

## Four-loop results on anomalous dimensions and splitting functions in QCD

---

### A. Vogt\*

*Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, UK*

*E-mail: Andreas.Vogt@liverpool.ac.uk*

### S. Moch

*II. Institut für Theoretische Physik, Universität Hamburg, D-22761 Hamburg, Germany*

*E-mail: sven-olaf.moch@desy.de*

### B. Ruijl

*Institute for Theoretical Physics, ETH Zürich, 8093 Zürich, Switzerland*

*E-mail: bruijl@phys.ethz.ch*

### T. Ueda, J.A.M. Vermaseren

*Nikhef Theory Group, Science Park 105, 1098 XG Amsterdam, The Netherlands*

*E-mails: tueda@nikhef.nl, t68@nikhef.nl*

We report on recent progress on the flavour non-singlet splitting functions in perturbative QCD. The exact four-loop ( $N^3\text{LO}$ ) contribution to these functions has been obtained in the planar limit of a large number of colours. Phenomenologically sufficient approximate expressions have been obtained for the parts not exactly known so far. Both cases include results for the four-loop cusp and virtual anomalous dimensions which are relevant well beyond the evolution of non-singlet quark distributions, for which an accuracy of (well) below 1% has now been reached.

*13th International Symposium on Radiative Corrections (Applications of Quantum Field Theory to Phenomenology)*

*25-29 September 2017, St. Gilgen, Austria*

---

\*Speaker.

## 1. Introduction

Up to power corrections, observables in  $ep$  and  $pp$  hard scattering can be schematically expressed as

$$O^{ep} = f_i \otimes c_i^0, \quad O^{pp} = f_i \otimes f_k \otimes c_{ik}^0 \quad (1.1)$$

in terms of the respective partonic cross sections (coefficient functions)  $c^0$  and the universal parton distribution functions (PDFs)  $f_i(x, \mu^2)$  of the proton at a (renormalization and factorization) scale  $\mu$  of the order of a physical hard scale, e.g.,  $M_H$  for the total cross section for the production of the Higgs boson. The dependence of the PDFs on the momentum fraction  $x$  is not calculable in perturbative QCD; their scale dependence is governed by the renormalization-group evolution equations

$$\frac{\partial}{\partial \ln \mu^2} f_i(x, \mu^2) = [P_{ik}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)](x) \quad (1.2)$$

where  $\otimes$  denotes the Mellin convolution. The splitting functions, which are closely related to the anomalous dimensions of twist-2 operators in the light-cone operator-product expansion (OPE), and the coefficient functions can be expanded in powers of the strong coupling  $a_s \equiv \alpha_s(\mu^2)/(4\pi)$ ,

$$P = a_s P^{(0)} + a_s^2 P^{(1)} + a_s^3 P^{(2)} + a_s^4 P^{(3)} + \dots, \quad (1.3)$$

$$c_a^0 = a_s^{n_a} [c_a^{(0)} + a_s c_a^{(1)} + a_s^2 c_a^{(2)} + a_s^3 c_a^{(3)} + \dots]. \quad (1.4)$$

Together the first three terms in eqs. (1.3) and (1.4) provide the next-to-next-to-leading order (N<sup>2</sup>LO) of perturbative QCD for the observables (1.1). This is now the standard approximation for many hard processes; see refs. [1–4] for the corresponding splitting functions.

Corrections beyond N<sup>2</sup>LO are of phenomenological interest where high precision is required, such as in determinations of  $\alpha_s$  from deep-inelastic scattering (DIS) (see refs. [5, 6] for the N<sup>3</sup>LO corrections to the most important structure functions), and where the perturbation series shows a slow convergence, such as for Higgs production via gluon-gluon fusion calculated in ref. [7] at N<sup>3</sup>LO. The size and structure of the corrections beyond N<sup>2</sup>LO are also of theoretical interest.

Here we briefly report about considerable recent progress on the three four-loop (N<sup>3</sup>LO) non-singlet splitting functions. We focus on the quantities  $P_{ns}^{\pm(3)}(x)$  for the evolution of flavour-differences  $q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k)$  of quark and antiquark distributions; for more details see ref. [8].

## 2. Diagram calculations of fixed- $N$ moments

Two methods have been applied for obtaining Mellin moments of the quantities  $P^{(3)}$  in eq. (1.3). Depending on the function, both can be used to determine the same even- $N$  or the odd- $N$  moments.

In the first one calculates, via the optical theorem and a dispersion relation in  $x$ , the unfactorized structure functions in DIS, as done at two and three loops in refs. [9–12]. The construction of the FORCER program [13] has facilitated the extension of those computations (which also provide moments of the coefficient functions) to four loops. For the hardest diagrams, the complexity of these computations rises quickly with  $N$ , hence only  $N \leq 6$  has been covered completely so far [14]. Much higher  $N$  can be accessed for simpler cases, e.g., values up to  $N > 40$  have been reached for high- $n_f$  parts. These were sufficient to determine the complete  $n_f^2$  and  $n_f^3$  parts of the non-singlet splitting functions  $P_{ns}^{(3)}(x)$  and the  $n_f^3$  parts of the corresponding flavour-singlet quantities [15].

The increase of the complexity of the Feynman integrals with  $N$  is more benign for the second method based on the OPE which was applied to the present non-singlet cases at NLO in ref. [16], see also ref. [17]. FORCER calculations in this framework have reached  $N = 16$  for all contributions to the functions  $P_{\text{ns}}^{(3)}$ ,  $N = 18$  for their  $n_f$  parts and  $N = 20$  for the complete limit of a large number of colours  $n_c$  [8]. See refs. [18–21] for earlier calculations of  $P_{\text{ns}}^{\pm(3)}$  at  $N \leq 4$ .

### 3. Towards all- $N$ expressions

If the anomalous dimensions  $\gamma_{\text{ns}}(N) = -P_{\text{ns}}(N)$  at  $N^{n>2}\text{LO}$  are analogous to the lower orders, then they can be expressed in terms of harmonic sums  $S_{\vec{w}}$  [22, 23] and denominators  $D_a^k \equiv (N+a)^{-k}$  as

$$\gamma_{\text{ns}}^{(n)}(N) = \sum_{w=0}^{2n+1} c_{00\vec{w}} S_{\vec{w}}(N) + \sum_a \sum_{k=1}^{2n+1} \sum_{w=0}^{2n+1-k} c_{ak\vec{w}} D_a^k S_{\vec{w}}(N). \quad (3.1)$$

The denominators at the calculated values of  $N$  indicate  $a = 0, 1$  for  $\gamma_{\text{ns}}^{\pm}$ , with coefficients  $c_{00\vec{w}}, c_{ak\vec{w}}$  that are integer modulo low powers of  $1/2$  and  $1/3$ . Sums up to weight  $w = 2n + 1$  occur at  $N^n\text{LO}$ .

Based on a conformal symmetry of QCD at an unphysical number of space-time dimensions  $D$ , it has been conjectured that the  $\overline{\text{MS}}$  functions  $\gamma_{\text{ns}}(N)$  are constrained by ‘self-tuning’ [24, 25],

$$\gamma_{\text{ns}}(N) = \gamma_{\text{u}}(N + \sigma \gamma_{\text{ns}}(N) - \beta(a_s)/a_s) \quad (3.2)$$

where  $\beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 - \dots$  is the beta function, for its present status see refs. [26, 27]. The initial-state (PDF) and final-state (fragmentation-function) anomalous dimensions are obtained for  $\sigma = -1$  and  $\sigma = 1$ , respectively, and the universal kernel  $\gamma_{\text{u}}$  is reciprocity respecting (RR), i.e., invariant under replacement  $N \rightarrow (1-N)$ . Eq. (3.2) implies that the non-RR parts and the spacelike/timelike difference are inherited from lower orders. Hence ‘only’  $\gamma_{\text{u}}$ , which includes  $2^{w-1}$  RR (combinations of) harmonic sums of weight  $w$ , needs to be determined at four loops.

Present information, given by the even- $N$  (odd- $N$ ) values  $N \leq 16$  (15) of  $\gamma_{\text{ns}}^{+(3)}(N)$  ( $\gamma_{\text{ns}}^{-(3)}(N)$ ) and endpoint constraints (see below), is insufficient to determine the  $n = 3$  coefficients in eq. (3.1). However,  $\gamma_{\text{ns}}^+ = \gamma_{\text{ns}}^-$  in the large- $n_c$  limit, hence the known even- $N$  and odd- $N$  values can be used. Moreover, alternating sums do not contribute to  $\gamma_{\text{ns}}^{\pm}$  in this limit, leaving 1, 1, 2, 3, 5, 8, 13 = Fibonacci( $w$ ) RR sums at weight  $w = 1, \dots, 7$  and a total of 87 basis functions for  $n = 3$  in eq. (3.1).

Large- $N$  and small- $x$  limits provide more than 40 constraints on their coefficients. At large- $N$ , the non-singlet anomalous dimensions have the form [33–35]

$$\gamma_{\text{ns}}^{(n-1)}(N) = A_n \ln \tilde{N} - B_n + N^{-1} \{ C_n \ln \tilde{N} - \tilde{D}_n + \frac{1}{2} A_n \} + O(N^{-2}) \quad (3.3)$$

with  $\ln \tilde{N} \equiv \ln N + \gamma_e$ , where  $\gamma_e$  denotes the Euler-Mascheroni constant.  $C_n$  and  $\tilde{D}_n$  are given by

$$C(a_s) = (A(a_s))^2, \quad \tilde{D}(a_s) = A(a_s) \cdot (B(a_s) - \beta(a_s)/a_s), \quad (3.4)$$

in terms of lower-order information on the cusp anomalous dimension  $A(a_s) = A_1 a_s + A_2 a_s^2 + \dots$  and the quantity  $B(a_s) = B_1 a_s + B_2 a_s^2 + \dots$  sometimes called the virtual anomalous dimension.

The resummation of small- $x$  double logarithms [28–31] provides the four-loop coefficients of  $x^a \ln^b x$  at  $4 \leq b \leq 6$  and all  $a$  in the large- $n_c$  limit (in full QCD, this holds only at even  $a$  for  $P_{\text{ns}}^+(x)$  and odd  $a$  for  $P_{\text{ns}}^-(x)$ ). Moreover, a relation leading to a single-logarithmic resummation at  $a = 0$ ,

$$\gamma_{\text{ns}}^+(N) \cdot (\gamma_{\text{ns}}^+(N) + N - \beta(a_s)/a_s) = O(1), \quad (3.5)$$

has been conjectured in ref. [32]. As far as it can be checked so far, this relation is found to be correct except for terms with  $\zeta_2 = \pi^2/6$  that vanish in the large- $n_c$  limit.

Taking into account all the above information, it is possible to set up systems of Diophantine equations for the coefficients  $c_{00\bar{w}}, c_{ak\bar{w}}$  of  $\gamma_{\text{ns}}^{\pm(3)}(N)$  in the large- $n_c$  limit that can be solved using the moments  $1 \leq N \leq 18$ , leaving the results of the diagram calculation at  $N = 19, 20$  as checks.

#### 4. All- $N$ anomalous dimension in the large- $n_c$ limit

The exact expressions for the new  $n_f^0$  and  $n_f^1$  parts cannot be shown here due to their length, they can be found in eq. (3.6) and (3.7) of ref. [8]. For the  $n_f^2$  and  $n_f^3$  terms see ref. [15]. The resulting large- $N$  coefficients  $A_{L,4}$  and  $B_{L,4}$  – the subscript  $L$  indicates the large- $n_c$  limit – are found to be

$$\begin{aligned} A_{L,4} = & C_F n_c^3 \left( \frac{84278}{81} - \frac{88832}{81} \zeta_2 + \frac{20992}{27} \zeta_3 + 1804 \zeta_4 - \frac{352}{3} \zeta_2 \zeta_3 - 352 \zeta_5 - 32 \zeta_3^2 - 876 \zeta_6 \right) \\ & - C_F n_c^2 n_f \left( \frac{39883}{81} - \frac{26692}{81} \zeta_2 + \frac{16252}{27} \zeta_3 + \frac{440}{3} \zeta_4 - \frac{256}{3} \zeta_2 \zeta_3 - 224 \zeta_5 \right) \\ & + C_F n_c n_f^2 \left( \frac{2119}{81} - \frac{608}{81} \zeta_2 + \frac{1280}{27} \zeta_3 - \frac{64}{3} \zeta_4 \right) - C_F n_f^3 \left( \frac{32}{81} - \frac{64}{27} \zeta_3 \right) \end{aligned} \quad (4.1)$$

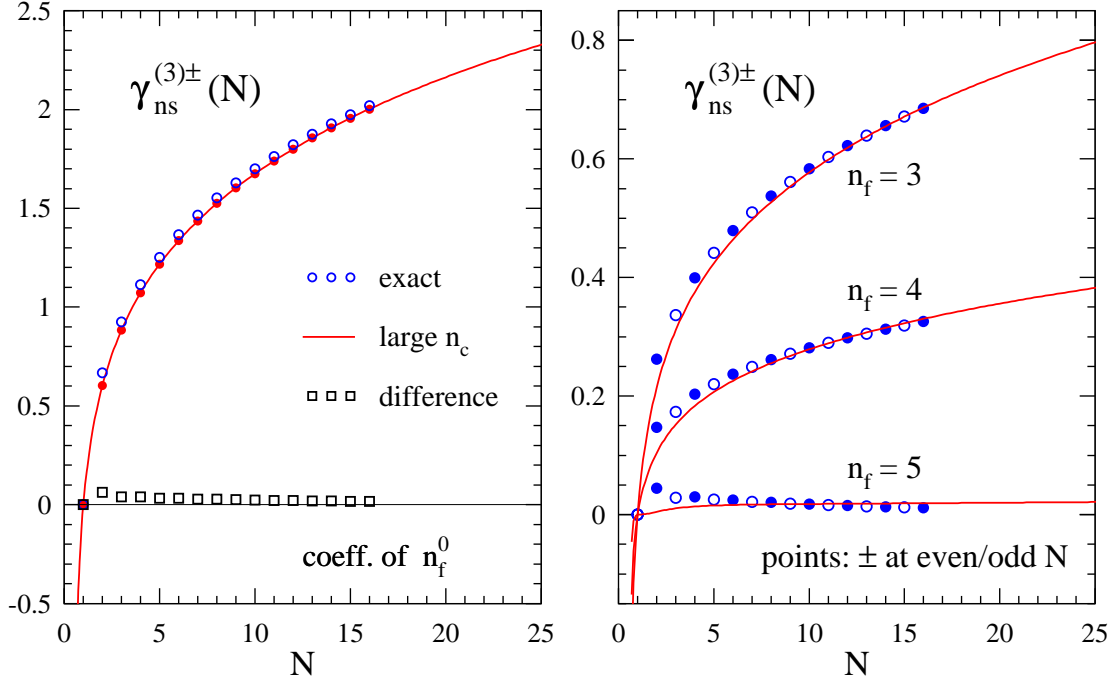
and

$$\begin{aligned} B_{L,4} = & C_F n_c^3 \left( -\frac{1379569}{5184} + \frac{24211}{27} \zeta_2 - \frac{9803}{162} \zeta_3 - \frac{9382}{9} \zeta_4 + \frac{838}{9} \zeta_2 \zeta_3 + 1002 \zeta_5 + \frac{16}{3} \zeta_3^2 \right. \\ & \left. + 135 \zeta_6 - 80 \zeta_2 \zeta_5 + 32 \zeta_3 \zeta_4 - 560 \zeta_7 \right) \\ & + C_F n_c^2 n_f \left( \frac{353}{3} - \frac{85175}{162} \zeta_2 - \frac{137}{9} \zeta_3 + \frac{16186}{27} \zeta_4 - \frac{584}{9} \zeta_2 \zeta_3 - \frac{248}{3} \zeta_5 - \frac{16}{3} \zeta_3^2 - 144 \zeta_6 \right) \\ & - C_F n_c n_f^2 \left( \frac{127}{18} - \frac{5036}{81} \zeta_2 + \frac{932}{27} \zeta_3 + \frac{1292}{27} \zeta_4 - \frac{160}{9} \zeta_2 \zeta_3 - \frac{32}{3} \zeta_5 \right) \\ & - C_F n_f^3 \left( \frac{131}{81} - \frac{32}{81} \zeta_2 - \frac{304}{81} \zeta_3 + \frac{32}{27} \zeta_4 \right). \end{aligned} \quad (4.2)$$

The agreement of the four-loop cusp anomalous dimension (4.1) with the result obtained from the large- $n_c$  photon-quark form factor [36,37] provides a further non-trivial check of the determination of the all- $N$  expressions from the moments at  $N \leq 18$ , and hence also of the relations (3.1) – (3.5).

The maximum-weight  $\zeta_3^2$  and  $\zeta_6$  parts of eq. (4.1) also agree with the result obtained in planar  $\mathcal{N} = 4$  maximally supersymmetric Yang-Mills theory (MSYM) obtained before in ref. [38]. There is no such direct connection between the four-loop virtual anomalous dimension (4.2) and its counterparts in planar  $\mathcal{N} = 4$  MSYM; see ref. [39] where the maximum-weight part of eq. (4.2) has been employed to derive the four-loop collinear anomalous dimension in planar  $\mathcal{N} = 4$  MSYM.

The all- $N$  large- $n_c$  limit of  $\gamma_{\text{ns}}^{\pm(3)}(N)$  is compared in fig. 1 with the integer- $N$  QCD results at  $N \leq 16$ . As illustrated in the left panel, the former are a decent approximation to the latter for the individual  $n_f^k$  contributions. However, as shown in the right panel, there are considerable cancellations between the these contributions. These cancellations are most pronounced for the physically relevant number of  $n_f = 5$  light quark flavours outside the large- $N$ /large- $x$  region. Hence the large- $n_c$  suppressed contributions – indicated by the subscript  $N$  below – need to be taken into account in phenomenological N<sup>3</sup>LO analyses.



**Figure 1:** The large- $n_c$  limit of the four-loop anomalous dimensions  $\gamma_{\text{ns}}^{\pm(3)}(N)$  (lines) compared to the QCD results for  $\gamma_{\text{ns}}^{+(3)}(N)$  at even  $N$  and  $\gamma_{\text{ns}}^{-(3)}(N)$  at odd  $N$  (points). Left: the  $n_f$ -independent contributions. Right: the results for physically relevant values of  $n_f$ . The values have been converted to an expansion in  $\alpha_s$ .

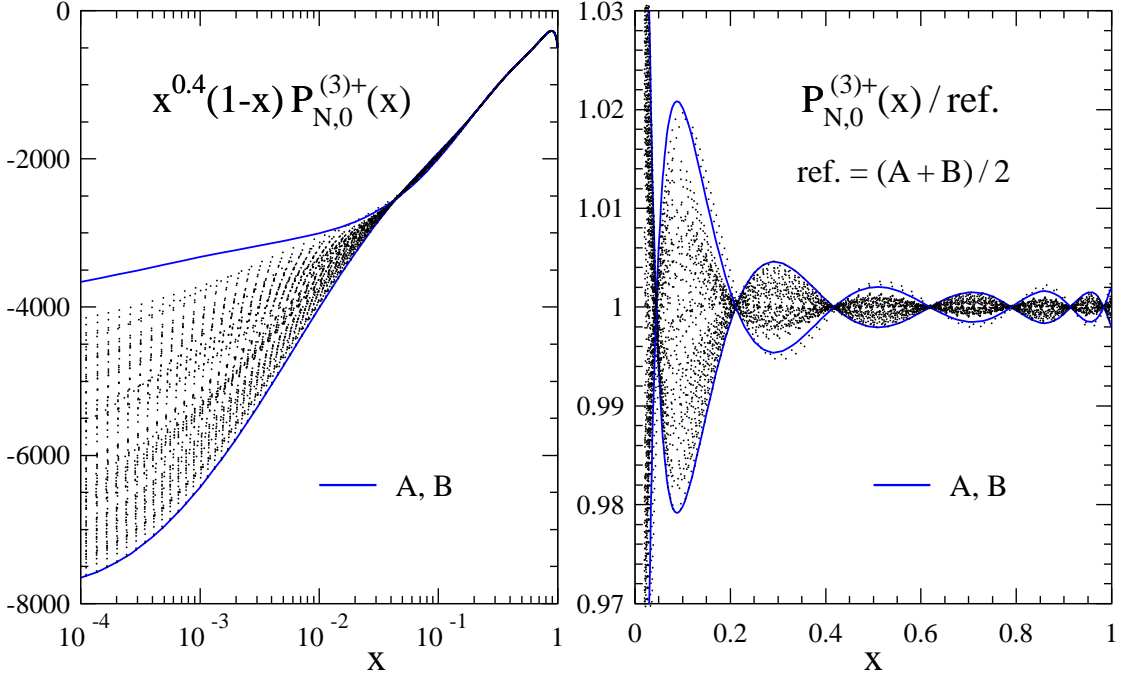
## 5. $x$ -space approximations of the large- $n_c$ suppressed parts

With eight integer- $N$  moments known for both  $P_{\text{ns}}^{+(3)}(x)$  and  $P_{\text{ns}}^{-(3)}(x)$  and the large- $x$  and small- $x$  knowledge discussed in section 2, it is possible to construct approximate  $x$ -space expressions which are analogous to (but more accurate than) those used before 2004 at N<sup>2</sup>LO, see refs. [40–43]. For this purpose an ansatz consisting of

- the two large- $x$  parameters  $A_4$  and  $B_4$  in eq. (3.3),
- two of three suppressed large- $x$  logs  $(1-x)\ln^k(1-x)$ ,  $k = 1, 2, 3$ ,
- one of ten two-parameter polynomials in  $x$  that vanish for  $x \rightarrow 1$ ,
- two of the three unknown small- $x$  logarithms  $\ln^k x$ ,  $k = 1, 2, 3$

is built for the large- $n_c$  suppressed  $n_f^0$  and  $n_f^1$  parts  $P_{N,0/1}^{+(3)}$  of  $P_{\text{ns}}^{+(3)}(x)$ . This results in 90 trial functions, the parameters of which can be fixed from the eight available moments. Of these functions, two representatives  $A$  and  $B$  are then chosen that indicate the remaining uncertainty, see fig. 2.

This non-rigorous procedure can be checked by comparing the same treatment for the large- $n_c$  parts to our exact results. Moreover, the trial functions lead to very similar values for the next moment, e.g.,  $N = 18$  for  $P_{\text{ns}}^{+(3)}$ . The residual uncertainty at this  $N$ -value is a consequence of the width of the band at large  $x$ , which in turn is correlated with the uncertainties at smaller  $x$ . If the spread of the result  $A$  and  $B$  would underestimate the true remaining uncertainties, then a comparison with an additional analytic result at this next value of  $N$  should reveal a discrepancy.



**Figure 2:** About 90 trial functions for the  $n_f$ -independent contribution to the large- $n_c$  suppressed part of splitting function  $P_{\text{ns}}^{+(3)}(x)$ , multiplied by  $x^{0.4}(1-x)$ . The two functions chosen to represent the remaining uncertainty are denoted by  $A$  and  $B$  and shown by solid (blue) lines. Due to the factor  $(1-x)$  the contribution  $A_{N,4}$  to the four-loop cusp anomalous dimension can be read off at  $x=1$ .

We were able to extend the diagram computations of the  $n_f^1$  parts of  $P_{\text{ns}}^{+(3)}(x)$  to  $N=18$  and find

$$P_{N,1}^{+(3)}(N=18) = 195.8888792_B < 195.8888857\dots_{\text{exact}} < 195.8888968_A . \quad (5.1)$$

A similar check for  $P_{N,0}^{+(3)}$  has been carried out by deriving a less accurate approximation using only seven moments and comparing the results to the now unused value at  $N=16$ .

The case of  $P_{\text{ns}}^{-(3)}(x)$  has been treated in the same manner, but taking into account that only its leading small- $x$  logarithm is known up to now [29]. See ref. [8] for the (large- $N$  suppressed) additional  $d^{abc}d_{abc}$  contribution  $P_{\text{ns}}^{s(3)}(x)$  to the splitting function for the total valence quark PDF.

## 6. Numerical results for the cusp and virtual anomalous dimensions

Combining the exact large- $n_c$  results, the approximations for the remaining  $n_f^0$  and  $n_f^1$  contributions and the complete high- $n_f$  contributions of ref. [15], the four-loop cusp anomalous dimension for QCD with  $n_f$  quark flavours are given by

$$A_4 = 20702(2) - 5171.9(2)n_f + 195.5772n_f^2 + 3.272344n_f^3 , \quad (6.1)$$

where the numbers in brackets represent a conservative estimate of the remaining uncertainty. The conversion of this result to an expansion in powers of  $\alpha_s$  leads to

$$\begin{aligned} A_q(\alpha_s, n_f=3) &= 0.42441 \alpha_s (1 + 0.72657 \alpha_s + 0.73405 a_s^2 + 0.6647(2) a_s^3 + \dots) , \\ A_q(\alpha_s, n_f=4) &= 0.42441 \alpha_s (1 + 0.63815 \alpha_s + 0.50998 a_s^2 + 0.3168(2) a_s^3 + \dots) , \\ A_q(\alpha_s, n_f=5) &= 0.42441 \alpha_s (1 + 0.54973 \alpha_s + 0.28403 a_s^2 + 0.0133(2) a_s^3 + \dots) . \end{aligned} \quad (6.2)$$

The corresponding results for the virtual anomalous dimension, i.e., the coefficient of  $\delta(1-x)$  show a similarly benign expansion with

$$B_4 = 23393(10) - 5551(1)n_f + 193.8554n_f^2 + 3.014982n_f^3 \quad (6.3)$$

and

$$\begin{aligned} B_q(\alpha_s, n_f=3) &= 0.31831 \alpha_s (1 + 0.99712 \alpha_s + 1.24116 a_s^2 + 1.0791(13) a_s^3 + \dots), \\ B_q(\alpha_s, n_f=4) &= 0.31831 \alpha_s (1 + 0.87192 \alpha_s + 0.97833 a_s^2 + 0.5649(13) a_s^3 + \dots), \\ B_q(\alpha_s, n_f=5) &= 0.31831 \alpha_s (1 + 1.74672 \alpha_s + 0.71907 a_s^2 + 0.1085(13) a_s^3 + \dots). \end{aligned} \quad (6.4)$$

Due to constraints by large- $N$  moments, the errors of  $A_4$  and  $B_4$  are fully correlated. The accuracy in eqs. (6.2) and (6.4) should be amply sufficient for phenomenological applications.

By repeating the approximation procedure in section 5 for individual colour factors, it is possible to obtain corresponding approximate coefficients for  $A_4$  and  $B_4$  which can be summarized as (for a table of the relevant group invariants see, e.g., appendix C of ref. [44])

	$A_4$	$B_4$
$C_F^4$	0	$197. \pm 3.$
$C_F^3 C_A$	0	$-687. \pm 10.$
$C_F^2 C_A^2$	0	$1219. \pm 12.$
$C_F C_A^3$	$610.3 \pm 0.3$	$295.6 \pm 2.4$
$d_R^{abcd} d_A^{abcd} / N_R$	$-507.5 \pm 6.0$	$-996. \pm 45.$
$n_f C_F^3$	$-31.00 \pm 0.4$	$81.4 \pm 2.2$
$n_f C_F^2 C_A$	$38.75 \pm 0.2$	$-455.7 \pm 1.1$
$n_f C_F C_A^2$	$-440.65 \pm 0.2$	$-274.4 \pm 1.1$
$n_f d_R^{abcd} d_A^{abcd} / N_R$	$-123.90 \pm 0.2$	$-143.5 \pm 1.2$
$n_f^2 C_F^2$	$-21.31439$	$-5.775288$
$n_f^2 C_F C_A$	$58.36737$	$51.03056$
$n_f^3 C_F$	$2.454258$	$2.261237$

where the exactly known  $n_f^2$  and  $n_f^3$  coefficients have been included for completeness. Due to the constraint provided by the exact large- $n_c$  limit, the errors in this table are highly correlated; for numerical applications in QCD eqs. (6.2) and (6.4) should be used instead. The above results show that both quartic group invariants definitely contribute to the four-loop cusp anomalous dimension, for this issue see also refs. [45–48] and references therein. This implies that the so-called Casimir scaling between the quark and gluon cases,  $A_q = C_F/C_A A_g$ , does not hold beyond three loops.

## 7. N<sup>3</sup>LO corrections to the evolution of non-singlet PDFs

The effect of the fourth-order contributions on the evolution of the non-singlet PDFs can be illustrated by considering the logarithmic derivatives of the respective combinations of quark PDFs with respect to the factorization scale,  $\dot{q}_{\text{ns}}^i \equiv d \ln q_{\text{ns}}^i / d \ln \mu_f^2$ , at a suitably chosen reference point.



As in ref. [1], we choose the schematic, order-independent initial conditions

$$xq_{\text{ns}}^{\pm,v}(x, \mu_0^2) = x^{0.5}(1-x)^3 \quad \text{and} \quad \alpha_s(\mu_0^2) = 0.2. \quad (7.1)$$

For  $\alpha_s(M_Z^2) = 0.114\dots 0.120$  this value for  $\alpha_s$  corresponds to  $\mu_0^2 \simeq 25\dots 50 \text{ GeV}^2$  beyond the leading order, a scale range typical for DIS at fixed-target experiments and at the  $ep$  collider HERA.

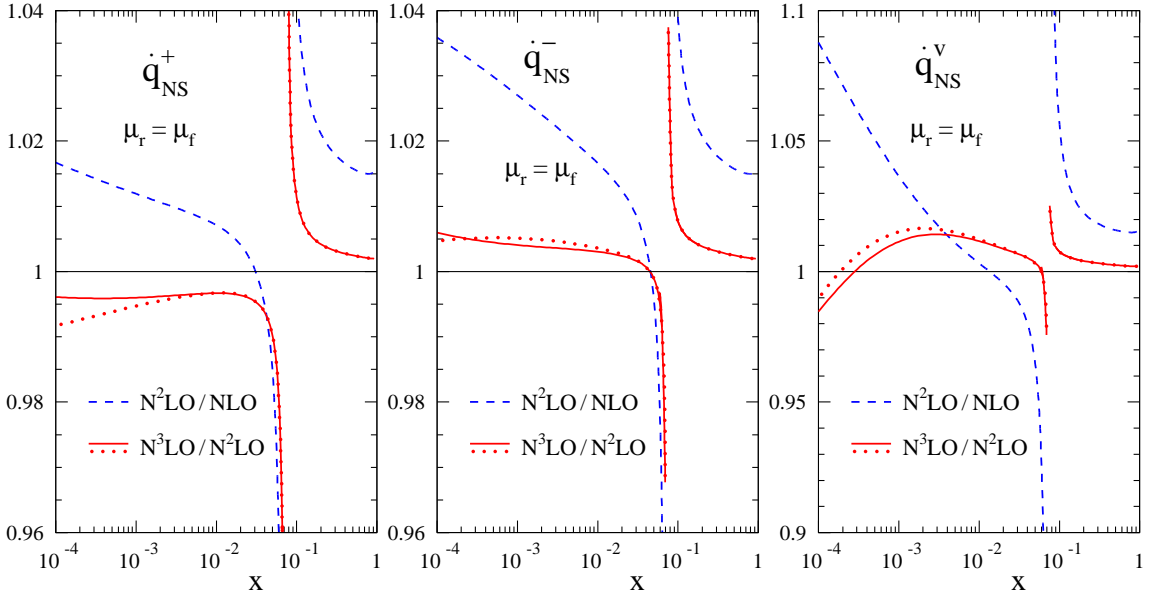
The new N<sup>3</sup>LO corrections to  $\dot{q}_{\text{ns}}^i$  are generally small, hence they are illustrated in fig. 3 by comparing their relative effect to that of the N<sup>2</sup>LO contributions for the standard identification  $\mu_r = \mu_f \equiv \mu$  of the renormalization scale with the factorization scale. Except close to the sign change of the scaling violations at  $x \simeq 0.07$ , the relative N<sup>3</sup>LO effects are (well) below 1% for the flavour-differences  $q_{\text{ns}}^+$  and  $q_{\text{ns}}^-$  (left and middle panel). The N<sup>2</sup>LO and N<sup>3</sup>LO corrections are larger for the valence distribution  $q_{\text{ns}}^v$  at  $x < 0.07$  due to the effect of the  $d^{abc}d_{abc}$  ‘sea’ contribution  $P_{\text{ns}}^s(x)$ , note the different scale of the right panel in fig. 3. Also in this case the N<sup>3</sup>LO evolution represents a clear improvement, and the relative four-loop corrections are below 2%.

The remaining uncertainty due to the approximate character of the four-loop splitting functions beyond the large- $n_c$  limit is indicated by the difference between the solid and dotted (red) curves in fig. 3 and fig. 4 below. Due to the small size of the four-loop contributions and the ‘ $x$ -averaging’ effect of the Mellin convolution,

$$[P_{\text{ns}} \otimes q_{\text{ns}}](x) = \int_x^1 \frac{dy}{y} P_{\text{ns}}(y) q_{\text{ns}}\left(\frac{x}{y}\right), \quad (7.2)$$

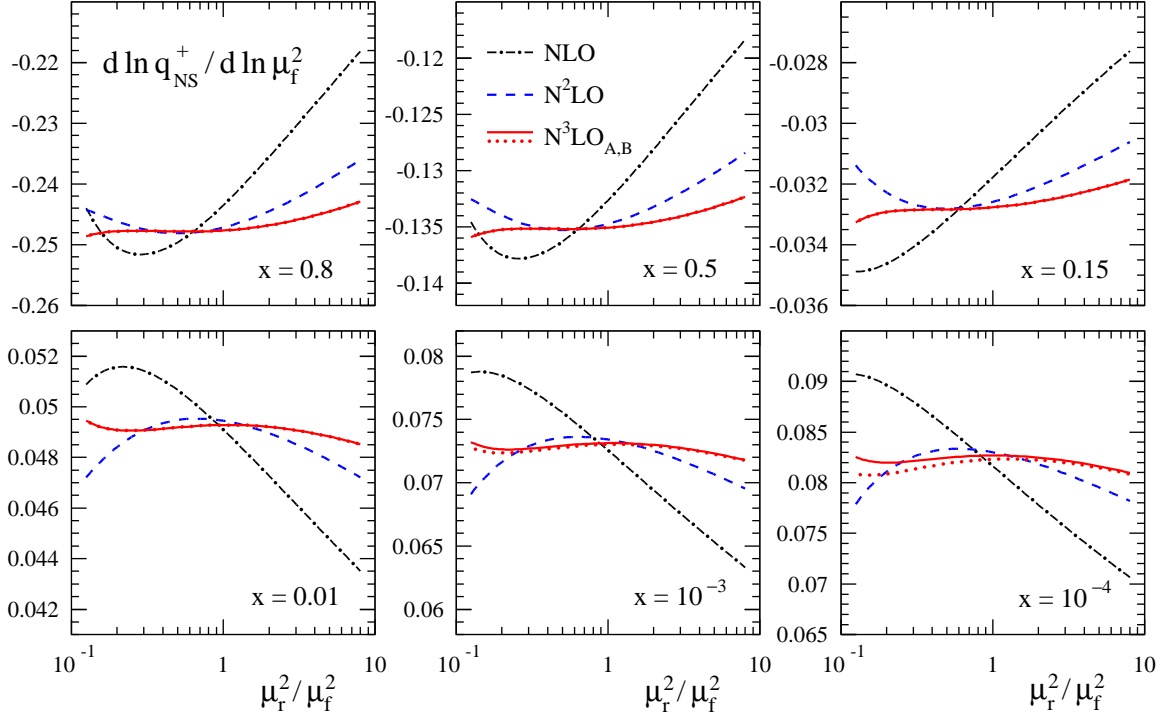
the results of section 4 are safely applicable to lower values of  $x$  than one might expect from fig. 2.

The stability of the NLO, N<sup>2</sup>LO and N<sup>3</sup>LO results under variation of the renormalization scale over the range  $\frac{1}{8}\mu_f^2 \leq \mu_r^2 \leq 8\mu_f^2$  is illustrated in fig. 4 at typical values of  $x$ . Except close to the sign change of  $\dot{q}_{\text{ns}}^+$ , the variation is well below 1% for the conventional interval  $\frac{1}{2}\mu_f \leq \mu_r \leq 2\mu_f$ .



**Figure 3:** The relative N<sup>2</sup>LO and N<sup>3</sup>LO corrections to the logarithmic scale derivative of the non-singlet combinations  $q_{\text{ns}}^a$  of quark PDFs for the schematic order-independent input (7.1) for  $n_f = 4$  at  $\mu_r = \mu_f$ .





**Figure 4:** The dependence of the NLO, N<sup>2</sup>LO and N<sup>3</sup>LO results for  $\dot{q}_{\text{ns}}^+ \equiv d \ln q_{\text{ns}}^+ / d \ln \mu_f^2$  on the renormalization scale  $\mu_r$ , at six typical values of  $x$  for the initial conditions (7.1) and  $n_f = 4$  flavours. The remaining uncertainty of the four-loop splitting function  $P_{\text{ns}}^{+(3)}(x)$  leads to the difference of the solid and dotted curves.

## 8. Summary and Outlook

The splitting functions for the non-singlet combinations of quark PDFs have been addressed at the fourth-order (N<sup>3</sup>LO) of perturbative QCD. The quantities  $P_{\text{ns}}^{\pm(3)}$  are now known exactly in the limit of a large number of colours  $n_c$ . Present results for the large- $n_c$  suppressed contributions with  $n_f^0$  and  $n_f^1$  are still approximate, but sufficiently accurate for phenomenological applications in deep-inelastic scattering and collider physics. FORM and FORTRAN files of these results can be obtained by downloading the source of ref. [8] from [arXiv.org](https://arxiv.org).

It would be desirable, mostly for theoretical purposes, to obtain also the analytic forms  $n_f^0$  and  $n_f^1$  parts of  $P_{\text{ns}}^{\pm(3)}$ . So far, only their contributions proportional to the values  $\zeta_4$  and  $\zeta_5$  of the Riemann  $\zeta$ -function have been completely determined, together with the (unpublished)  $\zeta_3$  part of the  $n_f^1$  contributions. The  $\zeta_4$  parts are particularly simple; in fact, it turns out that they (and other  $\pi^2$  terms) can be predicted via physical evolution kernels from lower-order quantities, see refs. [49,50].

The  $\zeta_5$  part of  $P_{\text{ns}}^{\pm(3)}$ , presented in appendix D of ref. [8], includes a (non large- $n_c$ ) contribution

$$- \frac{128}{3} \left\{ 3C_F^2 C_A^2 - 2C_F C_A^3 + 12d_F^{abcd} d_A^{abcd} / N_R \right\} 5\zeta_5 [S_1(N)]^2. \quad (8.1)$$

The resulting  $\ln^2 N$  large- $N$  behaviour needs to be compensated by non- $\zeta_5$  terms. Eq. (8.1) looks exactly like the  $\zeta_5$ -‘tail’ of the so-called wrapping correction in the anomalous dimensions in  $\mathcal{N} = 4$  maximally supersymmetric Yang-Mills theory, see refs. [51,52].

Phenomenologically, of course, one rather needs corresponding results for the flavour-singlet splitting functions  $P_{ij}^{(3)}(x)$ ,  $i, j = q, g$ . At present, it appears computationally too hard to obtain moments of all four functions beyond  $N = 6$  using the method of refs. [9–12]. Therefore one will need to resort to the OPE, which offers additional theoretical challenges in the massless flavour-singlet case, see refs. [53–55]. We hope to address this issue in a future publication.

### Acknowledgements

The research reported here has been supported by the *European Research Council* (ERC) Advanced Grant 320651, *HEPGAME*, the grant ST/L000431/1 of the *UK Science & Technology Facilities Council* (STFC), and the *Deutsche Forschungsgemeinschaft* (DFG) grant MO 1801/1-2 and SFB 676 project A3. Part of our computations have been performed on a computer cluster in Liverpool funded by the STFC grant ST/H008837/1.

### References

- [1] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B688 (2004) 101, hep-ph/0403192
- [2] A. Vogt, S. Moch and J.A.M. Vermaseren, Nucl. Phys. B691 (2004) 129, hep-ph/0404111
- [3] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B889 (2014) 351, arXiv:1409.5131
- [4] S. Moch, J.A.M. Vermaseren and A. Vogt, Phys. Lett. B748 (2015) 432, arXiv:1506.04517
- [5] J.A.M. Vermaseren, A. Vogt and S. Moch, Nucl. Phys. B724 (2005) 3, hep-ph/0504242
- [6] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B813 (2009) 220, arXiv:0812.4168 [hep-ph]
- [7] C. Anastasiou, C. Duhr, F. Dulat, E. Furlan, T. Gehrmann, F. Herzog, A. Lazopoulos, B. Mistlberger, JHEP 1605 (2016) 058, arXiv:1602.00695
- [8] S. Moch, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, JHEP 10 (2017) 041, arXiv:1707.08315
- [9] S.A. Larin and J.A.M. Vermaseren, Z. Phys. C57 (1993) 93
- [10] S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, Nucl. Phys. B427 (1994) 41
- [11] S. Larin, P. Nogueira, T. van Ritbergen, J. Vermaseren, Nucl. Phys. B492 (1997) 338, hep-ph/9605317
- [12] A. Retey and J.A.M. Vermaseren, Nucl. Phys. B604 (2001) 281, hep-ph/0007294
- [13] B. Ruijl, T. Ueda and J.A.M. Vermaseren, arXiv:1704.06650
- [14] B. Ruijl, T. Ueda, J. Vermaseren, J. Davies and A. Vogt, PoS LL2016 (2016) 071, arXiv:1605.08408
- [15] J. Davies, A. Vogt, B. Ruijl, T. Ueda, J. Vermaseren, Nucl. Phys. B915 (2017) 335, arXiv:1610.07477
- [16] E.G. Floratos, D.A. Ross and C.T. Sachrajda, Nucl. Phys. B129 (1977) 66
- [17] I. Bierenbaum, J. Blümlein and S. Klein, Nucl. Phys. B820 (2009) 417, arXiv:0904.3563
- [18] P.A. Baikov and K.G. Chetyrkin, Nucl. Phys. B (Proc. Suppl.) 160 (2006) 76
- [19] P.A. Baikov, K.G. Chetyrkin, J.H. Kühn, Nucl. Part. Phys. Proc. 261/2 (2015) 3, arXiv:1501.06739
- [20] V.N. Velizhanin, Nucl. Phys. B860 (2012) 288, arXiv:1112.3954
- [21] V.N. Velizhanin, arXiv:1411.1331
- [22] J.A.M. Vermaseren, Int. J. Mod. Phys. A14 (1999) 2037, hep-ph/9806280

- [23] J. Blümlein and S. Kurth, Phys. Rev. D60 (1999) 014018, hep-ph/9810241
- [24] B. Basso and G.P. Korchemsky, Nucl. Phys. B775 (2007) 1, hep-th/0612247
- [25] Yu.L. Dokshitzer and G. Marchesini, Phys. Lett. B646 (2007) 189, hep-th/0612248
- [26] P.A. Baikov, K.G. Chetyrkin and J. H. Kühn, Phys. Rev. Lett. 118 (2017) 082002, arXiv:1606.08659
- [27] F. Herzog, B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt, JHEP 1702 (2017) 090, arXiv:1701.01404
- [28] R. Kirschner and L.N. Lipatov, Nucl. Phys. B213 (1983) 122
- [29] J. Blümlein and A. Vogt, Phys. Lett. B370 (1996) 149, hep-ph/9510410
- [30] A. Vogt, C.H. Kom, N.A. Lo Presti, G. Soar, A.A. Almasy, S. Moch, J. Vermaseren and K. Yeats, PoS LL2012 (2012) 004, arXiv:1212.2932
- [31] J. Davies, C.-H. Kom, and A. Vogt, TTP18-002, LTH 1148
- [32] V.N. Velizhanin, Mod. Phys. Lett. A32 (2017) 1750213, arXiv:1412.7143v3
- [33] G.P. Korchemsky, Mod. Phys. Lett. A4 (1989) 1257
- [34] S. Albino and R.D. Ball, Phys. Lett. B513 (2001) 93, hep-ph/0011133
- [35] Yu.L. Dokshitzer, G. Marchesini and G.P. Salam, Phys. Lett. B634 (2006) 504, hep-ph/0511302
- [36] J.M. Henn, A.V. Smirnov, V.A. Smirnov, M. Steinhauser, JHEP 1605 (2016) 066, arXiv:1604.03126v2
- [37] J.M. Henn, A.V. Smirnov, V.A. Smirnov, M. Steinhauser and R.N. Lee, JHEP 03 (2017) 139, arXiv:1612.04389v2
- [38] Z. Bern, M. Czakon, L.J. Dixon, D.A. Kosower and V.A. Smirnov, Phys. Rev. D75 (2007) 085010, hep-th/0610248
- [39] L.J. Dixon, arXiv:1712.07274
- [40] W.L. van Neerven and A. Vogt, Nucl. Phys. B568 (2000) 263, hep-ph/9907472
- [41] W.L. van Neerven and A. Vogt, Nucl. Phys. B588 (2000) 345, hep-ph/0006154
- [42] W.L. van Neerven and A. Vogt, Phys. Lett. B490 (2000) 111, hep-ph/0007362
- [43] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B621 (2002) 413, hep-ph/0110331
- [44] B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, JHEP 06 (2017) 040, arXiv:1703.08532
- [45] A. Grozin, J.M. Henn, G.P. Korchemsky and P. Marquard, JHEP 1601 (2016) 140, arXiv:1510.07803
- [46] R.H. Boels, T. Huber and G. Yang, Phys. Rev. Lett. 119 (2017) 201601, arXiv:1705.03444
- [47] A. Grozin, J. Henn and M. Stahlhofen, JHEP 10 (2017) 052, arXiv:1708.01221
- [48] R.H. Boels, T. Huber and G. Yang, arXiv:1711.08449
- [49] M. Jamin and R. Miravitllas, arXiv:1711.00787
- [50] J. Davies and A. Vogt, Phys. Lett. B776 (2018) 189, arXiv:1711.05267
- [51] A.V. Kotikov, L.N. Lipatov, A. Rej, M. Staudacher, V.N. Velizhanin, J. Stat. Mech. 10 (2007) P10003, arXiv:0704.3586
- [52] Z. Bajnok, R.A. Janik and T. Lukowski, Nucl. Phys. B816 (2009) 376, arXiv:0811.4448
- [53] R. Hamberg and W.L. van Neerven, Nucl. Phys. B379 (1992) 143
- [54] J.C. Collins and R.J. Scalise, Phys. Rev. D50 (1994) 4117, hep-ph/9403231
- [55] Y. Matiounine, J. Smith and W.L. van Neerven, Phys. Rev. D57 (1998) 6701, hep-ph/9801224