

The asymptotic behavior of the heavy quark form factors at higher order

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Abstract

In the asymptotic limit $Q^2 \gg m^2$, the heavy quark form factors exhibit Sudakov behavior. We study the corresponding renormalization group equations of the heavy quark form factors which do not only govern the structure of infrared divergences, but also control the high energy logarithms. This enables us to obtain the complete logarithmic three-loop and partial four-loop contributions to the heavy quark form factors in perturbative Quantum Chromodynamics.

1 Introduction

Amplitudes for hard scattering processes in Quantum Chromodynamics (QCD) at higher order do provide precise phenomenological predictions for scattering processes and also a clear insight into underlying principles such as factorization or the universality of infrared (IR) singularities. The latter properties can be used to resum large logarithmic contributions by applying the corresponding evolution equations either globally or in particular kinematic regions. Especially for massless scattering amplitudes, remarkable progress has been made in the understanding of the structure of IR divergences due to the interplay of the soft- and collinear dynamics. A first step was taken in [1], where a prediction for the singularities of two-loop amplitudes was given except the single pole in the dimensional variable $\varepsilon = (4 - D)/2$. Later, generalizations of this result for multi-parton amplitudes have been obtained in [2–4] beyond two-loop order. The IR structure is more prominent and interesting especially in the case of the form factors. The interplay of the soft and collinear anomalous dimensions building up the singular structure of the form factors has first been noticed in [5] at two-loop order and has been later established at three-loop order in [6]. In many following publications this has been studied in detail, see e.g. [7] for references.

It is also of interest to generalize these considerations to the massive case. Here a first step has been taken in [8] by obtaining the IR singularities for one-loop scattering amplitudes with massive partons. Next, in the asymptotic limit $Q^2 \gg m^2$, a factorization theorem has been proposed in [9–11]. While a first step was taken in [12] to obtain the IR structure of a generic two-loop amplitude without considering the small mass limit, finally in [13] the general solution has been presented for the singular structure of a generic scattering amplitude containing massless and massive partons.

The universality of these IR singularities, along with the factorization of QCD amplitudes, presents a rich structure even for the massive case. Especially in the asymptotic limit, the amplitudes with massive partons exhibit the Sudakov behavior. It was first studied in [10] and a general factorization formula was presented. Recently a partial result for the poles in ε of the heavy quark vector form factor in the leading color limit has been obtained up to four-loop order in [14] solving the renormalization group equation (RGE).

In the present paper, we study this behavior of the different heavy quark form factors following the method proposed for massless cases in [15, 16] and in the massive case of Ref. [10]. From a phenomenological perspective, the closer understanding of heavy quark production plays a significant role in current studies in elementary particle physics, from precision measurements of the Standard Model parameters to the search for signals from beyond the Standard Model. Henceforth, there has been much attention during last decade to obtain precise theoretical predictions for physical quantities involving heavy quarks, see also [17–24]. Important elements to all these predictions are the form factors. Dedicated work is going on for decades to obtain the heavy quark form factors for different currents, namely vector, axial-vector, scalar and pseudo-scalar currents at two-loop order [25–30] and three-loop order [31–34] in perturbative QCD. Since the complete computations are very challenging, results obtained in certain kinematic limits form important checks to these calculations.

We present the complete three-loop results for the form factors in the asymptotic limit, retaining all logarithmic orders. For the $\mathcal{O}(\varepsilon^0 L^0)$ terms we obtain the leading color contributions. A partial result for the vector form factor has been given in [29] for three-loop order, where unknown coefficients K_I at three-loop order were yet missing. We also present yet approximate four-loop results, combining all ingredients known at present and line out which missing terms have still to be calculated.

2 The Sudakov behavior

We consider the renormalized form factors F_I arising in the decay of a colorless massive boson, I , of momentum Q to a pair of heavy quarks of mass m in the asymptotic limit $Q^2 \gg m^2$. Here Q^2 is the center-of-mass energy squared and $I = V, A, S$ and P indicates a vector, an axial-vector, a scalar and a pseudo-scalar boson, respectively. In the asymptotic limit the differences between the vector and axial-vector form factors and between scalar and pseudo-scalar form factors vanish. Also the magnetic form factor in the vector and axial-vector case vanish in this limit. Henceforth, we therefore only consider the electric form factor (F_V) for the vector current and the scalar form factor (F_S),

$$-iv_Q \delta^{cd} \gamma^\mu F_V^{cd} \quad (1)$$

$$-\frac{m}{v} s_Q \delta_{cd} F_S^{cd}, \quad (2)$$

where $v = (\sqrt{2}G_F)^{-1/2}$ is the vacuum expectation value, v_Q and s_Q are the heavy quark vector and scalar couplings, and c, d are color indices. We consider n_l light quarks and a single heavy quark and deal with the case of only massless QCD contributions to the heavy quark form factors, i.e. no internal massive lines.

In the asymptotic limit, the functions $\hat{F}_I(a_s(\mu), \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, \varepsilon)$ satisfy the following integro-differential equation [35]

$$\mu^2 \frac{\partial}{\partial \mu^2} \ln \hat{F}_I \left(\frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, a_s, \varepsilon \right) = \frac{1}{2} \left[K_I \left(\frac{m^2}{\mu^2}, a_s, \varepsilon \right) + G_I \left(\frac{Q^2}{\mu^2}, a_s, \varepsilon \right) \right]. \quad (3)$$

Here \hat{F}_I contains all logarithmic and infrared pole contributions of the respective form factor. The strong coupling constant a_s in the $\overline{\text{MS}}$ scheme obeys the scale evolution equation

$$\frac{da_s(\mu^2)}{d \ln \mu^2} = - \sum_{k=0}^{\infty} \beta_k a_s^{k+2}(\mu^2), \quad (4)$$

where β_k are the expansion coefficients of the QCD β -function [36–42], which depend on the Casimir operators $C_A = N_C, C_F = (N_C^2 - 1)/(2N_C)$ and $T_F = 1/2$ in $SU(N_C)$ and the number of flavors n_l up to three-loop order. The functions K_I incorporate the contributions from the heavy quark mass m and are process independent, whereas, the functions G_I are process dependent. The functions G_I and K_I obey the renormalization group equations

$$\mu^2 \frac{d}{d\mu^2} G_I \left(\frac{Q^2}{\mu^2}, a_s, \varepsilon \right) = - \lim_{m \rightarrow 0} \mu^2 \frac{d}{d\mu^2} K_I \left(\frac{m^2}{\mu^2}, a_s, \varepsilon \right) = A_q(a_s(\mu^2)), \quad (5)$$

cf. also [10]. Individually one obtains

$$\begin{aligned} K_I &= K_I(a_s(m^2), 1, \varepsilon) - \int_{\frac{m^2}{\mu^2}}^1 \frac{d\lambda}{\lambda} A_q(a_s(\lambda\mu^2)), \\ G_I &= G_I(a_s(Q^2), 1, \varepsilon) + \int_{\frac{Q^2}{\mu^2}}^1 \frac{d\lambda}{\lambda} A_q(a_s(\lambda\mu^2)). \end{aligned} \quad (6)$$

It is understood that all contributing functions obey series expansions of the kind

$$\mathcal{A}_I(a_s) = \sum_{k=1}^{\infty} a_s^k \mathcal{A}_I^{(k)}. \quad (7)$$

The coefficients $A_q^{(n)}$ are known up to three-loop order [43, 44]. The leading color and complete n_l^2 , n_l^3 contributions of $A_q^{(4)}$ have been obtained in [45–50] recently. The finite functions K_I , G_I and A_q depend on $a_s(k^2)$ with the corresponding scales $k^2 = m^2, Q^2, \lambda\mu^2$.

To solve Eq. (3), the following expansion is performed

$$\ln \hat{F}_I\left(a_s, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, \varepsilon\right) = \sum_{n=1}^{\infty} a_s^n \left(\frac{m^2}{\mu^2}\right)^{-n\varepsilon} \mathcal{F}_I^{(n)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}, \varepsilon\right). \quad (8)$$

In the following we use $L \equiv \ln(Q^2/m^2)$ by setting $\mu^2 = m^2$, for convenience, cf. also [14]. The μ^2 -dependence can be easily recovered. The non-logarithmic contributions to the form factor are not contained in \hat{F}_I . They are obtained by matching

$$F_I\left(a_s, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, \varepsilon\right) = \mathcal{C}_I(a_s, \varepsilon) \hat{F}_I\left(a_s, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, \varepsilon\right) \quad (9)$$

to the complete form factors.

To have a more compact representation, the functions $\mathcal{F}_I^{(k)}$ in Eq. (8) are presented for the case of the unrenormalized coupling \hat{a}_s here:

$$\begin{aligned} \mathcal{F}_I^{(1)} = & \frac{1}{\varepsilon} \left[\left(-\frac{1}{2}K_I^{(1)} - \frac{1}{2}G_I^{(1)} \right) + \frac{1}{2}A_I^{(1)}L \right] + \frac{1}{2}G_I^{(1)}L - \frac{1}{4}A_I^{(1)}L^2 + \varepsilon \left[-\frac{1}{4}G_I^{(1)}L^2 \right. \\ & \left. + \frac{1}{12}A_I^{(1)}L^3 \right] + \varepsilon^2 \left[\frac{1}{12}G_I^{(1)}L^3 - \frac{1}{48}A_I^{(1)}L^4 \right] + \varepsilon^3 \left[-\frac{1}{48}G_I^{(1)}L^4 + \frac{1}{240}A_I^{(1)}L^5 \right]. \end{aligned} \quad (10)$$

$$\begin{aligned} \mathcal{F}_I^{(2)} = & \frac{1}{\varepsilon^2} \left[\left(-\frac{1}{4}K_I^{(1)}\beta_0 - \frac{1}{4}G_I^{(1)}\beta_0 \right) + \frac{1}{4}A_I^{(1)}\beta_0L \right] + \frac{1}{\varepsilon} \left[\left(-\frac{1}{4}K_I^{(2)} - \frac{1}{4}G_I^{(2)} \right) + \left(\frac{1}{4}A_I^{(2)} \right. \right. \\ & \left. \left. + \frac{1}{2}G_I^{(1)}\beta_0 \right)L - \frac{1}{4}A_I^{(1)}\beta_0L^2 \right] + \frac{1}{2}G_I^{(2)}L + \left(-\frac{1}{4}A_I^{(2)} - \frac{1}{2}G_I^{(1)}\beta_0 \right)L^2 \\ & + \frac{1}{6}A_I^{(1)}\beta_0L^3 + \varepsilon \left[-\frac{1}{2}G_I^{(2)}L^2 + \left(\frac{1}{6}A_I^{(2)} + \frac{1}{3}G_I^{(1)}\beta_0 \right)L^3 - \frac{1}{12}A_I^{(1)}\beta_0L^4 \right] \\ & + \varepsilon^2 \left[\frac{1}{3}G_I^{(2)}L^3 + \left(-\frac{1}{12}A_I^{(2)} - \frac{1}{6}G_I^{(1)}\beta_0 \right)L^4 + \frac{1}{30}A_I^{(1)}\beta_0L^5 \right]. \end{aligned} \quad (11)$$

$$\begin{aligned} \mathcal{F}_I^{(3)} = & \frac{1}{\varepsilon^3} \left[\left(-\frac{1}{6}K_I^{(1)}\beta_0^2 - \frac{1}{6}G_I^{(1)}\beta_0^2 \right) + \frac{1}{6}A_I^{(1)}\beta_0^2L \right] + \frac{1}{\varepsilon^2} \left[\left(-\frac{1}{3}K_I^{(2)}\beta_0 - \frac{1}{12}K_I^{(1)}\beta_1 - \frac{1}{3}G_I^{(2)}\beta_0 \right. \right. \\ & \left. \left. - \frac{1}{12}G_I^{(1)}\beta_1 \right) + \left(\frac{1}{3}A_I^{(2)}\beta_0 + \frac{1}{12}A_I^{(1)}\beta_1 + \frac{1}{2}G_I^{(1)}\beta_0^2 \right)L - \frac{1}{4}A_I^{(1)}\beta_0^2L^2 \right] + \frac{1}{\varepsilon} \left[-\frac{1}{6}(K_I^{(3)} + G_I^{(3)}) \right. \\ & \left. + \left(\frac{1}{6}A_I^{(3)} + G_I^{(2)}\beta_0 + \frac{1}{4}G_I^{(1)}\beta_1 \right)L + \left(-\frac{1}{2}A_I^{(2)}\beta_0 - \frac{1}{8}A_I^{(1)}\beta_1 - \frac{3}{4}G_I^{(1)}\beta_0^2 \right)L^2 + \frac{1}{4}A_I^{(1)}\beta_0^2L^3 \right] \\ & + \frac{1}{2}G_I^{(3)}L + \left(-\frac{1}{4}A_I^{(3)} - \frac{3}{2}G_I^{(2)}\beta_0 - \frac{3}{8}G_I^{(1)}\beta_1 \right)L^2 + \left(\frac{1}{2}A_I^{(2)}\beta_0 + \frac{1}{8}A_I^{(1)}\beta_1 + \frac{3}{4}G_I^{(1)}\beta_0^2 \right)L^3 \\ & - \frac{3}{16}A_I^{(1)}\beta_0^2L^4 + \varepsilon \left[-\frac{3}{4}G_I^{(3)}L^2 + \left(\frac{1}{4}A_I^{(3)} + \frac{3}{2}G_I^{(2)}\beta_0 + \frac{3}{8}G_I^{(1)}\beta_1 \right)L^3 + \left(-\frac{3}{8}A_I^{(2)}\beta_0 \right. \right. \\ & \left. \left. - \frac{3}{32}A_I^{(1)}\beta_1 - \frac{9}{16}G_I^{(1)}\beta_0^2 \right)L^4 + \frac{9}{80}A_I^{(1)}\beta_0^2L^5 \right]. \end{aligned} \quad (12)$$

$$\begin{aligned} \mathcal{F}_I^{(4)} = & \frac{1}{\varepsilon^4} \left[\left(-\frac{1}{8}K_I^{(1)}\beta_0^3 - \frac{1}{8}G_I^{(1)}\beta_0^3 \right) + \frac{1}{8}A_I^{(1)}\beta_0^3L \right] + \frac{1}{\varepsilon^3} \left[\left(-\frac{3}{8}K_I^{(2)}\beta_0^2 - \frac{1}{6}K_I^{(1)}\beta_0\beta_1 - \frac{3}{8}G_I^{(2)}\beta_0^2 \right. \right. \\ & \left. \left. - \frac{1}{6}G_I^{(1)}\beta_0\beta_1 \right) + \left(\frac{3}{8}A_I^{(2)}\beta_0^2 + \frac{1}{6}A_I^{(1)}\beta_0\beta_1 + \frac{1}{2}G_I^{(1)}\beta_0^3 \right)L - \frac{1}{4}A_I^{(1)}\beta_0^3L^2 \right] + \frac{1}{\varepsilon^2} \left[\left(-\frac{3}{8}K_I^{(3)}\beta_0 \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{8}K_I^{(2)}\beta_1 - \frac{1}{24}K_I^{(1)}\beta_2 - \frac{3}{8}G_I^{(3)}\beta_0 - \frac{1}{8}G_I^{(2)}\beta_1 - \frac{1}{24}G_I^{(1)}\beta_2 \Big) + \left(\frac{3}{8}A_I^{(3)}\beta_0 + \frac{1}{8}A_I^{(2)}\beta_1 \right. \\
& + \left. \frac{1}{24}A_I^{(1)}\beta_2 + \frac{3}{2}G_I^{(2)}\beta_0^2 + \frac{2}{3}G_I^{(1)}\beta_0\beta_1 \right) L + \left(-\frac{3}{4}A_I^{(2)}\beta_0^2 - \frac{1}{3}A_I^{(1)}\beta_0\beta_1 - G_I^{(1)}\beta_0^3 \right) L^2 \\
& + \frac{1}{3}A_I^{(1)}\beta_0^3 L^3 \Big] + \frac{1}{\varepsilon} \left[\left(-\frac{1}{8}K_I^{(4)} - \frac{1}{8}G_I^{(4)} \right) + \left(\frac{1}{8}A_I^{(4)} + \frac{3}{2}G_I^{(3)}\beta_0 + \frac{1}{2}G_I^{(2)}\beta_1 + \frac{1}{6}G_I^{(1)}\beta_2 \right) L \right. \\
& + \left(-\frac{3}{4}A_I^{(3)}\beta_0 - \frac{1}{4}A_I^{(2)}\beta_1 - \frac{1}{12}A_I^{(1)}\beta_2 - 3G_I^{(2)}\beta_0^2 - \frac{4}{3}G_I^{(1)}\beta_0\beta_1 \right) L^2 + \left(A_I^{(2)}\beta_0^2 + \frac{4}{9}A_I^{(1)}\beta_0\beta_1 \right. \\
& + \left. \frac{4}{3}G_I^{(1)}\beta_0^3 \right) L^3 - \frac{1}{3}A_I^{(1)}\beta_0^3 L^4 \Big] + \frac{1}{2}G_I^{(4)}L + \left(-\frac{1}{4}A_I^{(4)} - 3G_I^{(3)}\beta_0 - G_I^{(2)}\beta_1 - \frac{1}{3}G_I^{(1)}\beta_2 \right) L^2 \\
& + \left(A_I^{(3)}\beta_0 + \frac{1}{3}A_I^{(2)}\beta_1 + \frac{1}{9}A_I^{(1)}\beta_2 + 4G_I^{(2)}\beta_0^2 + \frac{16}{9}G_I^{(1)}\beta_0\beta_1 \right) L^3 + \left(-A_I^{(2)}\beta_0^2 - \frac{4}{9}A_I^{(1)}\beta_0\beta_1 \right. \\
& \left. - \frac{4}{3}G_I^{(1)}\beta_0^3 \right) L^4 + \frac{4}{15}A_I^{(1)}\beta_0^3 L^5. \tag{13}
\end{aligned}$$

The corresponding expression for the running coupling a_s is obtained by replacing

$$\hat{a}_s = 1 - a_s \frac{\beta_0}{\varepsilon} + a_s^2 \left(\frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2\varepsilon} \right) + a_s^3 \left(-\frac{\beta_0^3}{\varepsilon^3} + \frac{7}{6\varepsilon^2}\beta_0\beta_1 - \frac{\beta_2}{3\varepsilon} \right) + \mathcal{O}(a_s^4) \tag{14}$$

in Eqs. (8,9).

In massless scenarios, the soft (f_i) and collinear (B_i) anomalous dimensions of massless QCD govern the infrared structures, where $i \equiv q, g$ label quarks and gluons, respectively. In the case of two-parton amplitudes, e.g. the massless form factors, $\gamma_i = B_i + \frac{f_i}{2}$ is obtained from the leading pole $1/\varepsilon$ and terms from lower orders. It is imperative that for the case of heavy quarks γ_q , along with similar contributions from the heavy quark anomalous dimension, will form the leading singularity. On the other hand, following soft-collinear effective theory, it was shown in [13] that the two-loop anomalous dimension matrix for two heavy parton correlations, which controls the singular structure, contains a heavy quark anomalous dimension γ_Q . Hence it is suggestive to form the following structure for the finite function

$$K_I^{(n)} = -2 \left(\gamma_q^{(n)} + \gamma_Q^{(n)} - \gamma_I^{(n-1)} \right), \tag{15}$$

where $\gamma_q = B_q + \frac{f_q}{2}$ and B_q and f_q are the collinear and soft anomalous dimension for massless quarks. They are known up to three-loop order [5, 43, 44]. Recently $\gamma_q^{(4)}$ in the color-planar limit has been calculated in [45, 47, 48] from the contributions to the four-loop massless form factor in addition to the n_l^2 and n_l^3 contributions. The anomalous dimension $\gamma_I^{(n-1)}$ is arising due to renormalization of the current. It vanishes for conserved currents, as the vector current. For the scalar current, the Yukawa coupling introduces the mass renormalization and hence $\gamma_S^{(n-1)} = \gamma_m^{(n-1)}$, being known up to four-loop order [51–57]. Furthermore, the renormalization constants $Z_{m,0S}$ and $Z_{2,0S}$ are given in [51, 53, 55, 58, 59], and [58–60], respectively.

The heavy quark anomalous dimension γ_Q is connected with the soft Wilson-line operator and was known up to two-loop [13, 61, 62]. Also, it constitutes the non-logarithmic contribution of the massive cusp anomalous dimension in the asymptotic limit. Thus, we can obtain γ_Q up to three-loop order from the known three-loop result [63]

$$\gamma_Q^{(0)} = -2C_F,$$

$$\begin{aligned}
\gamma_Q^{(1)} &= C_F C_A \left(-\frac{98}{9} + 4\zeta_2 - 4\zeta_3 \right) + C_F T_F n_l \left(\frac{40}{9} \right), \\
\gamma_Q^{(2)} &= C_F C_A^2 \left(-\frac{343}{9} + \frac{608}{9}\zeta_2 - \frac{88}{5}\zeta_2^2 - \frac{740}{9}\zeta_3 - 8\zeta_2\zeta_3 + 36\zeta_5 \right) + C_F C_A T_F n_l \left(\frac{356}{27} - \frac{160}{9}\zeta_2 \right. \\
&\quad \left. + \frac{496}{9}\zeta_3 \right) + C_F^2 T_F n_l \left(\frac{110}{3} - 32\zeta_3 \right) + C_F T_F^2 n_l^2 \left(\frac{32}{27} \right). \tag{16}
\end{aligned}$$

Here ζ_k , $k \geq 2, k \in \mathbb{N}$ are the values of the Riemann ζ -function at integer argument. Note that, as mentioned in [13], the non-abelian exponentiation theorem constrains the color structures of γ_Q . Hence, up to $\mathcal{O}(\alpha_s^3)$, the corresponding anomalous dimension for a massive color-octet boson (G) is given by $\gamma_G = \frac{C_A}{C_F} \gamma_Q$. The other finite function $G_I^{(n)}$ only contains the information about the process and depends on Q^2 and not the heavy quark mass m . It is similar to the massless case of the Drell-Yan form factor

$$G_I^{(n)} = 2(B_q^{(n)} - \gamma_I^{(n-1)}) + f_q^{(n)} + C_I^{(n)} + \sum_{k=1}^{\infty} \varepsilon^k g_I^{n,k}. \tag{17}$$

Up to four-loop order, $C_I^{(n)}$ are given by

$$\begin{aligned}
C_I^{(1)} &= 0, \\
C_I^{(2)} &= \beta_0 g_I^{1,1}, \\
C_I^{(3)} &= \beta_1 g_I^{1,1} + \beta_0 (g_I^{2,1} - \beta_0 g_I^{1,2}), \\
C_I^{(4)} &= \beta_2 g_I^{1,1} + \beta_1 (g_I^{2,1} - 2\beta_0 g_I^{1,2}) + \beta_0 (g_I^{3,1} - \beta_0 g_I^{2,2} + \beta_0^2 g_I^{1,3}). \tag{18}
\end{aligned}$$

The components $g_I^{n,k}$ have to be extracted from explicit result in this limit. Given the structural similarities, the functions $g_I^{n,k}$ are the same in both the massive and the massless case.

3 Phenomenological Results

Eqs. (12) and (13), along with (8) and (9), provide the three- and four-loop heavy quark form factors in the asymptotic limit. To obtain the vector and scalar form factors, we need $C_I^{(3)}$ and $C_I^{(4)}$, along with all the anomalous dimensions up to three and four loops and $g_I^{1,k}, g_I^{2,k}, g_I^{3,k}, C_I^{1,k}$ and $C_I^{2,k}$ to sufficient order k in ε . As noted earlier, the coefficients $g_I^{n,k}$ can be obtained from the massless results, cf. [64,65]. The ε^0 -part of $G_I^{(4)}$, as noted in Eq. (17), can also be obtained from the massless quark form factor. Below, we present $G_V^{(4)}$ and $G_S^{(4)}$. Note that $G_S^{(4)}$ constitutes the single pole in ε of the Higgs-quark massless form factor.

$$\begin{aligned}
G_V^4 &= N_C^4 \left(\frac{1913092765}{139968} + \frac{571537}{243}\zeta_2 - \frac{2308}{5}\zeta_2^2 + \frac{12350}{63}\zeta_2^3 - \frac{169547}{18}\zeta_3 + \frac{1622}{3}\zeta_3^2 + \frac{8480}{9}\zeta_3\zeta_2 \right. \\
&\quad \left. - \frac{656}{5}\zeta_3\zeta_2^2 + \frac{10100}{3}\zeta_5 - 288\zeta_5\zeta_2 - 1410\zeta_7 \right) + N_C^3 n_l T_F \left(-\frac{164021633}{11664} - \frac{1903985}{486}\zeta_2 + \frac{7547}{15}\zeta_2^2 \right. \\
&\quad \left. - \frac{29432}{315}\zeta_2^3 + \frac{406621}{81}\zeta_3 + \frac{32}{3}\zeta_3^2 - \frac{2432}{9}\zeta_3\zeta_2 + \frac{1340}{9}\zeta_5 \right) + C_F^2 n_l^2 T_F^2 \left(\frac{693446}{243} - \frac{3232}{3}\zeta_2 \right. \\
&\quad \left. - \frac{21376}{45}\zeta_2^2 + \frac{65632}{27}\zeta_3 - \frac{512}{3}\zeta_3\zeta_2 - \frac{832}{3}\zeta_5 \right) + C_F C_A n_l^2 T_F^2 \left(\frac{8996449}{1458} + \frac{919544}{243}\zeta_2 + \frac{2464}{15}\zeta_2^2 \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{172832}{81}\zeta_3 + \frac{1216}{9}\zeta_3\zeta_2 - \frac{4928}{9}\zeta_5 \Big) + C_F n_l^3 T_F^3 \left(-\frac{1117984}{2187} - \frac{25984}{81}\zeta_2 - \frac{1664}{45}\zeta_2^2 - \frac{640}{81}\zeta_3 \right) \\
& + \mathcal{O}(\varepsilon). \tag{19}
\end{aligned}$$

$$\begin{aligned}
G_S^4 = & N_C^4 \left(-\frac{11003395}{4374} + \frac{93937}{972}\zeta_2 - 290\zeta_2^2 + \frac{12350}{63}\zeta_2^3 - \frac{36592}{9}\zeta_3 + \frac{1622}{3}\zeta_3^2 + \frac{8480}{9}\zeta_3\zeta_2 \right. \\
& - \frac{656}{5}\zeta_3\zeta_2^2 + \frac{8780}{3}\zeta_5 - 288\zeta_5\zeta_2 - 1410\zeta_7 \Big) + N_C^3 n_l T_F \left(\frac{14211811}{11664} - \frac{497663}{486}\zeta_2 + \frac{5027}{15}\zeta_2^2 \right. \\
& - \frac{29432}{315}\zeta_2^3 + \frac{199909}{81}\zeta_3 + \frac{32}{3}\zeta_3^2 - \frac{2432}{9}\zeta_3\zeta_2 \\
& + \frac{620}{9}\zeta_5 \Big) + C_F^2 n_l^2 T_F^2 \left(\frac{201110}{243} - \frac{2944}{9}\zeta_2 - \frac{16192}{45}\zeta_2^2 + \frac{67360}{27}\zeta_3 - \frac{512}{3}\zeta_3\zeta_2 - \frac{832}{3}\zeta_5 \right) \\
& + C_F C_A n_l^2 T_F^2 \left(-\frac{823055}{1458} + \frac{293144}{243}\zeta_2 + \frac{2752}{15}\zeta_2^2 - \frac{136544}{81}\zeta_3 + \frac{1216}{9}\zeta_3\zeta_2 - \frac{4928}{9}\zeta_5 \right) \\
& + C_F n_l^3 T_F^3 \left(-\frac{41728}{2187} - \frac{6400}{81}\zeta_2 - \frac{1664}{45}\zeta_2^2 - \frac{640}{81}\zeta_3 \right) + \mathcal{O}(\varepsilon). \tag{20}
\end{aligned}$$

Due to the recent two-loop calculations in Ref. [30] we can extract the coefficients in

$$\mathcal{C}^{(i)} = \sum_{k=0}^{\infty} \varepsilon^k \mathcal{C}^{i,k} \tag{21}$$

up to the required order. They read

$$\begin{aligned}
\mathcal{C}_V^{1,0} &= C_F \left(4 + \zeta_2 \right), \\
\mathcal{C}_V^{1,1} &= C_F \left(8 + \frac{1}{2}\zeta_2 - \frac{2}{3}\zeta_3 \right), \\
\mathcal{C}_V^{1,2} &= C_F \left(16 + 2\zeta_2 + \frac{9}{20}\zeta_2^2 - \frac{1}{3}\zeta_3 \right) \\
\mathcal{C}_V^{1,3} &= C_F \left(32 + 4\zeta_2 + \frac{9}{40}\zeta_2^2 - \frac{4}{3}\zeta_3 - \frac{1}{3}\zeta_2\zeta_3 - \frac{2}{5}\zeta_5 \right), \\
\mathcal{C}_V^{2,0} &= C_F^2 \left(\frac{241}{8} + 26\zeta_2 - 48 \ln(2)\zeta_2 - \frac{163}{10}\zeta_2^2 - 6\zeta_3 \right) + C_F C_A \left(\frac{12877}{648} + \frac{323}{18}\zeta_2 + 24 \ln(2)\zeta_2 \right. \\
& \left. - \frac{47}{5}\zeta_2^2 + \frac{89}{9}\zeta_3 \right) + C_F T_F n_l \left(-\frac{1541}{162} - \frac{74}{9}\zeta_2 - \frac{52}{9}\zeta_3 \right), \\
\mathcal{C}_V^{2,1} &= C_F^2 \left(-\frac{557}{16} + 8c_1 + \frac{399}{4}\zeta_2 - 24 \ln(2)\zeta_2 - \frac{1363}{10}\zeta_2^2 + \frac{286}{3}\zeta_3 - \frac{146}{3}\zeta_2\zeta_3 - 6\zeta_5 \right) \\
& + C_F C_A \left(\frac{629821}{3888} - 4c_1 + \frac{7805}{108}\zeta_2 + 12 \ln(2)\zeta_2 + \frac{209}{2}\zeta_2^2 - \frac{1162}{27}\zeta_3 + \frac{83}{3}\zeta_2\zeta_3 - 208\zeta_5 \right) \\
& + C_F n_l T_F \left(-\frac{46205}{972} - \frac{673}{27}\zeta_2 - \frac{98}{5}\zeta_2^2 - \frac{364}{27}\zeta_3 \right), \\
\mathcal{C}_V^{2,2} &= C_F^2 \left(\frac{1817}{32} + 4c_1 + \frac{48}{5}c_2 + 8c_1\zeta_2 + \frac{5165}{8}\zeta_2 - 576\zeta_2 \ln(2) + \frac{393}{8}\zeta_2^2 - \frac{37137}{140}\zeta_2^3 - 144\zeta_2^2 \ln^2(2) \right. \\
& \left. + \frac{1889}{6}\zeta_3 - \frac{799}{9}\zeta_3^2 - \frac{494}{3}\zeta_3\zeta_2 + 168\zeta_3\zeta_2 \ln(2) - 1431\zeta_5 \right) + C_F C_A \left(\frac{10899301}{23328} - 2c_1 - 4c_1\zeta_2 \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{24}{5}c_2 + \frac{53045}{648}\zeta_2 + \frac{2513}{24}\zeta_2^2 - \frac{9839}{210}\zeta_2^3 + 288\zeta_2 \ln(2) + 72\zeta_2^2 \ln^2(2) - \frac{6517}{162}\zeta_3 + \frac{13}{2}\zeta_3^2 \\
& + \frac{1387}{9}\zeta_3\zeta_2 - 84\zeta_3\zeta_2 \ln(2) + \frac{23077}{30}\zeta_5 \Big) + C_F n_l T_F \left(-\frac{1063589}{5832} - \frac{15481}{162}\zeta_2 - \frac{1607}{30}\zeta_2^2 \right. \\
& \left. - \frac{3422}{81}\zeta_3 - \frac{224}{9}\zeta_3\zeta_2 - \frac{1444}{15}\zeta_5 \right) \\
\mathcal{C}_V^{3,0} = & 2C_F^2 T_F n_l \left(-\frac{78427}{972} - \frac{32}{9}c_1 - \frac{38267}{324}\zeta_2 + \frac{224}{3} \ln(2)\zeta_2 + \frac{17873}{135}\zeta_2^2 - \frac{5264}{81}\zeta_3 - \frac{154}{9}\zeta_2\zeta_3 \right. \\
& \left. + \frac{596}{9}\zeta_5 \right) + N_C^3 \left(\frac{10907077}{52488} + \frac{15763067}{46656}\zeta_2 - \frac{116957}{1080}\zeta_2^2 + \frac{145051}{3024}\zeta_2^3 - \frac{15743}{486}\zeta_3 \right. \\
& \left. + \frac{533}{18}\zeta_2\zeta_3 + \frac{298}{9}\zeta_3^2 - 357\zeta_5 \right) + 2C_F C_A T_F n_l \left(-\frac{533996}{6561} + \frac{16}{9}c_1 - \frac{102019}{729}\zeta_2 - \frac{112}{3} \ln(2)\zeta_2 \right. \\
& \left. - \frac{10942}{135}\zeta_2^2 - \frac{2696}{81}\zeta_3 + \frac{302}{3}\zeta_5 \right) + 4C_F T_F^2 n_l^2 \left(\frac{58883}{13122} + \frac{580}{81}\zeta_2 + \frac{884}{135}\zeta_2^2 + \frac{2144}{243}\zeta_3 \right) \quad (22)
\end{aligned}$$

$$\mathcal{C}_S^{1,0} = C_F \zeta_2$$

$$\mathcal{C}_S^{1,1} = C_F \left(-\zeta_2 - \frac{2}{3}\zeta_3 \right)$$

$$\mathcal{C}_S^{1,2} = C_F \left(\frac{9}{20}\zeta_2^2 + \frac{2}{3}\zeta_3 \right)$$

$$\mathcal{C}_S^{1,3} = C_F \left(-\frac{9}{20}\zeta_2^2 - \frac{1}{3}\zeta_2\zeta_3 - \frac{2}{5}\zeta_5 \right),$$

$$\begin{aligned}
\mathcal{C}_S^{2,0} = & C_F^2 \left(15 - 8\zeta_2 - \frac{163}{10}\zeta_2^2 - 18\zeta_3 \right) + C_F C_A \left(-\frac{2140}{81} + \frac{467}{18}\zeta_2 - \frac{47}{5}\zeta_2^2 + \frac{143}{9}\zeta_3 \right) \\
& + C_F n_l T_F \left(\frac{188}{81} - \frac{2}{9}\zeta_2 - \frac{52}{9}\zeta_3 \right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_S^{2,1} = & C_F^2 \left(-\frac{209}{2} - 76\zeta_2 + 264 \ln(2)\zeta_2 - 37\zeta_2^2 - 34\zeta_3 - \frac{146}{3}\zeta_2\zeta_3 - 6\zeta_5 \right) + C_F C_A \left(-\frac{8155}{486} \right. \\
& \left. + \frac{2561}{27}\zeta_2 - 132 \ln(2)\zeta_2 + \frac{541}{10}\zeta_2^2 + \frac{242}{27}\zeta_3 + \frac{83}{3}\zeta_2\zeta_3 - 208\zeta_5 \right) + C_F n_l T_F \left(\frac{214}{243} + \frac{200}{27}\zeta_2 \right. \\
& \left. - \frac{98}{5}\zeta_2^2 + \frac{68}{27}\zeta_3 \right),
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_S^{2,2} = & C_F^2 \left(-\frac{779}{4} - 44c_1 + 8c_1\zeta_2 - 17\zeta_2 + 480\zeta_2 \ln(2) + \frac{857}{2}\zeta_2^2 - \frac{37137}{140}\zeta_2^3 - 144\zeta_2^2 \ln^2(2) \right. \\
& \left. - 272\zeta_3 - \frac{799}{9}\zeta_3^2 - \frac{92}{3}\zeta_3\zeta_2 + 168\zeta_3\zeta_2 \ln(2) - 213\zeta_5 \right) + C_F C_A \left(-\frac{415831}{2916} + 22c_1 - 4c_1\zeta_2 \right. \\
& \left. + \frac{11818}{81}\zeta_2 - 240\zeta_2 \ln(2) - \frac{2098}{15}\zeta_2^2 - \frac{9839}{210}\zeta_2^3 + 72\zeta_2^2 \ln^2(2) + \frac{14422}{81}\zeta_3 + \frac{13}{2}\zeta_3^2 + \frac{793}{9}\zeta_3\zeta_2 \right. \\
& \left. - 84\zeta_3\zeta_2 \ln(2) + \frac{4807}{30}\zeta_5 \right) + C_F n_l T_F \left(-\frac{9049}{729} + \frac{940}{81}\zeta_2 + \frac{92}{15}\zeta_2^2 + \frac{880}{81}\zeta_3 - \frac{224}{9}\zeta_3\zeta_2 \right. \\
& \left. - \frac{1444}{15}\zeta_5 \right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_S^{3,0} = & N_C^3 \left(-\frac{10407317}{52488} + \frac{324905}{2916} \zeta_2 - \frac{77213}{1080} \zeta_2^2 + \frac{145051}{3024} \zeta_2^3 - \frac{28865}{486} \zeta_3 + \frac{1181}{18} \zeta_2 \zeta_3 + \frac{298}{9} \zeta_3^2 \right. \\
& \left. - 372 \zeta_5 \right) + 2C_F^2 T_F n_l \left(\frac{32351}{972} + \frac{9545}{162} \zeta_2 - 160 \ln(2) \zeta_2 + \frac{1969}{27} \zeta_2^2 + \frac{1612}{81} \zeta_3 - \frac{154}{9} \zeta_2 \zeta_3 \right. \\
& \left. + \frac{596}{9} \zeta_5 \right) + 2C_F C_A T_F n_l \left(\frac{842609}{13122} - \frac{45319}{729} \zeta_2 + 80 \ln(2) \zeta_2 - \frac{9574}{135} \zeta_2^2 - \frac{1094}{81} \zeta_3 + \frac{302}{3} \zeta_5 \right) \\
& + 4C_F T_F^2 n_l^2 \left(-\frac{2324}{6561} - \frac{356}{81} \zeta_2 + \frac{884}{135} \zeta_2^2 + \frac{632}{243} \zeta_3 \right). \tag{23}
\end{aligned}$$

The functions $\mathcal{C}_V^{3,0}$ and $\mathcal{C}_S^{3,0}$ have been obtained from the results in [31–34]. Here \mathcal{C}_V and \mathcal{C}_S have been calculated individually. One may check that they are related by

$$\mathcal{C}_S = \frac{Z_{m,\text{OS}}}{Z_{m,\text{MS}}} \mathcal{C}_V. \tag{24}$$

With all the available components, we can now present all the logarithmic contributions to the finite part for three-loop vector and scalar form factors in the asymptotic limit:

$$\begin{aligned}
F_V^{(3)} = & \frac{1}{\varepsilon^3} \left[-C_F^3 \frac{4}{3} (1-L)^3 - C_F^2 C_A \frac{22}{3} (1-L)^2 - C_F C_A^2 \frac{242}{27} (1-L) + C_F C_A n_l T_F \frac{176}{27} (1-L) \right. \\
& \left. + C_F^2 n_l T_F \frac{8}{3} (1-L)^2 - C_F n_l^2 T_F^2 \frac{32}{27} (1-L) \right] + \frac{1}{\varepsilon^2} \left[C_F^3 \left\{ -8 + 4\zeta_2 + (22 - 8\zeta_2)L \right. \right. \\
& \left. \left. + (-22 + 4\zeta_2)L^2 + 10L^3 - 2L^4 \right\} + C_F^2 C_A \left\{ -\frac{34}{9} + \frac{10}{3} \zeta_2 + 4\zeta_3 + \left(-\frac{1}{9} + \frac{2\zeta_2}{3} - 4\zeta_3 \right) L \right. \right. \\
& \left. \left. + \left(\frac{2}{9} - 4\zeta_2 \right) L^2 + \frac{11L^3}{3} \right\} + C_F C_A^2 \left\{ \frac{1690}{81} - \frac{44}{9} \zeta_2 + \frac{44}{9} \zeta_3 + \left(-\frac{2086}{81} + \frac{44\zeta_2}{9} \right) L \right\} \right. \\
& \left. + C_F C_A n_l T_F \left\{ -\frac{1192}{81} + \frac{16}{9} \zeta_2 - \frac{16}{9} \zeta_3 + \left(\frac{1336}{81} - \frac{16\zeta_2}{9} \right) L \right\} \right. \\
& \left. + C_F^2 n_l T_F \left\{ -\frac{16}{9} - \frac{8}{3} \zeta_2 + \left(\frac{20}{9} + \frac{8\zeta_2}{3} \right) L + \frac{8L^2}{9} - \frac{4L^3}{3} \right\} + C_F n_l^2 T_F^2 \frac{160}{81} (1-L) \right] \\
& + \frac{1}{\varepsilon} \left[C_F^3 \left\{ -76 - 82\zeta_2 + 96 \ln(2) \zeta_2 + \frac{236}{5} \zeta_2^2 + 80\zeta_3 + \left(129 + 88\zeta_2 - 96 \ln(2) \zeta_2 - \frac{236}{5} \zeta_2^2 \right. \right. \right. \\
& \left. \left. - 136\zeta_3 \right) L + \left(-89 + 56\zeta_3 \right) L^2 + \left(\frac{137}{3} - 6\zeta_2 \right) L^3 - \frac{34L^4}{3} + \frac{5L^5}{3} \right\} + C_F^2 C_A \left\{ \frac{2986}{27} - 10\zeta_2 \right. \\
& \left. - 48 \ln(2) \zeta_2 + \frac{26}{5} \zeta_2^2 - \frac{200}{3} \zeta_3 - 4\zeta_2 \zeta_3 + \left(-\frac{5396}{27} + \frac{5}{3} \zeta_2 + 48 \ln(2) \zeta_2 - \frac{26}{5} \zeta_2^2 + \frac{362}{3} \zeta_3 \right) L \right. \\
& \left. + \left(\frac{6107}{54} + \frac{19\zeta_2}{3} - 50\zeta_3 \right) L^2 + \left(-\frac{523}{18} + 6\zeta_2 \right) L^3 + \frac{11L^4}{9} \right\} + C_F C_A^2 \left\{ -\frac{343}{27} + \frac{608}{27} \zeta_2 \right. \\
& \left. - \frac{88}{15} \zeta_2^2 - \frac{740}{27} \zeta_3 - \frac{8}{3} \zeta_2 \zeta_3 + 12\zeta_5 + \left(\frac{245}{9} - \frac{536}{27} \zeta_2 + \frac{88}{15} \zeta_2^2 + \frac{44}{9} \zeta_3 \right) L \right\} + C_F C_A n_l T_F \left\{ \frac{356}{81} \right. \\
& \left. - \frac{160}{27} \zeta_2 + \frac{496}{27} \zeta_3 + \left(-\frac{836}{81} + \frac{160\zeta_2}{27} - \frac{112\zeta_3}{9} \right) L \right\} + C_F^2 n_l T_F \left\{ -\frac{470}{27} + 8\zeta_2 - \frac{16}{3} \zeta_3 \right. \\
& \left. + \left(\frac{1198}{27} - \frac{4\zeta_2}{3} + \frac{16\zeta_3}{3} \right) L + \left(-\frac{962}{27} - \frac{20\zeta_2}{3} \right) L^2 + \frac{82L^3}{9} - \frac{4L^4}{9} \right\} + C_F n_l^2 T_F^2 \frac{32}{81} (1-L) \left. \right]
\end{aligned} \tag{25}$$

$$\begin{aligned}
& + \left[X_{V,3}^{0,0} + N_C^3 \left\{ -\frac{554267}{2916} + \frac{23773\zeta_2}{216} - \frac{1727\zeta_2^2}{30} + \frac{8156\zeta_3^2}{315} + \frac{33197\zeta_3}{162} - \frac{16\zeta_3^2}{3} + \frac{113\zeta_3\zeta_2}{3} \right. \right. \\
& - \left. \frac{875\zeta_5}{3} \right\} + C_F^3 \left\{ \left(250 + 16c_1 + 585\zeta_2 - 192 \ln(2)\zeta_2 - \frac{1942}{5}\zeta_2^2 - 520\zeta_3 - 8\zeta_2\zeta_3 - 276\zeta_5 \right) L \right. \\
& - \left(332 + 123\zeta_2 - 48 \ln(2)\zeta_2 - \frac{302}{5}\zeta_2^2 - 340\zeta_3 \right) L^2 + \left(\frac{494}{3} + \frac{17\zeta_2}{3} - \frac{268\zeta_3}{3} \right) L^3 - \left(\frac{148}{3} \right. \\
& - \left. \frac{16\zeta_2}{3} \right) L^4 + \frac{17L^5}{2} - L^6 \left. \right\} + C_F^2 C_A \left\{ \left(-\frac{178337}{162} - 8c_1 - \frac{5959}{18}\zeta_2 + 96 \ln(2)\zeta_2 + \frac{4279}{15}\zeta_2^2 \right. \right. \\
& + \left. \frac{3706}{3}\zeta_3 - 46\zeta_2\zeta_3 - 194\zeta_5 \right) L + \left(\frac{64625}{81} + \frac{266}{3}\zeta_2 - 24 \ln(2)\zeta_2 - \frac{143}{5}\zeta_2^2 - \frac{3743}{9}\zeta_3 \right) L^2 \\
& + \left(-\frac{6260}{27} - \frac{97\zeta_2}{18} + \frac{232\zeta_3}{3} \right) L^3 + \left(\frac{4289}{108} - \frac{16\zeta_2}{3} \right) L^4 - \frac{11L^5}{4} \left. \right\} + C_F C_A^2 \left\{ \left(\frac{1045955}{1458} \right. \right. \\
& + \frac{17366}{81}\zeta_2 - \frac{94}{3}\zeta_2^2 - \frac{17464}{27}\zeta_3 + \frac{88}{3}\zeta_2\zeta_3 + 136\zeta_5 \left. \right) L + \left(-\frac{18682}{81} + \frac{26}{9}\zeta_2 - \frac{44}{5}\zeta_2^2 + 88\zeta_3 \right) L^2 \\
& + \left(\frac{2869}{81} - \frac{44\zeta_2}{9} \right) L^3 - \frac{121L^4}{54} \left. \right\} + C_F C_A n_l T_F \left\{ \frac{259150}{729} + \frac{32c_1}{9} + \frac{3008\zeta_2}{81} - \frac{7288\zeta_2^2}{45} \right. \\
& - \frac{224}{3}\zeta_2 \ln(2) - \frac{31120\zeta_3}{81} + \frac{8\zeta_3\zeta_2}{3} + \frac{596\zeta_5}{3} + \left(-\frac{309838}{729} - \frac{11728}{81}\zeta_2 + \frac{88}{15}\zeta_2^2 + \frac{1448}{9}\zeta_3 \right) L \\
& + \left(\frac{11752}{81} + \frac{32\zeta_2}{3} - 16\zeta_3 \right) L^2 + \left(-\frac{1948}{81} + \frac{16\zeta_2}{9} \right) L^3 + \frac{44L^4}{27} \left. \right\} + C_F^2 n_l T_F \left\{ -\frac{2011}{81} \right. \\
& - \frac{64c_1}{9} - \frac{962\zeta_2}{3} + \frac{12232\zeta_2^2}{45} + \frac{2752\zeta_3}{9} - 48\zeta_3\zeta_2 + 40\zeta_5 + \frac{448\zeta_2 \ln(2)}{3} + \left(\frac{18812}{81} + \frac{682}{9}\zeta_2 \right. \\
& - \left. \frac{392}{15}\zeta_2^2 - \frac{1976}{9}\zeta_3 \right) L + \left(-\frac{18817}{81} - \frac{100\zeta_2}{3} + \frac{232\zeta_3}{9} \right) L^2 + \left(\frac{2032}{27} + \frac{58\zeta_2}{9} \right) L^3 - \frac{355L^4}{27} \\
& + L^5 \left. \right\} + C_F n_l^2 T_F^2 \left\{ -\frac{29344}{729} - \frac{976\zeta_2}{81} + \frac{928\zeta_2^2}{45} + \frac{256\zeta_3}{9} + \left(\frac{39352}{729} + \frac{608\zeta_2}{27} + \frac{64\zeta_3}{27} \right) L \right. \\
& - \left. \left(\frac{1624}{81} + \frac{32\zeta_2}{9} \right) L^2 + \frac{304L^3}{81} - \frac{8L^4}{27} \right\}. \tag{26}
\end{aligned}$$

$$\begin{aligned}
F_S^{(3)} &= \frac{1}{\varepsilon^3} \left[-C_F^3 \frac{4}{3} (1-L)^3 - C_F^2 C_A \frac{22}{3} (1-L)^2 - C_F C_A^2 \frac{242}{27} (1-L) + C_F C_A n_l T_F \frac{176}{27} (1-L) \right. \\
& + \left. C_F^2 n_l T_F \frac{8}{3} (1-L)^2 - C_F n_l^2 T_F^2 \frac{32}{27} (1-L) \right] \\
& + \frac{1}{\varepsilon^2} \left[C_F^3 \left\{ -4 + 4\zeta_2 + (8 - 8\zeta_2)L + (-6 + 4\zeta_2)L^2 + 4L^3 - 2L^4 \right\} + C_F^2 C_A \left\{ \frac{32}{9} + \frac{10}{3}\zeta_2 \right. \right. \\
& + 4\zeta_3 + \left(-\frac{166}{9} + \frac{2\zeta_2}{3} - 4\zeta_3 \right) L + \left(\frac{101}{9} - 4\zeta_2 \right) L^2 + \frac{11L^3}{3} \left. \right\} + C_F C_A^2 \left\{ \frac{1690}{81} + \frac{44}{9}(\zeta_3 - \zeta_2) \right. \\
& + \left. \left(-\frac{2086}{81} + \frac{44\zeta_2}{9} \right) L \right\} + C_F C_A n_l T_F \left\{ -\frac{1192}{81} + \frac{16}{9}\zeta_2 - \frac{16}{9}\zeta_3 + \left(\frac{1336}{81} - \frac{16\zeta_2}{9} \right) L \right\} \\
& + C_F^2 n_l T_F \left\{ -\frac{40}{9} - \frac{8}{3}\zeta_2 + \left(\frac{80}{9} + \frac{8\zeta_2}{3} \right) L - \frac{28L^2}{9} - \frac{4L^3}{3} \right\} + C_F n_l^2 T_F^2 \frac{160}{81} (1-L) \left. \right] \\
& + \frac{1}{\varepsilon} \left[C_F^3 \left\{ -50 - 10\zeta_2 + \frac{236}{5}\zeta_2^2 + 104\zeta_3 + \left(62 + 34\zeta_2 - \frac{236}{5}\zeta_2^2 - 160\zeta_3 \right) L - \left(20 + 18\zeta_2 \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -56\zeta_3)L^2 + \left(\frac{26}{3} - 6\zeta_2\right)L^3 - \frac{7L^4}{3} + \frac{5L^5}{3} \Big\} + C_F^2 C_A \left\{ \frac{1636}{27} - 59\zeta_2 + \frac{26}{5}\zeta_2^2 - \frac{140}{3}\zeta_3 - 4\zeta_2\zeta_3 \right. \\
& + \left(-\frac{2030}{27} + \frac{134}{3}\zeta_2 - \frac{26}{5}\zeta_2^2 + \frac{308}{3}\zeta_3 \right)L + \left(\frac{835}{27} + \frac{37\zeta_2}{3} - 50\zeta_3 \right)L^2 + \left(-\frac{212}{9} + 6\zeta_2 \right)L^3 \\
& + \left. \frac{11L^4}{9} \right\} + C_F C_A^2 \left\{ -\frac{343}{27} + \frac{608}{27}\zeta_2 - \frac{88}{15}\zeta_2^2 - \frac{740}{27}\zeta_3 - \frac{8}{3}\zeta_2\zeta_3 + 12\zeta_5 + \left(\frac{245}{9} - \frac{536}{27}\zeta_2 \right. \right. \\
& + \left. \left. \frac{88}{15}\zeta_2^2 + \frac{44}{9}\zeta_3 \right)L \right\} + C_F C_A n_l T_F \left\{ \frac{356}{81} - \frac{160}{27}\zeta_2 + \frac{496}{27}\zeta_3 + \left(-\frac{836}{81} + \frac{160\zeta_2}{27} - \frac{112\zeta_3}{9} \right)L \right\} \\
& + C_F^2 n_l T_F \left\{ -\frac{38}{27} + 4\zeta_2 - \frac{16}{3}\zeta_3 + \left(\frac{190}{27} + \frac{8\zeta_2}{3} + \frac{16\zeta_3}{3} \right)L + \left(-\frac{332}{27} - \frac{20\zeta_2}{3} \right)L^2 + \frac{64L^3}{9} \right. \\
& - \left. \frac{4L^4}{9} \right\} + C_F n_l^2 T_F^2 \frac{32}{81}(1-L) + \left[X_{S,3}^{0,0} + N_C^3 \left\{ -\frac{133949}{2916} + \frac{6623\zeta_2}{108} - \frac{545\zeta_2^2}{12} + \frac{8156\zeta_2^3}{315} \right. \right. \\
& + \left. \left. \frac{3403\zeta_3}{324} - \frac{16\zeta_3^2}{3} + \frac{397\zeta_3\zeta_2}{6} - \frac{830\zeta_5}{3} \right\} + C_F^3 \left\{ \left(-131 + 14\zeta_2 + 528 \ln(2)\zeta_2 - \frac{616}{5}\zeta_2^2 \right. \right. \right. \\
& - \left. \left. 428\zeta_3 - 8\zeta_2\zeta_3 - 276\zeta_5 \right)L - \left(87 + 79\zeta_2 - \frac{302}{5}\zeta_2^2 - 244\zeta_3 \right)L^2 + \left(\frac{76}{3} + \frac{98\zeta_2}{3} - \frac{268\zeta_3}{3} \right)L^3 \right. \\
& + \left. \left(-\frac{17}{2} + \frac{16\zeta_2}{3} \right)L^4 + L^5 - L^6 \right\} + C_F^2 C_A \left\{ \left(-\frac{8569}{81} - \frac{1091}{9}\zeta_2 - 264 \ln(2)\zeta_2 + \frac{2524}{15}\zeta_2^2 \right. \right. \\
& + \left. \left. \frac{2686}{3}\zeta_3 - 46\zeta_2\zeta_3 - 194\zeta_5 \right)L + \left(\frac{9824}{81} + \frac{175}{6}\zeta_2 - \frac{143}{5}\zeta_2^2 - \frac{2312}{9}\zeta_3 \right)L^2 + \left(-\frac{1208}{27} \right. \right. \\
& - \left. \left. \frac{259\zeta_2}{18} + \frac{232\zeta_3}{3} \right)L^3 + \left(\frac{2309}{108} - \frac{16\zeta_2}{3} \right)L^4 - \frac{11L^5}{4} \right\} + C_F C_A^2 \left\{ \left(\frac{10289}{1458} + \frac{9644}{81}\zeta_2 - \frac{94}{3}\zeta_2^2 \right. \right. \\
& - \left. \left. \frac{13900}{27}\zeta_3 + \frac{88}{3}\zeta_2\zeta_3 + 136\zeta_5 \right)L + \left(-\frac{11939}{162} + \frac{26}{9}\zeta_2 - \frac{44}{5}\zeta_2^2 + 88\zeta_3 \right)L^2 + \left(\frac{1780}{81} \right. \right. \\
& - \left. \left. \frac{44\zeta_2}{9} \right)L^3 - \frac{121L^4}{54} \right\} + C_F C_A n_l T_F \left\{ \frac{35626}{729} - \frac{2680\zeta_2}{81} - \frac{1340\zeta_2^2}{9} + 160\zeta_2 \ln(2) - \frac{19348\zeta_3}{81} \right. \\
& + \left. \frac{8\zeta_3\zeta_2}{3} + \frac{596\zeta_5}{3} + \left(-\frac{14998}{729} - \frac{6544}{81}\zeta_2 + \frac{88}{15}\zeta_2^2 + \frac{1448}{9}\zeta_3 \right)L + \left(\frac{3454}{81} + \frac{32\zeta_2}{3} - 16\zeta_3 \right)L^2 \right. \\
& + \left. \left(-\frac{1156}{81} + \frac{16\zeta_2}{9} \right)L^3 + \frac{44L^4}{27} \right\} + C_F^2 n_l T_F \left\{ \frac{7727}{81} + 22\zeta_2 + \frac{1592}{9}\zeta_2^2 - 320\zeta_2 \ln(2) \right. \\
& + \left. \frac{3328}{9}\zeta_3 - 48\zeta_3\zeta_2 + 40\zeta_5 + \left(\frac{1415}{81} + \frac{820}{9}\zeta_2 - \frac{392}{15}\zeta_2^2 - \frac{1904}{9}\zeta_3 \right)L + \left(-\frac{2635}{81} - \frac{58\zeta_2}{3} \right. \right. \\
& + \left. \left. \frac{232\zeta_3}{9} \right)L^2 + \left(\frac{460}{27} + \frac{58\zeta_2}{9} \right)L^3 - \frac{175L^4}{27} + L^5 \right\} + C_F n_l^2 T_F^2 \left\{ -\frac{112}{729} - \frac{2272\zeta_2}{81} + \frac{928\zeta_2^2}{45} \right. \\
& + \left. \frac{64\zeta_3}{9} + \left(\frac{3712}{729} + \frac{320\zeta_2}{27} + \frac{64\zeta_3}{27} \right)L + \left(-\frac{400}{81} - \frac{32\zeta_2}{9} \right)L^2 + \frac{160L^3}{81} - \frac{8L^4}{27} \right\} \Big]. \quad (27)
\end{aligned}$$

Here we used the abbreviation

$$c_1 = 12\zeta_2 \ln^2(2) + \ln^4(2) + 24\text{Li}_4\left(\frac{1}{2}\right), \quad (28)$$

where $\text{Li}_k(x)$ denotes the polylogarithm [66–68]. The functions $X_{V,3}^{0,0}$ and $X_{S,3}^{0,0}$ are independent of L and contain the sub-leading contributions in N_C which can be obtained through an exact calculation only, including the non-planar color topologies. The leading pole contributions are the same in the vector and scalar cases, also for the four-loop terms given below.

At the four-loop level, we obtain the complete contributions to the $\frac{1}{\varepsilon^4}$, $\frac{1}{\varepsilon^3}$ and $\frac{1}{\varepsilon^2}$ poles, leading color contributions to L and complete $L^2 \dots L^{\text{Max}}$ of the single pole in ε and leading color contributions to L^2 and complete $L^3 \dots L^{\text{Max}}$ for the finite pieces. The yet unknown terms are denoted by $X_{I,n}^{k,j}$ indicating the $\varepsilon^k L^j$ -coefficient of $F_I^{(n)}$. Again the terms $X_{I,n}^{k,j}$ do not contain logarithmic contributions.

A partial prediction of the four-loop vector and scalar form factor is given by

$$\begin{aligned}
F_V^{(4)} = & \frac{1}{\varepsilon^4} \left[C_F^4 \frac{2}{3} (1-L)^4 + C_A C_F^3 \frac{22}{3} (1-L)^3 + C_A^2 C_F^2 \frac{1331}{54} (1-L)^2 + C_A^3 C_F \frac{1331}{54} (1-L) \right. \\
& - C_F^3 n_l T_F \frac{8}{3} (1-L)^3 - C_A C_F^2 n_l T_F \frac{484}{27} (1-L)^2 - C_A^2 C_F n_l T_F \frac{242}{9} (1-L) \\
& \left. + C_F^2 n_l^2 T_F^2 \frac{88}{27} (1-L)^2 + C_A C_F n_l^2 T_F^2 \frac{88}{9} (1-L) - C_F n_l^3 T_F^3 \frac{32}{27} (1-L) \right] \\
& + \frac{1}{\varepsilon^3} \left[C_F^4 \left\{ \left(\frac{16}{3} - \frac{8}{3} \zeta_2 \right) + \left(-20 + 8 \zeta_2 \right) L + \left(\frac{88}{3} - 8 \zeta_2 \right) L^2 + \left(-\frac{64}{3} + \frac{8}{3} \zeta_2 \right) L^3 + 8 L^4 \right. \right. \\
& - \left. \left. \frac{4}{3} L^5 \right\} + C_A C_F^3 \left\{ \left(\frac{166}{9} - 4 \zeta_3 - \frac{32}{3} \zeta_2 \right) + \left(-44 + 8 \zeta_3 + \frac{52}{3} \zeta_2 \right) L + \left(40 - 4 \zeta_3 - \frac{8}{3} \zeta_2 \right) L^2 \right. \right. \\
& - \left. \left. \left(\frac{196}{9} + 4 \zeta_2 \right) L^3 + \frac{22}{3} L^4 \right\} + C_A^2 C_F^2 \left\{ \left(-\frac{2093}{81} - \frac{154}{9} \zeta_3 - \frac{22}{27} \zeta_2 \right) + \left(\frac{6298}{81} + \frac{154}{9} \zeta_3 \right. \right. \\
& - \left. \left. \frac{440}{27} \zeta_2 \right) L + \left(-\frac{3479}{81} + \frac{154}{9} \zeta_2 \right) L^2 - \frac{242}{27} L^3 \right\} + C_A^3 C_F \left\{ \left(-\frac{12661}{162} - \frac{121}{9} \zeta_3 + \frac{121}{9} \zeta_2 \right) \right. \\
& + \left. \left(\frac{14839}{162} - \frac{121}{9} \zeta_2 \right) L \right\} + C_F^3 n_l T_F \left\{ \left(-\frac{8}{9} + \frac{16}{3} \zeta_2 \right) + \left(\frac{16}{3} - \frac{32}{3} \zeta_2 \right) L - \left(\frac{32}{3} - \frac{16}{3} \zeta_2 \right) L^2 \right. \\
& + \left. \frac{80}{9} L^3 - \frac{8}{3} L^4 \right\} + C_A C_F^2 n_l T_F \left\{ \left(\frac{2708}{81} + \frac{56}{9} \zeta_3 + \frac{184}{27} \zeta_2 \right) + \left(-\frac{5260}{81} - \frac{56}{9} \zeta_3 \right. \right. \\
& - \left. \left. \frac{16}{27} \zeta_2 \right) L + \left(\frac{2024}{81} - \frac{56}{9} \zeta_2 \right) L^2 + \frac{176}{27} L^3 \right\} + C_A^2 C_F n_l T_F \left\{ \left(\frac{730}{9} + \frac{88}{9} \zeta_3 - \frac{88}{9} \zeta_2 \right) \right. \\
& + \left. \left(-\frac{818}{9} + \frac{88}{9} \zeta_2 \right) L \right\} + C_F^2 n_l^2 T_F^2 \left\{ \left(-\frac{608}{81} - \frac{64}{27} \zeta_2 \right) + \left(\frac{880}{81} + \frac{64}{27} \zeta_2 \right) L - \frac{176}{81} L^2 \right. \\
& - \left. \frac{32}{27} L^3 \right\} + C_A C_F n_l^2 T_F^2 \left\{ \left(-\frac{664}{27} - \frac{16}{9} \zeta_3 + \frac{16}{9} \zeta_2 \right) + \left(\frac{712}{27} - \frac{16}{9} \zeta_2 \right) L \right\} \\
& + C_F n_l^3 T_F^3 \frac{160}{81} (1-L) \left. \right] + \frac{1}{\varepsilon^2} \left[C_F^4 \left\{ \frac{212}{3} - \frac{232}{3} \zeta_3 + \frac{250}{3} \zeta_2 - \frac{236}{5} \zeta_2^2 - 96 \ln(2) \zeta_2 \right. \right. \\
& + \left. \left. \left(-\frac{551}{3} + 208 \zeta_3 - \frac{524}{3} \zeta_2 + \frac{472}{5} \zeta_2^2 + 192 \ln(2) \zeta_2 \right) L + \left(185 - 184 \zeta_3 + 94 \zeta_2 - \frac{236}{5} \zeta_2^2 \right. \right. \\
& - \left. \left. 96 \ln(2) \zeta_2 \right) L^2 + \left(-\frac{991}{9} + \frac{160}{3} \zeta_3 + \frac{8}{3} \zeta_2 \right) L^3 + \left(48 - \frac{16}{3} \zeta_2 \right) L^4 - \frac{34}{3} L^5 + \frac{13}{9} L^6 \right\} \\
& + C_A C_F^3 \left\{ \frac{188}{27} - 88 \zeta_3 + \frac{1613}{9} \zeta_2 + 8 \zeta_2 \zeta_3 - \frac{1436}{15} \zeta_2^2 - 128 \ln(2) \zeta_2 + \left(\frac{851}{6} + 76 \zeta_3 - \frac{1987}{9} \zeta_2 \right. \right. \\
& - \left. \left. 8 \zeta_2 \zeta_3 + \frac{1574}{15} \zeta_2^2 + 80 \ln(2) \zeta_2 \right) L + \left(-\frac{671}{3} + 60 \zeta_3 + \frac{290}{9} \zeta_2 - \frac{46}{5} \zeta_2^2 + 48 \ln(2) \zeta_2 \right) L^2 \right. \\
& + \left. \left(\frac{5057}{54} - 48 \zeta_3 + \frac{4}{3} \zeta_2 \right) L^3 + \left(-\frac{305}{18} + 8 \zeta_2 \right) L^4 - \frac{11}{6} L^5 \right\} + C_A^2 C_F^2 \left\{ -\frac{10159}{54} + \frac{4700}{27} \zeta_3 \right. \\
& + \left. 2 \zeta_3^2 - \frac{1249}{81} \zeta_2 + \frac{100}{9} \zeta_2 \zeta_3 + \frac{79}{45} \zeta_2^2 + 88 \ln(2) \zeta_2 - 24 \zeta_5 + \left(\frac{47179}{162} - \frac{7778}{27} \zeta_3 + \frac{6595}{81} \zeta_2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{4}{3}\zeta_2\zeta_3 - \frac{697}{45}\zeta_2^2 - 88\ln(2)\zeta_2 + 24\zeta_5 \Big) L + \left(-\frac{8159}{54} + \frac{902}{9}\zeta_3 - \frac{572}{9}\zeta_2 + \frac{206}{15}\zeta_2^2 \right) L^2 \\
& + \left(\frac{9925}{162} - \frac{110}{9}\zeta_2 \right) L^3 - \frac{121}{81}L^4 \Big\} + C_A^3 C_F \left\{ \frac{4981}{54} + \frac{2341}{27}\zeta_3 - \frac{1978}{27}\zeta_2 + \frac{22}{3}\zeta_2\zeta_3 + \frac{242}{15}\zeta_2^2 \right. \\
& - 33\zeta_5 + \left(-\frac{287}{2} - \frac{121}{9}\zeta_3 + \frac{1780}{27}\zeta_2 - \frac{242}{15}\zeta_2^2 \right) L \Big\} + C_F^3 n_l T_F \left\{ -\frac{673}{27} + \frac{208}{3}\zeta_3 - \frac{652}{9}\zeta_2 \right. \\
& + \frac{472}{15}\zeta_2^2 + 64\ln(2)\zeta_2 + \left(\frac{41}{9} - \frac{368}{3}\zeta_3 + \frac{740}{9}\zeta_2 - \frac{472}{15}\zeta_2^2 - 64\ln(2)\zeta_2 \right) L + \left(\frac{392}{9} + \frac{160}{3}\zeta_3 \right. \\
& + \frac{8}{9}\zeta_2 \Big) L^2 + \left(-\frac{758}{27} - \frac{32}{3}\zeta_2 \right) L^3 + \frac{38}{9}L^4 + \frac{2}{3}L^5 \Big\} + C_A C_F^2 n_l T_F \left\{ \frac{5369}{81} - \frac{556}{9}\zeta_3 - \frac{1100}{81}\zeta_2 \right. \\
& - \frac{32}{9}\zeta_2\zeta_3 + \frac{196}{45}\zeta_2^2 - 32\ln(2)\zeta_2 + \left(-\frac{3883}{27} + \frac{1072}{9}\zeta_3 - \frac{1612}{81}\zeta_2 - \frac{196}{45}\zeta_2^2 + 32\ln(2)\zeta_2 \right) L \\
& + \left(\frac{9512}{81} - \frac{520}{9}\zeta_3 + \frac{880}{27}\zeta_2 \right) L^2 + \left(-\frac{3284}{81} + \frac{40}{9}\zeta_2 \right) L^3 + \frac{88}{81}L^4 \Big\} \\
& + C_A^2 C_F n_l T_F \left\{ -\frac{13241}{162} - \frac{2284}{27}\zeta_3 + \frac{1228}{27}\zeta_2 - \frac{8}{3}\zeta_2\zeta_3 - \frac{88}{15}\zeta_2^2 + 12\zeta_5 + \left(\frac{19313}{162} + \frac{352}{9}\zeta_3 \right. \right. \\
& - \frac{1156}{27}\zeta_2 + \frac{88}{15}\zeta_2^2 \Big) L \Big\} + C_F^2 n_l^2 T_F^2 \left\{ \frac{116}{81} - \frac{224}{27}\zeta_3 + \frac{464}{81}\zeta_2 + \left(\frac{1132}{81} + \frac{224}{27}\zeta_3 - \frac{80}{81}\zeta_2 \right) L \right. \\
& + \left(-\frac{1720}{81} - \frac{128}{27}\zeta_2 \right) L^2 + \frac{488}{81}L^3 - \frac{16}{81}L^4 \Big\} + C_A C_F n_l^2 T_F^2 \left\{ \frac{1105}{81} + \frac{496}{27}\zeta_3 - \frac{160}{27}\zeta_2 \right. \\
& + \left(-\frac{1585}{81} - \frac{112}{9}\zeta_3 + \frac{160}{27}\zeta_2 \right) L \Big\} + C_F n_l^3 T_F^3 \frac{32}{81}(1-L) \Big] + \frac{1}{\varepsilon} \left[X_{V,4}^{-1,0} + X_{V,4}^{-1,1} L \right. \\
& + N_C^4 \left(-\frac{2073967}{11664} - \frac{441}{2}\zeta_5 + \frac{178303}{648}\zeta_3 - \frac{1}{3}\zeta_2^2 + \frac{1141}{54}\zeta_2 + \frac{118}{9}\zeta_2\zeta_3 - \frac{12142}{135}\zeta_2^2 \right. \\
& + \frac{1013}{45}\zeta_2^3 \Big) L + N_C^3 n_l T_F \left(\frac{495349}{2916} + 82\zeta_5 - \frac{2767}{18}\zeta_3 + \frac{2531}{162}\zeta_2 - \frac{32}{9}\zeta_2\zeta_3 + \frac{694}{45}\zeta_2^2 \right) L \\
& + C_F^4 \left\{ \left(\frac{1972}{3} + 16c_1 - 516\zeta_5 - \frac{3244}{3}\zeta_3 + \frac{2974}{3}\zeta_2 + 8\zeta_2\zeta_3 - 504\zeta_2^2 - 432\ln(2)\zeta_2 \right) L^2 \right. \\
& + \left(-\frac{3025}{6} + \frac{4928}{9}\zeta_3 - \frac{638}{3}\zeta_2 + \frac{1232}{15}\zeta_2^2 + 96\ln(2)\zeta_2 \right) L^3 + \left(\frac{1942}{9} - \frac{992}{9}\zeta_3 + \frac{22}{3}\zeta_2 \right) L^4 \\
& + \left(-62 + 6\zeta_2 \right) L^5 + \frac{32}{3}L^6 - \frac{10}{9}L^7 \Big\} + C_A C_F^3 \left\{ \left(-\frac{80599}{36} - 8c_1 - 74\zeta_5 + \frac{7045}{3}\zeta_3 \right. \right. \\
& - \frac{15098}{27}\zeta_2 - 132\zeta_2\zeta_3 + \frac{760}{3}\zeta_2^2 + 216\ln(2)\zeta_2 \Big) L^2 + \left(\frac{101510}{81} - \frac{5404}{9}\zeta_3 + \frac{526}{9}\zeta_2 - 18\zeta_2^2 \right. \\
& - 48\ln(2)\zeta_2 \Big) L^3 + \left(-\frac{18415}{54} + \frac{305}{3}\zeta_3 + \frac{58}{9}\zeta_2 \right) L^4 + \left(\frac{161}{3} - 9\zeta_2 \right) L^5 - \frac{22}{9}L^6 \Big\} \\
& + C_A^2 C_F^2 \left\{ \left(\frac{3887225}{2916} + 260\zeta_5 - \frac{99733}{81}\zeta_3 + \frac{14699}{81}\zeta_2 + \frac{328}{3}\zeta_2\zeta_3 + \frac{1232}{45}\zeta_2^2 \right) L^2 \right. \\
& + \left(-\frac{190591}{486} + \frac{2024}{27}\zeta_3 + \frac{4829}{54}\zeta_2 - \frac{412}{15}\zeta_2^2 \right) L^3 + \left(\frac{38237}{972} - \frac{154}{27}\zeta_2 \right) L^4 - \frac{605}{324}L^5 \Big\} \\
& + C_F^3 n_l T_F \left\{ \left(\frac{9925}{27} - \frac{2560}{9}\zeta_3 + \frac{4120}{27}\zeta_2 + \frac{104}{15}\zeta_2^2 \right) L^2 + \left(-\frac{26932}{81} - \frac{64}{9}\zeta_3 - \frac{428}{9}\zeta_2 \right) L^3 \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2902}{27} + \frac{100}{9} \zeta_2 \right) L^4 - \frac{52}{3} L^5 + \frac{8}{9} L^6 \Big\} + C_A C_F^2 n_l T_F \Big\{ \left(-\frac{519812}{729} + \frac{17068}{81} \zeta_3 - \frac{1912}{9} \zeta_2 \right. \\
& - \frac{208}{45} \zeta_2^2 \Big) L^2 + \left(\frac{55448}{243} + \frac{416}{27} \zeta_3 - \frac{400}{27} \zeta_2 \right) L^3 + \left(-\frac{6446}{243} + \frac{56}{27} \zeta_2 \right) L^4 + \frac{110}{81} L^5 \Big\} \\
& + C_F^2 n_l^2 T_F^2 \Big\{ \left(-\frac{47536}{729} - \frac{1184}{81} \zeta_3 - \frac{1720}{27} \zeta_2 + \frac{3424}{135} \zeta_2^2 \right) L + \left(\frac{52316}{729} + \frac{512}{81} \zeta_3 + \frac{3568}{81} \zeta_2 \right) L^2 \\
& + \left(-\frac{6536}{243} - \frac{104}{27} \zeta_2 \right) L^3 + \frac{980}{243} L^4 - \frac{20}{81} L^5 \Big\} + C_A C_F n_l^2 T_F^2 \Big\{ \left(\frac{923}{162} + \frac{1120}{27} \zeta_3 - \frac{304}{81} \zeta_2 \right. \\
& - \frac{112}{15} \zeta_2^2 \Big) L \Big\} + C_F n_l^3 T_F^3 \Big\{ \left(-\frac{32}{81} + \frac{64}{27} \zeta_3 \right) L \Big\} + \left[X_{V,4}^{0,0} + X_{V,4}^{0,1} L + X_{V,4}^{0,2} L^2 + N_C^4 \left(\frac{201483937}{279936} \right. \right. \\
& - \frac{149}{3} \zeta_5 - \frac{308201}{486} \zeta_3 + \frac{1193}{12} \zeta_3^2 + \frac{2972341}{7776} \zeta_2 + \frac{839}{108} \zeta_2 \zeta_3 - \frac{3451}{80} \zeta_2^2 + \frac{761}{36} \zeta_2^3 \Big) L^2 \\
& + N_C^3 n_l T_F \left(-\frac{2276657}{34992} + \frac{314}{3} \zeta_5 + \frac{7945}{243} \zeta_3 - \frac{93347}{486} \zeta_2 - \frac{55}{27} \zeta_2 \zeta_3 - \frac{12979}{540} \zeta_2^2 \right) L^2 + C_F^4 \Big\{ \left(\right. \\
& - \frac{16252}{9} - 16c_1 + 1060\zeta_5 + \frac{27328}{9} \zeta_3 - \frac{16273}{9} \zeta_2 - \frac{1640}{9} \zeta_2 \zeta_3 + \frac{9596}{15} \zeta_2^2 + 368 \ln(2) \zeta_2 \Big) L^3 \\
& + \left(\frac{20759}{24} - 822\zeta_3 + \frac{571}{2} \zeta_2 - \frac{299}{3} \zeta_2^2 - 56 \ln(2) \zeta_2 \right) L^4 + \left(-\frac{47299}{180} + \frac{1144}{9} \zeta_3 - \frac{38}{3} \zeta_2 \right) L^5 \\
& + \left(\frac{2518}{45} - \frac{44}{9} \zeta_2 \right) L^6 - \frac{343}{45} L^7 + \frac{27}{40} L^8 \Big\} + C_A C_F^3 \Big\{ \left(\frac{3658325}{486} + 8c_1 - 370\zeta_5 - \frac{188444}{27} \zeta_3 \right. \\
& + \frac{60428}{27} \zeta_2 + 400\zeta_2 \zeta_3 - \frac{60979}{90} \zeta_2^2 - \frac{728}{3} \ln(2) \zeta_2 \Big) L^3 + \left(-\frac{1588517}{648} + \frac{10637}{9} \zeta_3 - \frac{4825}{27} \zeta_2 \right. \\
& + \frac{1531}{30} \zeta_2^2 + 28 \ln(2) \zeta_2 \Big) L^4 + \left(\frac{176119}{360} - 116\zeta_3 - 13\zeta_2 \right) L^5 + \left(-\frac{33517}{540} + \frac{22}{3} \zeta_2 \right) L^6 \\
& + \frac{209}{60} L^7 \Big\} + C_A^2 C_F^2 \Big\{ \left(-\frac{4578454}{729} - 540\zeta_5 + \frac{38435}{9} \zeta_3 - \frac{299959}{324} \zeta_2 - \frac{1976}{9} \zeta_2 \zeta_3 + \frac{6752}{45} \zeta_2^2 \right. \\
& + \frac{88}{3} \ln(2) \zeta_2 \Big) L^3 + \left(\frac{1220761}{972} - \frac{6413}{18} \zeta_3 - \frac{691}{9} \zeta_2 + \frac{1486}{45} \zeta_2^2 \right) L^4 - \left(\frac{1363133}{9720} - \frac{1001}{54} \zeta_2 \right) L^5 \\
& + \frac{3751}{540} L^6 \Big\} + C_A^3 C_F \Big\{ \left(\frac{467107}{486} - \frac{968}{3} \zeta_3 - \frac{694}{27} \zeta_2 + \frac{484}{15} \zeta_2^2 \right) L^3 + \left(-\frac{33803}{324} + \frac{121}{9} \zeta_2 \right) L^4 \\
& + \frac{1331}{270} L^5 \Big\} + C_F^3 n_l T_F \Big\{ \left(-\frac{427058}{243} + \frac{30664}{27} \zeta_3 - \frac{19252}{27} \zeta_2 + \frac{290}{9} \zeta_2^2 + \frac{64}{3} \ln(2) \zeta_2 \right) L^3 \\
& + \left(\frac{115585}{162} - \frac{896}{9} \zeta_3 + \frac{2660}{27} \zeta_2 \right) L^4 + \left(-\frac{14269}{90} - 8\zeta_2 \right) L^5 + \frac{2807}{135} L^6 - \frac{19}{15} L^7 \Big\} \\
& + C_A C_F^2 n_l T_F \Big\{ \left(\frac{2759932}{729} - \frac{98272}{81} \zeta_3 + \frac{64054}{81} \zeta_2 - \frac{160}{3} \zeta_2^2 - \frac{32}{3} \ln(2) \zeta_2 \right) L^3 \\
& + \left(-\frac{21805}{27} + 78\zeta_3 - \frac{1354}{81} \zeta_2 \right) L^4 + \left(\frac{116653}{1215} - \frac{182}{27} \zeta_2 \right) L^5 - \frac{682}{135} L^6 \Big\} \\
& + C_A^2 C_F n_l T_F \Big\{ \left(-\frac{25553}{27} + 176\zeta_3 - \frac{712}{27} \zeta_2 - \frac{176}{15} \zeta_2^2 \right) L^3 + \left(\frac{2920}{27} - \frac{88}{9} \zeta_2 \right) L^4 \\
& - \frac{242}{45} L^5 \Big\} + C_F^2 n_l^2 T_F^2 \Big\{ \left(\frac{2471390}{2187} - \frac{60640}{243} \zeta_3 + \frac{54284}{81} \zeta_2 + \frac{3596}{135} \zeta_2^2 \right) L^2
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{370744}{729} + \frac{1616}{81}\zeta_3 - \frac{12172}{81}\zeta_2 \right) L^3 + \left(\frac{28904}{243} + \frac{1064}{81}\zeta_2 \right) L^4 - \frac{18826}{1215} L^5 + \frac{124}{135} L^6 \Big\} \\
& + C_A C_F n_l^2 T_F^2 \left\{ \left(\frac{8996449}{2916} - \frac{2464}{9}\zeta_5 - \frac{86416}{81}\zeta_3 + \frac{459772}{243}\zeta_2 + \frac{608}{9}\zeta_2\zeta_3 + \frac{1232}{15}\zeta_2^2 \right) L \right. \\
& + \left(-\frac{308686}{243} + \frac{2032}{9}\zeta_3 - \frac{11440}{27}\zeta_2 + \frac{80}{3}\zeta_2^2 \right) L^2 + \left(\frac{23302}{81} - \frac{64}{3}\zeta_3 + \frac{736}{27}\zeta_2 \right) L^3 \\
& + \left(-\frac{968}{27} + \frac{16}{9}\zeta_2 \right) L^4 + \frac{88}{45} L^5 \Big\} + C_F n_l^3 T_F^3 \left\{ \left(-\frac{558992}{2187} - \frac{320}{81}\zeta_3 - \frac{12992}{81}\zeta_2 - \frac{832}{45}\zeta_2^2 \right) L \right. \\
& + \left. \left(\frac{79280}{729} + \frac{1216}{27}\zeta_2 \right) L^2 + \left(-\frac{6496}{243} - \frac{128}{27}\zeta_2 \right) L^3 + \frac{304}{81} L^4 - \frac{32}{135} L^5 \right\}. \tag{29}
\end{aligned}$$

$$\begin{aligned}
F_S^4 = & \frac{1}{\varepsilon^4} \left[C_F^4 \frac{2}{3} (1-L)^4 + C_A C_F^3 \frac{22}{3} (1-L)^3 + C_A^2 C_F^2 \frac{1331}{54} (1-L)^2 + C_A^3 C_F \frac{1331}{54} (1-L) \right. \\
& - C_F^3 n_l T_F \frac{8}{3} (1-L)^3 - C_A C_F^2 n_l T_F \frac{484}{27} (1-L)^2 - C_A^2 C_F n_l T_F \frac{242}{9} (1-L) \\
& + \left. C_F^2 n_l^2 T_F^2 \frac{88}{27} (1-L)^2 + C_A C_F n_l^2 T_F^2 \frac{88}{9} (1-L) - C_F n_l^3 T_F^3 \frac{32}{27} (1-L) \right] \\
& + \frac{1}{\varepsilon^3} \left[C_F^4 \left\{ \frac{8}{3} - \frac{8}{3}\zeta_2 - 8(1-\zeta_2)L + \left(\frac{28}{3} - 8\zeta_2 \right) L^2 + \left(-\frac{20}{3} + \frac{8}{3}\zeta_2 \right) L^3 + 4L^4 - \frac{4}{3}L^5 \right\} \right. \\
& + C_F^3 C_A \left\{ \frac{34}{9} - 4\zeta_3 - \frac{32}{3}\zeta_2 + \left(\frac{22}{3} + 8\zeta_3 + \frac{52}{3}\zeta_2 \right) L - \left(\frac{56}{3} + 4\zeta_3 + \frac{8}{3}\zeta_2 \right) L^2 + \left(\frac{2}{9} - 4\zeta_2 \right) L^3 \right. \\
& + \left. \frac{22}{3} L^4 \right\} + C_F^2 C_A^2 \left\{ -\frac{3545}{81} - \frac{154}{9}\zeta_3 - \frac{22}{27}\zeta_2 + \left(\frac{9928}{81} + \frac{154}{9}\zeta_3 - \frac{440}{27}\zeta_2 \right) L + \left(-\frac{5657}{81} \right. \right. \\
& + \left. \left. \frac{154}{9}\zeta_2 \right) L^2 - \frac{242}{27} L^3 \right\} + C_F C_A^3 \left\{ -\frac{12661}{162} - \frac{121}{9}\zeta_3 + \frac{121}{9}\zeta_2 + \left(\frac{14839}{162} - \frac{121}{9}\zeta_2 \right) L \right\} \\
& + C_F^3 n_l T_F \left\{ \frac{40}{9} + \frac{16}{3}\zeta_2 + \left(-\frac{40}{3} - \frac{32}{3}\zeta_2 \right) L + \left(\frac{32}{3} + \frac{16}{3}\zeta_2 \right) L^2 + \frac{8}{9} L^3 - \frac{8}{3} L^4 \right\} \\
& + C_F^2 C_A n_l T_F \left\{ \frac{3764}{81} + \frac{56}{9}\zeta_3 + \frac{184}{27}\zeta_2 + \left(-\frac{7900}{81} - \frac{56}{9}\zeta_3 - \frac{16}{27}\zeta_2 \right) L + \left(\frac{3608}{81} - \frac{56}{9}\zeta_2 \right) L^2 \right. \\
& + \left. \frac{176}{27} L^3 \right\} + C_F C_A^2 n_l T_F \left\{ \frac{730}{9} + \frac{88}{9}\zeta_3 - \frac{88}{9}\zeta_2 + \left(-\frac{818}{9} + \frac{88}{9}\zeta_2 \right) L \right\} + C_F^2 n_l^2 T_F^2 \left\{ -\frac{800}{81} \right. \\
& - \frac{64}{27}\zeta_2 + \left(\frac{1360}{81} + \frac{64}{27}\zeta_2 \right) L - \frac{464}{81} L^2 - \frac{32}{27} L^3 \Big\} + C_F C_A n_l^2 T_F^2 \left\{ -\frac{664}{27} - \frac{16}{9}\zeta_3 + \frac{16}{9}\zeta_2 \right. \\
& + \left. \left(\frac{712}{27} - \frac{16}{9}\zeta_2 \right) L \right\} + C_F n_l^3 T_F^3 \left\{ \frac{160}{81} - \frac{160}{81} L \right\} \Big] + \frac{1}{\varepsilon^2} \left[C_F^4 \left\{ \frac{142}{3} - \frac{304}{3}\zeta_3 + \frac{28}{3}\zeta_2 - \frac{236}{5}\zeta_2^2 \right. \right. \\
& + \left. \left(-\frac{308}{3} + 256\zeta_3 - \frac{128}{3}\zeta_2 + \frac{472}{5}\zeta_2^2 \right) L + \left(70 - 208\zeta_3 + 52\zeta_2 - \frac{236}{5}\zeta_2^2 \right) L^2 + \left(-\frac{196}{9} \right. \right. \\
& + \left. \frac{160}{3}\zeta_3 - \frac{40}{3}\zeta_2 \right) L^3 + \left(9 - \frac{16}{3}\zeta_2 \right) L^4 - \frac{10}{3} L^5 + \frac{13}{9} L^6 \Big\} + C_F^3 C_A \left\{ \frac{545}{27} - 148\zeta_3 + \frac{830}{9}\zeta_2 \right. \\
& + 8\zeta_2\zeta_3 - \frac{1436}{15}\zeta_2^2 + \left(\frac{431}{9} + 148\zeta_3 - \frac{1798}{9}\zeta_2 - 8\zeta_2\zeta_3 + \frac{1574}{15}\zeta_2^2 \right) L + \left(-\frac{808}{9} + 48\zeta_3 \right. \\
& + \left. \frac{776}{9}\zeta_2 - \frac{46}{5}\zeta_2^2 \right) L^2 + \left(\frac{1390}{27} - 48\zeta_3 + \frac{40}{3}\zeta_2 \right) L^3 + \left(-\frac{503}{18} + 8\zeta_2 \right) L^4 - \frac{11}{6} L^5 \Big\}
\end{aligned}$$

$$\begin{aligned}
& + C_F^2 C_A^2 \left\{ -\frac{13553}{162} - 24\zeta_5 + \frac{3776}{27}\zeta_3 + 2\zeta_3^2 + \frac{6374}{81}\zeta_2 + \frac{100}{9}\zeta_2\zeta_3 + \frac{79}{45}\zeta_2^2 + \left(\frac{254}{9} + 24\zeta_5 \right. \right. \\
& - \left. \frac{6986}{27}\zeta_3 + \frac{160}{81}\zeta_2 - \frac{4}{3}\zeta_2\zeta_3 - \frac{697}{45}\zeta_2^2 \right) L + \left(\frac{343}{18} + \frac{902}{9}\zeta_3 - \frac{704}{9}\zeta_2 + \frac{206}{15}\zeta_2^2 \right) L^2 + \left(\frac{4418}{81} \right. \\
& - \left. \frac{110}{9}\zeta_2 \right) L^3 - \frac{121}{81} L^4 \left. \right\} + C_F C_A^3 \left\{ \frac{4981}{54} - 33\zeta_5 + \frac{2341}{27}\zeta_3 - \frac{1978}{27}\zeta_2 + \frac{22}{3}\zeta_2\zeta_3 + \frac{242}{15}\zeta_2^2 \right. \\
& + \left. \left(-\frac{287}{2} - \frac{121}{9}\zeta_3 + \frac{1780}{27}\zeta_2 - \frac{242}{15}\zeta_2^2 \right) L \right\} + C_F^3 n_l T_F \left\{ -\frac{901}{27} + \frac{256}{3}\zeta_3 - \frac{184}{9}\zeta_2 + \frac{472}{15}\zeta_2^2 \right. \\
& + \left. \left(\frac{379}{9} - \frac{416}{3}\zeta_3 + \frac{344}{9}\zeta_2 - \frac{472}{15}\zeta_2^2 \right) L + \left(\frac{28}{9} + \frac{160}{3}\zeta_3 - \frac{64}{9}\zeta_2 \right) L^2 + \left(-\frac{560}{27} - \frac{32}{3}\zeta_2 \right) L^3 \right. \\
& + \left. \frac{74}{9} L^4 + \frac{2}{3} L^5 \right\} + C_F^2 C_A n_l T_F \left\{ -\frac{35}{9} - \frac{148}{3}\zeta_3 - \frac{3476}{81}\zeta_2 - \frac{32}{9}\zeta_2\zeta_3 + \frac{196}{45}\zeta_2^2 + \left(\frac{2219}{81} \right. \right. \\
& + \left. \frac{976}{9}\zeta_3 + \frac{332}{81}\zeta_2 - \frac{196}{45}\zeta_2^2 \right) L + \left(\frac{788}{81} - \frac{520}{9}\zeta_3 + \frac{1024}{27}\zeta_2 \right) L^2 + \left(-\frac{2888}{81} + \frac{40}{9}\zeta_2 \right) L^3 \\
& + \left. \frac{88}{81} L^4 \right\} + C_F C_A^2 n_l T_F \left\{ -\frac{13241}{162} + 12\zeta_5 - \frac{2284}{27}\zeta_3 + \frac{1228}{27}\zeta_2 - \frac{8}{3}\zeta_2\zeta_3 - \frac{88}{15}\zeta_2^2 \right. \\
& + \left. \left(\frac{19313}{162} + \frac{352}{9}\zeta_3 - \frac{1156}{27}\zeta_2 + \frac{88}{15}\zeta_2^2 \right) L \right\} + C_F^2 n_l^2 T_F^2 \left\{ \frac{964}{81} - \frac{224}{27}\zeta_3 + \frac{320}{81}\zeta_2 - \left(\frac{844}{81} \right. \right. \\
& - \left. \frac{224}{27}\zeta_3 - \frac{64}{81}\zeta_2 \right) L + \left(-\frac{520}{81} - \frac{128}{27}\zeta_2 \right) L^2 + \frac{416}{81} L^3 - \frac{16}{81} L^4 \left. \right\} + C_F C_A n_l^2 T_F^2 \left\{ \frac{1105}{81} + \frac{496}{27}\zeta_3 \right. \\
& - \left. \frac{160}{27}\zeta_2 + \left(-\frac{1585}{81} - \frac{112}{9}\zeta_3 + \frac{160}{27}\zeta_2 \right) L \right\} + C_F n_l^3 T_F^3 \left\{ \frac{32}{81} - \frac{32}{81} L \right\} + \frac{1}{\varepsilon} \left[X_{S,4}^{-1,0} + X_{S,4}^{-1,1} L \right. \\
& + N_C^4 \left(\frac{598439}{11664} - \frac{429}{2}\zeta_5 + \frac{42851}{324}\zeta_3 - \frac{1}{3}\zeta_3^2 + \frac{1327}{108}\zeta_2 + \frac{487}{9}\zeta_2\zeta_3 - \frac{40243}{540}\zeta_2^2 + \frac{1013}{45}\zeta_2^3 \right) L \\
& + N_C^3 n_l T_F \left(-\frac{117913}{5832} + 82\zeta_5 - \frac{2021}{18}\zeta_3 + \frac{6017}{162}\zeta_2 - \frac{32}{9}\zeta_2\zeta_3 - \frac{317}{45}\zeta_2^2 \right) L \\
& + C_F^4 \left\{ \left(-\frac{293}{3} - 516\zeta_5 - \frac{2068}{3}\zeta_3 + \frac{424}{3}\zeta_2 + 8\zeta_2\zeta_3 - 168\zeta_2^2 + 528 \ln(2)\zeta_2 \right) L^2 + \left(-118 \right. \right. \\
& + \left. \frac{3560}{9}\zeta_3 - \frac{388}{3}\zeta_2 + \frac{1232}{15}\zeta_2^2 \right) L^3 + \left(\frac{271}{9} - \frac{992}{9}\zeta_3 + \frac{118}{3}\zeta_2 \right) L^4 + \left(-\frac{29}{3} + 6\zeta_2 \right) L^5 + 2L^6 \\
& - \left. \frac{10}{9} L^7 \right\} - C_F^3 C_A \left\{ \left(\frac{5315}{27} + 74\zeta_5 - \frac{5002}{3}\zeta_3 + \frac{8681}{27}\zeta_2 + 132\zeta_2\zeta_3 - \frac{2342}{15}\zeta_2^2 + 264 \ln(2)\zeta_2 \right) L^2 \right. \\
& - \left. \left(\frac{16982}{81} - \frac{3244}{9}\zeta_3 - \frac{140}{9}\zeta_2 - 18\zeta_2^2 \right) L^3 + \left(\frac{4225}{54} - \frac{305}{3}\zeta_3 + \frac{158}{9}\zeta_2 \right) L^4 - \left(\frac{223}{6} - 9\zeta_2 \right) L^5 \right. \\
& + \left. \frac{22}{9} L^6 \right\} + C_F^2 C_A^2 \left\{ \left(\frac{96527}{729} + 260\zeta_5 - \frac{90526}{81}\zeta_3 + \frac{30721}{162}\zeta_2 + \frac{328}{3}\zeta_2\zeta_3 + \frac{88}{9}\zeta_2^2 \right) L^2 \right. \\
& + \left. \left(-\frac{53009}{243} + \frac{2024}{27}\zeta_3 + \frac{4631}{54}\zeta_2 - \frac{412}{15}\zeta_2^2 \right) L^3 + \left(\frac{27347}{972} - \frac{154}{27}\zeta_2 \right) L^4 - \frac{605}{324} L^5 \right\} \\
& + C_F^3 n_l T_F \left\{ \left(-\frac{1018}{27} - \frac{3280}{9}\zeta_3 + \frac{4084}{27}\zeta_2 + \frac{104}{15}\zeta_2^2 \right) L^2 + \left(-\frac{2272}{81} - \frac{64}{9}\zeta_3 - \frac{176}{9}\zeta_2 \right) L^3 \right. \\
& + \left. \left(\frac{754}{27} + \frac{100}{9}\zeta_2 \right) L^4 - \frac{34}{3} L^5 + \frac{8}{9} L^6 \right\} + C_F^2 C_A n_l T_F \left\{ \left(-\frac{59948}{729} + \frac{24088}{81}\zeta_3 - \frac{524}{3}\zeta_2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{208}{45}\zeta_2^2)L^2 + \left(\frac{27944}{243} + \frac{416}{27}\zeta_3 - \frac{364}{27}\zeta_2\right)L^3 + \left(-\frac{4466}{243} + \frac{56}{27}\zeta_2\right)L^4 + \frac{110}{81}L^5\left\{ \right. \\
& + C_F^2 n_l^2 T_F^2 \left\{ \left(\frac{18740}{729} - \frac{2624}{81}\zeta_3 - \frac{1840}{27}\zeta_2 + \frac{3424}{135}\zeta_2^2\right)L + \left(\frac{5984}{729} + \frac{512}{81}\zeta_3 + \frac{2632}{81}\zeta_2\right)L^2 \right. \\
& + \left(-\frac{2576}{243} - \frac{104}{27}\zeta_2\right)L^3 + \frac{620}{243}L^4 - \frac{20}{81}L^5\left. \right\} + C_F C_A n_l^2 T_F^2 \left\{ \left(\frac{923}{162} + \frac{1120}{27}\zeta_3 - \frac{304}{81}\zeta_2 \right. \right. \\
& \left. \left. - \frac{112}{15}\zeta_2^2\right)L\right\} + C_F n_l^3 T_F^3 \left\{ \left(-\frac{32}{81} + \frac{64}{27}\zeta_3\right)L\right\} + \left[X_{S,4}^{0,0} + X_{S,4}^{0,1}L + X_{S,4}^{0,2}L^2 + N_C^4 \left\{ \left(\right. \right. \right. \\
& \left. \left. - \frac{31524563}{69984} - \frac{455}{3}\zeta_5 + \frac{194209}{1944}\zeta_3 + \frac{1193}{12}\zeta_3^2 + \frac{26833}{486}\zeta_2 - \frac{187}{108}\zeta_2\zeta_3 - \frac{7343}{120}\zeta_2^2 + \frac{761}{36}\zeta_2^3\right)L^2\right\} \\
& + N_C^3 n_l T_F \left\{ \left(\frac{472810}{2187} + \frac{314}{3}\zeta_5 - \frac{23231}{243}\zeta_3 + \frac{23}{243}\zeta_2 - \frac{55}{27}\zeta_2\zeta_3 - \frac{271}{540}\zeta_2^2\right)L^2\right\} + C_F^4 \left\{ \left(\frac{239}{9} \right. \right. \\
& + 1060\zeta_5 + \frac{11132}{9}\zeta_3 - \frac{3964}{9}\zeta_2 - \frac{1640}{9}\zeta_2\zeta_3 + \frac{2852}{15}\zeta_2^2 - 528\ln(2)\zeta_2\left. \right\} L^3 + \left(\frac{275}{2} - \frac{1388}{3}\zeta_3 \right. \\
& + 171\zeta_2 - \frac{299}{3}\zeta_2^2\left. \right) L^4 + \left(-\frac{1321}{45} + \frac{1144}{9}\zeta_3 - \frac{146}{3}\zeta_2\right)L^5 + \left(\frac{731}{90} - \frac{44}{9}\zeta_2\right)L^6 - \frac{43}{45}L^7 \\
& + \frac{27}{40}L^8\left. \right\} + C_F^3 C_A \left\{ \left(\frac{133270}{243} - 370\zeta_5 - \frac{116066}{27}\zeta_3 + \frac{36173}{27}\zeta_2 + 400\zeta_2\zeta_3 - \frac{40567}{90}\zeta_2^2 \right. \right. \\
& + 264\ln(2)\zeta_2\left. \right\} L^3 + \left(-\frac{49319}{162} + \frac{6470}{9}\zeta_3 - \frac{3502}{27}\zeta_2 + \frac{1531}{30}\zeta_2^2\right)L^4 + \left(\frac{7771}{90} - 116\zeta_3 \right. \\
& + 14\zeta_2\left. \right) L^5 + \left(-\frac{16687}{540} + \frac{22}{3}\zeta_2\right)L^6 + \frac{209}{60}L^7\left. \right\} + C_F^2 C_A^2 \left\{ \left(-\frac{411373}{729} - 540\zeta_5 + \frac{28843}{9}\zeta_3 \right. \right. \\
& \left. \left. - \frac{52132}{81}\zeta_2 - \frac{1976}{9}\zeta_2\zeta_3 + \frac{8336}{45}\zeta_2^2\right)L^3 + \left(\frac{376009}{972} - \frac{6413}{18}\zeta_3 - \frac{383}{9}\zeta_2 + \frac{1486}{45}\zeta_2^2\right)L^4 \right. \\
& + \left(-\frac{203297}{2430} + \frac{1001}{54}\zeta_2\right)L^5 + \frac{3751}{540}L^6\left. \right\} + C_F C_A^3 \left\{ \left(\frac{83618}{243} - \frac{968}{3}\zeta_3 - \frac{694}{27}\zeta_2 + \frac{484}{15}\zeta_2^2\right)L^3 \right. \\
& + \left(-\frac{5456}{81} + \frac{121}{9}\zeta_2\right)L^4 + \frac{1331}{270}L^5\left. \right\} + C_F^3 n_l T_F \left\{ \left(-\frac{13454}{243} + \frac{30448}{27}\zeta_3 - \frac{11980}{27}\zeta_2 \right. \right. \\
& + \frac{290}{9}\zeta_2^2\left. \right) L^3 + \left(\frac{5243}{81} - \frac{896}{9}\zeta_3 + \frac{1724}{27}\zeta_2\right)L^4 + \left(-\frac{1352}{45} - 8\zeta_2\right)L^5 + \frac{1277}{135}L^6 - \frac{19}{15}L^7\left. \right\} \\
& + C_F^2 C_A n_l T_F \left\{ \left(\frac{269308}{729} - \frac{87616}{81}\zeta_3 + \frac{40528}{81}\zeta_2 - \frac{160}{3}\zeta_2^2\right)L^3 + \left(-\frac{18701}{81} + 78\zeta_3 \right. \right. \\
& \left. \left. - \frac{2362}{81}\zeta_2\right)L^4 + \left(\frac{66658}{1215} - \frac{182}{27}\zeta_2\right)L^5 - \frac{682}{135}L^6\right\} + C_F C_A^2 n_l T_F \left\{ \left(-\frac{8693}{27} + 176\zeta_3 - \frac{712}{27}\zeta_2 \right. \right. \\
& \left. \left. - \frac{176}{15}\zeta_2^2\right)L^3 + \left(\frac{1831}{27} - \frac{88}{9}\zeta_2\right)L^4 - \frac{242}{45}L^5\right\} + C_F^2 n_l^2 T_F^2 \left\{ \left(-\frac{47206}{2187} - \frac{78928}{243}\zeta_3 + \frac{22760}{81}\zeta_2 \right. \right. \\
& + \frac{3596}{135}\zeta_2^2\left. \right) L^2 + \left(-\frac{37744}{729} + \frac{1616}{81}\zeta_3 - \frac{7168}{81}\zeta_2\right)L^3 + \left(\frac{7256}{243} + \frac{1064}{81}\zeta_2\right)L^4 - \frac{9736}{1215}L^5 \\
& + \frac{124}{135}L^6\left. \right\} + C_F C_A n_l^2 T_F^2 \left\{ \left(-\frac{823055}{2916} + 96\zeta_4 - \frac{2464}{9}\zeta_5 - \frac{68272}{81}\zeta_3 + \frac{146572}{243}\zeta_2 + \frac{608}{9}\zeta_2\zeta_3 \right. \right. \\
& + \frac{160}{3}\zeta_2^2\left. \right) L + \left(-\frac{26293}{243} + \frac{2032}{9}\zeta_3 - \frac{6400}{27}\zeta_2 + \frac{80}{3}\zeta_2^2\right)L^2 + \left(\frac{7030}{81} - \frac{64}{3}\zeta_3 + \frac{736}{27}\zeta_2\right)L^3
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{572}{27} + \frac{16}{9}\zeta_2 \right) L^4 + \frac{88}{45} L^5 \Big\} + C_F n_l^3 T_F^3 \left\{ \left(-\frac{20864}{2187} - \frac{320}{81}\zeta_3 - \frac{3200}{81}\zeta_2 - \frac{832}{45}\zeta_2^2 \right) L \right. \\
& \left. + \left(\frac{8000}{729} + \frac{640}{27}\zeta_2 \right) L^2 + \left(-\frac{1600}{243} - \frac{128}{27}\zeta_2 \right) L^3 + \frac{160}{81} L^4 - \frac{32}{135} L^5 \right\}. \tag{30}
\end{aligned}$$

To obtain $X_{I,4}^{-1,0}$ for $I = V, S$ one needs the massive quark anomalous dimension at four-loop order, $\gamma_Q^{(3)}$, and the complete contributions from three-loop order are necessary as well. On the other hand, $X_{I,4}^{-1,1}$ lacks the non-planar contributions only, which can be obtained once the massless cusp anomalous dimension $A_q^{(4)}$ is known completely. It similarly follows for the remaining of the functions $X_{I,4}^{0,k}$.

The exact result for the color-planar and complete light quark contributions for the vector and scalar form factors are available [31–34] now. We successfully cross-checked our results in the corresponding limit. An interesting point to note is that similar to the coefficients $g_I^{n,k}$, the coefficients $C_I^{n,k}$ are also in accordance with principle of leading transcendentality, *i.e.* the leading transcendental terms for each order in the ε -expansion are the same in both the vector and scalar cases. This aspect is of importance considering form factors in super-symmetric $\mathcal{N} = 4$ SYM theories.

Finally, we would like to mention that the Q^2 -dependent parts of \hat{F}_I and the massless form factor $\bar{F}_I = F_I(m = 0)$ are the same. The universal function $Z_I^{(m|0)}$ [10] is then given by

$$\begin{aligned}
Z_I^{(m|0)} \left(\frac{m^2}{\mu^2}, a_s, \varepsilon \right) &= F_I \left(\frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, a_s, \varepsilon \right) \left(\bar{F}_I \left(\frac{Q^2}{\mu^2}, a_s, \varepsilon \right) \right)^{-1} \\
&= C_I(a_s, \varepsilon) \hat{F}_I \left(\frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}, a_s, \varepsilon \right) \left(\bar{F}_I \left(\frac{Q^2}{\mu^2}, a_s, \varepsilon \right) \right)^{-1}. \tag{31}
\end{aligned}$$

4 Conclusion

We presented a systematic study of the massive form factors at $Q^2 \gg m^2$ at three-loop order by solving the associated evolution equations both in the vector and scalar cases. The universal structure of the IR singularities, along with the interplay of the various anomalous dimensions, has enabled us to obtain all asymptotic corrections at three-loop order for all logarithmic contributions. We also obtained partial four-loop results, still containing pieces, which can only be determined by performing a four-loop calculation. The Dirac axial-vector and vector form factor, and likewise the pseudo-scalar and scalar form factors, agree in this limit, while the Pauli vector form factors vanish. We remark that there are additional corrections due to massive internal quark loops, which have not been considered in the present paper and are the subject of further investigations. The present results constrain future calculations and may serve as important checks.

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