

Four-loop wave function renormalization in QCD and QED

Peter Marquard^a, Alexander V. Smirnov^b,
Vladimir A. Smirnov^c, Matthias Steinhauser^d

(a) *Deutsches Elektronen-Synchrotron, DESY 15738 Zeuthen, Germany*

(b) *Research Computing Center, Moscow State University
119991, Moscow, Russia*

(c) *Skobeltsyn Institute of Nuclear Physics of Moscow State University
119991, Moscow, Russia*

(d) *Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT)
76128 Karlsruhe, Germany*

Abstract

We compute the on-shell wave function renormalization constant to four-loop order in QCD and present numerical results for all coefficients of the $SU(N_c)$ colour factors. We extract the four-loop HQET anomalous dimension of the heavy quark field and also discuss the application of our result to QED.

1 Introduction

Heavy quarks play an important role in modern particle physics, in particular in the context of Quantum Chromodynamics (QCD). This concerns both virtual effects, the production of massive quarks at collider experiments and the study of bound state effects of heavy quark-anti-quark pairs.

Processes which involve heavy quarks require the renormalization constants for the heavy quark mass and, when they appear as external particles, also for the quark wave function. The mass renormalization constant in the on-shell scheme, Z_m^{OS} , has been computed to four-loop order in Refs. [1,2]. In this work we compute the wave function renormalization constant in the on-shell scheme, Z_2^{OS} , to the same order in perturbation theory. Z_2^{OS} is needed for all processes involving external heavy quarks to obtain properly normalized Green's functions as dictated by the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula. Currently there is no immediate application for the four-loop term of Z_2^{OS} . However, it is an important building block for future applications. For example, it enters all processes which involve the massive four-loop form factor. Z_2^{OS} is also needed for the five-loop corrections to static properties like the anomalous magnetic moment of quarks or, in the case of QED, of leptons.

The calculation of Z_2^{OS} is for several reasons more involved than the one of Z_m^{OS} . First of all, one has to compute the derivative of the fermion self energy which leads to higher powers of propagators and thus to a more involved reduction problem. Furthermore, Z_2^{OS} contains both ultraviolet and infrared divergences. Thus, dividing Z_2^{OS} by its $\overline{\text{MS}}$ counterpart does not lead to a finite quantity as in the case of Z_m^{OS} . Z_2^{OS} also depends on the QCD gauge parameter whereas Z_m^{OS} does not.

The on-shell renormalization constants Z_2^{OS} and Z_m^{OS} can be extracted from the quark propagator by demanding that the quark two-point function has a zero at the position of the on-shell mass and that the residue of the propagator is $-i$. In the following we briefly sketch the derivation of the relations between the heavy quark self energy and Z_2^{OS} and Z_m^{OS} .

The renormalized quark propagator is given by

$$S_F(q) = \frac{-iZ_2^{\text{OS}}}{\not{q} - m^0 + \Sigma(q, M)}, \quad (1)$$

where the renormalization constants are defined as

$$\begin{aligned} m^0 &= Z_m^{\text{OS}} M, \\ \psi^0 &= \sqrt{Z_2^{\text{OS}}} \psi. \end{aligned} \quad (2)$$

ψ is the quark field with mass m , M is the on-shell mass and bare quantities are denoted by a superscript 0. Σ denotes the quark self energy which is conveniently decomposed as

$$\Sigma(q, m) = m \Sigma_1(q^2, m) + (\not{q} - m) \Sigma_2(q^2, m). \quad (3)$$

In the limit $q^2 \rightarrow M^2$ we require

$$S_F(q) \xrightarrow{q^2 \rightarrow M^2} \frac{-i}{\not{q} - M}. \quad (4)$$

The calculation outlined in Ref. [3] for the evaluation of Z_m^{OS} and Z_2^{OS} reduces all occurring Feynman diagrams to the evaluation of on-shell integrals at the bare mass scale. In particular, it avoids the introduction of explicit counterterm diagrams. We find it more convenient to follow the more direct approach described in Refs. [4, 5], which requires the calculation of diagrams with mass counterterm insertion.

Following Refs. [3–6] we expand Σ around $q^2 = M^2$ and obtain

$$\begin{aligned} \Sigma(q, M) &\approx M \Sigma_1(M^2, M) + (\not{q} - M) \Sigma_2(M^2, M) \\ &\quad + M \frac{\partial}{\partial q^2} \Sigma_1(q^2, M) \Big|_{q^2=M^2} (q^2 - M^2) + \dots \\ &\approx M \Sigma_1(M^2, M) \\ &\quad + (\not{q} - M) \left(2M^2 \frac{\partial}{\partial q^2} \Sigma_1(q^2, M) \Big|_{q^2=M^2} + \Sigma_2(M^2, M) \right) + \dots \end{aligned} \quad (5)$$

Inserting Eq. (5) into Eq. (1) and comparing to Eq. (4) leads to the following formulae for the renormalization constants

$$\begin{aligned} Z_m^{\text{OS}} &= 1 + \Sigma_1(M^2, M), \\ (Z_2^{\text{OS}})^{-1} &= 1 + 2M^2 \frac{\partial}{\partial q^2} \Sigma_1(q^2, M) \Big|_{q^2=M^2} + \Sigma_2(M^2, M). \end{aligned} \quad (6)$$

Thus, Z_m^{OS} is obtained from Σ_1 for $q^2 = M^2$. To calculate Z_2^{OS} , one has to compute the first derivative of the self-energy diagrams. The mass renormalization is taken into account iteratively by calculating lower-loop diagrams with zero-momentum insertions.

It is convenient to introduce $q = Q(1+t)$ with $Q^2 = M^2$ and re-write the self energy as

$$\Sigma(q, M) = M \Sigma_1(q^2, M) + (Q - M) \Sigma_2(q^2, M) + tQ \Sigma_2(q^2, M). \quad (7)$$

Let us now consider the quantity $\text{Tr}\left\{\frac{Q+M}{4M^2}\Sigma\right\}$ and expand it to first order in t which leads to

$$\begin{aligned} \text{Tr}\left\{\frac{Q+M}{4M^2}\Sigma(q, M)\right\} &= \Sigma_1(q^2, M) + t \Sigma_2(q^2, M) \\ &= \Sigma_1(M^2, M) + \left(2M^2 \frac{\partial}{\partial q^2} \Sigma_1(q^2, M) \Big|_{q^2=M^2} + \Sigma_2(M^2, M) \right) t \\ &\quad + \mathcal{O}(t^2). \end{aligned} \quad (8)$$

The comparison to Eq. (6) shows that the leading term provides Z_m^{OS} and the coefficient of the linear term in t leads to Z_2^{OS} .

In the next Section we present results for Z_2^{OS} up to four loops and in Section 3 we discuss consistency checks which are obtained from matching full QCD to Heavy Quark Effective Theory (HQET). Section 4 contains a brief summary and our conclusions.

2 Results for Z_2^{OS}

The wave function renormalization constant is conveniently cast into the form

$$Z_2^{\text{OS}} = 1 + \sum_{j \geq 1} \left(\frac{\alpha_s^0(\mu)}{\pi} \right)^j \left(\frac{e^{\gamma_E}}{4\pi} \right)^{-j\epsilon} \left(\frac{\mu^2}{M^2} \right)^{j\epsilon} \delta Z_2^{(j)}, \quad (9)$$

where the bare strong coupling constant α_s^0 has been used for the parametrization. Note that $\delta Z_2^{(i)}$ for $i \geq 3$ depend on the bare QCD gauge parameter ξ which is introduced in the gluon propagator via

$$D_g^{\mu\nu}(q) = -i \frac{g^{\mu\nu} - \xi \frac{q^\mu q^\nu}{q^2}}{q^2 + i\epsilon}. \quad (10)$$

With these choices we can define the coefficients $\delta Z_2^{(i)}$ such that they do not contain $\log(\mu^2/M^2)$ terms. In fact, they can be combined to the factors $(\mu^2/M^2)^{j\epsilon}$ where j is the loop order (cf. Eq. (9)). The renormalization of α_s and $(\xi - 1)$ is multiplicative so that, if required, α_s^0 and ξ^0 can be replaced in a straightforward way by their renormalized counterparts using the relations

$$\begin{aligned} \alpha_s^0 &= (\mu^2)^{2\epsilon} Z_{\alpha_s} \alpha_s, \\ \xi^0 - 1 &= Z_3 (\xi - 1), \end{aligned} \quad (11)$$

where

$$\begin{aligned} Z_{\alpha_s} &= 1 + \frac{1}{\epsilon} \left(\frac{n_f}{6} - \frac{11}{12} C_A \right) \frac{\alpha_s}{\pi} + \dots, \\ Z_3 &= 1 + \frac{1}{\epsilon} \left[-\frac{n_f}{6} + \left(\frac{5}{12} + \frac{1}{8} \xi \right) C_A \right] \frac{\alpha_s}{\pi} + \dots \end{aligned} \quad (12)$$

$C_A = 3$ is a SU(3) colour factor and n_f is the number of active quarks. The ellipses denote higher order terms in α_s . To obtain the ultraviolet-renormalized version of Z_2^{OS} we need Z_{α_s} to three loops and Z_3 to one-loop order. Note that in Eq. (9) it is assumed that the heavy quark mass is renormalized on-shell, i.e., all mass renormalization counterterms from lower-order diagrams are included.

For the calculation of the four-loop diagrams we proceeded in the same way as for the calculation of the mass renormalization constant [1,2] and the muon anomalous magnetic moment [7] and thus refer to [2] for more details. Let us still describe some complications. After a tensor reduction we obtain Feynman integrals from the same hundred families with 14 indices as in [1,2]. The maximal number of positive indices is eleven. One can describe the complexity of integrals of a given sector (determined by a decomposition of the set of indices into subsets of positive and non-positive indices) by the number $\sum |a_i - n_i|$, where the indices $n_i = 1$ or 0 characterize a given sector. What is most crucial for the

feasibility of an integration-by-parts (IBP) reduction is the complexity of input integrals in the top sector, i.e. with $n_i = 1$ for $i = 1, 2, \dots, 11$ and $n_i = 0$ for $i = 12, 13, 14$. In the present calculation, this number was up to six while in our previous calculation it was five. Therefore, the reduction procedure performed with **FIRE** [8–10] coupled with **LiteRed** [11, 12] and **Crusher** [13] was essentially more complicated as compared to that of [1, 2].

As in [1, 2] we revealed additional relations between master integrals of different families using symmetries and applied the code **tsort** which is part of the latest **FIRE** version [10]. In most cases, the master integrals were computed numerically with the help of **FIESTA** [14–16]. For some master integrals, we used analytic results obtained by a straightforward loop-by-loop integration at general dimension d and also used analytical results obtained for the 13 non-trivial four-loop on-shell master integrals computed in [17]. As it is described in detail in [2], we also applied Mellin-Barnes representations [18–21]. In the case of one-fold Mellin-Barnes representations, it is possible to obtain a very high precision (up to 1000 digits) so that analytic results can be recovered using the PSLQ algorithm [22]. Often the two-, three- and higher-fold MB representation provide a better precision than **FIESTA**. Recently a subset of the master integrals has been calculated either analytically or with high numerical precision, in the context of the anomalous magnetic moment of the electron [23]. However, these results are not available to us.

The more complicated IBP reduction resulted in higher ϵ poles in the coefficients of some of the master integrals, so that the corresponding results are needed to higher powers in ϵ . Depending on the integral, we either straightforwardly evaluated more terms with **FIESTA**, or obtained more analytical terms, or more numerical terms via Mellin-Barnes integrals.

Let us mention that we compute the self energies on the right-hand side of Eq. (6) including terms of order ξ^2 . We did not evaluate the ξ^3 , ξ^4 and ξ^5 contributions. Some diagrams develop ξ^6 terms which we reduced to master integrals and we could show that their contributions to Z_2^{OS} add up to zero. Thus, in our final result for Z_2^{OS} contains ξ^2 terms. We cannot exclude that also higher order ξ terms are present but we do not expect that there are ξ^n terms present in Z_2^{OS} for $n \geq 4$.

Let us in a first step turn to the one-, two- and three-loop results for Z_2^{OS} which are available from Refs. [5, 6, 24]. We have added higher order ϵ terms which are necessary to obtain Z_2^{OS} at four loops. In Appendix B we present results which in particular include the $\mathcal{O}(\epsilon)$ terms of the three-loop coefficient.

In the following we present results for all 23 $\text{SU}(N_c)$ colour structures which occur at four-loop order. It is convenient to decompose $\delta Z_2^{(4)}$ as

$$\begin{aligned} \delta Z_2^{(4)} = & C_F^4 \delta Z_2^{FFFF} + C_F^3 C_A \delta Z_2^{FFFA} + C_F^2 C_A^2 \delta Z_2^{FFAA} + C_F C_A^3 \delta Z_2^{FAAA} \\ & + \frac{d_F^{abcd} d_A^{abcd}}{N_c} \delta Z_2^{dFA} + n_l \frac{d_F^{abcd} d_F^{abcd}}{N_c} \delta Z_2^{dFFL} + n_h \frac{d_F^{abcd} d_F^{abcd}}{N_c} \delta Z_2^{dFFH} \\ & + C_F^3 T n_l \delta Z_2^{FFFLL} + C_F^2 C_A T n_l \delta Z_2^{FFAL} + C_F C_A^2 T n_l \delta Z_2^{FAAL} \end{aligned}$$

$$\begin{aligned}
& + C_F^2 T^2 n_l^2 \delta Z_2^{FLL} + C_F C_A T^2 n_l^2 \delta Z_2^{FALL} + C_F T^3 n_l^3 \delta Z_2^{FLLL} \\
& + C_F^3 T n_h \delta Z_2^{FFFH} + C_F^2 C_A T n_h \delta Z_2^{FFAH} + C_F C_A^2 T n_h \delta Z_2^{FAAH} \\
& + C_F^2 T^2 n_h^2 \delta Z_2^{FFHH} + C_F C_A T^2 n_h^2 \delta Z_2^{FAHH} + C_F T^3 n_h^3 \delta Z_2^{FHHH} \\
& + C_F^2 T^2 n_l n_h \delta Z_2^{FLH} + C_F C_A T^2 n_l n_h \delta Z_2^{FALH} + C_F T^3 n_l^2 n_h \delta Z_2^{FLLH} \\
& + C_F T^3 n_l n_h^2 \delta Z_2^{FLHH} , \tag{13}
\end{aligned}$$

where C_F, C_A, T, n_l and n_h are defined after Eq. (24) in Appendix B. The new colour factors at four loops are the symmetrized traces of four generators in the fundamental and adjoint representation denoted by d_F^{abcd} and d_A^{abcd} , respectively.

In Tabs. 5, 6, 7 and 8 (see Appendix A) we show the numerical results for the coefficients introduced in Eq. (13). The numerical uncertainties have been obtained by adding the uncertainties from each individual master integral in quadrature and multiplying the result by a security factor 10. This approach is quite conservative, however, we observed that there are rare cases where the uncertainty from numerical integration is underestimated by several standard deviations. A factor 10 covers all cases which we have experienced (see also discussion in Ref. [2]). All coefficients which have a non-zero numerical uncertainty are truncated in such a way that two digits of the uncertainty are shown; otherwise we present (at least) five significant digits. Note that the n_l^3 and n_h^2 terms are known analytically [17]. None of the other coefficients are known analytically to us although for some of them the uncertainty is very small, see, e.g., $C_F n_h^3$.

Let us start with the discussion of Tab. 5. Most of the coefficients are known with an uncertainty of a few per cent or below. An exception are the C_F^4 and $C_F^3 C_A$ colour factors, where the uncertainty is about 30%. In the case of $n_h (d_F^{abcd})^2$ the uncertainty is larger than the central value and we are not able to decide whether the corresponding coefficient is zero or numerically small. For some colour structures our precision is below a per mille level, in particular for the most non-abelian colour factor $C_F C_A^3$ which provides the numerically largest contribution.

There are some coefficients in the pole parts where the numerical uncertainty is larger than the central value. In these cases no definite conclusion can be drawn. Within our (conservative) uncertainty estimate the results are compatible with zero. Still, in these cases we cannot exclude a small non-zero result. Note, however, that in most cases the uncertainty is much smaller than the central value. In particular, all colour structures except those involving d_F^{abcd} or d_A^{abcd} have a non-zero $1/\epsilon^4$ pole. In fact, we expect that the colour structures involving d_F^{abcd} and d_A^{abcd} only have a $1/\epsilon$ poles which is consistent with our result.

The coefficients in Tab. 6 representing the linear ξ terms are in general much smaller than for $\xi = 0$ and the situation is similar as for the pole terms of Tab. 5: We can conclude that the colour structures $C_F^2 C_A^2$, $C_F C_A^3$, $d_F^{abcd} d_A^{abcd}$, $C_F C_A^2 n_l$, $C_F^2 C_A n_h$, $C_F C_A^2 n_h$, $C_F C_A n_h^2$ and $C_F C_A n_l n_h$ have non-zero coefficients. Within our precision the coefficient of $C_F^3 C_A$ is zero: the central value is of order 10^{-4} and furthermore ten times smaller than the uncertainty. However, a closer look into this contribution shows that non-trivial master integrals are

involved which combine to the numerical result given in Tab. 6. Since the master integrals are linear independent and since they are beyond “three-loop complexity” (i.e., they are neither products of lower-loop integrals nor contain simple one-loop insertions) we would expect a non-zero coefficient unless there are accidental cancellations. Note that at three-loop order there are two colour structures which have ξ dependent coefficients: $C_F C_A^2$ and $C_F C_A n_h$.

In Tab. 7, which contains the ξ^2 terms, there are non-zero coefficients for the colour structures $C_F C_A^3$, $d_F^{abcd} d_A^{abcd}$ and $C_F C_A^2 n_h$.

It is interesting to check the cancellations between the bare four-loop expression and the mass counterterm contributions (which are known analytically and can be found in the ancillary file to this paper [25]). For this reason we show in Tab. 8 the bare four-loop coefficients. The comparison with the corresponding entries in Tab. 5 shows that the coefficients of some of the colour structures suffer from large cancellations which in some cases is even more than two orders of magnitude (see, e.g., the $C_F^3 n_l$ term). Note that the numerically dominant colour structure $C_F C_A^3$ is not affected by mass renormalization.

In Ref. [26] the pole of the colour structure $n_l (d_F^{abcd})^2$ has been determined from the requirement that a certain combination of renormalization constants in full QCD and HQET are finite (see also discussion in Section 3 below). Its analytic expression in our notation reads

$$\begin{aligned} \delta Z_2^{d_{FF}L} &= -\frac{1}{\epsilon} \left(\frac{1}{8} + \frac{\pi^2}{12} - \frac{\zeta_3}{8} - \frac{\pi^2 \zeta_3}{12} + \frac{5\zeta_5}{32} \right) + \dots \\ &\approx \frac{0.0294223}{\epsilon} + \dots, \end{aligned} \tag{14}$$

which has to be compared to our numerical result $(0.011 \pm 0.064)/\epsilon + \dots$ (see Tab. 5). The result in Eq. (14) agrees with our result within the uncertainty. Note, however, that the absolute value of this contribution is quite small which explains our large relative uncertainty.

It is interesting to insert the numerical values of the colour factors and evaluate $\delta Z_2^{(4)}$ for $N_c = 3$. To obtain the corresponding expression we choose $N_c = 3$ after inserting the master integrals but before combining the uncertainties from the various ϵ expansion coefficients of the colour factors. The results for the various powers of n_l are given in Tab. 1. Note that for $\xi = 0$ (top) all uncertainties are of order 10^{-4} . Furthermore, for all powers of n_l we observe non-zero coefficients in the poles up to fourth order. For completeness we present in Tab. 1 also results for the ξ^1 and ξ^2 terms. For the linear ξ coefficients we observe non-zero entries only for the n_l^0 and the linear- n_l term. The coefficients of ξ^2 are only non-zero for the n_l^0 contribution.

Finally, we discuss the wave function renormalization for QED. It is obtained from the QCD result by adopting the following values for the QCD colour factors

$$C_F \rightarrow 1, \quad C_A \rightarrow 0, \quad T \rightarrow 1, \quad d_F^{abcd} \rightarrow 1, \quad d_A^{abcd} \rightarrow 0, \quad N_c \rightarrow 1. \tag{15}$$

$\xi = 0$	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
n_l^0	-1.77242 ± 0.00040	-27.6674 ± 0.0041	-317.093 ± 0.029	-3142.15 ± 0.33	-28709.9 ± 3.2
n_l^1	0.460936 ± 0.000016	6.69143 ± 0.00023	74.6540 ± 0.0013	696.6612 ± 0.0076	6174.290 ± 0.084
n_l^2	-0.039931	-0.51572	-5.5055	-48.777	-418.93
n_l^3	0.00115741	0.0125386	0.126757	1.07105	8.9160

ξ^1	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
n_l^0	-0.018555 ± 0.000011	0.034239 ± 0.000089	-0.05678 ± 0.00052	5.2230 ± 0.0028	36.820 ± 0.017
n_l^1	0.00173611	-0.0052083	0.0224269	-0.34863	-1.61105

ξ^2	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
n_l^0	0.0000002 ± 0.0000038	0.001952 ± 0.000026	-0.03022 ± 0.00012	-0.18686 ± 0.00061	-2.9266 ± 0.0028

Table 1: Results for the coefficients of $\delta Z_2^{(4)}$ after choosing $N_c = 3$. The $\xi = 0$, ξ^1 and ξ^2 contributions are shown in the top, middle and bottom table. A security factor 10 has been applied to the uncertainties.

	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
n_l^0	0.20500 ± 0.00037	0.5980 ± 0.0027	-0.895 ± 0.021	-6.18 ± 0.17	-17.4 ± 1.6
n_l^1	0.17058 ± 0.00011	0.9556 ± 0.0014	2.9397 ± 0.0079	10.480 ± 0.064	25.92 ± 0.80
n_l^2	0.056424	0.46123	3.03509	18.7456	105.069
n_l^3	0.0069444	0.075231	0.76054	6.4263	53.496

Table 2: Results for Z_2^{OS} specified to QED.

We furthermore set $n_h = 1$ but keep the dependence on n_l . Note that $n_l = 0$ corresponds to the case of a massive electron and $n_l = 1$ describes the case of a massive muon and a massless electron. Our results are shown in Tab. 2. For the n_l -independent part we have an uncertainty of about 10%, the n_l^1 term is determined with a 3% accuracy.

The on-shell wave function renormalization constant in QED has to be independent of ξ [5, 27] which is fulfilled in our result as can be seen from the absence of all abelian coefficients in Tabs. 6 and 7; they are analytically zero. Note that the gauge parameter dependence only cancels after adding the mass counterterm contributions.

3 Checks and HQET wave function renormalization

In this Section we describe several checks of our results. In particular, we discuss the relation to the wave function renormalization constant in HQET.

We start with the discussion of the $\overline{\text{MS}}$ wave function renormalization constant $Z_2^{\overline{\text{MS}}}$ which has been obtained to five-loop accuracy in Refs. [28, 29]. In these papers also the full ξ -dependence at four loops has been computed which is crucial for our application. By definition it only contains ultraviolet poles. On the other hand, as discussed in the Introduction, Z_2^{OS} , contains both ultraviolet and infrared poles since it has to take care of both types of divergences in processes containing external heavy quarks. The ultraviolet

divergences of Z_2^{OS} have to agree with the ones of $Z_2^{\overline{\text{MS}}}$ and thus $Z_2^{\overline{\text{MS}}}/Z_2^{\text{OS}}$ only contains infrared poles. Note that the latter have to agree with the ultraviolet poles of the wave function renormalization constant in HQET, Z_2^{HQET} , which can be seen as follows (see also discussion in Ref. [5]): The off-shell heavy-quark propagator is infrared finite and contains only ultraviolet divergences, which can be renormalized in the $\overline{\text{MS}}$ scheme, i.e., they are taken care of by $Z_2^{\overline{\text{MS}}}$. If one applies an asymptotic expansion [30,31] around the on-shell limit one obtains two contributions. The first one corresponds to a naive Taylor expansion of on-shell integrals which have to be evaluated in full QCD. It develops both ultraviolet and infrared divergences, as discussed above for the case of Z_2^{OS} . The second contribution corresponds to HQET integrals and only has ultraviolet poles which have to cancel the infrared poles of the QCD contribution. Note that the wave functions Z_2^{OS} and Z_2^{HQET} considered in this paper correspond to the leading term in the expansion and thus $Z_2^{\text{HQET}}/Z_2^{\text{OS}}$ has to be infrared finite. As a consequence, the following combination of renormalization constants

$$\frac{Z_2^{\overline{\text{MS}}}}{Z_2^{\text{OS}}} Z_2^{\text{HQET}} \quad (16)$$

has to be finite (see also discussion in Ref. [32]). We will use this fact to determine the poles of Z_2^{HQET} .

HQET describes the limit of QCD where the mass of the heavy quark goes to infinity. The heavy quark field is integrated out from the Lagrange density. Thus, it is not a dynamical degree of freedom any more. As a consequence, HQET contains as parameters the strong coupling constant and gauge parameter defined in the n_l -flavour theory, $\alpha_s^{(n_l)}$ and $\xi^{(n_l)}$.¹ Furthermore, there are no closed heavy quark loops, i.e., colour factors involving n_h are absent. Thus, when constructing (16) we can check that in the ratio $Z_2^{\overline{\text{MS}}}/Z_2^{\text{OS}}$ all colour structures containing n_h are finite after using the decoupling relations for α_s and ξ [33]. At two- and three-loop order this check can be performed analytically. At four loops we observe that $Z_2^{\overline{\text{MS}}}/Z_2^{\text{OS}}$ is finite within our numerical precision. Note that this concerns the eleven colour structures in Eq. (13) which are proportional to n_h , n_h^2 or n_h^3 . Let us mention that all coefficients are zero within three standard deviations of the original FIESTA uncertainty which means that in this case a security factor 3 would be sufficient.

The remaining twelve four-loop colour structures are present in Z_2^{HQET} and the corresponding pole term can be extracted from Eq. (16). Before presenting the results we remark that Z_2^{HQET} exponentiates according to [5,34]

$$\begin{aligned} Z_2^{\text{HQET}} = \exp \left\{ & x_1 C_F \left(\frac{\alpha_s}{\pi} \right) + C_F [x_2 C_A + x_3 T n_l] \left(\frac{\alpha_s}{\pi} \right)^2 + C_F [x_4 C_A^2 + x_5 C_A T n_l \right. \\ & + x_6 T^2 n_l^2 + x_7 C_F T n_l] \left(\frac{\alpha_s}{\pi} \right)^3 + [C_F (x_8 C_A^3 + x_9 C_A^2 T n_l + x_{10} C_A T^2 n_l^2 \\ & + x_{11} T^3 n_l^3 + x_{12} C_F^2 T n_l + x_{13} C_F C_A T n_l + x_{14} C_F T^2 n_l^2) \\ & \left. + x_{15} d_F^{abcd} d_A^{abcd} / N_c + x_{16} d_F^{abcd} d_F^{abcd} n_l / N_c] \left(\frac{\alpha_s}{\pi} \right)^4 + \dots \right\}, \quad (17) \end{aligned}$$

¹Note that all quantities discussed in Section 2 depend on $n_f = n_l + n_h$ flavours.

and thus there are only nine genuinely new colour coefficients at four loops (x_8, \dots, x_{16}) and the remaining three contributions proportional to $C_F^4, C_F^3 C_A$ and $C_F^2 C_A^2$ can be predicted from lower loop orders. The comparison with the explicit calculation provides a strong check on our calculation. Note that the predictions of the $C_F^4, C_F^3 C_A$ and $C_F^2 C_A^2$ contributions are available in analytic form.

In our practical calculations we proceed as follows: In a first step we use Eq. (16) to obtain a result for Z_2^{HQET} from the requirement that the combination of the three quantities is finite. Afterwards we use this result and compare to the expanded version of Eq. (17) to determine the coefficients x_i . Finally, we use Eq. (17) to predict the $C_F^4, C_F^3 C_A$ and $C_F^2 C_A^2$ of Z_2^{HQET} .

We refrain from providing explicit results for Z_2^{HQET} but provide our results for x_i in the ancillary file to this paper [25]. Furthermore, we present the expressions for the corresponding anomalous dimension, which is given by

$$\begin{aligned}\gamma_{\text{HQET}} &= \frac{d \log Z_2^{\text{HQET}}}{d \log \mu^2} \\ &= \sum_{n \geq 1} \gamma_{\text{HQET}}^{(n)} \left(\frac{\alpha_s(\mu)}{\pi} \right)^n.\end{aligned}\tag{18}$$

Since our four-loop expression for Z_2^{HQET} is only known numerically, we have spurious ϵ poles in γ_{HQET} . However, all of them are zero within two standard deviations of the uncertainty provided by FIESTA, which constitutes another useful cross check for our calculation.

Let us in the following present our results for γ_{HQET} . Up to three-loop order we have

$$\begin{aligned}\gamma_{\text{HQET}}^{(1)} &= -\frac{C_F}{2} \left(1 + \frac{\xi^{(n_l)}}{2} \right), \\ \gamma_{\text{HQET}}^{(2)} &= C_F C_A \left(-\frac{19}{24} - \frac{5\xi^{(n_l)}}{32} + \frac{(\xi^{(n_l)})^2}{64} \right) + \frac{C_F T n_l}{3}, \\ \gamma_{\text{HQET}}^{(3)} &= C_F C_A^2 \left[-\frac{19495}{27648} - \frac{3\zeta_3}{16} - \frac{\pi^4}{360} + \xi^{(n_l)} \left(-\frac{379}{2048} - \frac{15\zeta_3}{256} + \frac{\pi^4}{1440} \right) \right. \\ &\quad \left. + (\xi^{(n_l)})^2 \left(\frac{69}{2048} + \frac{3\zeta_3}{512} \right) - \frac{5(\xi^{(n_l)})^3}{1024} \right] + C_F C_A T n_l \left(\frac{1105}{6912} + \frac{3\zeta_3}{4} + \frac{17\xi^{(n_l)}}{256} \right) \\ &\quad + C_F^2 T n_l \left(\frac{51}{64} - \frac{3\zeta_3}{4} \right) + \frac{5C_F T^2 n_l^2}{108},\end{aligned}\tag{19}$$

which agree with Refs. [5, 34].

The four-loop terms to γ_{HQET} can be found in Tab. 3 where for each colour factor the coefficients of the $(\xi^{(n_l)})^k$ terms are shown together with their uncertainty. As for Z_2^{OS} in Section 2 we have introduced a security factor 10. Note that the coefficients of $(\xi^{(n_l)})^k$ with $k \geq 3$ have not been computed.

	$(\xi^{(n_l)})^0$	$(\xi^{(n_l)})^1$	$(\xi^{(n_l)})^2$
$FAAA$	-2.03 ± 0.35	-0.29037 ± 0.00052	0.07083 ± 0.00010
d_{FA}	1.53 ± 0.84	0.5083 ± 0.0098	-0.1031 ± 0.0024
d_{FFL}	0.54 ± 0.26		
$FFFL$	0.1894 ± 0.0030		
$FFAL$	-0.4566 ± 0.0055	-0.0076630	
$FAAL$	2.576 ± 0.010	0.25147	-0.0103348
$FFLL$	0.25725		
$FALL$	-0.53745	-0.0077460	
$FLLL$	-0.048262		

Table 3: Results for the different colour factors of $\gamma_{\text{HQET}}^{(4)}$. In the columns two to four the coefficients of different powers of $\xi^{(n_l)}$ are given. In the uncertainties a security factor 10 has been introduced.

We have the worst precision of about 50% for the colour factors $d_F^{abcd} d_A^{abcd}$ and $n_l d_F^{abcd} d_F^{abcd}$ followed by $C_F C_A^3$ which is 17%. The relative uncertainty of the remaining n_l terms is much smaller. Note that the n_l^2 and n_l^3 terms are known analytically. They are obtained in a straightforward way for the corresponding analytic results for Z_2^{OS} from Ref. [17]. Our results read

$$\begin{aligned}
\gamma_{\text{HQET}}^{(4),FFLL} &= \frac{3\zeta_3}{4} - \frac{\pi^4}{240} - \frac{103}{432}, \\
\gamma_{\text{HQET}}^{(4),FALL} &= -\frac{35\zeta_3}{48} + \frac{\pi^4}{240} - \frac{4157}{62208} + \xi^{(n_l)} \left(-\frac{\zeta_3}{48} + \frac{269}{15552} \right), \\
\gamma_{\text{HQET}}^{(4),FLLL} &= \frac{1}{54} - \frac{\zeta_3}{18}.
\end{aligned} \tag{20}$$

The expression for $\gamma_{\text{HQET}}^{(4),FFLL}$ agrees with Ref. [35,36] and $\gamma_{\text{HQET}}^{(4),FLLL}$ can be found in Ref. [37]. $\gamma_{\text{HQET}}^{(4),FALL}$ is new.

Recently also for the $n_l d_F^{abcd} d_F^{abcd}$ colour structure analytic results have been obtained [26]. Their result reads

$$\gamma_{\text{HQET}}^{(4),d_{FFL}} = -\frac{5}{8}\zeta_5 + \frac{1}{3}\pi^2\zeta_3 + \frac{1}{2}\zeta_3 - \frac{1}{3}\pi^2 \approx 0.617689\dots, \tag{21}$$

and agrees well with our findings $\gamma_{\text{HQET}}^{(4),d_{FFL}} \approx 0.54 \pm 0.26$. Note that here a security factor 2 would have been sufficient.

There are no contributions from the colour structures C_F^4 , $C_F^3 C_A$ and $C_F^2 C_A^2$ to $\gamma_{\text{HQET}}^{(4)}$ as it is obvious by inspecting Eq. (17): the four-loop C_F^4 , $C_F^3 C_A$ and $C_F^2 C_A^2$ terms are generated by products of lower-order contributions. Since all coefficients x_i only contain poles in ϵ , the $1/\epsilon$ pole of Z_2^{HQET} does not involve C_F^4 , $C_F^3 C_A$ and $C_F^2 C_A^2$.

Let us finally compare the predicted C_F^4 , $C_F^3 C_A$ and $C_F^2 C_A^2$ contributions to Z_2^{HQET} to the ones we obtain by an explicit calculation. Tab. 4 contains coefficients of $(\xi^{(n_l)})^k \epsilon^n$

	$(\xi^{(n_i)})^0$	$(\xi^{(n_i)})^1$	$(\xi^{(n_i)})^2$
C_F^4			
$1/\epsilon^4$	0.0026042	0.0052083	0.0039063
	0.0025932 ± 0.000025	0.0052083	0.0039063
$1/\epsilon^3$	0.00000	0.00000	0.00000
	0.00013049 ± 0.00019	0.00000	0.00000
$C_F^3 C_A$			
$1/\epsilon^4$	0.035156	0.044922	0.016602
	0.035190 ± 0.00005	0.044922	0.016602
$1/\epsilon^3$	-0.049479	-0.059245	-0.021159
	-0.049878 ± 0.00044	-0.059245 ± 0.00000006	-0.021159
$1/\epsilon^2$	0.00000	0.00000	0.00000
	0.0029893 ± 0.0041	-0.0000002 ± 0.0000020	0.00000
$C_F^2 C_A^2$			
$1/\epsilon^4$	0.130914	0.085558	0.0027262
	0.130887 ± 0.00004	0.085558 ± 0.00000002	0.0027262
$1/\epsilon^3$	-0.31170	-0.191497	-0.0081380
	-0.31133 ± 0.00035	-0.191497 ± 0.0000002	-0.0081380
$1/\epsilon^2$	0.27852	0.162322	0.0088241
	0.27669 ± 0.0033	0.162323 ± 0.000002	0.0088241
$1/\epsilon^1$	0.00000	0.00000	0.00000
	0.046 ± 0.031	-0.000014 ± 0.000022	0.00000

Table 4: Contributions of the colour structures C_F^4 , $C_F^3 C_A$ and $C_F^2 C_A^2$ to Z_2^{HQET} . The coefficients of $(\xi^{(n_i)})^0$, $(\xi^{(n_i)})^1$ and $(\xi^{(n_i)})^2$ are given in the rows two to four. For each power of ϵ the first row corresponds to the numerical evaluation of the analytic result and the second row to the numerical result of our explicit calculation of Z_2^{OS} . Relative uncertainties below 10^{-5} are set to zero. Note that the uncertainties in this paper are not multiplied by a security factor 10.

for $k = 0, 1$ and 2 and for values of $n = -4, -3, \dots$ up to one unit higher than the order up to which the corresponding colour structure has a non-zero contribution. The last ϵ order is shown as a check and demonstrates how well we can reproduce the 0. Note that in this table the displayed uncertainties are not multiplied by a security factor but correspond to the quadratically combined FIESTA uncertainties. In some case the relative uncertainty is very small and thus not shown at all. In all cases shown in Fig. 4 the numerical results agree within 1.5 sigma with the analytic predictions from Eq. (17). Note the colour factors C_F^4 , $C_F^3 C_A$ and $C_F^2 C_A^2$ get contributions from the most complicated master integrals and thus the above comparison provides a strong check on the numerical setup of our calculation.

4 Conclusions

We have computed four-loop QCD corrections to the wave function renormalization constant of heavy quarks, Z_2^{OS} . Besides the on-shell quark mass renormalization constant and the leptonic anomalous magnetic moment, which have been considered in Refs. [1,2] and [7], respectively, this constitutes a third “classical” application of four-loop on-shell integrals. In the present calculation we could largely profit from the previous calculations. However, we had to deal with a more involved reduction to master integrals. Furthermore, we observed higher ϵ poles in the prefactors of some of the master integrals which forced us to either change the basis or to expand the corresponding master integrals to higher order in ϵ .

Z_2^{OS} is neither gauge parameter independent nor infrared finite which excludes two important checks used for Z_m^{OS} and the anomalous magnetic moment. However, a number of cross checks are provided by the relation to the wave function renormalization constant of HQET.

In physical applications Z_2^{OS} enters, among other quantities, as building block. Most likely in the evaluation of the other pieces numerical methods play an important role as well and thus various numerical pieces have to be combined to arrive at physical cross sections or decay rates. It might be that numerical cancellations take place and thus, to date, it is not clear whether the numerical precision reached for Z_2^{OS} (which is of the order of 10^{-4} for $N_c = 3$) is sufficient for phenomenological applications. However, the results obtained in this paper serve for sure as important cross checks for future more precise or even analytic calculations.

In future, it would, of course, be desirable to obtain analytic results for fundamental quantities like on-shell QCD renormalization constants as Z_2^{OS} , which is considered in this paper, and Z_m^{OS} from Refs. [1,2]. First steps in this direction have been undertaken in Ref. [23] where a semi-analytic approach has been used to obtain a high-precision result for the anomalous magnetic moment of the electron. One could imagine to extend this analysis to the QCD-like master integrals.

Acknowledgements

The work of A.S. and V.S. is supported by RFBR, grant 17-02-00175A. We thank the High Performance Computing Center Stuttgart (HLRS) for providing computing time used for the numerical computations with FIESTA. The research is carried out using the equipment of the shared research facilities of HPC computing resources at Lomonosov Moscow State University. P.M was supported in part by the EU Network HIGGSTOOLS PITN-GA-2012-316704. We thank Andrey Grozin for carefully reading the manuscript and many useful comments.

A Numerical results for Z_2^{OS}

Tables 5, 6, 7 and 8 contain the numerical results for the coefficients of the individual colour factors contributing to Z_2^{OS} .

B Z_2^{OS} to three loops

In this appendix we provide results for the coefficients of Z_2^{OS} as defined in Eq. (9) up to three loops including higher order terms in ϵ : the n -loop expression contains terms up to order ϵ^{4-n} . Note that in Eq. (9) the quark mass M is renormalized on-shell but α_s is bare. Our results read

$$\delta Z_2^{(1)} = \left(\frac{\zeta_3}{3} - \frac{3\pi^4}{640} - \frac{\pi^2}{6} - 8\right)\epsilon^3 C_F + \left(\frac{\zeta_3}{4} - \frac{\pi^2}{12} - 4\right)\epsilon^2 C_F + \left(-2 - \frac{\pi^2}{16}\right)\epsilon C_F - \frac{3C_F}{4\epsilon} - C_F, \quad (22)$$

$$\begin{aligned} \delta Z_2^{(2)} = & \epsilon \left(C_A C_F \left(12a_4 + \frac{199\zeta_3}{24} - \frac{7\pi^4}{40} + \frac{227\pi^2}{384} - \frac{4241}{256} + \frac{\log^4(2)}{2} \right. \right. \\ & \left. \left. + \pi^2 \log^2(2) - \frac{23}{8}\pi^2 \log(2) \right) \right. \\ & \left. + C_F^2 \left(-24a_4 - \frac{297\zeta_3}{16} + \frac{7\pi^4}{20} - \frac{339\pi^2}{128} + \frac{211}{256} - \log^4(2) \right. \right. \\ & \left. \left. - 2\pi^2 \log^2(2) + \frac{23}{4}\pi^2 \log(2) \right) \right. \\ & \left. + C_F \left(-\frac{43\zeta_3}{6} - \frac{437\pi^2}{288} + \frac{20275}{1728} \right. \right. \\ & \left. \left. + 2\pi^2 \log(2) \right) T_F n_h + \left(\frac{11\zeta_3}{12} + \frac{15\pi^2}{32} + \frac{369}{64} \right) C_F T_F n_l \right) \\ & + \epsilon^2 \left(C_A C_F \left(69a_4 + 72a_5 - \frac{11\pi^2\zeta_3}{8} + \frac{2561\zeta_3}{96} - \frac{609\zeta_5}{8} - \frac{7229\pi^4}{11520} \right. \right. \\ & \left. \left. + \frac{2005\pi^2}{768} - \frac{30163}{512} - \frac{3\log^5(2)}{5} + \frac{23\log^4(2)}{8} - 2\pi^2 \log^3(2) + \frac{23}{4}\pi^2 \log^2(2) \right. \right. \\ & \left. \left. + \frac{13}{30}\pi^4 \log(2) - \frac{41}{4}\pi^2 \log(2) \right) \right. \\ & \left. + C_F^2 \left(-138a_4 - 144a_5 + \frac{11\pi^2\zeta_3}{4} - \frac{2069\zeta_3}{32} + \frac{609\zeta_5}{4} + \frac{3901\pi^4}{3840} \right. \right. \\ & \left. \left. - \frac{8851\pi^2}{768} + \frac{4889}{512} + \frac{6\log^5(2)}{5} - \frac{23\log^4(2)}{4} + 4\pi^2 \log^3(2) - \frac{23}{2}\pi^2 \log^2(2) \right. \right. \\ & \left. \left. - \frac{13}{15}\pi^4 \log(2) + \frac{41}{2}\pi^2 \log(2) \right) \right) \end{aligned}$$

$$\begin{aligned}
& +C_F T_F n_h \left(-48a_4 - \frac{2413\zeta_3}{72} + \frac{47\pi^4}{160} - \frac{8509\pi^2}{1728} \right. \\
& + \frac{450395}{10368} - 2\log^4(2) - 4\pi^2 \log^2(2) + \frac{19}{2}\pi^2 \log(2) \left. \right) \\
& + \left(\frac{33\zeta_3}{8} + \frac{101\pi^4}{960} + \frac{295\pi^2}{192} + \frac{2259}{128} \right) C_F T_F n_l \\
& + C_A C_F \left(\frac{3\zeta_3}{4} + \frac{49\pi^2}{192} - \frac{803}{128} - \frac{1}{2}\pi^2 \log(2) \right) \\
& + C_F^2 \left(-\frac{3\zeta_3}{2} - \frac{49\pi^2}{64} + \frac{433}{128} + \pi^2 \log(2) \right) + \left(\frac{1139}{288} - \frac{7\pi^2}{24} \right) C_F T_F n_h \\
& + \left(\frac{59}{32} + \frac{5\pi^2}{48} \right) C_F T_F n_l \\
& + \frac{-\frac{11C_A C_F}{32} + \frac{1}{4}C_F T_F n_h + \frac{1}{8}C_F T_F n_l + \frac{9C_F^2}{32}}{\epsilon^2} \\
& + \frac{-\frac{101C_A C_F}{64} + \frac{19}{48}C_F T_F n_h + \frac{9}{16}C_F T_F n_l + \frac{51C_F^2}{64}}{\epsilon}, \tag{23}
\end{aligned}$$

$$\begin{aligned}
\delta Z_2^{(3)} = & \left(-10a_4 + \frac{\pi^2\zeta_3}{8} - \frac{739\zeta_3}{128} - \frac{5\zeta_5}{16} - \frac{5\log^4(2)}{12} + 3\pi^2 \log^2(2) \right. \\
& + \frac{685}{48}\pi^2 \log(2) - \frac{41\pi^4}{120} - \frac{58321\pi^2}{9216} - \frac{10823}{3072} \left. \right) C_F^3 \\
& + C_A \left(-\frac{319a_4}{6} - \frac{45\pi^2\zeta_3}{16} - \frac{19981\zeta_3}{384} + \frac{145\zeta_5}{16} - \frac{319\log^4(2)}{144} - \frac{499}{72}\pi^2 \log^2(2) \right. \\
& + \frac{2281}{288}\pi^2 \log(2) + \frac{20053\pi^4}{17280} - \frac{15053\pi^2}{9216} + \frac{150871}{9216} \left. \right) C_F^2 \\
& + T_F n_l \left(\frac{64a_4}{3} + \frac{1661\zeta_3}{96} + \frac{8\log^4(2)}{9} + \frac{16}{9}\pi^2 \log^2(2) - \frac{58}{9}\pi^2 \log(2) - \frac{733\pi^4}{2160} \right. \\
& + \left. \frac{6931\pi^2}{2304} - \frac{3773}{2304} \right) C_F^2 \\
& + T_F n_h \left(28a_4 + \frac{5327\zeta_3}{288} + \frac{7\log^4(2)}{6} + \frac{5}{6}\pi^2 \log^2(2) - \frac{31}{9}\pi^2 \log(2) \right. \\
& - \left. \frac{137\pi^4}{720} + \frac{25223\pi^2}{20736} - \frac{78967}{6912} \right) C_F^2 \\
& + T_F^2 n_l^2 \left(-\frac{37\zeta_3}{36} - \frac{23\pi^2}{48} - \frac{4025}{972} \right) C_F \\
& + T_F^2 n_h n_l \left(\frac{49\zeta_3}{12} - \frac{4}{3}\pi^2 \log(2) + \frac{77\pi^2}{72} - \frac{1168}{81} \right) C_F \\
& + T_F^2 n_h^2 \left(\frac{85\zeta_3}{12} - \frac{4}{3}\pi^2 \log(2) + \frac{767\pi^2}{720} - \frac{6887}{648} \right) C_F \\
& + C_A T_F n_h \left(-16a_4 + \xi \left(-\frac{7\zeta_3}{192} + \frac{\pi^2}{256} + \frac{407}{1728} \right) + \frac{11\pi^2\zeta_3}{48} - \frac{3359\zeta_3}{144} - \frac{15\zeta_5}{16} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{2\log^4(2)}{3} - \frac{1}{3}\pi^2\log^2(2) + \frac{521}{36}\pi^2\log(2) + \frac{5\pi^4}{72} - \frac{105359\pi^2}{10368} + \frac{32257}{648})C_F \\
& + C_A T_F n_l \left(-\frac{32a_4}{3} - \frac{301\zeta_3}{72} - \frac{4\log^4(2)}{9} - \frac{8}{9}\pi^2\log^2(2) + \frac{29}{9}\pi^2\log(2) \right. \\
& \left. + \frac{29\pi^4}{216} + \frac{2413\pi^2}{3456} + \frac{416405}{15552} \right) C_F \\
& + C_A^2 \left(\frac{349a_4}{12} + \frac{127\pi^2\zeta_3}{72} + \frac{3623\zeta_3}{144} + \xi \left(\frac{\pi^2\zeta_3}{144} - \frac{13\zeta_3}{256} + \frac{7\zeta_5}{384} + \frac{17\pi^4}{27648} \right. \right. \\
& \left. \left. - \frac{\pi^2}{256} - \frac{13}{768} \right) - \frac{37\zeta_5}{6} + \frac{349\log^4(2)}{288} + \frac{391}{144}\pi^2\log^2(2) - \frac{271}{36}\pi^2\log(2) \right. \\
& \left. - \frac{10811\pi^4}{23040} - \frac{107\pi^2}{864} - \frac{2551697}{62208} \right) C_F \\
& + \frac{1}{\epsilon^3} \left[-\frac{9C_F^3}{128} + \frac{33}{128}C_A C_F^2 - \frac{3}{16}T_F n_h C_F^2 - \frac{3}{32}T_F n_l C_F^2 - \frac{121}{576}C_A^2 C_F \right. \\
& \left. - \frac{1}{12}T_F^2 n_h^2 C_F - \frac{1}{36}T_F^2 n_l^2 C_F + \frac{11}{72}C_A T_F n_l C_F - \frac{1}{12}T_F^2 n_h n_l C_F \right. \\
& \left. + C_A T_F n_h \left(\frac{15}{64} - \frac{\xi}{192} \right) C_F \right] \\
& + \frac{1}{\epsilon^2} \left[-\frac{81C_F^3}{256} + \frac{1217}{768}C_A C_F^2 - \frac{91}{192}T_F n_h C_F^2 - \frac{103}{192}T_F n_l C_F^2 - \frac{1501}{864}C_A^2 C_F \right. \\
& \left. - \frac{5}{36}T_F^2 n_h^2 C_F - \frac{23}{108}T_F^2 n_l^2 C_F + \frac{1069}{864}C_A T_F n_l C_F - \frac{7}{18}T_F^2 n_h n_l C_F \right. \\
& \left. + C_A T_F n_h \left(\frac{\xi}{64} + \frac{353}{288} \right) C_F \right] \\
& + \frac{1}{\epsilon} \left[\left(\frac{9\zeta_3}{8} - \frac{3}{4}\pi^2\log(2) + \frac{303\pi^2}{512} - \frac{1039}{512} \right) C_F^3 + T_F n_l \left(\frac{3\zeta_3}{4} - \frac{2}{3}\pi^2\log(2) \right) \right. \\
& \left. + \frac{175\pi^2}{384} - \frac{351}{128} \right) C_F^2 + T_F n_h \left(\zeta_3 - \frac{2}{3}\pi^2\log(2) + \frac{143\pi^2}{192} - \frac{1525}{384} \right) C_F^2 \\
& \left. + C_A \left(-\frac{27\zeta_3}{8} + \frac{53}{24}\pi^2\log(2) - \frac{2549\pi^2}{1536} + \frac{14887}{1536} \right) C_F^2 \right. \\
& \left. + C_A^2 \left(\xi \left(-\frac{3\zeta_3}{256} + \frac{\pi^4}{4320} - \frac{1}{768} \right) \right. \right. \\
& \left. \left. + \frac{173\zeta_3}{128} - \frac{11}{12}\pi^2\log(2) - \frac{\pi^4}{1080} + \frac{1199\pi^2}{2304} - \frac{55945}{5184} \right) C_F \right. \\
& \left. + C_A T_F n_h \left(\left(-\frac{35}{576} - \frac{\pi^2}{768} \right) \xi - \frac{\zeta_3}{2} + \frac{1}{3}\pi^2\log(2) - \frac{1753\pi^2}{2304} + \frac{503}{48} \right) C_F \right. \\
& \left. + C_A T_F n_l \left(-\frac{\zeta_3}{4} + \frac{1}{3}\pi^2\log(2) - \frac{5\pi^2}{288} + \frac{550}{81} \right) C_F + T_F^2 n_h^2 \left(-\frac{131}{54} + \frac{29\pi^2}{144} \right) C_F \right. \\
& \left. + T_F^2 n_h n_l \left(-\frac{31}{9} + \frac{7\pi^2}{48} \right) C_F + T_F^2 n_l^2 \left(-\frac{325}{324} - \frac{\pi^2}{16} \right) C_F \right] \\
& + \epsilon \left(\left(\frac{29\zeta_3^2}{32} + 14\pi^2\log(2)\zeta_3 + \frac{5267\pi^2\zeta_3}{288} - \frac{69511\zeta_3}{256} - \frac{4267a_4}{6} - \frac{116a_5}{3} \right) \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{1403\zeta_5}{16} + \frac{29\log^5(2)}{90} + \frac{2}{3}\pi^2\log^4(2) - \frac{4267\log^4(2)}{144} - \frac{196}{27}\pi^2\log^3(2) \\
& -\frac{2}{3}\pi^4\log^2(2) - \frac{3997}{72}\pi^2\log^2(2) - \frac{2351\pi^4\log(2)}{2160} + \frac{1345}{8}\pi^2\log(2) - \frac{899\pi^6}{5670} \\
& + \frac{450893\pi^4}{552960} + 16a_4\pi^2 - \frac{56455\pi^2}{2048} - \frac{108677}{6144})C_F^3 \\
& + T_F n_l \left(\frac{1856a_4}{9} + \frac{512a_5}{3} - \frac{69\pi^2\zeta_3}{16} + \frac{57581\zeta_3}{576} - \frac{2245\zeta_5}{12} - \frac{64\log^5(2)}{45} \right. \\
& + \frac{232\log^4(2)}{27} - \frac{128}{27}\pi^2\log^3(2) + \frac{464}{27}\pi^2\log^2(2) + \frac{139}{270}\pi^4\log(2) - \frac{908}{27}\pi^2\log(2) \\
& \left. - \frac{610451\pi^4}{414720} + \frac{251107\pi^2}{13824} - \frac{36677}{13824} \right) C_F^2 \\
& + T_F n_h \left(\frac{539a_4}{6} + 168a_5 - \frac{287\pi^2\zeta_3}{24} + \frac{1087\zeta_3}{96} - \frac{899\zeta_5}{8} - \frac{7\log^5(2)}{5} \right. \\
& + \frac{539\log^4(2)}{144} - \frac{5}{3}\pi^2\log^3(2) - \frac{683}{144}\pi^2\log^2(2) + \frac{247}{180}\pi^4\log(2) + \frac{673}{24}\pi^2\log(2) \\
& \left. - \frac{32857\pi^4}{69120} - \frac{341735\pi^2}{41472} - \frac{143029}{13824} \right) C_F^2 \\
& + C_A \left(-\frac{143\zeta_3^2}{4} + \frac{21}{4}\pi^2\log(2)\zeta_3 - \frac{5777\pi^2\zeta_3}{384} - \frac{188083\zeta_3}{2304} - \frac{3787a_4}{36} \right. \\
& - \frac{1441a_5}{3} + \frac{254581\zeta_5}{384} + \frac{1441\log^5(2)}{360} + \frac{1}{4}\pi^2\log^4(2) - \frac{3787\log^4(2)}{864} \\
& + \frac{2089}{108}\pi^2\log^3(2) - \frac{1}{4}\pi^4\log^2(2) - \frac{25441}{432}\pi^2\log^2(2) - \frac{559}{540}\pi^4\log(2) \\
& - \frac{5623}{216}\pi^2\log(2) - \frac{27331\pi^6}{181440} + \frac{11810161\pi^4}{1658880} + 6a_4\pi^2 \\
& \left. - \frac{1265393\pi^2}{55296} + \frac{4824655}{55296} \right) C_F^2 \\
& + T_F^2 n_l^2 \left(-\frac{851\zeta_3}{108} - \frac{535\pi^4}{3456} - \frac{325\pi^2}{144} - \frac{23789}{1458} \right) C_F \\
& + T_F^2 n_h n_l \left(\frac{128a_4}{3} + \frac{595\zeta_3}{18} + \frac{16\log^4(2)}{9} + \frac{32}{9}\pi^2\log^2(2) - \frac{92}{9}\pi^2\log(2) \right. \\
& \left. - \frac{341\pi^4}{3456} + \frac{145\pi^2}{36} - \frac{17740}{243} \right) C_F \\
& + T_F^2 n_h^2 \left(\frac{152a_4}{3} + \frac{25723\zeta_3}{720} + \frac{19\log^4(2)}{9} + \frac{17}{9}\pi^2\log^2(2) - \frac{33}{5}\pi^2\log(2) \right. \\
& \left. - \frac{391\pi^4}{1152} + \frac{83227\pi^2}{16200} - \frac{1023397}{19440} \right) C_F \\
& + C_A^2 \left(\frac{7451\zeta_3^2}{384} - \frac{49}{8}\pi^2\log(2)\zeta_3 + \frac{3299\pi^2\zeta_3}{4608} + \frac{236171\zeta_3}{3456} + \frac{4147a_4}{18} + \frac{1499a_5}{6} \right. \\
& \left. + \xi \left(\frac{25\zeta_3^2}{384} + \frac{77\pi^2\zeta_3}{9216} - \frac{63\zeta_3}{256} - \frac{149\zeta_5}{768} + \frac{49\pi^6}{51840} + \frac{383\pi^4}{138240} - \frac{35\pi^2}{1024} - \frac{35}{256} \right) \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{77297\zeta_5}{256} - \frac{1499\log^5(2)}{720} - \frac{7}{24}\pi^2\log^4(2) + \frac{4147\log^4(2)}{432} - \frac{1697}{216}\pi^2\log^3(2) \\
& + \frac{7}{24}\pi^4\log^2(2) + \frac{4679}{108}\pi^2\log^2(2) + \frac{6823\pi^4\log(2)}{8640} - \frac{25069}{864}\pi^2\log(2) \\
& + \frac{45047\pi^6}{362880} - \frac{2973217\pi^4}{829440} - 7a_4\pi^2 + \frac{26425\pi^2}{20736} - \frac{74570603}{373248} \Big) C_F \\
& + C_A T_F n_l \left(-\frac{928a_4}{9} - \frac{256a_5}{3} + \frac{77\pi^2\zeta_3}{48} - \frac{11159\zeta_3}{864} + \frac{2077\zeta_5}{24} + \frac{32\log^5(2)}{45} \right. \\
& - \frac{116\log^4(2)}{27} + \frac{64}{27}\pi^2\log^3(2) - \frac{232}{27}\pi^2\log^2(2) - \frac{139}{540}\pi^4\log(2) + \frac{454}{27}\pi^2\log(2) \\
& \left. + \frac{122131\pi^4}{103680} + \frac{10535\pi^2}{2592} + \frac{10805903}{93312} \right) C_F \\
& + C_A T_F n_h \left(\frac{181\zeta_3^2}{32} + 7\pi^2\log(2)\zeta_3 + \frac{1823\pi^2\zeta_3}{192} - \frac{357881\zeta_3}{1152} - \frac{5855a_4}{12} - 96a_5 \right. \\
& + \xi \left(\frac{7\zeta_3}{64} + \frac{173\pi^4}{92160} - \frac{35\pi^2}{2304} - \frac{4859}{5184} \right) + \frac{2697\zeta_5}{64} + \frac{4\log^5(2)}{5} + \frac{1}{3}\pi^2\log^4(2) \\
& - \frac{5855\log^4(2)}{288} + \frac{2}{3}\pi^2\log^3(2) - \frac{1}{3}\pi^4\log^2(2) - \frac{21349}{288}\pi^2\log^2(2) - \frac{149}{180}\pi^4\log(2) \\
& + \frac{3973}{36}\pi^2\log(2) - \frac{1501\pi^6}{15120} + \frac{173713\pi^4}{55296} + 8a_4\pi^2 \\
& \left. - \frac{163981\pi^2}{5184} + \frac{1004165}{3888} \right) C_F, \tag{24}
\end{aligned}$$

where $C_F = (N_c^2 - 1)/(2N_c)$ and $C_A = N_c$ are the eigenvalues of the quadratic Casimir operators of the fundamental and adjoint representation for the $SU(N_c)$ colour group, respectively, $T = 1/2$ is the index of the fundamental representation, and n_l and n_h count the number of massless and massive (with mass M) quarks. It is convenient to keep the variable n_h as a parameter although in our case we have $n_h = 1$. Computer-readable expressions of $\delta Z_2^{(1)}$, $\delta Z_2^{(2)}$ and $\delta Z_2^{(3)}$ can be found in [25].

References

- [1] P. Marquard, A. V. Smirnov, V. A. Smirnov and M. Steinhauser, Phys. Rev. Lett. **114** (2015) 14, 142002 doi:10.1103/PhysRevLett.114.142002 [arXiv:1502.01030 [hep-ph]].
- [2] P. Marquard, A. V. Smirnov, V. A. Smirnov, M. Steinhauser and D. Wellmann, Phys. Rev. D **94** (2016) no.7, 074025 doi:10.1103/PhysRevD.94.074025 [arXiv:1606.06754 [hep-ph]].
- [3] N. Gray, D. J. Broadhurst, W. Grafe and K. Schilcher, Z. Phys. C **48** (1990) 673.
- [4] K. Melnikov and T. v. Ritbergen, Phys. Lett. B **482** (2000) 99 [hep-ph/9912391].

- [5] K. Melnikov and T. van Ritbergen, Nucl. Phys. B **591** (2000) 515 doi:10.1016/S0550-3213(00)00526-5 [hep-ph/0005131].
- [6] P. Marquard, L. Mihaila, J. H. Piclum and M. Steinhauser, Nucl. Phys. B **773** (2007) 1 doi:10.1016/j.nuclphysb.2007.03.010 [hep-ph/0702185].
- [7] P. Marquard, A. V. Smirnov, V. A. Smirnov, M. Steinhauser and D. Wellmann, arXiv:1708.07138 [hep-ph].
- [8] A. V. Smirnov, JHEP **0810** (2008) 107 [arXiv:0807.3243 [hep-ph]].
- [9] A. V. Smirnov and V. A. Smirnov, Comput. Phys. Commun. **184** (2013) 2820 [arXiv:1302.5885 [hep-ph]].
- [10] A. V. Smirnov, Comput. Phys. Commun. **189** (2015) 182 doi:10.1016/j.cpc.2014.11.024 [arXiv:1408.2372 [hep-ph]].
- [11] R. N. Lee, arXiv:1212.2685 [hep-ph].
- [12] R. N. Lee, J. Phys. Conf. Ser. **523** (2014) 012059 doi:10.1088/1742-6596/523/1/012059 [arXiv:1310.1145 [hep-ph]].
- [13] P. Marquard, D. Seidel, unpublished.
- [14] A. V. Smirnov and M. N. Tentyukov, Comput. Phys. Commun. **180** (2009) 735 [arXiv:0807.4129 [hep-ph]].
- [15] A. V. Smirnov, V. A. Smirnov and M. Tentyukov, Comput. Phys. Commun. **182** (2011) 790 [arXiv:0912.0158 [hep-ph]].
- [16] A. V. Smirnov, Comput. Phys. Commun. **185** (2014) 2090 [arXiv:1312.3186 [hep-ph]].
- [17] R. Lee, P. Marquard, A. V. Smirnov, V. A. Smirnov and M. Steinhauser, JHEP **1303** (2013) 162 [arXiv:1301.6481 [hep-ph]].
- [18] V. A. Smirnov, Phys. Lett. B **460** (1999) 397 doi:10.1016/S0370-2693(99)00777-7 [hep-ph/9905323].
- [19] J. B. Tausk, Phys. Lett. B **469** (1999) 225 doi:10.1016/S0370-2693(99)01277-0 [hep-ph/9909506].
- [20] M. Czakon, Comput. Phys. Commun. **175** (2006) 559 doi:10.1016/j.cpc.2006.07.002 [hep-ph/0511200].
- [21] A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **62** (2009) 445 doi:10.1140/epjc/s10052-009-1039-6 [arXiv:0901.0386 [hep-ph]].
- [22] H.R.P. Ferguson and D.H. Bailey, RNR Technical Report, RNR-91-032; H.R.P. Ferguson, D.H. Bailey and S. Arno, NASA Technical Report, NAS-96-005.

- [23] S. Laporta, Phys. Lett. B **772** (2017) 232 doi:10.1016/j.physletb.2017.06.056 [arXiv:1704.06996 [hep-ph]].
- [24] D. J. Broadhurst, N. Gray and K. Schilcher, Z. Phys. C **52** (1991) 111. doi:10.1007/BF01412333
- [25] <https://www.ttp.kit.edu/preprints/2018/ttp18-007/>.
- [26] A. Grozin, J. Henn and M. Stahlhofen, JHEP **1710** (2017) 052 doi:10.1007/JHEP10(2017)052 [arXiv:1708.01221 [hep-ph]].
- [27] K. Johnson and B. Zumino, Phys. Rev. Lett. **3** (1959) 351. doi:10.1103/PhysRevLett.3.351
- [28] T. Luthe, A. Maier, P. Marquard and Y. Schroder, JHEP **1701** (2017) 081 doi:10.1007/JHEP01(2017)081 [arXiv:1612.05512 [hep-ph]].
- [29] P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, JHEP **1704** (2017) 119 doi:10.1007/JHEP04(2017)119 [arXiv:1702.01458 [hep-ph]].
- [30] M. Beneke and V. A. Smirnov, Nucl. Phys. B **522** (1998) 321 doi:10.1016/S0550-3213(98)00138-2 [hep-ph/9711391].
- [31] V. A. Smirnov, Springer Tracts Mod. Phys. **250** (2012) 1. doi:10.1007/978-3-642-34886-0 [28]
- [32] A. G. Grozin, Phys. Lett. B **692** (2010) 161 doi:10.1016/j.physletb.2010.07.032 [arXiv:1004.2662 [hep-ph]].
- [33] K. G. Chetyrkin, B. A. Kniehl and M. Steinhauser, Nucl. Phys. B **510** (1998) 61 doi:10.1016/S0550-3213(98)81004-3, 10.1016/S0550-3213(97)00649-4 [hep-ph/9708255].
- [34] K. G. Chetyrkin and A. G. Grozin, Nucl. Phys. B **666** (2003) 289 doi:10.1016/S0550-3213(03)00490-5 [hep-ph/0303113].
- [35] A. Grozin, J. M. Henn, G. P. Korchemsky and P. Marquard, JHEP **1601** (2016) 140 doi:10.1007/JHEP01(2016)140 [arXiv:1510.07803 [hep-ph]].
- [36] A. Grozin, PoS LL **2016** (2016) 053 [arXiv:1605.03886 [hep-ph]].
- [37] D. J. Broadhurst and A. G. Grozin, Phys. Rev. D **52** (1995) 4082 doi:10.1103/PhysRevD.52.4082 [hep-ph/9410240].

	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
<i>FFFF</i>	0.01317 ± 0.00025	0.0836 ± 0.0019	-0.084 ± 0.017	-1.96 ± 0.16	-4.1 ± 1.5
<i>FFFA</i>	-0.09665 ± 0.00053	-0.7611 ± 0.0044	-1.275 ± 0.041	1.10 ± 0.38	-9.8 ± 3.6
<i>FFAA</i>	0.21661 ± 0.00040	2.1150 ± 0.0035	9.698 ± 0.033	57.52 ± 0.31	324.5 ± 2.9
<i>FAAA</i>	-0.14442 ± 0.00011	-1.76642 ± 0.00096	-14.4491 ± 0.0092	-123.354 ± 0.086	-1007.40 ± 0.82
d_{FA}	-0.00002 ± 0.00029	0.0006 ± 0.0033	-0.002 ± 0.024	0.40 ± 0.21	9.4 ± 2.1
d_{FFL}	0.00001 ± 0.00011	-0.0001 ± 0.0014	0.0000 ± 0.0079	0.011 ± 0.064	-2.18 ± 0.80
d_{FFH}	-0.00001 ± 0.00023	0.0001 ± 0.0015	-0.001 ± 0.011	-0.120 ± 0.076	0.10 ± 0.50
<i>FFFL</i>	0.0351561 ± 0.0000013	0.2499987 ± 0.0000092	0.496651 ± 0.000077	0.39174 ± 0.00074	1.3920 ± 0.0067
<i>FFAL</i>	-0.1575519 ± 0.0000033	-1.457029 ± 0.000022	-7.60181 ± 0.00016	-46.0162 ± 0.0014	-236.417 ± 0.012
<i>FAAL</i>	0.1575515 ± 0.0000052	1.889980 ± 0.000070	17.10515 ± 0.00039	145.3220 ± 0.0026	1190.195 ± 0.031
<i>FFLL</i>	0.0286458	0.244792	1.37840	8.3824	40.329
<i>FALL</i>	-0.057292	-0.66059	-6.3943	-54.229	-447.65
<i>FLLL</i>	0.0069444	0.075231	0.76054	6.4263	53.496
<i>FFFH</i>	0.070313 ± 0.000023	0.255860 ± 0.000093	-0.65497 ± 0.00055	-3.8002 ± 0.0036	-5.953 ± 0.019
<i>FFAH</i>	-0.26173 ± 0.00010	-1.58102 ± 0.00044	-3.2136 ± 0.0021	-11.729 ± 0.013	-26.860 ± 0.083
<i>FAAH</i>	0.215336 ± 0.000061	1.95402 ± 0.00027	11.5396 ± 0.0014	70.3186 ± 0.0091	424.301 ± 0.056
<i>FFHH</i>	0.0937498 ± 0.0000014	0.2109378 ± 0.0000059	-0.329095 ± 0.000035	-0.57438 ± 0.00013	-7.99681 ± 0.00079
<i>FAHH</i>	-0.117186 ± 0.000011	-0.681863 ± 0.000054	-2.52735 ± 0.00029	-10.3208 ± 0.0012	-40.2646 ± 0.0062
<i>FHHH</i>	0.0277778	0.047454	0.173582	0.276902	0.61212
<i>FFLH</i>	0.093750	0.50781	1.3923245 ± 0.0000012	5.834231 ± 0.000010	8.990228 ± 0.000074
<i>FALH</i>	-0.1545138 ± 0.0000011	-1.3179979 ± 0.0000063	-9.088033 ± 0.000034	-56.32679 ± 0.00020	-344.7315 ± 0.0015
<i>FLLH</i>	0.0277778	0.216435	1.65669	10.3632	64.740
<i>FLHH</i>	0.041667	0.197917	1.05074	4.2433	17.7160

Table 5: Results for the coefficients of $\delta Z_2^{(4)}$ as defined in Eq. (13) for $\xi = 0$. A security factor 10 has been applied to the uncertainties.

	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
<i>FFFA</i>	0	0	-0.000000 ± 0.000020	0.00001 ± 0.00020	-0.0001 ± 0.0011
<i>FFAA</i>	0	-0.0000001 ± 0.0000016	-0.005369 ± 0.000022	-0.03679 ± 0.00022	-0.3166 ± 0.0012
<i>FAAA</i>	0	-0.0000001 ± 0.0000039	0.013200 ± 0.000023	0.11976 ± 0.00013	1.42164 ± 0.00076
d_{FA}	-0.0000003 ± 0.0000100	0.000005 ± 0.000088	-0.00000 ± 0.00051	0.1135 ± 0.0025	0.147 ± 0.013
<i>FAAL</i>	0	0	-0.0035799	-0.033281	-0.40121
<i>FFAH</i>	0.0039062	-0.0094401	0.0069760 ± 0.0000032	0.037345 ± 0.000035	-0.76089 ± 0.00024
<i>FAAH</i>	-0.0052626	0.0112034	-0.0854353 ± 0.0000018	0.216644 ± 0.000018	-1.40360 ± 0.00013
<i>FAHH</i>	0.00260417	-0.0078125	0.047919	-0.182917	0.78980
<i>FALH</i>	0.00173611	-0.0052083	0.043906	-0.148948	0.79619

Table 6: Same as in Tab. 5 but the coefficients of the linear ξ terms.

	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
$FAAA$	0	0.0000000 ± 0.0000011	-0.0006711 ± 0.0000052	-0.005817 ± 0.000025	-0.07062 ± 0.00012
d_{FA}	0.0000002 ± 0.0000038	-0.000001 ± 0.000026	-0.00000 ± 0.00012	-0.01250 ± 0.00061	-0.0748 ± 0.0028
$FAAH$	0	0.00032552	-0.00100945	0.0089715	-0.032896

Table 7: Same as in Tab. 5 but the coefficients of ξ^2 .

	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
<i>FFFF</i>	-2.73194 ± 0.00025	1.9450 ± 0.0019	-22.265 ± 0.017	93.41 ± 0.16	167.9 ± 1.5
<i>FFFA</i>	-2.62790 ± 0.00053	-4.6110 ± 0.0044	-48.022 ± 0.041	-74.51 ± 0.38	-618.5 ± 3.6
<i>FFAA</i>	-1.02981 ± 0.00040	-5.9684 ± 0.0035	-44.525 ± 0.033	-255.18 ± 0.31	-1664.9 ± 2.9
<i>FAAA</i>	-0.14442 ± 0.00011	-1.76642 ± 0.00096	-14.4491 ± 0.0092	-123.354 ± 0.086	-1007.40 ± 0.82
<i>d_{FA}</i>	-0.00002 ± 0.00029	0.0006 ± 0.0033	-0.002 ± 0.024	0.40 ± 0.21	9.4 ± 2.1
<i>d_{FFL}</i>	0.00001 ± 0.00011	-0.0001 ± 0.0014	0.0000 ± 0.0079	0.011 ± 0.064	-2.18 ± 0.80
<i>d_{FFH}</i>	-0.00001 ± 0.00023	0.0001 ± 0.0015	-0.001 ± 0.011	-0.120 ± 0.076	0.10 ± 0.50
<i>FFFL</i>	1.3124999 ± 0.0000013	2.6503893 ± 0.0000092	31.883671 ± 0.000077	55.08300 ± 0.00074	477.6787 ± 0.0067
<i>FFAL</i>	0.8750002 ± 0.0000033	4.729724 ± 0.000022	42.82731 ± 0.00016	217.1492 ± 0.0014	1568.153 ± 0.012
<i>FAAL</i>	0.1575515 ± 0.0000052	1.889980 ± 0.000070	17.10515 ± 0.00039	145.3220 ± 0.0026	1190.195 ± 0.031
<i>FLLL</i>	-0.179688	-0.88281	-9.3076	-43.714	-340.67
<i>FALL</i>	-0.057292	-0.66059	-6.3943	-54.229	-447.65
<i>FLLL</i>	0.0069444	0.075231	0.76054	6.4263	53.496
<i>FFFH</i>	2.156250 ± 0.000023	1.492579 ± 0.000093	20.69169 ± 0.00055	9.3957 ± 0.0036	92.254 ± 0.019
<i>FFAH</i>	1.22916 ± 0.00010	4.77852 ± 0.00044	26.4927 ± 0.0021	101.634 ± 0.013	487.440 ± 0.083
<i>FAAH</i>	0.215336 ± 0.000061	1.95402 ± 0.00027	11.5396 ± 0.0014	70.3186 ± 0.0091	424.301 ± 0.056
<i>FFHH</i>	-0.4166669 ± 0.0000014	-0.6484372 ± 0.0000059	-3.496296 ± 0.000035	-7.42824 ± 0.00013	-20.15261 ± 0.00079
<i>FAHH</i>	-0.117186 ± 0.000011	-0.681863 ± 0.000054	-2.52735 ± 0.00029	-10.3208 ± 0.0012	-40.2646 ± 0.0062
<i>FHHH</i>	0.0277778	0.047454	0.173582	0.276902	0.61212
<i>FFLH</i>	-0.50521	-1.68750	$-13.1485794 \pm 0.0000012$	-44.062536 ± 0.000010	-243.607004 ± 0.000074
<i>FALH</i>	-0.1545138 ± 0.0000011	-1.3179979 ± 0.0000063	-9.088033 ± 0.000034	-56.32679 ± 0.00020	-344.7315 ± 0.0015
<i>FLLH</i>	0.0277778	0.216435	1.65669	10.3632	64.740
<i>FLHH</i>	0.041667	0.197917	1.05074	4.2433	17.7160

Table 8: Results for the coefficients of $\delta Z_2^{(4)}$ as defined in Eq. (13) for $\xi = 0$ and without taking into account the mass counterterms from lower loop orders. A security factor 10 has been applied to the uncertainties.