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Scalar 1-loop Feynman integrals as meromorphic functions in space-time dimension d

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Abstract

The long-standing problem of representing the general massive one-loop Feynman integral as a meromorphic function of the space-time dimension d has been solved for the basis of scalar one- to four-point functions with indices one. In 2003 the solution of difference equations in the space-time dimension allowed to determine the necessary classes of special functions: self-energies need ordinary logarithms and Gauss hypergeometric functions ${}_2F_1$, vertices need additionally Kampé de Fériet-Appell functions F_1, and box integrals also Lauricella-Saran functions F_S. In this study, alternative recursive Mellin-Barnes representations are used for the representation of n-point functions in terms of $(n-1)$-point functions. The approach enabled the first derivation of explicit solutions for the Feynman integrals at arbitrary kinematics. In this article, we scetch our new representations for the general massive vertex and box Feynman integrals and derive a numerical approach for the necessary Appell functions F_1 and Saran functions F_S at arbitrary kinematical arguments.

1. Introduction

We are studying scalar one-loop Feynman integrals,

$$J_n(d) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_n^{\nu_n}}, \tag{1}$$

with inverse propagators $D_i = (k + q_i)^2 - m_i^2 + i\varepsilon$. We assume $\nu_i = 1$ as well as momentum conservation and all external momenta to be incoming, $\sum_{e=1}^n p_e = 0$. The q_i are loop momenta shifts and will be expressed for applications by the external momenta p_e. Dimensions $d = 4 + 2n - 2\epsilon$ with $n \geq 0$ are of physical interest because tensor one-loop Feynman integrals of rank r in $4 - 2\epsilon$ dimensions may be expressed by scalar integrals taken in higher dimensions up to $d = 4 + 2r - 2\epsilon$ [1]. Higher indices ν_i will also appear in the representation, but may be eliminated, so that a complete reduction basis of higher-dimensional scalar one- to four point integrals may be derived. For two- to seven-point tensor functions this has been worked out in [2, 3].

The first terms of the ϵ-expansion of one- to four-point scalar functions for $d = 4 - 2\epsilon$, until including the constant term, was given by G. t'Hooft and M. Veltman in 1978 [4]. A systematic numerical treatment of the next terms of order ϵ-terms was performed in 1992 [5], and a systematic numerical approach was worked out in 2001 [6]. It has been shown in 2003 [7, 8] that representations in general dimension d, including $d = 4 - 2\epsilon$, will rely on certain multiple hypergeometric functions of the type ${}_2F_1, F_1, F_S$. Though, the explicit solutions for arbitrary kinematics could not be found.

A scetch of the Feynman integrals at arbitrary kinematics in terms of ${}_2F_1, F_1, F_S$ and their explicit numerical determination are the subject of this letter. The dependence on the external momenta p_e will be contained exclusively in the functions R_n:

$$R_n \equiv R_{12\ldots n} = -\frac{\lambda_n}{G_n} - i\varepsilon. \tag{2}$$

The R_n carry the causal regulator $-i\varepsilon$. The Cayley matrix $\lambda_{12\ldots n}$ was introduced in [9]. It is composed of the variables Y_{ij}, and its determinant λ_n is:

$$\lambda_n \equiv \det(\lambda_{12\ldots n}) = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix}, \tag{3}$$

with

$$Y_{ij} = Y_{ji} = m_i^2 + m_j^2 - (q_i - q_j)^2. \tag{4}$$

Further, we use the $(n-1) \times (n-1)$ dimensional Gram determinant G_n,

$$G_n \equiv -2^n \det(G_{12\cdots n}), \tag{5}$$

and

$$\det(G_{12\cdots n}) = \tag{6}$$

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$$\begin{vmatrix} (q_1-q_n)^2 & \dots & (q_1-q_n)(q_{n-1}-q_n) \\ (q_1-q_n)(q_2-q_n) & \dots & (q_2-q_n)(q_{n-1}-q_n) \\ \vdots & \ddots & \vdots \\ (q_1-q_n)(q_{n-1}-q_n) & \dots & (q_{n-1}-q_n)^2 \end{vmatrix}.$$

We use the special assignment for tadpoles:

$$G_1 = -2. \tag{7}$$

Both determinants λ_n and G_n are independent of a common shift of the internal momenta q_i. Further, we introduce the notion $R(i)$,

$$R(i) \equiv r(i) - i\varepsilon \;\equiv\; -\det(\lambda_i)/G_1 - i\varepsilon \;=\; m_i^2 - i\varepsilon, \tag{8}$$

and use, wherever it is unique from the context,

$$R_1 \equiv R(i). \tag{9}$$

We derived in [10] a new ansatz, a recursion relation for the Feynman integrals defined in (1),

$$J_n(d) = \frac{-1}{2\pi i} \int\limits_{c_0-i\infty}^{c_0+i\infty} ds \frac{\Gamma(-s)\Gamma(\frac{d-n+1}{2}+s)}{2\Gamma(\frac{d-n+1}{2})} \times \Gamma(s+1) R_n^{-s-1} \sum_{k=1}^{n} \partial_k R_n \, \mathbf{k}^- J_n(d+2s), \tag{10}$$

and its solution by a sequence of Mellin-Barnes representations. We use the representation $\partial_k R_n$ for the co-factor of the Cayley matrix, also called signed minors in e.g. [9]:

$$\partial_k R_n \;=\; \frac{\partial R_n}{\partial m_k^2} \;=\; \begin{pmatrix} 0 \\ k \end{pmatrix}_n. \tag{11}$$

The operator $\mathbf{k}^-$ reduces an n-point Feynman integral $J_n(d)$ to $(n-1)$-point integrals $J_{n-1}(d+2s)$ by shrinking the k^{th} propagator, $1/D_k$:

$$\mathbf{k}^- \; J_n(d) \;\;=\;\; \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{\prod_{j\neq k, j=1}^{n} D_j}. \tag{12}$$

The recurrence relation (10) is the master integral for one-loop n-point functions in space-time dimension d, representing them by n integrals over $(n-1)$-point functions with a shifted, continuous dimension $d+2s$. The recurrence starts at $n=2$ with the tadpole $J_1(d)$ in the integrand:

$$J_1(d; m_i^2) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 - m_i^2 + i\varepsilon} = -\frac{\Gamma(1-d/2)}{(m_i^2 - i\varepsilon)^{1-d/2}} \equiv -\frac{\Gamma(1-d/2)}{{R_1}^{1-d/2}}. \tag{13}$$

Eqn. (10) contains for $n=2$ the term $\int ds (\frac{R_1}{R_2})^s$, multiplied by Γ-matrices with arguments depending on s, and is formally a *Mellin-Barnes integral*. Our representation is an alternative to Eq. (19) of [8]. There, an *infinite sum* over a *discrete dimensional parameter s* was derived in order to represent an n-point function $J_n(d)$ by integrals $J_{n-1}(d+2s)$.

The further evaluations will depend, concerning the kinematics, exclusively on the R_1, R_2, etc. introduced in (2). Although, there will arise exceptional cases when the specific choice of the external scalars $(p_{e_i} p_{e_j})$ or of internal mass squares m_i^2 will lead to vanishing or divergent determinants λ_n or G_n. In such cases, one has to go back to intermediate definitions and look for specific solutions.[1] See also the remarks in [11].

2. Massive vertex and box functions

Representations of the massive self-energy, vertex and box integrals can be derived iteratively from (10) by closing the integration contours of the Mellin-Barnes integrals e.g. to the right and taking the two series of residues of the corresponding Γ-functions with arguments $(-s+\cdots)$. One Cauchy sum constitutes the analogue of the so-called *boundary or b-terms* of [8], the other one has a genuine d-dependence. Both sums together represent the Feynman integrals. In our approach, closed analytical expressions could be determined for arbitrary kinematics.

The general massive vertex and box integrals $J_3(d), J_4(d)$ have first been published at LL2018 [12]. An alternative, instructive version of the vertex is

$$J_3(d) = J_{123} + J_{231} + J_{312}, \tag{14}$$

with short notations $R_3 = R_{123}, R_2 = R_{12}$ etc., and:

$$\begin{aligned} J_{123} = {} & \Gamma\left(2-\frac{d}{2}\right) \frac{\partial_3 R_3}{R_3} \frac{\partial_2 R_2}{R_2} \frac{R_2}{2\sqrt{1-R_1/R_2}} \\ & \left[-R_2^{\frac{d}{2}-2} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)} \, {}_2F_1\left(\frac{d-2}{2}, 1; \frac{d-1}{2}; \frac{R_2}{R_3}\right) + R_3^{\frac{d}{2}-2} \, {}_2F_1\left(1,1;\frac{3}{2};\frac{R_2}{R_3}\right)\right] \\ & + \Gamma\left(2-\frac{d}{2}\right) \frac{\partial_3 R_3}{R_3} \frac{\partial_2 R_2}{R_2} \frac{R_1}{4\sqrt{1-R_1/R_2}} \\ & \left[+ \frac{2 R_1^{\frac{d}{2}-2}}{d-2} F_1\left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; \frac{R_1}{R_3}, \frac{R_1}{R_2}\right) - R_3^{\frac{d}{2}-2} F_1\left(1; 1, \frac{1}{2}; 2; \frac{R_1}{R_3}, \frac{R_1}{R_2}\right)\right] \\ & + \;\; (R_1(1) \leftrightarrow R_1(2)). \end{aligned} \tag{15}$$

We use the abbreviation (11). For $d \to 4$, both the sums of expressions with ${}_2F_1$ and F_1 in square brackets in (15) approach zero, thus compensating the pole factor $\Gamma(2-d/2)$ in this limit. The J_3 stays finite at $d=4$, as it should be for any massive 3-point function. And the ϵ expansion for J_{123} to order n needs, in this case, the evaluation of the components to order $(n+1)$.

[1] A complete analysis of the exceptional kinematical cases has been performed by K.H.P; to be published elsewhere.

The corresponding massive four-point function is:

$$J_4(d) = J_{1234} + J_{2341} + J_{3412} + J_{4123}, \tag{16}$$

with $R_4 = R_{1234}, R_3 = R_{123}, R_2 = R_{12}$ etc., and:

$$
\begin{aligned}
J_{1234} &= \Gamma\left(2-\frac{d}{2}\right)\frac{\partial_4 R_4}{R_4}\Bigg\{ \\
&\left[\frac{b_{123}}{2}\left(-\,R_3^{\frac{d}{2}-2}\;{}_2F_1\left(\frac{d-3}{2},1;\frac{d-2}{2};\frac{R_2}{R_3}\right)\right.\right. \\
&+\;R_4^{\frac{d}{2}-2}\,\sqrt{\pi}\,\frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-3}{2}\right)} \\
&\times\;{}_2F_1\left.\left.\left(\frac{1}{2},1;1;\frac{R_2}{R_3}\right)\right)\right] \\
&+\;\frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-3}{2}\right)}\,\frac{\sqrt{\pi}}{4}\,\frac{\partial_3 R_3}{R_3}\frac{\partial_2 R_2}{\sqrt{1-R_1/R_2}} \\
&\times\;{}_2F_1\left(\frac{1}{2},1;1;\frac{R_2}{R_3}\right) \\
&\Bigg[+\frac{R_2^{\frac{d}{2}-2}}{d-3}F_1\left(\frac{d-3}{2};1,\frac{1}{2};\frac{d-1}{2};\frac{R_2}{R_4},\frac{R_2}{R_3}\right) \\
&-\;R_4^{\frac{d}{2}-2}F_1\left(\frac{1}{2};1,\frac{1}{2};\frac{3}{2};\frac{R_2}{R_4},\frac{R_2}{R_3}\right)\Bigg] \\
&+\;\frac{R_1}{8}\,\frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-3}{2}\right)}\frac{\partial_3 R_3}{R_3}\frac{\partial_2 R_2}{R_2}\frac{1}{1-R_1/R_3}\frac{1}{1-R_1/R_2} \\
&\Bigg[-\,R_1^{\frac{d}{2}-2}\frac{\Gamma\left(\frac{d-3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \\
&\times F_S\left(\frac{d-3}{2},1,1;1,1,\frac{1}{2};\frac{d}{2},\frac{d}{2},\frac{d}{2};\frac{R_1}{R_4},\cdots\right. \\
&\qquad\qquad\qquad\left.\frac{R_1}{R_1-R_3},\frac{R_1}{R_1-R_2}\right) \\
&+\;R_4^{\frac{d}{2}-2}\,\sqrt{\pi} \\
&\times F_S(\frac{1}{2},1,1;1,1,\frac{1}{2};2,2,2,\frac{R_1}{R_4},\frac{R_1}{R_1-R_3},\frac{R_1}{R_1-R_2})\Bigg] \\
&+\;(R_1(1)\leftrightarrow R_1(2))\Bigg\} \\
&+\;(2,3,1)+(3,1,2),
\end{aligned} \tag{17}
$$

where the function b_{123} is independent of d,

$$
\begin{aligned}
b_{123} &= \frac{1}{2}\frac{\partial_3 R_3}{R_3}\frac{\partial_2 R_2}{R_2}\left[\frac{R_2}{\sqrt{1-\frac{R_1}{R_2}}}\;{}_2F_1\left(1,1;\frac{3}{2};\frac{R_2}{R_3}\right)\right. \\
&-\;\left.\frac{1}{2}\frac{R_1}{\sqrt{1-\frac{R_1}{R_2}}}\,F_1\left(1;1,\frac{1}{2};2;\frac{R_1}{R_3},\frac{R_1}{R_2}\right)\right]+(1\leftrightarrow 2).
\end{aligned} \tag{18}
$$

Here, it is $R_1 = R_1(1)$ and (11) defines derivatives like $\partial_2 r_2$. The term b_{123}, when multiplied with $\Gamma(-\frac{d-4}{2})R_3^{\frac{d}{2}-2}$, equals the term of J_{123} in (15) with d-independent F_1 and F_S. It replaces the so-called b_3-term of the vertex integral in [8] for arbitrary kinematics, while the d-dimensional parts of J_{1234} agree.

For $d \to 4$, all the expressions in square brackets in (17) approach zero, thus compensating the pole of $\Gamma(2-d/2)$ in this limit. As a result, the J_4 stays finite at $d = 4$, as it should be for any massive 4-point function. And the ϵ expansion for J_{1234} to order n needs, in this case, the evalution of the components to order $(n+1)$.

The derivations of J_{123} and J_{1234} were done under the assumption that the kinematical arguments x, y, z of the ${}_2F_1, F_1, F_S$ fulfill $|x|, |y|, |z| < 1$. Nevertheless, the above formulae are valid at arbitrary kinematical arguments, for massive vertices at $\Re e(d) > 2$ and for box integrals at $\Re e(d) > 3$. In Appendix A to Appendix C we will show how to calculate the various F_1 and F_S for arbitrary complex arguments; for ${}_2F_1$ we assume that such calculations are well-known.

3. Numerical results

The scalar one-loop basis consists of one- to four-point functions. Our two-point function $J_2(d)$ was reproduced in [12] and is in complete agreement with [8], while for $J_3(d)$ and $J_4(d)$ our results are novel. Concerning numerical results for the 3-point functions we refer to several tables in [13, 10]. The kinematics was chosen such that the results of [8] could be compared.[2] Another numerical comparison, for a box integral $J_4(d)$ with vanishing Gram determinant, may be found in [11, 14].

In Table 1 we show few examples of four-point functions in comparison to other packages. We did not aim at maximal accuracy and claim essentially eight safe digits. Further, one propagator is massive and $d = 4$ or $d = 5$, and we can also allow for complex masses at the internal lines. A true sample ε-expansion is reproduced for the generalized hypergeometric function F_1 in Table B.2.

For the safe numerical calculation of massive vertices J_3 and massive box integrals J_4 we collect stable numerical representations for the generalized hypergeometric functions F_1 and F_S in the Appendices.

4. Discussion

The massive oneloop Feynman integrals have been represented as meromorphic functions of space-time d in terms of generalized hypergeometric functions. Many details left out here will be published elsewhere. The Feynman integrals can be calculated numerically at arbitrary kinematics and arbitrary dimension d, including potential pole locations at $d = 4 + 2n$. For phenomenological or multi-loop applications, it is wishful to have the pole expansions in closed analytical form. Their derivation is subject of a subsequent study.

The new recursion (10) has a unique feature. It allows to derive n-dimensional Mellin-Barnes integrals for n-point Feynman integrals. Generally, n-dimensional integrals are obtained

[2] We would like to thank Oleg Tarasov for a helpful discussion concerning this issue.

Tab. 1: Comparison of the box integral J_4 defined in (17) with the LoopTools function `D0` $(p_1^2, p_2^2, p_3^2, p_4^2, (p_1{+}p_2)^2, (p_2{+}p_3)^2, m_1^2, m_2^2, m_3^2, m_4^2)$ **[15, 16] at $m_2^2 = m_3^2 = m_4^2 = 0$. Further numerical references are the packages K.H.P_D0 (PHK, unpublished) and MBOneLoop [14]. External invariants:** $(p_1^2 = \pm 1, p_2^2 = \pm 5, p_3^2 = \pm 2, p_4^2 = \pm 7, s = \pm 20, t = \pm 1)$.

$(p_1^2, p_2^2, p_3^2, p_4^2, s, t)$	4-point integral
$(-,-,-,-,-,-)$	$d = 4,\ \ m_1^2 = 100$
J_4	0.00917867
LoopTools	0.0091786707
MBOneLoop	0.0091786707
$(+,+,+,+,+,+)$	$d = 4,\ \ m_1^2 = 100$
J_4	$-0.0115927 - 0.00040603\, i$
LoopTools	$-0.0115917 - 0.00040602\, i$
MBOneLoop	$-0.0115917369 - 0.0004060243\, i$
$(-,-,-,-,-,-)$	$d = 5,\ \ m_1^2 = 100$
J_4	0.00926895
K.H.P_D0	0.00926888
MBOneLoop	0.0092689488
$(+,+,+,+,+,+)$	$d = 5,\ \ m_1^2 = 100$
J_4	$-0.00272889 + 0.0126488\, i$
K.H.P_D0	(–)
MBOneLoop	$-0.0027284242 + 0.0126488134\, i$
$(-,-,-,-,-,-)$	$d = 5,\ \ m_1^2 = 100 - 10\, i$
J_4	$0.00920065 + 0.000782308\, i$
K.H.P_D0	$0.0092006\ \ + 0.000782301\, i$
MBOneLoop	$0.0092006481 + 0.0007823090\, i$
$(+,+,+,+,+,+)$	$d = 5,\ \ m_1^2 = 100 - 10\, i$
J_4	$-0.00398725 + 0.012067\, i$
K.H.P_D0	$-0.00398723 + 0.012069\, i$
MBOneLoop	$-0.0039867702 + 0.0120670388\, i$

by sector decomposition methods, while in the Mellin-Barnes approach, as it is advocated in numerical loop calculations, the number of dimension grows faster. Within the MBsuite, AMBRE generates for the most general massive n-point one-loop function an $N_n = \frac{1}{2}n(n+1)$-dimensional MB-integral; according to the number of entries Y_{ij} in the second Symanzik polynomial, $F(x) = \frac{1}{2}x_i Y_{ij} x_j - i\varepsilon$. For a vertex or box, $N_3 = 6, N_4 = 10$. In the present approach, it is only $N_3' = 3, N_4' = 4$. Evidently, a replacement of the original kinematical invariants $m_i^2, (p_{e,i}p_{e,j})$ or Y_{ij} by the alternatives $R_n = -\lambda_n/G_n$ is an essential building block and it might well be possible to find similar lower-dimensional MB-representations also for more involved multi-loop integrals.

Basic numerical features of the new n-dimensional MB-representation (10) have been studied in [17, 14] in comparison with [2], with the package MBOneLoop,including cases of small or vanishing Gram determinant.

It is interesting to compare our results for $J_3(d)$ and $J_4(d)$ with the earlier study [8]. The d-dependent part of $J_3(d)$ as well as much of the d-dependent part of $J_4(d)$ agree with our results. Further, the expressions for the b-terms in [8] differ from our d-independent parts, although in certain kinematical regions they do agree numerically for $J_3(d)$. We find no agreement for $J_4(d)$, due to the various contributing b-terms.

Appendix A. The Appell function F_1 and Lauricella-Saran function F_S

Numerical calculations of specific Gauss hypergeometric functions ${}_2F_1$, Appell functions F_1 (Eqn. (1) of [18]), and Lauricella-Saran functions F_S (Eqn. (2.9) of [19]) are needed for the scalar one-loop Feynman integrals:

$$ {}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!\,(c)_k}\, x^k, \tag{A.1} $$

$$ F_1(a;b,b';c;y,z) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{m!\,n!\,(c)_{m+n}}\, y^m z^n, \tag{A.2} $$

$$ F_S(a_1,a_2,a_2;b_1,b_2,b_3;c,c,c;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m (a_2)_{n+p} (b_1)_m (b_2)_n (b_3)_p}{m!\,n!\,p!\,(c)_{m+n+p}}\, x^m y^n z^p. \tag{A.3} $$

The $(a)_k$ is the Pochhammer symbol. The series converge for $|x|, |y|, |z| < 1$, but the functions are needed for arbitrary arguments. All the ${}_2F_1, F_1, F_S$ are finite and have no pole terms in ϵ. Practically all aspects of ${}_2F_1$ are well-known and implemented in computer algebra systems, in Mathematica as built-in symbol `Hypergeometric2F1[a,b,c,z]`. There is no public F_S-package, while the Appell function $F_1(a; b_1, b_2; c; x, y)$ [18] is implemented in Mathematica as built-in symbol `AppellF1[a,b1,b2,c,x,y]` [20]. Another public package is `f1` [21, 22], and a wrapper package for `f1` is `appell` [23]. All the implementations mentioned have systematic limitations.

One approach to the numerics of F_1 and F_S may be based on Mellin-Barnes representations. For the Gauss function ${}_2F_1$ and the Appell function F_1, Mellin-Barnes representations are known. See Eqn. (1.6.1.6) in [24],

$$ {}_2F_1(a,b;c;z) = \frac{1}{2\pi i}\,\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \times \int_{-i\infty}^{+i\infty} ds\, (-z)^s\, \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}, \tag{A.4} $$

and Eqn. (10) in [18], which is a two-dimensional MB-integral:

$$ F_1(a;b,b';c;x,y) = \frac{1}{2\pi i}\,\frac{\Gamma(c)}{\Gamma(a)\Gamma(b')} \times \int_{-i\infty}^{+i\infty} dt\, (-y)^t\, {}_2F_1(a+t,b;c+t,x) \times \frac{\Gamma(a+t)\Gamma(b'+t)\Gamma(-t)}{\Gamma(c+t)}. \tag{A.5} $$

For the Lauricella-Saran function F_S we derived the following, new, three-dimensional MB-integral:

$$F_S(a_1, a_2, a_2; b_1, b_2, b_3; c, c, c; x, y, z) \tag{A.6}$$
$$= \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(b_1)} \int_{-i\infty}^{+i\infty} dt\, F_1(a_2; b_2, b_3; c+t; y, z)$$
$$\times (-x)^t \frac{\Gamma(a_1+t)\Gamma(b_1+t)\Gamma(-t)}{\Gamma(c+t)}.$$

A general numerical evaluation of these representations deserves some sophistication. Let us mention the simple one-loop massive QED vertex for which no trivial MB method exists when the kinematics is Minkowskian, a problem discussed e.g. in [25] and solved in [26]. It was demonstrated in [27] that MB-OneLoop, a fork of the package MBnumerics [28, 11, 14] may be used to solve (A.4) to (A.6) at arbitrary kinematics with high precision.

One might also try to approach the generalized hypergeometric functions using Pochhammer's double loop contours [29, 30], or study the defining differential equations [31, 32, 33], etc. After several trials, we decided to base our numerics on the integral representations of F_1 proposed in [34] and F_S proposed in [35]; see Appendix B and Appendix C. Astonishing enough, it will (nearly) suffice to use mathematics known to well-educated German gymnasiasts.

Appendix B. The Appel functions F_1

A one-dimensional integral representation for F_1 [34] is quoted in Eqn. (9) of [18]:

$$F_1(a; b, b'; c; x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \tag{B.1}$$
$$\times \int_0^1 du \frac{u^{a-1}(1-u)^{c-a-1}}{(1-xu)^b(1-yu)^{b'}}.$$

We need three specific cases, taken at $d \geq 4$. Namely for vertices:

$$F_1^v(d) \equiv F_1\left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; x_c, y_c\right) \tag{B.2}$$
$$= \frac{1}{2}(d-2) \int_0^1 \frac{du\, u^{\frac{d}{2}-2}}{(1-x_c u)\sqrt{1-y_c u}}.$$

Integrability is violated at $u = 0$ if not $\Re e(d) > 2$. Similarly, for box integrals:

$$F_1^b(d) \equiv F_1(\frac{d-3}{2}; 1, \frac{1}{2}; \frac{d-1}{2}; x_c, y_c) \tag{B.3}$$
$$= \frac{1}{2}(d-3) \int_0^1 \frac{du\, u^{d/2-5/2}}{(1-x_c u)\sqrt{1-y_c u}}$$
$$= F_1^v(d-1),$$

Integrability is violated at $u = 0$ if not $\Re e(d) > 3$. Finally for the definition of the box Saran function F_S (C.1):

$$F_1^S(y_c, z_c) \equiv F_1(1; 1, \frac{1}{2}, \frac{3}{2}; y_c, z_c) \tag{B.4}$$
$$= \frac{1}{2} \int_0^1 \frac{u\, du}{\sqrt{1-u}(1-x_c u)\sqrt{1-y_c u}}.$$

The singularity at $u = 1$ is integrable.

Appendix B.1. Specific values of ${}_2F_1$ and F_1 at $d = 4$

The vertex function (15) contains ${}_2F_1$ and F_1 with specific values at $d = 4$:

$${}_2F_1\left(1, 1; \frac{3}{2}; x_c\right) = \frac{\mathrm{ArcSin}(\sqrt{x_c})}{\sqrt{1-x_c}\sqrt{x_c}} \tag{B.5}$$

and

$$F_1\left(1; 1, \frac{1}{2}; 2; x_c, y_c\right) = 2\frac{\mathrm{ArcTanh}\left[\frac{\sqrt{x_c}\sqrt{1-y_c}}{\sqrt{x_c-y_c}}\right]}{\sqrt{x_c}\sqrt{x_c-y_c}}$$
$$- 2\frac{\mathrm{ArcTanh}\left[\frac{\sqrt{x_c}}{\sqrt{x_c-y_c}}\right]}{\sqrt{x_c}\sqrt{x_c-y_c}}. \tag{B.6}$$

Using logarithms only, $\mathrm{ArcSin}(z) = -i\ln(iz + \sqrt{1-z^2})$ and $\mathrm{ArcTanh}(z) = \frac{1}{2}[\ln(1+z) - \ln(1-z)]$. Eqn. (B.6) is only valid if $(x_c - y_c)$ has a well-defined imaginary part. For $x_c = x - i\varepsilon_x$ and $y_c = y - i\varepsilon_y$ this is not necessarily the case if ε_x and ε_y are independent and both infinitesimal. So (B.6) has to be used with a grain of care.

The box function (17) contains additional ${}_2F_1$ and F_1 with specific values at $d = 4$:

$${}_2F_1\left(\frac{1}{2}, 1; 1; x_c\right) = \frac{1}{\sqrt{1-x_c}} \tag{B.7}$$

and

$$F_1\left(\frac{1}{2}; 1, \frac{1}{2}; \frac{3}{2}; x_c, y_c\right) = \frac{1}{\sqrt{1-y_c}}\ {}_2F_1(\frac{1}{2}; 1, \frac{3}{2}; \frac{x_c-y_c}{1-y_c})$$
$$= \frac{\mathrm{ArcTanh}\left(\sqrt{\frac{x_c-y_c}{1-y_c}}\right)}{\sqrt{x_c-y_c}}. \tag{B.8}$$

Eqn. (B.8) is only valid if $(x_c - y_c)$ has a well-defined imaginary part. Finally, we like to mention that we have no analogue to (B.7) and (B.8) for F_S at $d = 4$, namely $F_S(\frac{1}{2}, 1, 1; 1, 1, \frac{1}{2}; 2, 2, 2; x_c, y_c, z_c)$.

The Appell function $F_1^S = F_1(1; 1, \frac{1}{2}; \frac{3}{2}; y_c, z_c)$ used in the integrand of the definition of the Saran function (C.1) can also be simplified:

$$F_1(1; 1, \frac{1}{2}; \frac{3}{2}; y_c, z_c) = \frac{1}{1-z_c}\ {}_2F_1\left(1, 1; \frac{3}{2}; \frac{y_c-z_c}{1-z_c}\right)$$
$$= \frac{\mathrm{ArcSin}\sqrt{\frac{y_c-z_c}{1-z_c}}}{\sqrt{(y_c-z_c)(1-z_c)}}. \tag{B.9}$$

Both representations in (B.9) are only valid when the imaginary part of the difference $(y_c - z_c)$ is well-defined.

For the Feynman integrals studied here, we have to take into account that x_c, y_c and z_c may have, in general, *uncorrelated infinitesimal* imaginary parts, and so their difference may be *not* well-defined. Let us remind that $x_c = R_1/R_4$, and $y_c = R_1/(R_1 - R_3)$, and $z_c = R_1/(R_1 - R_2)$. Here, all the R_n have, according to (2), identical imaginary parts $-i\varepsilon$. This leads to different infinitesimal imaginary parts $-\varepsilon_x, -\varepsilon_y, -\varepsilon_z$, with potentially different signs. So, one has basically two equivalent options. Either one treats $\varepsilon_x, \varepsilon_y$ and ε_z as independent quantities and avoids the appearance of terms like $(x_c - y_c)$ and $(y_c - z_c)$. Or one uses the exact knowledge of the imaginary parts of the R_n from their definitions and arrives at well-defined imaginary parts of these $(x_c - y_c)$ and $(y_c - z_c)$.

Appendix B.2. Numerical calculation of $F_1^v(d)$

For $x_c = x - iX$ and $y_c = y - iY$, Eqn. (B.2) may be used for numerics if $(X, Y) \geq$ const. > 0 or if $(x, y) < 1$. The remaining cases $(X = -\varepsilon_x, Y = -i\varepsilon_y) \to +0$ deserve a closer inspection. They appear from Feynman integrals.

We exemplify here the first one of the two more involved cases: $1 < x < y$ and $1 < y < x$.

We introduc an auxiliary split parameter

$$u_m = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{x}\right) \text{ with } 0 < \frac{1}{y} < u_m < \frac{1}{x} < 1. \tag{B.10}$$

In the integrand of F_1 there will a cut begin at $u = \frac{1}{y}$ and a pole arise at $u = \frac{1}{x}$ for infinitesimal $\varepsilon_x, \varepsilon_y$. A split of the integral at u_m,

$$\int_0^1 du = \int_0^{u_m} du + \int_{u_m}^1 du \equiv i_L + i_R, \tag{B.11}$$

will lead to a separation of the singularities. In both integrals at the right hand side, the integrand is regular with *one* exclusion. We discuss now several opportunities of calculations, all of them with an accuracy of eight safe digits or better.

Our most careful approach persued the following ansatz with additional splittings:

$$\begin{aligned} F_1^v(d) &= I_A + I_0 + I_C + I_D + I_B + I_E \\ &= \lim_{R\to+0}\left[\int_0^{\frac{1}{y}-R} + \int_{\frac{1}{y}-R}^{\frac{1}{y}+R} + \int_{\frac{1}{y}+R}^{u_m} \right. \\ &\left. + \int_{u_m}^{\frac{1}{x}-R} + \int_{\frac{1}{x}-R}^{\frac{1}{x}+R} + \int_{\frac{1}{x}+R}^{1}\right] \end{aligned} \tag{B.12}$$

After performing the limit $R \to 0$ wherever possible, the integrals A and B will give real contributions, and the others are purely imaginary:

$$\begin{aligned} F_1^v(d) &= [\Re e F_1^v(d)] + i\,[\Im m F_1^v(d)] \\ &= [A + \text{sign}(\varepsilon_x)\text{sign}(\varepsilon_y)\, B] \\ &\quad + i\,[\text{sign}(\varepsilon_y)\,(-C + D + E)]\,. \end{aligned} \tag{B.13}$$

It is

$$I_0 = 0, \tag{B.14}$$

$$A = \frac{d-2}{2}\int_0^{\frac{1}{y}} \frac{du\, u^{d/2-2}}{(1-xu)\sqrt{1-yu}}, \tag{B.15}$$

$$B = \frac{d-2}{2}\,\pi\,\frac{1}{x\sqrt{\frac{y}{x}-1}\,x^{d/2-2}}, \tag{B.16}$$

$$C = \frac{d-2}{2}\int_{\frac{1}{y}}^{u_m} \frac{du\, u^{d/2-2}}{(1-xu)\sqrt{yu-1}}, \tag{B.17}$$

$$\begin{aligned} D &= \frac{d-2}{2}\int_{u_m}^{\frac{1}{x}} \frac{du}{1-xu} \\ &\left(\frac{u^{d/2-2}}{\sqrt{yu-1}} - \frac{x^{-d/2+2}}{\sqrt{\frac{y}{x}-1}}\right) \\ &+ \frac{d-2}{2}\frac{1}{\sqrt{\frac{y}{x}-1}\,x^{d/2-2}} \\ &\left[\ln(R) - \ln(\frac{1}{2x} - \frac{1}{2y})\right], \end{aligned} \tag{B.18}$$

$$\begin{aligned} E &= \frac{d-2}{2}\int_{\frac{1}{x}}^{1} \frac{du}{1-xu} \\ &\left(\frac{u^{d/2-2}}{\sqrt{yu-1}} - \frac{x^{-d/2+2}}{\sqrt{\frac{y}{x}-1}}\right) \\ &+ \frac{d-2}{2}\frac{1}{\sqrt{\frac{y}{x}-1}\,x^{d/2-2}} \\ &\left[-\ln(R) + \ln(1 - \frac{1}{x})\right]. \end{aligned} \tag{B.19}$$

The remaining R-dependences in (B.18) and (B.19) drop out in the sum of D and E.

Alternatively, with a subtraction in each of the two partial integrals in (B.11), one may regularize the integrand of $F_1^v(d)$ as follows:

$$i_L = \int_0^{u_m} du\, \frac{g_x(u) - g_x\left(\frac{1}{y}\right)}{\sqrt{1-yu}} + i_L^{ana}, \tag{B.20}$$

$$i_R = \int_{u_m}^{1} du\, \frac{g_y(u) - g_y\left(\frac{1}{x}\right)}{1-xu} + i_R^{ana}, \tag{B.21}$$

with

$$\begin{aligned} i_L^{ana} &= -\,2\,\frac{g_x\left(\frac{1}{y}\right)}{y_c}\left[\sqrt{1-y_c u_m} - 1\right] \\ &\to -\,2\,\frac{g_x\left(\frac{1}{y}\right)}{y_c}\left[-1 + i\,\text{sign}(\varepsilon_y)\sqrt{yu_m - 1}\right], \end{aligned} \tag{B.22}$$

$$\begin{aligned} i_R^{ana} &= -\frac{g_y\left(\frac{1}{x}\right)}{x_c}\ln\left(\frac{1-x_c}{1-x_c u_m}\right) \\ &\to -\frac{g_y\left(\frac{1}{x}\right)}{x}\left[\ln\left(\frac{x-1}{1-xu_m}\right) + i\pi\,\text{sign}\,(\varepsilon_x)\right]. \end{aligned} \tag{B.23}$$

Finally, a simplest approach will also do a reasonable numerics: Perform mean value integrals, like e.g. the built-in function of Mathematica:

$$F_1^v(d) = \lim_{\epsilon\to+0}\left[\left(\int_0^{\frac{1}{y}-\epsilon}+\int_{\frac{1}{y}+\epsilon}^{u_m}\right)+\left(\int_{u_m}^{\frac{1}{x}-\epsilon}+\int_{\frac{1}{x}-\epsilon}^{1}\right)\right]. \tag{B.24}$$

Of course, a calculation with, say, more than eight safe digits, will deserve an explicit control of the algorithmic details.

Numerical examples for $F_1^v(d)$ are collected in Tables B.1 and (B.2).

Appendix B.3. Numerical calculation of the box Appell function $F_1^b(d)$

For the calculation of four-point Feynman integrals, one needs $F_1^b(d)$ as introduced in (B.3), both for $d = 4$ and for $d = 4 + n - 2\varepsilon$. The box F_1-function is related to the vertex function $F_1^v(d)$ by (B.3). Consequently, the numerics of the foregoing subsections may be taken over.

Appendix C. The Lauricella-Saran function F_s

For the calculation of the 4-point Feynman integrals, one needs the Lauricella-Saran function F_S [35]. Saran defines F_S as three-fold sum (A.3), see Eqn. (2.9) in [35]. He derives a 3-fold integral representation in Eqn. (2.15) and a 2-fold integral in Eqn. (2.16). We will use the following quite useful representation, derived at p. 304 of [35]:

$$\begin{aligned} &F_S(a_1, a_2, a_2; b_1, b_2, b_3; c, c, c, x, y, z) \qquad \text{(C.1)}\\ &= \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(c-a_1)}\\ &\int_0^1 dt \frac{t^{c-a_1-1}(1-t)^{a_1-1}}{(1-x+tx)^{b_1}} F_1(a_2; b_2, b_3; c-a_1; ty, tz). \end{aligned}$$

In our case, this becomes

$$\begin{aligned} F_S^b(d) &= F_S\left(\frac{d-3}{2}, 1, 1; 1, 1, \frac{1}{2}; \frac{d}{2}, \frac{d}{2}, \frac{d}{2}, x_c, y_c, z_c\right)\\ &= \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-3}{2})\Gamma(\frac{3}{2})} \qquad \text{(C.2)}\\ &\times \int_0^1 dt \frac{\sqrt{t}(1-t)^{\frac{d-5}{2}}}{(1-x_c+x_c t)} F_1(1; 1, \frac{1}{2}; \frac{3}{2}; y_c t, z_c t) \end{aligned}$$

Eqn. (C.2) is valid if $\Re e(d) > 3$. With a grain of care one may often use (B.9) for F_1^S. Because the F_1 under the t-integral is finite and smooth, we have to concentrate only on the term $1/(1-x_c+x_c t)$, which develops a pole in the integration region at $t_x = (1-x)/x$ if $\Re e(x_c) = x > 1$ and if $\Im m(x_c) = -\varepsilon_x$ is infinitesimal.

Appendix C.1. Case (i) $F_S^b(d)$ at $x \le 1$

For $x = 1$, the integral (C.2) is not well-defined. If $x < 1$, a direct, stable numerical integration of F_S is trivial once F_1 is known.

Appendix C.2. Case (ii) $F_S^b(d)$ at $x > 1$

If $x > 1$, one has to apply a regularization procedure to $F_S^b(d)$, as it is described in (Appendix B.2), and will get a stable result for F_S. The calculation of the F_1 in the integrand in (C.2) is discussed in Appendix B.1.

One now has to study the singularity structure of the t-integral as a function of x_c with regular F_1^S. Introduce

$$F_S^b(d) = \int_0^1 dt \frac{g_S(t) - g_S(t_x)}{1-x+xt} + g_S(t_x)\, I_S^{reg}(x_c), \tag{C.3}$$

with

$$g_S(t) = \sqrt{t}\,(1-t)^{(d-5)/2}\, F_1^S(y_c t, z_c t) \tag{C.4}$$

and

$$t_x = 1 - \frac{1}{x}. \tag{C.5}$$

The first integral in (C.3) is numerically stable, and what remains is to calculate analytically the integral

$$I_S^{reg}(x_c) = +\frac{1}{x_c}\int_0^1 \frac{dt}{t - t_{x_c}} = \frac{1}{x_c}\ln\left(1 - \frac{1}{t_{x_c}}\right). \tag{C.6}$$

For infinitesimal ε_x, we get

$$I_S^{reg}(x_c) \to \frac{1}{x}\left[-\ln(x-1) + i\pi\,\mathrm{sign}(\varepsilon_x)\right]. \tag{C.7}$$

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References

[1] A. I. Davydychev, A simple formula for reducing Feynman diagrams to scalar integrals, Phys. Lett. B263 (1991) 107–111, `http://www.higgs.de/~davyd/preprints/tensor1.pdf`. `doi:10.1016/0370-2693(91)91715-8`.

[2] J. Fleischer, T. Riemann, A Complete algebraic reduction of one-loop tensor Feynman integrals, Phys. Rev. D83 (2011) 073004. `arXiv:1009.4436`, `doi:10.1103/PhysRevD.83.073004`.

Tab. B.1: The Appell function F_1 of the massive vertex integrals as defined in (B.2)**. As a proof of principle, only the constant term of the expansion in $d = 4 - 2\varepsilon$ is shown, $F_1(1;1,\frac{1}{2};2;x,y)$. Upper values: this calculation,** (Appendix B.2)**, lower values:** (B.6)**.**

$x - i\varepsilon_x$	$y - i\varepsilon_y$	$F_1(1;1,\frac{1}{2};2;x,y)$	
$+11.1 - 10^{-12}i$	$+12.1 - 10^{-12}i$	-0.1750442480735	$-0.0542281294732\,i$
		$-0.17504424807351877884498289912 6$	$-0.05422812947330402788209764116 7\,i$
$+11.1 - 10^{-12}i$	$+12.1 + 10^{-12}i$	$+1.7108545293244$	$+0.0542281294732\,i$
		$+1.7108545293243355713483820417 5$	$+0.05422812947148217381589270924\,i$
$+11.1 + 10^{-12}i$	$+12.1 - 10^{-12}i$	$+1.7108545304114$	$-0.0542281294732\,i$
		$+1.7108545293243355713483820417 5$	$-0.05422812947148217381589270924\,i$
$+11.1 + 10^{-12}\,i$	$+12.1 + 10^{-12}\,i$	-0.1750442480735	$+0.0542281294733\,i$
		$-0.17504424807351877884498289912 6$	$+0.05422812947330402788209764116 7\,i$
$+12.1 - 10^{-15}\,i$	$+11.1 - 10^{-15}\,i$	-0.1700827166484	$-0.0518684846037\,i$
$+12.1 - 10^{-10}\,i$	$+11.1 - 10^{-15}\,i$	$-0.1700827166480005810116574927 92$	$-0.05186848460465674976556525621\,i$
$+12.1 - 10^{-15}\,i$	$+11.1 + 10^{-15}\,i$	-0.1700827166484	$-1.7544202909955\,i$
		$-0.1700827166484402564726881739 9$	$-1.7544202909955768873584256203 8\,i$
$+12.1 + 10^{-15}\,i$	$+11.1 - 10^{-15}\,i$	-0.1700827166484	$+1.7544202909955\,i$
		$-0.1700827166484402564726881739 9$	$+1.7544202909955768873584256203 8\,i$
$+12.1 + 10^{-15}\,i$	$+11.1 + 10^{-15}\,i$	-0.1700827166484	$+0.0518684846037\,i$
$+12.1 - 10^{-10}\,i$	$+11.1 - 10^{-15}\,i$	$-0.1700827166480005810116574927 92$	$+0.05186848460465674976556525621\,i$
$+11.1 - 10^{-15}\,i$	-12.1	-0.0533705146518	$-0.1957692111557\,i$
		$-0.05337051465189944473349401152 1$	$-0.19576921115573398538892083369 3\,i$
$+11.1 + 10^{-15}\,i$	-12.1	-0.0533705146518	$+0.1957692111557\,i$
		$-0.05337051465189944473349401152 1$	$+0.19576921115573398538892083369 3\,i$
-11.1	$+12.1 - 10^{-12}\,i$	$+0.1060864084662$	$-0.1447440700082i$
		$+0.10608640847651064287133527599 4$	$-0.14474407002133340716734961908 8\,i$
-11.1	$+12.1 + 10^{-12}\,i$	$+0.1060864084662$	$+0.1447440700082i$
		$+0.10608640847651064287133527599 4$	$+0.14474407002133340716734961908 8\,i$
-12.1	-11.1	$+0.122456767687224028$	
		$+0.12245676768722402506513395161 30$	

[3] J. Fleischer, J. Gluza, A. Almasy, T. Riemann, Replacing 1-loop tensor reduction by contractions, talk held by T. Riemann at 11^{th} International Symposium on Radiative Corrections – Applications of Quantum Field Theory to Phenomenology – RADCOR 2013, 22 - 27 Sep 2013, Lumley Castle, UK, unpublished. See (11) and following equations. `https://conference.ippp.dur.ac.uk/event/341/session/8/contribution/56/material/slides/0.pdf`.

[4] G. 't Hooft, M. Veltman, Scalar one loop integrals, Nucl. Phys. B153 (1979) 365–401. `doi:10.1016/0550-3213(79)90605-9`.

[5] U. Nierste, D. Müller, M. Böhm, Two loop relevant parts of D-dimensional massive scalar one loop integrals, Z. Phys. C57 (1993) 605–614. `doi:10.1007/BF01561479`.

[6] G. Passarino, An approach toward the numerical evaluation of multiloop Feynman diagrams, Nucl. Phys. B619 (2001) 257–312. `arXiv:hep-ph/0108252`, `doi:10.1016/S0550-3213(01)00528-4`.

[7] O. Tarasov, Application and explicit solution of recurrence relations with respect to space-time dimension, Nucl. Phys. Proc. Suppl. 89 (2000) 237. `arXiv:hep-ph/0102271`, `doi:10.1016/S0920-5632(00)00849-5`.

[8] J. Fleischer, F. Jegerlehner, O. Tarasov, A new hypergeometric representation of one loop scalar integrals in d dimensions, Nucl. Phys. B672 (2003) 303. `arXiv:hep-ph/0307113`, `doi:10.1016/j.nuclphysb.2003.09.004`.

[9] D. B. Melrose, Reduction of Feynman diagrams, Nuovo Cim. 40 (1965) 181–213. `doi:10.1007/BF028329`.

[10] K. H. Phan, J. Blümlein, T. Riemann, Scalar one-loop vertex integrals as meromorphic functions of space-time dimension d, Acta Phys. Polon. B48 (2017) 2313. `arXiv:1711.05510`, `doi:10.5506/APhysPolB.48.2313`.

[11] J. Usovitsch, I. Dubovyk, T. Riemann, MBnumerics: Numerical integration of Mellin-Barnes integrals in physical regions, PoS LL2018 (2018) 046, `https://pos.sissa.it/303/046/pdf`. `arXiv:1810.04580`, `doi:10.22323/1.303.0046`.

[12] K. Phan, J. Blümlein and T. Riemann, Scalar one-loop Feynman integrals in arbitrary space-time dimension, talk held by T. Riemann at 14^{th} Workshop Loops and Legs in Quantum Field Theory (LL2018), April 29 - May 4, 2018, St. Goar, Germany, unpublished. `https://indico.desy.de/indico/event/16613/session/12/contribution/24/material/slides/0.pdf`.

Tab. B.2: The Appell function $F_1(1-\epsilon;1,\frac{1}{2};2-\epsilon;x_c,y_c)$ as defined in (B.2), needed for $d=4-2\varepsilon$.

$x - i\varepsilon_x$	$y - i\varepsilon_y$	$F_1(1-\epsilon;1,\frac{1}{2};2-\epsilon;x,y)$	
$+11.1-10^{-12}i$	$+12.1-10^{-12}i$	$+(-0.17504424807358806571$	$-0.05422812947328981004i)$
		$+(-0.00861885859131501092$	$-0.39051763820462137566i)\epsilon$
		$+(+0.37518853545319785781$	$-0.34047477405516524129i)\epsilon^2$
		$+(+0.49765461883470790694$	$-0.00717399489427550385i)\epsilon^3$
		$+(+0.32835724868237320395$	$+0.23005850008124251183i)\epsilon^4$
		$+(+0.11199125312340825478$	$+0.25409725390712356585i)\epsilon^5$
		$+(-0.00954795237038318610$	$+0.17050760870656256341i)\epsilon^6$
		$+(-0.04217861994524757518 5$	$+0.08576862780838478972 4i)\epsilon^7$

[13] K. H. Phan, J. Blümlein, T. Riemann, Scalar one-loop vertex integrals as meromorphic functions of space-time dimension d, slides of talk held at 41st International Conference of Theoretical Physics: Matter to the Deepest: Podlesice, Poland, September 4-8, 2017. http://indico.if.us.edu.pl/event/4/contribution/32/material/slides/0.pdf.

[14] J. Usovitsch, I. Dubovyk, T. Riemann, The MBnumerics project, section E.2 in [36].

[15] T. Hahn, M. Perez-Victoria, Automatized one loop calculations in four-dimensions and D-dimensions, Comput. Phys. Commun. 118 (1999) 153–165. arXiv:hep-ph/9807565, doi:10.1016/S0010-4655(98)00173-8.

[16] G. van Oldenborgh, FF: A package to evaluate one loop Feynman diagrams, Comput. Phys. Commun. 66 (1991) 1–15, https://lib-extopc.kek.jp/preprints/PDF/1990/9004/9004168.pdf. doi:10.1016/0010-4655(91)90002-3.

[17] J. Usovitsch, T. Riemann, New approach to Mellin-Barnes integrals for massive one-loop Feynman integrals, section E.6 in [36].

[18] P. Appell, Sur les fonctions hypergéométriques de plusieurs variables, les polynômes d'Hermite et autres fonctions sphériques dans l'hyperespace, Mémorial des sciences mathématiques, fascicule 3 (1925), p. 82. http://www.numdam.org/item?id=MSM_1925__3__1_0.

[19] S. Saran, Transformation of certain hypergeometric functions of three variables, Acta Math. 93 (1955) 293. doi:10.1007/BF02392525.

[20] Lauricella functions are generalizations of hypergeometric functions with more than one argument, see http://mathworld.wolfram.com/AppellHypergeometricFunction.html. Among them are $F_A^n, F_B^n, F_C^n, F_D^n$, studied by Lauricella, and later also by Campe de Ferrie. For n=2, these functions become the Appell functions F_2, F_3, F_4, F_1, respectively, and are the first four in the set of Horn functions. The F_1 function is implemented in the Wolfram Language as AppellF1[a,b1,b2,c,x,y] in two ways, see https://mathematica.stackexchange.com/questions/139810/how-is-the-function-appellf1-implemented-internally and the entry appellf1() in https://docs.sympy.org/0.7.5/modules/mpmath/functions/hypergeometric.html (22 Dec 2018) where it is claimed: Both representations ... are being used. But one can see, by looking at the pattern conditions, that the function is not implemented for all cases.

[21] F. Colavecchia, G. Gasaneo, J. Miraglia, Numerical evaluation of Appell's F_1 hypergeometric function, Comput. Phys. Commun. 138 (2001) 29–43.

[22] F. Colavecchia, G. Gasaneo, f1: a code to compute Appell's F1 hypergeometric function, Comput. Phys. Commun. 157 (2004) 32, catalogue identifier ADSJ in the CPC Program Library, http://www.cpc.cs.qub.ac.uk/. doi:10.1016/S0010-4655(03)00490-9.

[23] Daniel Sabanes Bove, Appell's F1 hypergeometric function, a wrapper for the Fortran code [22], https://cran.r-project.org/web/packages/appell/appell.pdf.

[24] L. Slater, Generalized hypergeometric functions, Cambridge University Press, Cambridge, 1966.

[25] M. Czakon, Automatized analytic continuation of Mellin-Barnes integrals, Comput. Phys. Commun. 175 (2006) 559. arXiv:hep-ph/0511200, doi:10.1016/j.cpc.2006.07.002.

[26] I. Dubovyk, J. Gluza, T. Riemann, J. Usovitsch, Numerical integration of massive two-loop Mellin-Barnes integrals in Minkowskian regions, PoS LL2016 (2016) 034, https://pos.sissa.it/260/034/pdf. arXiv:1607.07538.

[27] J. Usovitsch and T. Riemann, New approach to Mellin-Barnes integrals for massive one-loop Feynman integrals, section E.6 in [36]. arXiv:1809.01830.

[28] J. Usovitsch, MBnumerics, a Mathematica/Fortran package for the numerical calculation of multiple MB-integral representations for Feynman integrals at arbitrary kinematics, to be published at http://prac.us.edu.pl/~gluza/ambre/.

[29] L. Pochhammer, Über hypergeometrische Funktionen höherer Ordnungen, Crelle's Journal für die reine und angewandte Mathematik 71 (1870) 316. doi:10.1515/crll.1870.71.316.

[30] A. Erdelyi, Hypergeometric functions of two variables, Acta Math. 83 (1950), 131. https://projecteuclid.org/download/pdf_1/euclid.acta/1485888583.

[31] V. Bytev, M. Kalmykov, B. A. Kniehl, HYPERDIRE, HYPERgeometric functions DIfferential REduction: MATHEMATICA-based packages for differential reduction of generalized hypergeometric functions ${}_pF_{p-1}$, F_1,F_2,F_3,F_4, Comput. Phys. Commun. 184 (2013) 2332–2342. arXiv:1105.3565, doi:10.1016/j.cpc.2013.05.009.

[32] V. Bytev, B. A. Kniehl, HYPERDIRE, HYPERgeometric functions DIfferential REduction: Mathematica-based packages for the differential reduction of generalized hypergeometric functions: Horn-type hypergeometric functions of two variables, Comput. Phys. Commun. 189 (2015) 128–154. arXiv:1309.2806, doi:10.1016/j.cpc.2014.11.022.

[33] V. Bytev, M. Kalmykov, S. Moch, HYPERgeometric functions DIfferential REduction, (HYPERDIRE): MATHEMATICA based packages for differential reduction of generalized hypergeometric functions: F_D and F_S Horn-type hypergeometric functions of three variables, Comput. Phys. Commun. 185 (2014) 3041–3058. arXiv:1312.5777, doi:10.1016/j.cpc.2014.07.014.

[34] E. Picard, sur une extension aux fonctions de deux variables du problème de Riemann relatif aux fonctions hypergéométriques. Annales scientifiques de l'É.N.S. 2 e série 10, 305- 322 (1881). http://www.numdam.org/item?id=ASENS_1881_2_10__305_0.

[35] S. Saran, Transformations of certain hypergeometric functions of three variables, Acta Math. 93 (1955) 293–312. doi:10.1007/BF02392525. URL https://doi.org/10.1007/BF02392525

[36] A. Blondel, J. Gluza, S. Jadach, P. Janot, T. Riemann (eds.), Standard Model Theory for the FCC-ee: The Tera-Z, report on the mini workshop on precision EW and QCD calculations for the FCC studies: methods and techniques, CERN, Geneva, Switzerland, January 12-13, 2018; subm. as CERN Yellow Report. arXiv:1809.01830.