

Exact instantons via worldline deformations

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The imaginary part of the one loop effective action in external backgrounds can be efficiently computed using worldline instantons which are closed periodic paths in spacetime. Exact solutions for nonstatic backgrounds are only known in certain cases. In this paper, we propose a novel technique allowing the construction of further exactly solvable models. In order to do so, we introduce a deformation function which maps the worldline instantons for a given model to the closed periodic stationary paths of a new model. Executing this procedure iteratively results in a chain of infinitely many solvable models. Similar ideas were applied to topological and nontopological defects in quantum field theory. We explicitly discuss the tunneling exponential in the Schwinger pair creation rate and illustrate the validity of the proposed technique for well-known cases.

I. INTRODUCTION

Pair creation from the vacuum in the presence of macroscopic gauge fields is one of the few nonperturbative results in quantum field theory with a clear physical prediction [1]. Despite its longstanding prediction, the Schwinger effect has so far eluded a direct experimental verification. In the weak field and weak coupling limit, the creation rate is determined by the imaginary part of the one loop Euler-Heisenberg effective action. In the simplest scenario, i.e. when the system is subjected to a static electric background (standard Schwinger mechanism), the pair creation rate \mathcal{R} [2]

$$\mathcal{R} \simeq \frac{E^2}{(2\pi)^3} e^{-\mathcal{W}_0}, \quad \mathcal{W}_0 = \pi \frac{E_S}{E} \quad (1)$$

is dramatically suppressed for field strengths E much below the critical value $E_S = m^2$, where m denotes the mass of the particles. Notably, the nonanalytic dependence $\mathcal{W}_0 \sim 1/E$ in the exponent clearly manifests the characteristic nonperturbative behavior. In the case of nonstatic Abelian backgrounds, it has been shown that \mathcal{R} is very sensitive to the background shape. Generally, large background inhomogeneities tend to trigger an enormous enhancement even for $E \ll E_S$ [3–5]. Backgrounds of this type may, for instance, be realized with upcoming lasers.

Beyond the static limit, preferably for purely electric backgrounds, investigations mostly rely on numerical computation techniques, e.g. [6–11]. Analytic investigations have so far been performed only for certain configurations, e.g. [3–5, 12–19]. Nevertheless, a deeper understanding of the impact of more complicated backgrounds is necessary to better understand the formation process, especially, to unveil interesting insights about critical points [20], nonlocal effects [4] and also universal features [21]. Technically, this task is highly challenging and cannot be pursued with conventional field theoretic methods.

A powerful technique relies on the first-quantized string inspired formulation [22]. In particular, the semiclassical treatment via worldline instantons proves highly efficient, see e.g. [4, 7, 11, 18, 20, 23–26] and also [27–30] for applications in other contexts. Within this approach, the imaginary part of the effective action just becomes determined by closed periodic paths in spacetime. Principally, the basic task then is to find the associated worldline instantons. There exist only a few cases in the literature for which the stationary worldline paths have been computed exactly [4]. Even the numerical treatment is highly nontrivial due to the underlying nonlinear equations [7, 11].

In this paper, we propose a novel technique which enables the construction of arbitrary many solvable models for which the corresponding worldline instantons can be obtained by deforming a known solution according to a particularly constructed map. Similar techniques have been utilized to generate solvable models giving rise to topological and nontopological defects in quantum field theory [31, 32].

The remaining part of this paper is structured as follows: in section II, we introduce the Schwinger pair creation rate and the corresponding instanton equations. In section III, we recap the deformation procedure in the context of defects in field theory. In sections IV and V, being the main parts of this paper, we first reshape the underlying saddle point equations and propose the deformation procedure for worldline instantons. In section VI, we discuss our results and validate our predictions for certain solvable models. We finish with a brief summary and outlook in section VII.

II. WORLDLINE INSTANTONS

We start with a brief discussion of vacuum decay in the presence of an external Abelian background. The general expression for the decay probability \mathcal{P} against the creation of matter-antimatter pairs is [1]

$$\mathcal{P} \simeq 1 - e^{-2\mathcal{R}}. \quad (2)$$

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As usual, we neglect the contribution coming from the dynamical part of the vector field. Then one can rewrite the logarithm of the kinetic operator determinant, here given in scalar quantum electrodynamics, and apply the quantum mechanical path integral representation for the resulting trace using position eigenstates. The outcome, after analytically continuing to Euclidean spacetime, is the following worldline representation

$$\mathcal{R} \simeq \int_0^\infty \frac{ds}{s} e^{-m^2 s} \oint \mathcal{D}x e^{-\int_0^s d\tau \left[\frac{\dot{x}^2(\tau)}{4} + iA \cdot \dot{x}(\tau) \right]}, \quad (3)$$

where $\mathcal{D}x$ denotes the path integral measure over the closed worldlines with periodic boundaries, A^μ is the external vector potential and s denotes the proper time.

Performing a saddle point analysis in s and expanding the resulting worldline action

$$\mathcal{W} \simeq ma + i \int_0^1 du A \cdot \dot{x}(u) \quad (4)$$

in the fluctuations over the saddle points, called worldline instantons, obeying the boundary condition $x_\mu(0) = x_\mu(1)$, yields

$$\mathcal{R} \simeq \frac{1}{\sqrt{\text{Det}M}} e^{-\mathcal{W}_0}, \quad \mathcal{W}_0 \equiv \mathcal{W}[\text{instanton}], \quad (5)$$

where $M_{\mu\nu}$ is the second order variation operator and $\dot{x}^2 = \text{const.} \equiv a^2$ is an internal invariant on the worldline [23]. The expression in front of the exponential leads to the quantum fluctuation prefactor. For the present discussion we only focus on the tunneling exponential.

Minimizing the resulting exponent shows that the closed periodic instanton path is determined by the following set of equations

$$m\ddot{x}^\mu = ia\mathcal{F}^{\mu\nu}\dot{x}^\nu. \quad (6)$$

In this paper, we consider time dependent backgrounds with general the form

$$\mathcal{A}_3 = -iEF(x_4) \quad (7)$$

where F is assumed to be an odd function in x_4 . After inserting \mathcal{A}_3 into the instanton equations from (6), we end up with the following system of coupled differential equations

$$\begin{aligned} \ddot{x}_3 &= -\frac{aE}{m}F'(x_4)\dot{x}_4, \\ \ddot{x}_4 &= +\frac{aE}{m}F'(x_4)\dot{x}_3. \end{aligned} \quad (8)$$

Evaluating the worldline action \mathcal{W} on the solution of the coupled differential equations in (8) gives the exponential part in \mathcal{R} . In what follows, we briefly comment on some ideas regarding defect solutions in field theory.

III. DEFORMING DEFECTS

Defects are important in high energy [33, 34] as well as condensed matter physics [35], e.g. in the context of gravity with warped geometry involving extra dimensions, and gap opening in the mass spectrum of charge carriers, respectively. Defects can be of topological and nontopological type. Quantum field theories which support spontaneous symmetry breaking usually give rise to the former kind. Examples are kinks, domain walls and monopoles to name a few.

Models in terms of real scalar fields usually have kinklike or lumplike defects [36, 37]. Interestingly, in a spatially one-dimensional solvable model of such kind, a particularly realized deformation of defects can lead to new models with defects expressed in terms of the original defect solutions [31]. For instance, consider a scalar field theory in 1 + 1 dimensions where fields and coordinates are taken to be dimensionless. Let its Lagrangian be

$$\mathcal{L}_\chi = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V_\chi(\chi) \quad (9)$$

where V_χ is the specific potential in this model that we call the χ model. The corresponding equations of motion determining the associated static solution are

$$\begin{aligned} \frac{d^2\chi}{du^2} &= \frac{dV_\chi(\chi)}{d\chi}, \\ \left[\frac{d\chi}{du}\right]^2 &= 2V_\chi(\chi). \end{aligned} \quad (10)$$

For the second line, it has been assumed that χ satisfies the following boundary conditions

$$\chi \rightarrow \bar{\chi} \quad \wedge \quad \chi' \rightarrow 0 \quad (u \rightarrow -\infty). \quad (11)$$

Here, $\bar{\chi}$ denotes the critical point of the potential, i.e. $V'_\chi(\bar{\chi}) = 0$, where $V_\chi(\bar{\chi}) = 0$.

Let us now suppose a second model described by the Lagrangian

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_\phi(\phi) \quad (12)$$

for which the static solution is determined by

$$\begin{aligned} \frac{d^2\phi}{du^2} &= \frac{dV_\phi(\phi)}{d\phi}, \\ \left[\frac{d\phi}{du}\right]^2 &= 2V_\phi(\phi), \end{aligned} \quad (13)$$

similar to before, but now depending on a different potential, i.e. $V_\phi \neq V_\chi$.

Assume that the former χ model described by \mathcal{L}_χ is exactly solvable and supports finite energy defects. Then, we can write the potential V_ϕ for the ϕ model in the following specific form [31]

$$V_\phi(\phi) = \frac{V_\chi(\chi \rightarrow f(\phi))}{[df/d\phi]^2} \quad (14)$$

where f is taken to be a real bijective function with nonzero first order derivative. This particular deformation has the interesting consequence that the solution in the deformed ϕ model described by \mathcal{L}_ϕ is exactly given by

$$\phi = f^{-1}(\chi) \quad (15)$$

where f^{-1} denotes the inverse of the deforming map and χ exactly solves the stationary equations in the original theory.

IV. RECASTING THE SADDLE POINT EQUATIONS

Treating worldline instantons in an analogous way as described above, we aim to work out a relation similar to (14), but now we want to relate two distinct models with different stationary worldline solutions. Note that in this case, we deal with worldline components instead of a real quantum field as in section III. Nevertheless, we will show below that a close similarity is indeed present. To start with, we first need to bring the corresponding stationary equations into an appropriate shape. We integrate the first line in the saddle point equations (8) to obtain

$$\dot{x}_3 = -\frac{aE}{m}F(x_4). \quad (16)$$

The latter we then insert into the second expression which leads to the relation,

$$\ddot{x}_4 = -\frac{d}{dx_4} \left[\frac{aE}{\sqrt{2}m}F(x_4) \right]^2. \quad (17)$$

It is advantageous to define a potential of the form

$$V(x_4) \equiv -\left[\frac{aE}{\sqrt{2}m}F(x_4) \right]^2 \quad (18)$$

in order to write down

$$\dot{x}_4^2 = a^2 + 2V(x_4) \quad (19)$$

which follows from the second line in (6).

Expressing the right-hand side of (8) in terms of the potential in (18), we can reformulate the corresponding stationary equations in the following way

$$\begin{aligned} \frac{d^2x_4}{du^2} &= \frac{dV(x_4)}{dx_4}, \\ \left[\frac{dx_4}{du} \right]^2 &= a^2 + 2V(x_4). \end{aligned} \quad (20)$$

Note that the latter are quite similar to the equations in (10) and (13), respectively, with the occurrence of an additional constant term in the second equation being the only difference. Here, we only treat the Euclidean time coordinate. The spatial one can be computed from the latter.

V. WORLDLINE DEFORMATIONS

The similarity between the stationary equations (10) and (20) encourages one to proceed as in section III. In order to stick to the same convention, we first rename the time coordinate,

$$x_4 \rightarrow \chi. \quad (21)$$

Then, assuming a solvable model is given, we first write down the equations for its stationary solution

$$\begin{aligned} \frac{d^2\chi}{du^2} &= \frac{dV_\chi(\chi)}{d\chi}, \\ \left[\frac{d\chi}{du} \right]^2 &= a_\chi^2 + 2V_\chi(\chi) \end{aligned} \quad (22)$$

where $a_\chi^2 = \dot{\chi}^2 + \dot{x}_3^2$ is the corresponding invariant.

Similarly, we consider a second model depending on some other potential V_ϕ for which we set

$$x_4 \rightarrow \phi. \quad (23)$$

The associated stationary solution is determined by the equations

$$\begin{aligned} \frac{d^2\phi}{du^2} &= \frac{dV_\phi(\phi)}{d\phi}, \\ \left[\frac{d\phi}{du} \right]^2 &= a_\phi^2 + 2V_\phi(\phi) \end{aligned} \quad (24)$$

with the invariant $a_\phi^2 = \dot{\phi}^2 + \dot{x}_3^2$. The main observation is that if we impose a deformation prescription of the form

$$a_\phi^2 + 2V_\phi(\phi) = \frac{a_\chi^2 + 2V_\chi(\chi \rightarrow f(\phi))}{[df/d\phi]^2} \quad (25)$$

then the stationary solution in the ϕ model in (24) can be entirely expressed in terms of χ determined by (22), as in (15).

In other words, if we know the exact worldline instantons in the χ model, then the worldline instantons in the deformed ϕ model with potential V_ϕ that is constructed according to the prescription in (25), where f is an appropriately chosen deformation function, are also known.

In order to show that $\phi = f^{-1}(\chi)$ is the exact deformed solution, we start by expressing the potential V_ϕ in terms of a_χ, a_ϕ, V_χ and f using the relation (25), in the first line of (24). The resulting right-hand side then reads

$$\frac{d^2\phi}{du^2} = \frac{V'_\chi(f(\phi))}{f'(\phi)} - \frac{f''(\phi)}{[f'(\phi)]^3} [a_\chi^2 + 2V_\chi(f(\phi))] \quad (26)$$

where we have used $\partial_u a_\phi = 0$. On the other hand, the direct computation of the second derivative via (15) yields

$$\frac{d^2\phi}{du^2} = \frac{1}{f'(\phi)} \left[\frac{d^2\chi}{du^2} \right] - \frac{f''(\phi)}{[f'(\phi)]^3} \left[\frac{d^2\chi}{du^2} \right]^2. \quad (27)$$

Replacing the first bracket by the first line in (22) and the second bracket by the second line, we immediately end up with equation (26) after setting $\chi = f(\phi)$. This completes the proof.

VI. DISCUSSION

In this section, the deformation technique is discussed for certain examples.

A. General aspects

For simplifying reasons, we first introduce the following definitions

$$\tilde{F}(\chi) \equiv \omega F(\chi), \quad \gamma \equiv \frac{m\omega}{E} \quad (28)$$

where ω denotes the frequency. Here, γ (usually referred to as the Keldysh parameter) is a useful dimensionless parameter regulating the background inhomogeneity. With these definitions, the potential introduced in (18) can be rewritten as

$$V(\chi) = -\frac{a^2}{2\gamma^2} \tilde{F}^2(\chi). \quad (29)$$

$$f(\phi) = \left(\int^{\chi} \frac{d\chi'}{\sqrt{\gamma^2 - \tilde{F}_\chi^2(\chi')}} \right)^{-1} \left(\left[\frac{a_\chi}{a_\phi} \right] \left[\int^{\phi} \frac{d\phi'}{\sqrt{\gamma^2 - \tilde{F}_\phi^2(\phi')}} \right] \right), \quad (32)$$

$$f^{-1}(\chi) = \left(\int^{\phi} \frac{d\phi'}{\sqrt{\gamma^2 - \tilde{F}_\phi^2(\phi')}} \right)^{-1} \left(\left[\frac{a_\phi}{a_\chi} \right] \left[\int^{\chi} \frac{d\chi'}{\sqrt{\gamma^2 - \tilde{F}_\chi^2(\chi')}} \right] \right). \quad (33)$$

As can be clearly seen, one has to compute the inverse of some integral function with the general form

$$H(x) \equiv \int^x h(u) du. \quad (34)$$

The latter can be also written in terms of the integrand. Namely, using basic arguments, it can be shown that the first order derivative of the inverse has to satisfy the following differential equation

$$(H^{-1})'(x) = 1/[h(H^{-1}(x))]. \quad (35)$$

However, in this way the direct computation of the deformation function f and its inverse f^{-1} for some given potentials V_χ and V_ϕ is still difficult to achieve.

Note that the deformation technique substantially simplifies the problem of finding a suitable model for which the worldline instantons are known exactly. According to the deformation technique, we do not need to compute the underlying integrals in (33) for that purpose. Instead, one may start with an appropriately chosen deformation function in order to construct a suitable potential V_ϕ by differentiating the deformation function. This procedure

Then, after plugging the latter into (25), we may write down the deformation function in terms of \tilde{F}_χ , \tilde{F}_ϕ and the internal invariants a_χ , a_ϕ leading to the expression

$$[f'(\phi)]^2 = \left[\frac{a_\chi}{a_\phi} \right]^2 \left[\frac{\gamma^2 - \tilde{F}_\chi^2(f(\phi))}{\gamma^2 - \tilde{F}_\phi^2(\phi)} \right]. \quad (30)$$

Similarly, one can formulate the corresponding equation for the inverse deformation function,

$$[(f^{-1})'(\chi)]^{-2} = \left[\frac{a_\chi}{a_\phi} \right]^2 \left[\frac{\gamma^2 - \tilde{F}_\chi^2(\chi)}{\gamma^2 - \tilde{F}_\phi^2(f^{-1}(\chi))} \right]. \quad (31)$$

The formal solutions for the latter equations are of the following form

is substantially easier to undertake. The difference between both strategies is sketched in figure 1 below. Tak-

$$V_\phi \xrightarrow{\int \frac{d\phi}{\sqrt{\gamma^2 - \tilde{F}_\phi^2(\phi)}}} f$$

$$f \xrightarrow{\partial_\phi f} V_\phi$$

FIG. 1. Deformation technique (below) compared to the standard route (top). The task of computing a complicated integral is replaced by the computation of a first order derivative.

ing that into account, the stationary worldline action for the deformed model depending on the constructed potential V_ϕ can be compactly written as

$$\mathcal{W}_0 = ma_\phi \left[1 + \frac{1}{\gamma^2} \int_0^1 du \tilde{F}_\phi^2(f^{-1}(\chi)) \right]. \quad (36)$$

B. Constructing exactly solvable models

The deformation strategy can be repeated once a new solvable model has been constructed. It is therefore possible to generate many other exactly solvable models. The required steps can be specified in the following order:

1. Choose an appropriate deformation function f and compute its inverse f^{-1} .
2. Choose an exactly solvable model depending on some potential V_χ .
3. Following the prescription of (25), deform the initial solvable model to construct a new model with potential V_ϕ .
4. The exact instanton solution for the new model is then given by $\phi = f^{-1}(\chi)$.
5. Repeat the latter steps (2 - 4) to construct a chain of arbitrary many exactly solvable models.

A separate chain of solvable models is possible if one operates with another suitable deformation function $\tilde{f} \neq f$ as sketched in figure 2. Here, one should notice that \tilde{f} principally has to be normalized (in each iteration step), since the underlying quantities are in general not dimensionless. This is necessary to generate a reasonable background. In the following, we discuss certain well-known

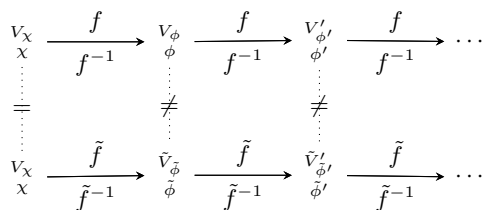


FIG. 2. Two separate chains of exactly solvable models generated by applying two distinct deformation functions f and \tilde{f} , respectively.

solvable models to confirm the validity of the proposed prescription (25). We choose both models so that f as well as f^{-1} can be computed directly. This we require in order to test the predicted solution of the form (15). To do so, we assume a static electric background described by $\mathbf{E} = E\hat{x}_3$ for the undeformed χ model. The corresponding potential function reads

$$\tilde{F}_\chi(\chi) = \omega\chi. \quad (37)$$

As mentioned previously, this is the simplest scenario and was first considered by Affleck et al. within the semiclassical worldline approach. The stationary solutions are the well known circular worldline instantons determined by [23]

$$\begin{aligned} \chi &= (\gamma/\omega) \sin(2\pi nu), \\ a_\chi &= (\gamma/\omega) 2\pi n. \end{aligned} \quad (38)$$

Here, $n \in \mathbb{N}$ is usually called the winding number where integers $n > 1$ correspond to higher order instanton contributions which can be understood as the coherent creation of n pairs [38].

C. A trivial example

In the following, we compute f directly. As noted before, the main advantage of the deformation technique is basically the opposite direction, means choosing f and constructing V_ϕ afterwards. However, for illustrative reasons, we choose a known model with exact solutions. A trivial example is given when we assume a static background for the deformed model, i.e.

$$\tilde{F}_\phi(\phi) = \omega\phi. \quad (39)$$

Then, plugging the potential functions (37) and (39) into (25) results in the following deformation function

$$f(\phi) = \frac{\gamma \tan\left(\frac{a_\chi}{a_\phi} \tan^{-1}\left(\frac{\omega\phi}{\sqrt{\gamma^2 - (\omega\phi)^2}}\right)\right)}{\omega \sqrt{1 + \tan^2\left(\frac{a_\chi}{a_\phi} \tan^{-1}\left(\frac{\omega\phi}{\sqrt{\gamma^2 - (\omega\phi)^2}}\right)\right)}}. \quad (40)$$

Its inverse reads

$$f^{-1}(\chi) = \frac{\gamma \tan\left(\frac{a_\phi}{a_\chi} \tan^{-1}\left(\frac{\omega\chi}{\sqrt{\gamma^2 - (\omega\chi)^2}}\right)\right)}{\omega \sqrt{1 + \tan^2\left(\frac{a_\phi}{a_\chi} \tan^{-1}\left(\frac{\omega\chi}{\sqrt{\gamma^2 - (\omega\chi)^2}}\right)\right)}}. \quad (41)$$

It is straightforward to verify that inserting the expressions from (38) into f^{-1} yields the expected circular worldline instanton determined by

$$\begin{aligned} \phi &= (\gamma/\omega) \sin(2\pi nu), \\ a_\phi &= (\gamma/\omega) 2\pi n. \end{aligned} \quad (42)$$

Thus, in agreement with our expectation, the deformation function correctly maps between the two stationary worldline paths.

D. Sinusoidal background

Another model that can be exactly solved depends on a sinusoidal background of the form $\mathbf{E} = E \cos(\omega t)\hat{x}_3$ [4]. The corresponding potential function reads

$$\tilde{F}_\phi(\phi) = \sinh(\omega\phi). \quad (43)$$

As before, inserting (37) and (43) into (25), we find

$$f(\phi) = \frac{\gamma \tan\left(\frac{ia_\chi}{\gamma a_\phi} \mathbf{F}\left(i\phi\omega \left| -\frac{1}{\gamma^2} \right.\right)\right)}{\omega \sqrt{1 + \tan^2\left(\frac{ia_\chi}{\gamma a_\phi} \mathbf{F}\left(i\phi\omega \left| -\frac{1}{\gamma^2} \right.\right)\right)}}. \quad (44)$$

where \mathbf{F} is the incomplete elliptic integral of the first kind. The inverse deformation function is given by

$$f^{-1}(\chi) = \frac{i}{\omega} \mathbf{am} \left(\frac{\gamma a_\phi}{a_\chi} \tanh^{-1} \left(\frac{\omega \chi}{\sqrt{(\omega \chi)^2 - \gamma^2}} \right) \middle| -\frac{1}{\gamma^2} \right) \quad (45)$$

with \mathbf{am} denoting the Jacobi amplitude. Again, inserting the expressions from (38) into f^{-1} shows that

$$\begin{aligned} \phi &= \frac{i}{\omega} \mathbf{am} \left(\frac{2\gamma \tan^{-1} \left(\frac{\sin(2n\pi u)}{\sqrt{\cos^2(2n\pi u)}} \right) \mathbf{K} \left(\frac{\gamma^2}{1+\gamma^2} \right)}{i\pi \sqrt{1+\gamma^2}} \middle| -\frac{1}{\gamma^2} \right), \\ a_\phi &= \frac{4\gamma n \mathbf{K} \left(\frac{\gamma^2}{1+\gamma^2} \right)}{\omega \sqrt{1+\gamma^2}} \end{aligned} \quad (46)$$

where \mathbf{K} is the complete elliptic integral of the first kind. As can be easily checked, the solution ϕ and invariant a_ϕ coincide with the findings in [4]. This again confirms that the deformation function based on (25) perfectly maps between the two distinct instanton paths.

VII. SUMMARY AND OUTLOOK

We have proposed a novel technique permitting the construction of exactly solvable models coupled to external backgrounds whose stationary worldline paths

(worldline instantons) determine the Schwinger pair creation rate.

Motivated by the found similarities with earlier ideas on defect solutions in quantum field theories, we have proposed a deformation function which maps between the distinct stationary worldline solutions of two separate models. Starting with a given exactly solvable model, a sequence of iterative deformations generates a chain of (infinitely many) new exactly solvable models. The proposed technique may have many advantageous for exact analytical studies of nonstatic backgrounds in the Schwinger effect.

In the present work, we have focused on purely time-dependent electric backgrounds and confirmed the validity of our results in well-known solvable cases. The construction of new solvable models is anticipated in a forthcoming publication. A potential extension may be the analogous treatment of spatially inhomogeneous backgrounds. Moreover, our results may allow an examination of the quantum fluctuation prefactor in more complicated backgrounds. The proposed technique may also help to better understand the mechanism of dynamical assistance. We plan to investigate these subjects elsewhere.

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