi^{\nu}_{12}} & \frac{1}{\sqrt{2}}\widetilde{m}^{\nu}_{13} e^{i\varphi^{\nu}_{13}} \\ \frac{1}{\sqrt{2}}\widetilde{m}^{\nu}_{12} e^{i\varphi^{\nu}_{12}} & \widetilde{m}^{\nu}_{22} e^{i\varphi^{\nu}_{22}} & \frac{1}{\sqrt{2}}\widetilde{m}^{\nu}_{23} e^{i\varphi^{\nu}_{23}} \\ \frac{1}{\sqrt{2}}\widetilde{m}^{\nu}_{13} e^{i\varphi^{\nu}_{13}} & \frac{1}{\sqrt{2}}\widetilde{m}^{\nu}_{23} e^{i\varphi^{\nu}_{23}} & \widetilde{m}^{\nu}_{33} e^{i\varphi_{33}} \end{pmatrix}. \tag{53}$$

From the two remaining unphysical degrees of freedom, we can induce several texture zeros in the neutrino masses, e. g. with $\omega^{\nu}_4 = 0$ and $\omega^{\nu}_3 = \frac{\pi}{2}$ we find

$$\boldsymbol{M}_{\nu} = \begin{pmatrix} 0 & A_{\nu} e^{-i\alpha_1} & 0 \\ A_{\nu} e^{-i\alpha_1} & B_{\nu} & C_{\nu} e^{-i\alpha_2} \\ 0 & C_{\nu} e^{-i\alpha_2} & D_{\nu} \end{pmatrix}. \tag{54}$$

It is outside the scope of this work to provide an exhaustive list of different weak basis matrix forms. Therefore, the take-home message lies in the simplicity of the spherical mass matrix interpretation on studying matrices in different weak bases.

## 6.2 Phenomenological application: the Altarelli–Feruglio model

The charged lepton basis is ideal to get further insights into the masses or mixing of neutrinos, as everything is extracted from their mass matrix. In this regard, the famous Altarelli–Feruglio model provides us with a good example [25, 26]. The model implements the $A_4$ non-Abelian and discrete symmetry group inside a Frogatt–Nielsen framework. It naturally implies tribimaximal mixing (TBM) for the PMNS matrix [27]:

$$\boldsymbol{U}_{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \tag{55}$$

Its weak point, however, is that the reactor mixing angle is predicted to be exactly zero, $\theta_{13}^{\nu} = 0$, so it is in the meantime excluded by experimental observation [28–30]. Nevertheless, the main ingredient of the model, the underlying tribimaximal mixing, still can be relevant for a partial diagonalisation. The fact, that neutrino masses are less hierarchical than charged fermion masses, suggest a more democratic flavour pattern, which is related to tribimaximal mixing.

Three main features characterize the Altarelli–Feruglio mass matrix: $m_{12}^{\nu} = m_{13}^{\nu}$, $m_{22}^{\nu} = m_{33}^{\nu}$, and $m_{22}^{\nu} = -2m_{12}^{\nu}$. Under the spherical mass matrix interpretation we look for the consequences of implementing them starting from the charged lepton basis.

We assume the following assignment of the real matrix elements:

$$\widetilde{m}_{11}^{\nu} = R^{\nu} \sin\chi^{\nu} \sin\omega_1^{\nu} \sin\omega_2^{\nu} \cos\omega_3^{\nu}, \tag{56a}$$

$$\widetilde{m}_{12}^{\nu} = R^{\nu} \sin\chi^{\nu} \sin\omega_1^{\nu} \sin\omega_2^{\nu} \sin\omega_3^{\nu} \sin\omega_4^{\nu}, \tag{56b}$$

$$\widetilde{m}_{13}^{\nu} = R^{\nu} \sin\chi^{\nu} \sin\omega_1^{\nu} \sin\omega_2^{\nu} \sin\omega_3^{\nu} \cos\omega_4^{\nu}, \tag{56c}$$

$$\widetilde{m}_{22}^{\nu} = R^{\nu} \sin\chi^{\nu} \sin\omega_1^{\nu} \cos\omega_2^{\nu}, \tag{56d}$$

$$\widetilde{m}_{23}^{\nu} = R^{\nu} \sin\chi^{\nu} \cos\omega_1^{\nu}, \tag{56e}$$

$$\widetilde{m}_{33}^{\nu} = R^{\nu} \cos\chi^{\nu}. \tag{56f}$$

The equality of $\widetilde{m}_{12}^{\nu} = \widetilde{m}_{13}^{\nu}$ implies a basis choice in which $\omega_4^{\nu} = \frac{\pi}{4}$. On the other hand, with $\widetilde{m}_{33}^{\nu} = \widetilde{m}_{22}^{\nu}$ one needs $\tan\chi^{\nu} \geq 1$ and thus $\chi^{\nu} \in [\frac{\pi}{4}, \frac{\pi}{2})$. Note how one may identify the particular choice of the elements with a particular orientation of the mass vector in the flavour basis. Last, we require $\widetilde{m}_{22}^{\nu} = -2\widetilde{m}_{12}^{\nu}$ and see that it is only fulfilled with $\omega_3^{\nu} = \frac{3\pi}{2}$ and

$\omega_2^\nu = \frac{5\pi}{4}$. Under these conditions one gets the following mass matrix,

$$|\boldsymbol{M}_\nu| = \begin{pmatrix} 0 & a^\nu & a^\nu \\ a^\nu & -2a^\nu & b^\nu \\ a^\nu & b^\nu & -2a^\nu \end{pmatrix}, \tag{57}$$

where we have $a^\nu = \frac{R^\nu}{2\sqrt{2}} \sin\chi^\nu \sin\omega_1^\nu$ and $b^\nu = \frac{R^\nu}{\sqrt{2}} \sin\chi^\nu \cos\omega_1^\nu$, and the relation

$$\tan\chi^\nu \sin\omega_1^\nu = -\sqrt{2}. \tag{58}$$

Notice that it does not reproduce the full Altarelli–Feruglio mass matrix (e.g. $m_{11}^\nu \neq 0$). Therefore, we expect a deviation from tribimaximal mixing, which is actually required by experiment. The vanishing 1-1 element in our case is a direct consequence of the spherical mass matrix interpretation as the individual elements are not fully independent.

Let us decompose the mass matrix into a democratic part and a remainder which only has 2-3 mixing

$$|\boldsymbol{M}_\nu| = \begin{pmatrix} a^\nu & a^\nu & a^\nu \\ a^\nu & a^\nu & a^\nu \\ a^\nu & a^\nu & a^\nu \end{pmatrix} + \begin{pmatrix} -a^\nu & 0 & 0 \\ 0 & -3a^\nu & b^\nu - a^\nu \\ 0 & b^\nu - a^\nu & -3a^\nu \end{pmatrix}. \tag{59}$$

The first term gets diagonalised by the tribimaximal mixing matrix. After that, we have

$$|\boldsymbol{M}'_\nu| = \begin{pmatrix} \frac{1}{3}(-6a^\nu + b^\nu) & \frac{\sqrt{2}}{3}(3a^\nu - b^\nu) & 0 \\ \frac{\sqrt{2}}{3}(3a^\nu - b^\nu) & \frac{2b^\nu}{3} & 0 \\ 0 & 0 & -2a^\nu - b^\nu \end{pmatrix}, \tag{60}$$

which still requires a further diagonalisation. This, however, can be done trivially. The full PMNS matrix is then given by the initial tribimaximal mixing matrix, corrected with the diagonalisation of Eq. (60). Since there are furthermore only two independent parameters, $a^\nu$ and $b^\nu$, the mass spectrum as well as the neutrino mixing matrix can be fully determined by a fit to the experimentally known mass squared differences only. With the most recent results of [31][10]

$$\Delta m_{21}^2 = 7.40 \times 10^{-5}\,\text{eV}^2, \qquad \text{and} \qquad \Delta m_{31}^2 = 2.494 \times 10^{-3}\,\text{eV}^2,$$

[10] Similar results can be found in other sources like [32]. We only perform a proof of principle here and also do no error analysis, just to see whether we roughly get the right numbers.

we obtain, assuming normal hierarchy and ignoring the errorbars, we get two real and positive solutions for $a^\nu$ and $b^\nu$, that are very close

$$a^\nu \in \{0.0127, 0.0138\}\,\mathrm{eV}, \qquad \text{and} \tag{61a}$$
$$b^\nu \in \{0.0274, 0.0257\}\,\mathrm{eV}. \tag{61b}$$

This determines the neutrino mass spectrum to be for the two solutions

$$m_3^\nu = \{0.0527, 0.0533\}\,\mathrm{eV}, \quad m_2^\nu = \{0.0190, 0.0205\}\,\mathrm{eV}, \quad m_1^\nu = \{0.0169, 0.0186\}\,\mathrm{eV}, \tag{62}$$

and the PMNS matrix for both the cases

$$|\boldsymbol{U}_{\mathrm{PMNS}}| = \left\{ \begin{pmatrix} 0.727 & 0.686 & 0 \\ 0.485 & 0.514 & 0.707 \\ 0.485 & 0.514 & 0.707 \end{pmatrix}, \begin{pmatrix} 0.724 & 0.690 & 0 \\ 0.488 & 0.512 & 0.707 \\ 0.488 & 0.512 & 0.707 \end{pmatrix} \right\}. \tag{63}$$

Apparently, this PMNS matrix cannot describe the true neutrino phenomenology, which is also not surprising: the Altarelli–Feruglio models were invented to predict a $\theta_{13}^\nu = 0$, and staying within the underlying pattern for the mass matrix, we cannot generate a non-vanishing entry there.

It is, however, astonishingly simple to correct for a non-vanishing 1-3 mixing. The Altarelli–Feruglio matrix cannot have a 1-3 mixing: from Eq. (59), we see that the non-democratic part of the mass matrix does not mix the first and third generation. We can nevertheless accomodate for it by a small *misalignment* of the two elements $\widetilde{m}_{12}^\nu$ and $\widetilde{m}_{13}^\nu$, simply with the choice $\omega_4^\nu = \frac{\pi}{4} + \varepsilon$, leading to a corrected mass matrix

$$|\boldsymbol{M}_\nu| = \begin{pmatrix} 0 & a^\nu + \delta^\nu & a^\nu - \delta^\nu \\ a^\nu + \delta^\nu & -2a^\nu & b^\nu \\ a^\nu - \delta^\nu & b^\nu & -2a^\nu \end{pmatrix} + \mathcal{O}(\varepsilon^2), \tag{64}$$

with $\delta^\nu = a^\nu \varepsilon$. With $\delta^\nu$, we have a handle on $\theta_{13}^\nu$ and in order to generate $\sin\theta_{13}^\nu \approx 0.15$, we find $\delta^\nu = 0.005\,\mathrm{eV}$ and one set of solutions with

$$a^\nu = 0.0126\,\mathrm{eV}, \qquad \text{and} \qquad b^\nu = 0.0263\,\mathrm{eV}, \tag{65}$$

resulting in a slightly modified mass spectrum

$$m_3^\nu = 0.0526\,\mathrm{eV}, \quad m_2^\nu = 0.0187\,\mathrm{eV} \quad \text{and} \quad m_1^\nu = 0.0166\,\mathrm{eV}. \tag{66}$$

This naïve correction still has some tension in comparison with the global fit values of the PMNS matrix. We find

$$|\boldsymbol{U}_{\mathrm{PMNS}}| = \begin{pmatrix} 0.696 & 0.702 & 0.150 \\ 0.398 & 0.551 & 0.733 \\ 0.598 & 0.451 & 0.663 \end{pmatrix}. \tag{67}$$

We nowadays have strong hints of $CP$ violation in neutrino oscillations, besides the fact that the third mixing angle is definitely non-zero. Furthermore, recent global fits tend towards a rather maximal $CP$-phase in the Standard Parametrisation ($\delta_{CP} = 234^{+43\,\circ}_{-31}$ [31]), which is compatible with $\delta_{CP} \approx -90^\circ$. TBM mixing is thus ruled out and the Altarelli–Feruglio model has to be adjusted for this, including $CP$ violation. This easily can be accommodated within the approach presented above. Let us consider an *imaginary* perturbation, $\omega_4^\nu = \frac{\pi}{4} + \mathrm{i}\,\varepsilon$, and thus $\sin(\omega_4^\nu) \approx (1 + \mathrm{i}\,\epsilon)\sqrt{2}$, we can simply multiply $\delta^\nu$ with a maximal complex phase $e^{\mathrm{i}\,\pi/2}$. Keeping $\delta^\nu = 0.005$, to achieve a large $\sin\theta_{13}^\nu \approx 0.15$, this modifies slightly the mass eigenvalues. Hence, to reproduce the proper $\Delta m^2$, we have to refit the $a^\nu$ and $b^\nu$ parameters and find

$$a^\nu = 0.0127\,\mathrm{eV}, \qquad \text{and} \qquad b^\nu = 0.0285\,\mathrm{eV}, \tag{68}$$

and correspondingly

$$m_3^\nu = 0.0528\,\mathrm{eV}, \quad m_2^\nu = 0.0193\,\mathrm{eV} \quad \text{and} \quad m_1^\nu = 0.0172\,\mathrm{eV}. \tag{69}$$

The PMNS matrix now has a complex phase and is given by

$$\boldsymbol{U}_{\mathrm{PMNS}} = \begin{pmatrix} 0.742 & 0.668 & -0.00715 + 0.148\mathrm{i} \\ -0.463 + 0.101\mathrm{i} & 0.524 + 0.0456\mathrm{i} & -0.696 - 0.0673 \\ -0.463 - 0.101\mathrm{i} & 0.524 - 0.0456\mathrm{i} & 0.699 \end{pmatrix}. \tag{70}$$

This has surprisingly a $CP$-phase $\delta_{CP} = -0.485\pi$ in accordance with the global fit.

# 7 Small angles and hierarchies

Generically, it is believed that any kind of hierarchy in the eigenvalues (singular values) of a mass matrix has to be already coded in the hierarchical structure of the individual matrix elements, as was proposed by Froggatt and Nielsen [16]

$$-\mathcal{L}_{\mathrm{FN}}=\sum_{n,\psi}\bar{\psi}_{L,i}\psi_{R,j}H\lambda_{ij}^{\psi}\left(\frac{\varphi}{\Lambda}\right)^{n_{ij}}+\mathrm{h.\,c.}\,, \tag{71}$$

where $\psi_i$ are generic fermions with $i=1,2,3$ counting the number of generations, $H$ being the SM Higgs doublet breaking electroweak gauge symmetry and $\varphi$ a flavon field breaking the continous and global flavour symmetry. The flavour symmetry is assumed to be an Abelian $\mathrm{U}(1)_F$ global symmetry and the "coupling constants" $\lambda_{ij}^{\psi}$ are supposed to be *arbitrary* $\mathcal{O}(1)$ numbers, where the additional scale $\Lambda$ refers to a larger scale at which new degrees of freedom are integrated out. So the final "Yukawa couplings" as effective couplings of the SM fermions to the SM Higgs are given by

$$Y_{ij}^{f}=\lambda_{ij}^{f}\left(\frac{\langle\varphi\rangle}{\Lambda}\right)^{n_{ij}}, \tag{72}$$

with $n_{ij}\in\mathbb{N}$ being the sum of the corresponding $\mathrm{U}(1)_F$ charges. The hierarchical fermion masses are then encoded in powers of a small parameter $\varepsilon=\langle\varphi\rangle/\Lambda$. As numerical example: take $\Lambda$ to be 10 TeV and $\langle\varphi\rangle$ to be of the electroweak scale $\sim 100\,\mathrm{GeV}$, then $\varepsilon\simeq 10^{-2}$.

Therefore, apparently, a hierarchical matrix configuration can only be attached to the idea of a complicated mechanism fully responsible for it. The art of finding a viable UV-completion of this model typically leads to vastly extended sets of matter and scalar fields and may not be called aesthetic. In the following, we explore a different route to arrive at a very similar suppression of small numbers by high powers employing the spherical mass matrix interpretation. The small numbers then arise from a small misalignment of the mass vector with respect to the underlying flavour basis.

Let us consider all the angles very close to zero, so the actual vector in the 9-dimensional space points along the $m_{33}$-axis. Surprisingly, one finds immediately Froggatt–Nielsen-like structures. Let us take all angles to be of the same order, say $\varepsilon\equiv\chi\sim\phi_k^{a(b)}\ll 1$, and we get

$$|\boldsymbol{M}|\sim R\begin{pmatrix}\varepsilon^8 & \varepsilon^7 & \varepsilon^6\\ \varepsilon^5 & \varepsilon^4 & \varepsilon^3\\ \varepsilon^2 & \varepsilon & 1\end{pmatrix}, \tag{73}$$

*without* referring to a Froggatt–Nielsen (FN) mechanism of Eq. (71). Notice also, that the pattern of Eq. (73) is not unique and, moreover, there is no reason not to treat individual angles individually. A very obvious transformation of this kind is $\chi \to \chi - \frac{\pi}{2}$, then the 3-3 element becomes $\sim \varepsilon$ and the power of epsilons is reduced by one on the other elements. The key part in this construction is, that—depending on the alignment in the abstract high dimensional space—hierarchies can be generated by the choice of the basis and a hierarchical basis as suggested by the FN mechanism does not imply hierarchy of new physics scales. Finally, the relevant object to construct the mixing matrix is the left-hermitian product

$$|\boldsymbol{M}\boldsymbol{M}^\dagger| \sim R^2 \begin{pmatrix} \varepsilon^{12} + \mathcal{O}(\varepsilon^{14}) & \varepsilon^9 + \mathcal{O}(\varepsilon^{11}) & \varepsilon^6 + \mathcal{O}(\varepsilon^8) \\ \varepsilon^9 + \mathcal{O}(\varepsilon^{11}) & \varepsilon^6 + \mathcal{O}(\varepsilon^8) & \varepsilon^3 + \mathcal{O}(\varepsilon^5) \\ \varepsilon^6 + \mathcal{O}(\varepsilon^8) & \varepsilon^3 + \mathcal{O}(\varepsilon^5) & 1 + \varepsilon^2 + \mathcal{O}(\varepsilon^4) \end{pmatrix} \tag{74}$$

which shows a *strong* hierarchical structure.

Now, let us give a twist to the story. As previously noted, hierarchical mass ratios are a direct consequence of *only* two small angles, if we assign spherical coordinates to the singular values in a similar manner. Accordingly, there is no need to have all the eight angles as small numbers, $\phi_i, \chi \ll 1$. So to produce mass hierarchies, we actually do not need such a very strong suppression in all matrix elements. It is sufficient to have the following kind of mild hierarchical structures:

$$|\boldsymbol{M}| \sim R \begin{pmatrix} \varepsilon^2 & \varepsilon^2 & \varepsilon^2 \\ \varepsilon^2 & \varepsilon^2 & \varepsilon \\ \varepsilon & \varepsilon & 1 \end{pmatrix} \qquad \Rightarrow \qquad |\boldsymbol{M}\boldsymbol{M}^\dagger| \sim R^2 \begin{pmatrix} \varepsilon^4 & \varepsilon^3 & \varepsilon^2 \\ \varepsilon^3 & \varepsilon^2 & \varepsilon \\ \varepsilon^2 & \varepsilon & 1+\varepsilon^2 \end{pmatrix}. \tag{75}$$

# 8 Conclusions

We have introduced a new and innovative interpretation of the fermion mass matrix elements in the SM. This interpretation allows cross-relations to weak basis transformations. The key element is found in one of the matrix invariants involving the trace of the left-hermitian product. Its equation simultaneously describes the surface of a nine dimensional hypersphere with its radius equal to the Frobenius norm of the mass matrix. This interpretation is trivially not constrained to three families but applies to all $n \times n$ mass matrices. The idea of assigning to each matrix element a basis of spherical coordinates, provides a framework to correlate their magnitudes in a very simple manner. Moreover, it can be seen from this approach that individual matrix elements cannot be set to zero without affecting also others. There are eight angles needed in the spherical mass matrix interpretation which can be furthermore related to the weak basis angles and the singular values of the mass matrices. Therefore, this interpretation also allows to relate introduction of null elements, so called texture zeros, to a geometrical alignment in the underlying flavour space.

A very compelling application of this approach has been found in the neutrino sector. The main characteristics of the neutrino mass matrix in the Altarelli–Feruglio can be mapped to a set of conditions for the angles in the spherical mass matrix interpretation. Within the Altarelli–Feruglio model, we have been able to fully determine the mass spectrum as well as the neutrino mixing matrix. By virtue of a small correction in terms of a perturbation of one of the angles, we furthermore could reproduce a large reactor angle which is initially zero in that model. Moreover, with a purely imaginary perturbation, the value of the Dirac $CP$-phase in the PMNS matrix turns out to be close to the value favoured by the global fit.

In the same approach, with a small misalignment, it is easy to reproduce Froggatt–Nielsen like patterns for hierarchical mass matrices without the need of introducing a new physics scale or a complicated UV-completion for such suppression. Nevertheless, the mechanism behind this misalignment stays unclear at this stage. The spherical mass matrix interpretation is not to be seen as a dynamical model of flavour but shall rather help to simplify model assumptions behind such models. With the interpretation of aligning or misaligning individual mass matrix elements with a certain direction in flavour space, it might be possible to draw conclusions going further than texture zeros. We want to remind, that actually the flavour bases for the up and down sector are not fully independent in the spherical mass matrix interpretation and thus, in a more deeper analysis, relations between up- and down-type fermion masses shall be revealed.


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