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The Heavy Fermion Contributions to the Massive Three Loop Form Factors

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Abstract

We compute the n_h terms to the massive three loop vector-, axialvector-, scalar- and pseudoscalar form factors in a direct analytic calculation using the method of large moments. This method has the advantage, that the master integrals have to be dealt with only in their moment representation, allowing to also consider quantities which obey differential equations, which are not first order factorizable (elliptic and higher), already at this level. To obtain all the associated recursions, up to 8000 moments had to be calculated. A new technique has been applied to solve the associated differential equation systems. Here the decoupling is performed such, that only minimal depth ε–expansions had to be performed for non–first-order factorizing systems, minimizing the calculation of initial values. The pole terms in the dimensional parameter ε can be completely predicted using renormalization group methods, as confirmed by the present results. A series of contributions at $O(\varepsilon^0)$ have first order factorizable representations. For a smaller number of color–zeta projections this is not the case. All first order factorizing terms can be represented by harmonic polylogarithms. We also obtain analytic results for the non–first-order factorizing terms by Taylor series in a variable x, for which we have calculated at least 2000 expansion coefficients, in an approximation. Based on this representation the form factors can be given in the Euclidean region and in the region $q^2 \approx 0$. Numerical results are presented.

1 Introduction

The knowledge of the massive three–loop form factor is essential ingredient to the calculation for a series of massive processes at e^+e^- and hadron colliders, determined by vector, axialvector, scalar and pseudoscalar currents. It has been calculated to two–loop order in Refs. [1–6]. At three–loop order the color planar contributions have been computed in Refs. [7–12] and its asymptotic behaviour has been studied in [13, 14], including partial results at four–loop order.

In the present paper, we compute the n_h contributions of the massive three–loop form factor for vector, axialvector, scalar and pseudoscalar currents. As the basic computational method we use the method of arbitrarily large moments [15]. Here the differential equations given by the integration by parts (IBP) relations [16–23] are transformed into recursions, through which a large number of moments for the master integrals and the form factors are generated using the package `SolveCoupledSystems` [15]. These moments are sequences in $\mathbb{Q}$ parameterized by multiple zeta values (MZVs) [25] and color factors. Using the method of guessing [26][1] we determine the associated difference equations, which are finally solved using `Sigma` [24,28]. In the expansion of the form factors to master integrals usually higher order terms in $\varepsilon = 2 - D/2$ are contributing. These are containing, however, elliptic and more involved contributions. Although being present, these terms cannot be distinguished from the simpler contributions considering moments since they appear only encoded as rational numbers. The advantage of the present method is that it always allows to obtain difference equations for all MZV and color projections. In the case of the pole terms and a large number of contributions at $O(\varepsilon^0)$ the corresponding difference equations are first order factorizable and can therefore be solved by `Sigma`. In case of the remaining terms we are able to factorize the first order factors. The remainder terms need other methods to be solved. The first order factorizable contributions are given by iterative integrals, cf. [29]. In the present case these iterative integrals are harmonic polylogarithms (HPLs) [30].

The paper is organized as follows. After some basic definitions given in Section 2 main steps of the calculation are described in Section 3. In Section 4 the universal infrared structure of the form factors is presented which is later compared with the unrenormalized three–loop form factors providing a check of the calculation. In Section 5 we describe a new decoupling strategy, which allows to work with a minimal–depth expansion concerning the initial values. A brief summary on the recurrences, which are not factorizing to first order, is given in Section 6. In Section 7 we present the analytic results for the n_h–contributions to the different form factors and also give numerical illustrations. Section 8 contains the conclusions. In the appendix we present a series of deeper ε–expansions for some integrals defining the initial conditions.

2 The Form Factors

The basic structure of the massive form factors has been described in Ref. [6] before. We consider vector, axialvector, scalar and pseudoscalar currents coupling to a heavy quark pair of mass m

$$\overline{u}_c(q_1) X_{cd} v_d(q_2), \tag{2.1}$$

with $q = q_1 + q_2$. The main variable considered is x given by

$$\frac{q^2}{m^2} = z = -\frac{(1-x)^2}{x}. \tag{2.2}$$

[1]For an early application of this method in perturbative calculations in Quantum Field Theory, cf. [27].

We work in $D = 4 - 2\varepsilon$ dimensions. For the treatment of γ_5 in the axialvector and pseudoscalar case we consider here only the non-singlet contributions, where γ_5 can be treated anticommuting.

We consider the decay amplitude (Γ^μ) of the Z-boson into a pair of heavy quarks. The general structure of Γ^μ consists of six form factors, two of which are CP odd. As we consider only higher order QCD effects and Standard Model (SM) neutral current interactions to lowest order, the CP invariance holds. This implies that Γ^μ has four form factors $F_{V,i}(s), F_{A,i}(s)$ $i = 1, 2$ comprising the following general form

$$\begin{aligned}\Gamma^\mu_{cd} &= \Gamma^\mu_{V,cd} + \Gamma^\mu_{A,cd}\\ &= -i\delta_{cd}\Big[v_Q\Big(\gamma^\mu\ F_{V,1} + \frac{i}{2m}\sigma^{\mu\nu}q_\nu\ F_{V,2}\Big) + a_Q\Big(\gamma^\mu\gamma_5\ F_{A,1} + \frac{1}{2m}q^\mu\gamma_5\ F_{A,2}\Big)\Big] \qquad (2.3)\end{aligned}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, $q = q_1 + q_2$, and v_Q and a_Q are the SM vector and axial-vector coupling constants as defined by

$$v_Q = \frac{e}{\sin\theta_w\cos\theta_w}\Big(\frac{T_3^Q}{2} - \sin^2\theta_w Q_Q\Big), \qquad a_Q = -\frac{e}{\sin\theta_w\cos\theta_w}\frac{T_3^Q}{2}\,. \qquad (2.4)$$

e is the charge of positron, θ_w is the weak mixing angle, T_3^Q is the third component of the weak isospin, and Q_Q is the charge of the heavy quark.

In case of the vector and axialvector form factors it is convenient to use their decomposition into two parts, respectively, which are labeled by the functions

$$g^1_{V,1} = \frac{x}{4(1-\varepsilon)(1+x)^2} \qquad g^2_{V,1} = \frac{(3-2\varepsilon)x^2}{(1-\varepsilon)(1+x)^4} \qquad (2.5)$$

$$g^1_{V,2} = \frac{x^2}{(1-\varepsilon)(1-x^2)^2} \qquad g^2_{V,2} = \frac{2x^2[-1+\varepsilon(1-x)^2+(4-x)x]}{(1-\varepsilon)(1-x)^2(1+x)^4} \qquad (2.6)$$

$$g^1_{A,1} = \frac{x}{4(1-\varepsilon)(1+x)^2} \qquad g^2_{A,1} = \frac{x^2}{(1-\varepsilon)(1-x^2)^2} \qquad (2.7)$$

$$g^1_{A,2} = \frac{x^2}{(1-\varepsilon)(1-x^2)^2} \qquad g^2_{A,2} = \frac{2x^2[1-\varepsilon(1+x)^2+x(4+x)]}{(1-\varepsilon)(1-x)^4(1+x)^2}. \qquad (2.8)$$

We will use this decomposition throughout the present calculation.

Furthermore, we consider a general neutral particle h that couples to heavy quarks through the following Yukawa interaction

$$\mathcal{L}_{int} = -\frac{m}{v}\Big[s_Q\bar{Q}Q + ip_Q\bar{Q}\gamma_5 Q\Big]h, \qquad (2.9)$$

where m denotes the heavy quark mass, $v = (\sqrt{2}G_F)^{-1/2}$ is the SM Higgs vacuum expectation value, with G_F being the Fermi constant, s_Q and p_Q are the scalar/pseudo-scalar coupling, respectively, and Q and h are the heavy quark and scalar and pseudo-scalar field, respectively. The decay amplitude of $h \to \bar{Q} + Q$, $X_{cd} \equiv \Gamma_{cd}$, consists of two form factors with the following general structure

$$\begin{aligned}\Gamma_{cd} &= \Gamma_{S,cd} + \Gamma_{P,cd}\\ &= -\frac{m}{v}\delta_{ij}\,\Big[s_Q\,F_S + ip_Q\gamma_5\,F_P\Big], \qquad (2.10)\end{aligned}$$

where F_S and F_P denote the renormalized scalar and pseudo-scalar form factors, respectively. The form factors obey the expansion

$$F_{i,l}(x, a_s) = \delta_{l,1} + \sum_{k=1}^{\infty} a_s^k F_i^{(k)}(x), \qquad (2.11)$$

with $i = V, A, S, P$ and $l = 1, 2$ for $i = A, V$ and $a_s = \alpha_s/(4\pi)$ denotes the strong coupling constant.

3 The Calculation

The diagrams of the n_h–contributions of the different massive three loop form factors are generated using `QGRAF` [31] and the color structures are evaluated using `Color` [32]. Furthermore, we used `q2e/exp` [33,34] and perform the Dirac-algebra using `Form` [35,36]. In the present paper we perform the calculation of the form factor for QCD, setting $C_A = N_c = 3, C_F = (N_c^2-1)/(2N_c) = 4/3$ and $T_F = 1/2$, to reduce the complexity of the problem. The only free parameters are the numbers of equal mass heavy flavors n_h and massless flavors n_l, which partly occur together with zeta-values. In the expansions, in which we mainly work, this decomposition is unique. By transforming to x–space additional ζ–terms may occur e.g. due to regularizations. The IBP reduction is performed using the package `Crusher` [23] and systems of linear ordinary differential equations are obtained for the master integrals. To the n_h case 14 families with a total of 103 master integrals contribute. We map to the new variable y

$$x = 1 - y \tag{3.1}$$

in which the master integrals obey the Taylor expansions

$$M_k(\varepsilon, x) = \sum_{l=0}^{\infty} \tilde{m}_{k,l}(\varepsilon) y^l. \tag{3.2}$$

The systems of differential equations are thus mapped into associated systems of difference equations. To work with moments for the master integrals has the advantage that also general non–first order factorizing cases can be treated. The difference equations are now solved consecutively starting from the highest pole in ε working through to the required power in the dimensional parameter ε. The way of decoupling is here very important, cf. [37] and Section 5 for details, since the required depth in expanding in ε for the associated initial values, may easily go beyond the level, which is currently known. We have performed those extensions in a series of cases, see Appendix A. In general those extensions are costly and time consuming. In the present case they could be avoided and the initial values are either known from the literature [38] or having been calculated for other previous applications, cf. [39–42], turned out to be sufficient. We could work with a minimal number of additional terms in the ε-expansion, here in the non–first order factorizing case.

For each color–zeta projection of the given massive form factor one obtains a series of rational numbers $m_{k,l,n}$, where $-n$ labels the power in ε. We seek now a minimal recurrence determined by the set

$$\{m_{k,l,n} | l = 0..N_{\max}\}. \tag{3.3}$$

For each family we obtain a system of differential equations, the largest of which has a coefficient matrix of 7×7. The form factor can then be rewritten as

$$F^{(3)}(x) = \sum_{k=3}^{0} \frac{1}{\varepsilon^k} F_{-k}^{(3)}(y), \qquad F_{-k}^{(3)}(y) = \sum_{l=0}^{\infty} a_{-k,l} y^l. \tag{3.4}$$

Here the expansion coefficients $a_{-k,l}$ obey difference equations in l for $k \in \{3, 2, 1, 0\}$, which are parameterized by polynomials of color factors and multiple zeta values (MZVs) [25] over

$\mathbb{Q}$, irrespective of the fact whether in the general l representation elliptic or higher structures contribute or not, see also [43, 44].

Also the master integrals are rewritten in moment–form (3.4). Their recurrences are finally used to calculate a large set of moments, assembled to $a_{k,l}$, using the method of large moments of Ref. [15], which are projected to the different contributing monomials in MZVs and n_l, n_h. The number of moments, now given by sequences of rational numbers, needs to be large enough to allow the determination of their recurrence by the method of guessing [26], implemented in `Sage` [45], which has been successfully applied in different calculations in Refs. [15, 27] before.

In the case of the pole terms of the three–loop form factors all the corresponding recurrences have to be solvable in difference rings [24, 28, 46–58], since they are expected to factorize at first order. The solution can therefore be found using the package `Sigma` [24, 28]. This may be as well the case for some of the MZV-factors contributing to the constant term, as in the case for the massive operator matrix element $A_{Qg}^{(3)}$, cf. Ref. [15].

As result one obtains representations of $a_{k,l}$ in terms of harmonic sums [59,60] and generalized harmonic sums [61, 70] in l. The latter ones occur because of the necessary transformation $x \rightarrow (1 - y)$ to also deal with the logarithmic contributions $\propto \ln^m(x)$. Cyclotomic or finite binomial sums [62, 63] do not contribute.

The infinite sums appearing in (3.4) can now be performed using a series of procedures of the package `HarmonicSums` [59–67] and one obtains harmonic polylogarithms (HPLs) [30] or Kummer–Poincaré iterated integrals [61, 68–70] in the variable y. The transformation $y = 1 - x$ then yields representations in terms of HPLs in x, which are further reduced to suitable bases, to reduce the numbers of contributing functions as much as possible.

We remark, that the different master integrals for the present representation partly need deep expansion in the dimensional variable ε. If we needed to write them in real terms, also elliptic and even higher integral representations would be needed in explicit form. It is an advantage of the present method that these contributions do earliest show up in the factorization of some of the recursions to be solved for the physical quantity under consideration, but cancel otherwise. The automated solution of differential equations over general bases, presented in Ref. [12], working in the case of first order factorization, can therefore not be applied here.

In the following we discuss a sample calculation to illustrate the general method. For all computation we used the `qftquad`–cluster equipped with Xeon Gold 6128 and Xeon 6C E5-2643v4 processors. For this we choose all contributions to the massive three–loop vector form factors $\propto n_l$. They have been calculated up to $O(\varepsilon^0)$ by using different methods in Refs. [10, 12]. By setting $N_c = 3$, 60 different recurrences have to be found and solved. A linear combination of these quantities yields the vector current form factors $F_{V,1}$ and $F_{V,2}$. The reduction to master integrals led to maximally 5×5 systems, which were decoupled by using the Gauss approach implemented in the package `Oresys` [108] with a decoupling time of 3.1 min. 28 integrals contribute. The largest depth of the initial values to be provided were 10 moments with an ε-expansion up to ε^3.

The most demanding terms were $g_1 n_l$ and $g_2 n_l$. All other contributions are significantly simpler. For $g_1 n_l$ 3000 generated moments were not enough to determine the associated recurrence. Therefore we generated 4000 moments, which took 12.88 days instead of 9.96 days. The guessing time amounted to 16.1 min and 23.2 min and led to recurrences of degree and order $(\mathsf{d} = 474, \mathsf{o} = 22), (\mathsf{d} = 537, \mathsf{o} = 35)$ for the two largest cases, respectively. Their solution using `Sigma` and representation in terms of HPLs in the variable x took 3.28 days. In the representation in the variable y 13 harmonic sums and 185 generalized harmonic sums, transcendent to each other, contributed. In x–space we obtain representations in terms of 55 HPLs. The results agree with those given in Refs. [10, 12].

The needed computational times of the present example let it appear feasible to compute the pole terms and some of the terms of ε^0 of the massive three–loop form factor. Here a wider range of initial values, to high order in ε, is needed. We therefore had to extend the results given in [38] significantly.

4 The universal infrared structure of QCD amplitudes

Scattering amplitudes in perturbative QCD contain infrared singularities arising from soft gluon contributions and collinear parton divergences. In this respect much work has been performed for massless scattering amplitudes [71–76]. Especially in the case of massless QCD amplitudes with two partons, i.e. the form factors, the infrared (IR) structure becomes interesting due to its prominent form in terms of anomalous dimensions. The interplay of the collinear and soft anomalous dimensions to shape the singular structure of the massless form factors was noticed in [76] at two–loop order and later established at three–loop order in [77].

In the case of massive form factors, the finding of a Sudakov type integro-differential equation was a challenge as the massive form factors do not exponentiate. However, in the asymptotic limit i.e in the limit where the quark mass is small compared to the center of mass energy, the massless QCD corrections to the massive form factor do exponentiate. In [78], the first step was taken by obtaining the singular behavior of massive QCD amplitudes in the asymptotic limit. Meanwhile, a factorization theorem was also proposed in [79, 80], also in the asymptotic limit. Recently, following the method proposed for massless form factors in [81, 82], a rigorous study has been performed in [14] in the asymptotic limit to obtain all the poles and also all logarithmic contributions to finite pieces of the three loop heavy quark form factors for vector, axial-vector, scalar and pseudo-scalar currents. In this scenario, one can relate the massless form factors to the massive ones and hence use the massless results [83, 84] to obtain these predictions.

A general IR structure is needed for the exact computation, which was obtained in [85], following a soft–collinear effective theory (SCET) approach at two-loop. However, the argument can be extended to three–loop appropriately. The IR singularities of the massive form factors can be factorized as a multiplicative renormalization factor, whose structure is constrained by the renormalization group equation (RGE), as follows,

$$F_I(\alpha_s, x, \varepsilon) = Z(\alpha_s, x, \varepsilon, \mu) F_I^{\rm fin}(\alpha_s, x, \varepsilon, \mu)\,, \tag{4.1}$$

where $F_I^{\rm fin}$ is finite as $\varepsilon \to 0$. We note that Z does not carry any process dependent information (I). Here μ is the scale introduced corresponding to this particular factorization. Now one can write down the renormalization group equation (RGE) for Z which is characterized by the massive cusp anomalous dimension, Γ. However, before we proceed, we note that Γ does not contain any contributions from internal heavy quark loops, i.e. like the QCD β function, or light quark mass anomalous dimension, the massive cusp anomalous dimension also has been computed considering the massless QCD corrections with n_l light quark flavors. On the other hand, the form factors are defined for (n_l+1) flavors. Hence, Γ cannot describe the singularities arising in case of a massive quark loop contributions to the heavy quark form factors. To overcome the hurdle, the immediate solution is to use the decoupling relations [86–91]. To obtain these decoupling relations, one constructs an effective theory with n_l light quark flavors and then demands consistency with the full theory of $n_l + n_h$ flavors by relating the couplings and light quark masses in the two cases. For our case, we need the decoupling relation for the strong coupling constant i.e. the relation between $\bar{\alpha}_s$ and α_s, where $\bar{\alpha}_s$ is defined for an effective theory with n_l light quark only and α_s is defined for the full theory with $n_l + n_h$ quark flavors.

Keeping this in mind, we now write down the RGE for $\bar{Z}$, the equivalent of Z in the effective theory with n_l light quark, which reads

$$\frac{d}{d\ln\mu}\ln\bar{Z}(\alpha_s,x,\varepsilon,\mu) = -\Gamma(\alpha_s,x,\mu)\,. \tag{4.2}$$

Γ is by now available up to the three–loop level [92–95]. Both $\bar{Z}$ and Γ can be expanded in a perturbative series in α_s as follows

$$\bar{Z} = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n \bar{Z}^{(n)}\,, \qquad \Gamma = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^{n+1}\Gamma_n\,. \tag{4.3}$$

Next, we solve the RGE, Eq. (4.2), in massless QCD, i.e. considering only n_l light quarks.

$$\begin{aligned}\bar{Z} = 1 &+ \left(\frac{\bar{\alpha}_s}{4\pi}\right)\left[\frac{\Gamma_0}{2\varepsilon}\right] + \left(\frac{\bar{\alpha}_s}{4\pi}\right)^2\left[\frac{1}{\varepsilon^2}\Big(\frac{\Gamma_0^2}{8} - \frac{\bar{\beta}_0\Gamma_0}{4}\Big) + \frac{\Gamma_1}{4\varepsilon}\right] \\ &+ \left(\frac{\bar{\alpha}_s}{4\pi}\right)^3\left[\frac{1}{\varepsilon^3}\left(\frac{\Gamma_0^3}{48} - \frac{\bar{\beta}_0\Gamma_0^2}{8} + \frac{\bar{\beta}_0^2\Gamma_0}{6}\right) + \frac{1}{\varepsilon^2}\left(\frac{\Gamma_0\Gamma_1}{8} - \frac{\bar{\beta}_1\Gamma_0 + \bar{\beta}_0\Gamma_1}{6}\right) + \frac{1}{\varepsilon}\left(\frac{\Gamma_2}{6}\right)\right] + \mathcal{O}(\bar{\alpha}_s^4)\,.\end{aligned} \tag{4.4}$$

$\bar{\beta}$ is the QCD β function for n_l light quark flavors. Now to obtain Z from $\bar{Z}$. We use the following decoupling relation obtained using the background field method [91,96,97] to obtain the relation between $\bar{\alpha}_s$ and α_s

$$\bar{\alpha}_s = \alpha_s\Big[1 - \left(\frac{\alpha_s}{4\pi}\right)\left(\frac{2}{3}\zeta_2 n_h T_F\varepsilon + \mathcal{O}(\varepsilon^2)\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{32}{9}C_A T_F n_h - 15 C_F T_F n_h + \mathcal{O}(\varepsilon)\right) + \mathcal{O}(\alpha_s^3)\Big] \tag{4.5}$$

and obtain

$$\begin{aligned}Z &= \bar{Z}\big|_{\bar{\alpha}_s\to\alpha_s} \\ &= 1 + \left(\frac{\alpha_s}{4\pi}\right)\left[\frac{\Gamma_0}{2\varepsilon}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{\varepsilon^2}\Big(\frac{\Gamma_0^2}{8} - \frac{\bar{\beta}_0\Gamma_0}{4}\Big) + \frac{\Gamma_1}{4\varepsilon}\right] \\ &+ \left(\frac{\alpha_s}{4\pi}\right)^3\left[\frac{1}{\varepsilon^3}\left(\frac{\Gamma_0^3}{48} - \frac{\bar{\beta}_0\Gamma_0^2}{8} + \frac{\bar{\beta}_0^2\Gamma_0}{6}\right) + \frac{1}{\varepsilon^2}\left(\frac{\Gamma_0\Gamma_1}{8} - \frac{\bar{\beta}_1\Gamma_0 + \bar{\beta}_0\Gamma_1}{6}\right)\right. \\ &+ \left.\frac{1}{\varepsilon}\left(\frac{\Gamma_2}{6} + 2\Big(\frac{\Gamma_0^2}{8} - \frac{\bar{\beta}_0\Gamma_0}{4}\Big)\Big(-\frac{2}{3}\zeta_2 n_h T_F\Big) + \frac{\Gamma_0}{2}\Big(\frac{32}{9}C_A T_F n_h - 15 C_F T_F n_h\Big)\right)\right] + \mathcal{O}(\bar{\alpha}_s^4)\,.\end{aligned} \tag{4.6}$$

The above representation leads to the prediction of the pole terms for all the massive form factors to three–loop order, which has been an open problem in Ref. [78].

5 Refined versions of the large moment method

We will now describe a general toolbox that enables one to calculate large numbers of moments in the integer variable, say $n = 0, 1, 2, \ldots, \mu$, for a finite number of Feynman integrals $F_i(n,\varepsilon)$ with $1 \le i \le \lambda$. Here the moments $F_i(n,\varepsilon)$ depend also on the dimensional parameter ε and the corresponding ε-expansion

$$F_i(n,\varepsilon) = \sum_{k=l}^{r_i} F_{i,k}(n)\,\varepsilon^k + O(\varepsilon^{r_i+1}) \tag{5.1}$$

is calculated for each moment $n = 0, \ldots, \mu$ up to the order r_i. Standard procedures like `Mincer` [98] or `MATAD` [99] allow the calculation of a comparable small number of moments, e.g., $\mu \lesssim 50$. The idea to calculate expansions from differential equations has already frequently been used, see e.g. [100–106]. Only recently, we obtained a new method in [15] that can compute thousands of such moments. In general, this method assumes that their (formal) power series representations

$$f_i(x,\varepsilon) = \sum_{n=0}^{\infty} F_i(n,\varepsilon) x^n \tag{5.2}$$

are a solution of a given coupled system

$$D_x \begin{pmatrix} f_1(x,\varepsilon) \\ \vdots \\ f_\lambda(x,\varepsilon) \end{pmatrix} = A(x,\varepsilon) \begin{pmatrix} f_1(x,\varepsilon) \\ \vdots \\ f_\lambda(x,\varepsilon) \end{pmatrix} + \begin{pmatrix} g_1(x,\varepsilon) \\ \vdots \\ g_\lambda(x,\varepsilon) \end{pmatrix}, \tag{5.3}$$

with $A(x,\varepsilon)$ being an invertible $\lambda \times \lambda$ matrix with entries from the polynomial ring[2] $\mathbb{K}[x,\varepsilon]$ and where the inhomogeneous parts $g_i(x,\varepsilon)$ are given in terms of linear combinations of simpler master integrals. Here we assume that their moments $G_{i,k}(n)$, $n = 1, \ldots, \mu$, with

$$\begin{aligned} g_i(x,\varepsilon) &= \sum_{n=0}^{\infty} G_i(x,\varepsilon) x^n, \\ G_i(n,\varepsilon) &= \sum_{k=l}^{r'_i} G_{i,k}(n)\, ep^k + O(\varepsilon^{r'_i+1}) \end{aligned} \tag{5.4}$$

can be determined by

- other coupled systems to which the large moment method under consideration is applied recursively;
- symbolic summation or integration methods [24,28,107] that yield representations in terms of indefinite nested sums or integrals from which one can produce a large number of moments;
- by standard procedures like `Mincer` [98] or `MATAD` [99] if only a small number of moments contributes.

Summarizing, it is assumed that already $\mu + 1$ moments for the inhomogeneous parts $g_i(x,\varepsilon)$ in (5.3) are computed. Then given such an input, we propose the following strategy to compute the first $\mu + 1$ moments for $f_1(x,\varepsilon), \ldots, f_\lambda(x,\varepsilon)$.

Strategy 1:

1. Uncouple the system (5.3). Experiments showed that the implementation of Gauss's elimination method in `OreSys` [108] is an excellent choice. In general one obtains k linear

[2]We suppose that $\mathbb{K}$ is a computable field containing the rational numbers $\mathbb{Q}$ as a sub–field.

differential equations of the form

$$
\begin{aligned}
\sum_{i=0}^{o_1} b_{1,i}(x,\varepsilon) D_x^i f_1(x,\varepsilon) &= \sum_{i,j} d_{1,i,j}(x,\varepsilon) D_x^i g_j(x,\varepsilon)\\
&\vdots\\
\sum_{i=0}^{o_k} b_{k,i}(x,\varepsilon) D_x^i f_k(x,\varepsilon) &= \sum_{i,j} d_{k,i,j}(x,\varepsilon) D_x^i g_j(x,\varepsilon) + \sum_{j=1}^{k-1}\sum_i e_{k,i,j}(x,\varepsilon) D_x^i f_j(x,\varepsilon) \qquad (5.5)\\
&\vdots\\
\sum_{i=0}^{o_\lambda} b_{\lambda,i}(x,\varepsilon) D_x^i f_\lambda(x,\varepsilon) &= \sum_{i,j} d_{\lambda,i,j}(x,\varepsilon) D_x^i g_j(x,\varepsilon) + \sum_{j=1}^{\lambda-1}\sum_i e_{\lambda,i,j}(x,\varepsilon) D_x^i f_j(x,\varepsilon)
\end{aligned}
$$

for explicitly given polynomials $b_{k,i}(x,\varepsilon) \in \mathbb{K}[x,\varepsilon]$ and rational functions $d_{k,i,j}(x,\varepsilon)$, $e_{k,i,j}(x,\varepsilon) \in \mathbb{K}(x,\varepsilon)$. Here the kth equation, $1 \le k \le \lambda$, is considered as a linear differential equation of order o_k in $f_k(x,\varepsilon)$ where the right hand side is given in terms of the inhomogeneous parts $g_j(x,\varepsilon)$, the functions $f_1(x,\varepsilon),\dots,f_{k-1}(x,\varepsilon)$, that will be treated already within our iterative method, and their derivatives. In general it can happen that the orders o_i might be larger than λ. However, in all our calculations we ended up at surprisingly nice orders; see also Remark 5.1 below.

2. We suppose that the greatest common divisor of the coefficients $b_{1,0}(x,\varepsilon),\dots,b_{1,o_1}(x,\varepsilon)$ in the first equation (5.5) is 1, i.e., there is no common polynomial factor that depends on x or ε. This implies that there is at least one i such that $b_{1,i}(x,0) \neq 0$. In addition, let $p_1(x) \in \mathbb{K}[x]$ be the greatest common divisor of $b_{1,0}(x,0),\dots,b_{1,o_1}(x,0)$, i.e., $p_1(x)$ contains all common polynomial factors in x of the $b_{1,i}(x,0)$. Dividing the first equation of (5.5) by $p_1(x)$ yields

$$\sum_{i=0}^{o_1} \frac{b_{1,i}(x,\varepsilon)}{p_1(x)} D_x^i f_1(x,\varepsilon) = \sum_{i,j} \frac{d_{1,i,j}(x,\varepsilon)}{p_1(x)} D_x^i g_j(x,\varepsilon). \qquad (5.6)$$

Plugging (5.4) into the right-hand side of (5.6) (with r_1' sufficiently large) and performing its ε-expansion up to r_i and its x-expansion up to μ produces[3]

$$\sum_{i=0}^{o_1} \frac{b_{1,i}(x,\varepsilon)}{p_1(x)} D_x^i f_1(x,\varepsilon) = \sum_{k=l}^{r_1}\sum_{n=0}^{\mu} \tilde{G}_k(n)\varepsilon^k x^n + O(\varepsilon^{r_1+1}x^{\mu+1}). \qquad (5.7)$$

Consequently, by coefficient comparison in (5.7) w.r.t. ε^l we obtain the linear differential equation

$$\sum_{i=0}^{o_1} \tilde{b}_i(x) D_x^i \sum_{n=0}^{\infty} F_{1,l}(n)x^n = \sum_{n=0}^{\infty} \tilde{G}_l(n)\, x^n$$

with

$$\tilde{b}_i(x) = \frac{b_i(x,0)}{p_i(x)} \in \mathbb{K}[x],$$

[3]Efficient (and parallelized) methods have been implemented in the package `SolveCoupledSystem.m` in order to calculate the moments $\tilde{G}_k(n)$ efficiently from the given moments $G_{i,k}(n)$ in (5.4).

not all 0. Finally, by coefficient comparison w.r.t. x^n in the last equation we get a linear recurrence relation

$$\sum_{i=0}^{\nu_1} \beta_i(n) F_{1,l}(n+i) = \tilde{G}_{l+l_1}(n) \tag{5.8}$$

of order ν_1 for some polynomials $\beta_i(n) \in \mathbb{K}[n]$ with $\beta_{\nu_1}(n) \neq 0$ and an integer $l_1 \in \mathbb{Z}$. In particular, the following bound on the order of the recurrence holds:

$$\nu_1 \leq o_1 + \max_i \deg_x(\tilde{b}_i(x)) = o_1 + \max_i \deg_x(b_{1,i}(x,0)) - \deg_x(p_1(x)); \tag{5.9}$$

in the generic case we have equality – in any case the above bound is a good indication which order we can expect. Using this recurrence plus the first ν_1 initial values of $F_{1,l}$, one can now calculate[4] in linear time the moments for $F_{1,l}(n)$ for $n = 0, \dots, \mu$.

3. Next, we plug in these moments $F_{1,l}(n)$ with $n = 0, \dots, \mu$ into (5.7) and update its right-hand side. In this way, l is replaced by $l+1$ and $F_{1,l+1}$ takes over the role of $F_{1,l}$. Now we repeat this method for $k = l+1, \dots, r_1$ to get the moments of the remaining ε-contributions; we remark that the coefficients on the left-hand side of the recurrence (5.8) remain unchanged - only the inhomogeneous part on the right-hand side has to be updated.

4. Finally, we repeat the steps (2)–(3) for the second equation ($k = 2$) in (5.5) using the moments of $F_{1,k}(n)$. Together with sufficiently many initial values, say ν_2, we can calculate the moments $F_2(n, \varepsilon)$ for $n = 0 \dots, \mu$. Similarly, we calculate iteratively the moments for $F_k(n, \varepsilon)$ with $k = 3, \dots, \lambda$.

Note that the orders ν_i (for $i = 1$ see (5.8)) are usually small (in many cases ≤ 5 and in harder cases ~ 25), but μ can be arbitrarily large (e.g., $\mu = 2000$ or $\mu = 8000$).

Remark 5.1. In the original version of the large moment method [15] we followed a slightly different approach. (1) For the uncoupling task we used Zürcher's algorithm [109] from `OreSys` [108]. This method provides a system (5.3) where usually only the first equation is of higher-order (namely of order λ). All other equations are of order 0, i.e., $o_2 = \dots = o_\lambda = 0$. In general, such an uncoupled system leads to a simpler method to calculate the desired moments. On the other side, in all our calculations it turned out that the Gaussian elimination method delivered much smaller orders than λ. As a consequence, switching to the Gaussian elimination tactic, the recurrence orders can be reduced considerably (compare the order bound (5.9)). In addition, using the Gaussian uncoupling method, the coefficients $b_{1,i}(x, \varepsilon)$ are less complicated and factors of the form $\frac{1}{\varepsilon^q}$ arise with smaller $q \in \mathbb{N}$. As a consequence, using the Gaussian uncoupling strategy we could decrease notably the necessary orders r_i of the ε-expansions.
(2) Furthermore, our original method from [15] differs as follows: the first differential equation in (5.5) (and usually the only differential equation) is directly transformed to a linear recurrence in n whose coefficients also depend on ε. This allows one to utilize a rather efficient machinery [15, 110] to compute the moments $F_{1,k}(n)$. However, in order to compute such a recurrence as a preprocessing step, operations have to be carried out in $\mathbb{K}(x, \varepsilon)$ which are rather costly. In addition, this approach misses the opportunity to cancel the polynomial $p_1(x)$ (which in applications is usually large) and thus to reduce the recurrence order substantially (compare again the order bound (5.9)).
Summarizing, with our improved Strategy 1, the recurrence orders could be reduced significantly and the required ε-orders r_i can be kept rather small.

[4]Special care has to be taken if $\beta_{n_1}(n)$ has integer zeroes: this might require extra initial values. Furthermore, it might happen that $|l_1|$ extra moments for $G_{1,l}(n)$ are needed.

Strategy 1 has been implemented in the package `SolveCoupledSystem.m` and works for the calculations of the n_h contributions in many cases highly efficient.

Example. For a system (5.3) with $\lambda = 4$ we needed the moments (5.1) for $n = 0, \ldots, \mu$ up to the orders $r_1 = r_3 = r_4 = 4$ and $r_2 = 3$. The uncoupled system has the form (5.5) with $o_1 = 4$, $o_2 = 2$, $o_3 = 0$ and $o_4 = 0$. Using this output, we obtain a recurrence of the form (5.8) for $F_{1,l}(n)$ of order $\nu_1 = 4$. We note that this small recurrence order was possible by sneaking in the polynomial $p_1(x) \in \mathbb{Q}[x]$ of degree 13 within the linear differential equation (5.7); setting $p_1(x) = 1$ would have delivered a recurrence of order $4+13 = 17$. Similarly, we obtain recurrences for $F_{2,l}(n), F_{3,l}(n), F_{4,l}(n)$ of orders $2, 0, 0$, respectively (again we reduced the recurrence order from 9 to 2 for $F_2(n, \varepsilon)$ by factoring out a polynomial $p_2(x)$ of degree 7). Finally, with the corresponding initial values, we calculated the moments (5.1) for $n = 0, \ldots, \mu$. E.g., for $\mu = 4000$ we needed 14000 seconds (257723 CPU seconds[5]), for $\mu = 6000$ we needed 25598 seconds (257723 CPU seconds) and for $\mu = 8000$ we needed 62063 seconds (678321 CPU seconds) in order to get the moments of the rational contribution (ignoring the moments that depend on ζ_2, ζ_3 etc.).

In general, Strategy 1 turned out to be optimal for systems with the dimension $\lambda \leq 4$. However, if $\lambda \geq 5$, the uncoupling step (in particular, using Gaussian elimination or Zürcher's algorithm) failed by space-time resources or produced a not digestible output: the degrees of the polynomials $b_{i,j}(x, \varepsilon)$ in (5.5) were very high yielding linear recurrences with orders close to 1000.

For these more complicated systems $\lambda \geq 5$ we developed another variant of our large moment method, that is also implemented in our package `SolveCoupledSystem.m`.

Strategy 2. Here we assume that not only the matrix $A(x, \varepsilon)$ in (5.3) but also $A(x, 0)$ is invertible. Consider the Laurent series expansions

$$f_i(x, \varepsilon) = \sum_{k=l}^{\infty} f_{i,k}(x)\varepsilon^k,$$
$$g_i(x, \varepsilon) = \sum_{k=l}^{\infty} g_{i,k}(x)\varepsilon^k$$

for $1 \leq i \leq \lambda$ with $l \in \mathbb{Z}$. Then by coefficient comparison w.r.t. ε^l in (5.3) we get the system

$$D_x \begin{pmatrix} f_{1,l}(x) \\ \vdots \\ f_{\lambda,l}(x) \end{pmatrix} = A(x, 0) \begin{pmatrix} f_{1,l}(x) \\ \vdots \\ f_{\lambda,1}(x) \end{pmatrix} + \begin{pmatrix} g_{1,l}(x) \\ \vdots \\ g_{\lambda,l}(x) \end{pmatrix} \qquad (5.10)$$

which is free of ε; see also [12]. Now we carry out steps[6] (1)–(2) of Strategy 1 to this simpler system. This leads to two central improvements: In step (1) the uncoupling method only deals with univariate rational functions in $\mathbb{K}(x)$ and not in the very expensive multivariate case $\mathbb{K}(x, \varepsilon)$. Furthermore, the rational functions in (5.5) (now free of ε!) are much smaller and usually have lower degree. As a consequence, this leads in step (2) to linear recurrences with rather small orders. Furthermore, no factors of the form $\frac{1}{\varepsilon^q}$ can occur, i.e., the ε–order cannot be increased by carrying out steps (1) and (2). After calculating the moments $F_{i,l}(n)$ for $n = 0, \ldots, \mu$ and $1 \leq i \leq \lambda$, one plugs them into (5.3) and obtains a new system (similar to (5.10)) where $f_{i,l+1}(x)$

[5]Adding up the calculation time of the 15 CPUs that we used in our parallelized implementation.

[6]Since $A(x, 0)$ is invertible, the available uncoupling methods from `OreSys`, in particular Gauss' method, are applicable.

takes over the role $f_{i,l}(x)$ and the right hand side is adapted accordingly by taking into account the computed moments of $F_{i,l}(n)$. Now we repeat this calculation iteratively looping through $k = l + 1, \dots, \rho$ with

$$\rho = \max(r_1, \dots, r_\lambda) \tag{5.11}$$

in order to get the moments for $F_{i,k}(n)$, $n = 0, \dots, \mu$.

Example. For a system (5.3) with $\lambda = 6$ we needed the moments (5.1) for $n = 0, \dots, \mu$ up to the orders $r_i = 4$ for $1 \le i \le 6$. The uncoupled system of (5.10) (free of ε) has the form (5.5) with $o_1 = o_2 = 4$, $o_3 = 2$ and $o_4 = o_5 = o_6 = 0$. Using this information, we obtain recurrences for $F_{i,l}(n)$ of orders $\nu_1 = 28$, $\nu_2 = 11$, $\nu_3 = 10$ and $\nu_4 = \nu_5 = \nu_6 = 0$ with $l = -3$. Finally, with the corresponding initial values, we can calculate the moments $F_{i,l}(n)$ for $1 \le i \le 6$. Plugging them into (5.3) one obtains a new system of the form (5.3) and repeats this process for $k = l+1, \dots, 4$. E.g., for $\mu = 4000$ we needed 115209 seconds (637678 CPU seconds), for $\mu = 6000$ we needed 262156 seconds ($1.764 \cdot 10^6$ CPU seconds), and for $\mu = 8000$ we needed $3.724 \cdot 10^6$ seconds ($5.161 \cdot 10^6$ CPU seconds) in order to get the moments of the rational contribution (ignoring the moments that depend on ζ_2, ζ_3 etc.).

Remark 5.2. For all systems (5.3) with $\lambda \ge 5$ the following modification was sufficient to obtain a matrix $A(x, \varepsilon)$ such that $A(x, 0)$ was invertible. We simply carried out a substitution $f_i(x, \varepsilon) \mapsto \frac{1}{\varepsilon^{\lambda_i}} \bar{f}_i(x, \varepsilon)$ for some suitable $\lambda_i \in \mathbb{N}$ with $\bar{f}_i(x, \varepsilon) := x^{\lambda_i} f_i(x, \varepsilon)$, cleared the arising denominators and cancelled common factors. As a side-effect, this modification increased the necessary ε-orders r_i for $\bar{f}_i(x, \varepsilon)$ to obtain in the end the required ε-order for $f_i(x, \varepsilon)$. However, this phenomenon arises only in this initialization phase when the matrix A is usually simple. In our cases the λ_i turned out to be rather small ($\lambda_i \le 2$).

If Strategy 1 was applicable (which was the case for $\lambda \le 4$), it was superior to Strategy 2: One has to loop up simultaneously for all $f_1(x, \varepsilon), \dots, f_\lambda(x, \varepsilon)$ to ε^ρ with (5.11), while in Strategy 1 one computes the ε-expansions for smaller recurrences taking care individually to which order r_i the $f_i(x, \varepsilon)$ have to be expanded. The latter approach with individual r_i reduces substantially the calculation cost. Furthermore, taking ρ with (5.11) (instead of considering the individual orders r_i) often implied that the right hand sides $g_i(x, \varepsilon)$ in (5.3), and thus the simpler master integrals arising in the $g_i(x, \varepsilon)$, have to be calculated to a higher ε-expansion. As it turns out, this enlargement of the required ε-orders even raised to a higher power when one applies the large moment method iteratively to the recursively defined systems coming from IBP relations.

In order to tackle the master integrals coming from the n_h-contributions, we applied the two strategies (Strategy 1 if the dimension of the system is ≤ 4 and Strategy 2 if the dimension is ≥ 5) to 41 systems, more precisely, we tackled 16 systems with $\lambda = 1$, 15 with $\lambda = 2$, 2 with $\lambda = 3$, 3 with $\lambda = 4$, 3 with $\lambda = 5$, 1 with $\lambda = 6$, and 1 with $\lambda = 7$. Here the systems depend recursively on each other: the inhomogeneous parts $g_i(x, \varepsilon)$ in (5.3) depend on master integrals that are solutions of simpler systems. The calculation time of the moments up to the required ε-order for these 92 master integrals can be summarized as follows. For $\mu = 2000$ moments we needed in total 438432 second ($2.630 \cdot 10^6$ CPU seconds). From these 2000 moments we produced the corresponding number of moments of the color-factors and succeeded in guessing recurrences for almost all cases: only the recurrences for the ζ_3-contributions and the constant free contributions could not be guessed with the given number of moments. Thus we restarted our large moment method to produce $\mu = 4000$ moments in 720874 seconds ($6.751 \cdot 10^6$ CPU seconds) ignoring all constants that have been tackled already. This time we succeeded in computing the recurrences for the ζ_3-contribution but not yet for the constant-free terms. Therefore we restarted our method for $\mu = 6000$ moments ignoring also the ζ_3 contributions and obtained them in $1.735 \cdot 10^6$ seconds

($2.091 \cdot 10^6$ CPU seconds). However, we failed to derive further recurrences. Finally, we produced 8000 moments in $3.724 \cdot 10^6$ seconds ($5.161 \cdot 10^6$ CPU seconds) and succeeded in guessing the remaining recurrences.

6 Non-First Order Factorizing Contributions

Using the method of arbitrarily high moments [15] and guessing [26], the recurrences for all pole terms and of a series of color-zeta contributions at the $O(\varepsilon^0)$ can be obtained using 2000 moments, while for other color-zeta projections the corresponding recurrences can be obtained from 4000 moments. For the purely rational contributions we tried the guessing method first with 6000 moments, which were not sufficient. As the next step, we generated 8000 moments by which we obtained the recurrences for the purely rational terms.

	color	degree	order	remaining order
F_V	$g_1 n_h$	1288	54	15
	$g_1 n_h \zeta_3$	409	29	10
	$g_1 n_h \zeta_2$	295	24	6
	$g_2 n_h$	1324	55	15
	$g_2 n_h \zeta_3$	430	30	10
	$g_2 n_h \zeta_2$	273	23	6
F_A	$g_1 n_h$	1314	54	15
	$g_1 n_h \zeta_3$	419	29	10
	$g_1 n_h \zeta_2$	280	23	6
	$g_2 n_h$	1130	52	15
	$g_2 n_h \zeta_3$	352	28	10
	$g_2 n_h \zeta_2$	232	23	6
F_S	n_h	1114	50	15
	$n_h \zeta_3$	350	27	10
	$n_h \zeta_2$	230	22	6
F_P	n_h	1130	52	15
	$n_h \zeta_3$	352	28	10
	$n_h \zeta_2$	232	23	6

Table 1: Structure of the recurrences for the non–first order contributions

While for many of these projections the corresponding recurrences are first order factorizable and can thus be solved using the difference ring methods encoded in the package `Sigma`, for a smaller number non–first order factorizing terms contribute. In Table 1 we characterize those recurrences. Separating the first order factorizing parts, we find remaining non–first order factorizing contributions of order $o = 6, 10$ and 15. This is uniformly the case for all currents. The remaining recurrences have to be studied with other techniques. One may translate these remaining equations into systems of ordinary differential equations again. In other cases it has been observed [44, 111] that systems of this kind may decompose into series of smaller systems. This has to be investigated in further studies.

We also analyzed the leading color case for the scalar current. Here 10 color-ζ structures contribute, out of which 8 have a representation, which results from a difference equation which factorizes at first order. The largest recurrences could be found using 6000 moments. In Table 2 we summarize the characteristics for the recurrences for the constant term, which contain non–first order factorizing parts of order $\mathsf{o} = 5$ and $\mathsf{o} = 4$, respectively. The solution for the ζ_2–term at leading color, unlike in the full color case, can be represented in terms of nested sums.

	color	degree	order	remaining order
F_S	$N_c^2 n_h$	901	46	5
	$N_c^2 n_h \zeta_3$	257	23	4

Table 2: Structure of the recurrences for the non-first order contributions in leading color approximation for the scalar three loop massive form factor

The corresponding recurrence is of degree $\mathsf{d} = 150$ and order $\mathsf{o} = 17$ and one obtains

$$
\begin{aligned}
F_S^{n_h N_c^2 \zeta_2} = & -\frac{N_C^2 n_h \zeta_2}{2(1+x)^2}\Biggl\{\frac{8R_{28}}{9x(1+x)^4}\ln(2) + \frac{R_{27}}{54(1-x)^2(1+x)^4} + \Biggl[-\frac{8R_{23}}{3(1-x)(1+x)^3}\ln(2) \\
& -\frac{R_{29}}{18(1-x)^3(1+x)^4} + \left(\frac{16R_2}{(1-x^2)}\ln(2) - \frac{R_{24}}{3(1-x)(1+x)^3}\right)H_{-1} \\
& +\frac{8R_8}{3(1-x^2)}H_{-1}^2 - \frac{4R_{17}}{3(1-x)^2(1+x)}\zeta_2 + \frac{48x^2}{(1-x^2)}\zeta_3\Biggr]H_0 + \Biggl(\frac{24x^2}{(1-x^2)}\zeta_2 \\
& +\frac{R_{26}}{36(1-x)^2(1+x)^3} + \frac{8R_{15}}{(1-x)^2(1+x)}\ln(2) - \frac{8R_{14}}{3(1-x)^2(1+x)}H_{-1}\Biggr)H_0^2 \\
& +\frac{4R_{16}}{9(1-x)^2(1+x)}H_0^3 - \frac{2x^2}{(1-x^2)}H_0^4 + \Biggl[\Biggl(-\frac{80(1+x)\big(1+x^2\big)}{(1-x)}\ln(2) \\
& -\frac{8R_{22}}{9(1-x)(1+x)^3} - \frac{16R_5}{3(1-x^2)}H_{-1}\Biggr)H_0 + \frac{8\big(5-13x+54x^2-x^3-x^4\big)}{3(1-x^2)}H_0^2 \\
& -\frac{16R_6}{3(1-x^2)}\zeta_2\Biggr]H_1 - \frac{32(1+x)\big(1-3x+x^2\big)}{3(1-x)}H_0H_1^2 + \Biggl(-\frac{8R_{28}}{9x(1+x)^4} \\
& -\frac{16R_7}{3(1-x^2)}\zeta_2\Biggr)H_{-1} + \Biggl(\frac{80(1+x)\big(1+x^2\big)}{(1-x)}\ln(2) + \frac{8R_{22}}{9(1-x)(1+x)^3} \\
& -\frac{8R_{19}}{3(1-x)^2(1+x)}H_0 - \frac{32x^2H_0^2}{(1-x^2)} + \frac{64(1+x)\big(1-3x+x^2\big)}{3(1-x)}H_1 + \frac{128R_1}{3(1-x^2)} \\
& \times H_{-1}\Biggr)H_{0,1} + \Biggl(\frac{8R_{11}}{3(1-x)^2}H_0 - \frac{16R_2}{(1-x^2)}\ln(2) - \frac{R_{21}}{3(1-x)(1+x)^3} + \frac{64x^2}{(1-x^2)}H_0^2 \\
& +\frac{32R_4}{3(1-x^2)}H_1 - \frac{32R_9}{3(1-x^2)}H_{-1}\Biggr)H_{0,-1} - \frac{16R_2}{(1-x^2)}H_0H_{-1,1} \\
& +\left(\frac{16xR_{13}}{3(1-x)^2(1+x)} + \frac{192x^2}{(1-x^2)}H_0\right)H_{0,0,1} - \left(\frac{16R_{20}}{3(1-x)^2(1+x)} + \frac{288x^2}{(1-x^2)}H_0\right)
\end{aligned}
$$

$$
\times H_{0,0,-1} - \frac{64(1+x)\big(1-3x+x^2\big)}{3(1-x)} H_{0,1,1} - \frac{16R_{10}}{3(1-x^2)} H_{0,1,-1} - \frac{32R_4}{3(1-x^2)} H_{0,-1,1}
+\frac{16R_{12}}{3(1-x^2)} H_{0,-1,-1} - \frac{384x^2}{(1-x^2)} H_{0,0,0,1} + \frac{480x^2}{(1-x^2)} H_{0,0,0,-1} + \Bigg(\frac{16R_3}{(1-x^2)}\ln(2)
-\frac{R_{25}}{18(1-x)(1+x)^3}\Bigg)\zeta_2 - \frac{72x^2}{5(1-x^2)}\zeta_2^2 + \frac{2R_{18}}{3(1-x)^2(1+x)}\zeta_3\Bigg\}, \qquad (6.1)
$$

with the polynomials

$$R_1 = 7x^4+3x^3+28x^2+3x+7, \qquad (6.2)$$
$$R_2 = 11x^4+4x^3+34x^2+4x+11, \qquad (6.3)$$
$$R_3 = 17x^4-2x^3+58x^2-2x+17, \qquad (6.4)$$
$$R_4 = 19x^4+21x^3+76x^2+21x+19, \qquad (6.5)$$
$$R_5 = 23x^4+12x^3+122x^2+12x+23, \qquad (6.6)$$
$$R_6 = 23x^4+17x^3+60x^2+17x+23, \qquad (6.7)$$
$$R_7 = 24x^4+5x^3+130x^2+5x+24, \qquad (6.8)$$
$$R_8 = 31x^4+26x^3+86x^2+26x+31, \qquad (6.9)$$
$$R_9 = 32x^4+19x^3+94x^2+19x+32, \qquad (6.10)$$
$$R_{10} = 71x^4+54x^3+254x^2+54x+71, \qquad (6.11)$$
$$R_{11} = 83x^4-96x^3+214x^2-96x+83, \qquad (6.12)$$
$$R_{12} = 97x^4+50x^3+290x^2+50x+97, \qquad (6.13)$$
$$R_{13} = 100x^4-19x^3+197x^2-105x+11, \qquad (6.14)$$
$$R_{14} = 3x^5+23x^4-22x^3+148x^2+7x+57, \qquad (6.15)$$
$$R_{15} = 15x^5-3x^4+14x^3-16x^2-x-1, \qquad (6.16)$$
$$R_{16} = 19x^5-5x^4+212x^3-112x^2+43x+19, \qquad (6.17)$$
$$R_{17} = 31x^5+39x^4+258x^3-70x^2+91x+27, \qquad (6.18)$$
$$R_{18} = 89x^5-289x^4+22x^3-134x^2-55x-1, \qquad (6.19)$$
$$R_{19} = 101x^5-19x^4+142x^3-38x^2-7x+5, \qquad (6.20)$$
$$R_{20} = 125x^5-45x^4+182x^3-78x^2-23x+23, \qquad (6.21)$$
$$R_{21} = -617x^6-690x^5+2033x^4+3572x^3+2033x^2-690x-617, \qquad (6.22)$$
$$R_{22} = 45x^6+398x^5-481x^4-2148x^3-481x^2+398x+45, \qquad (6.23)$$
$$R_{23} = 48x^6+131x^5-28x^4+18x^3-28x^2+131x+48, \qquad (6.24)$$
$$R_{24} = 233x^6-358x^5-1809x^4-3716x^3-1809x^2-358x+233, \qquad (6.25)$$
$$R_{25} = 2571x^6+8438x^5-13795x^4-45084x^3-13795x^2+8438x+2571, \qquad (6.26)$$
$$R_{26} = 917x^7+12853x^6-3767x^5-24311x^4-11289x^3+9223x^2-325x-709, \qquad (6.27)$$
$$R_{27} = -100331x^8-173294x^7-40776x^6+237806x^5+226918x^4+237806x^3 -40776x^2-173294x-100331, \qquad (6.28)$$
$$R_{28} = 124x^8-5x^7-609x^6-2635x^5-3950x^4-2635x^3-609x^2-5x+124, \qquad (6.29)$$
$$R_{29} = 1984x^{10}-5471x^9-11702x^8-28981x^7+9401x^6+43521x^5+12817x^4 -24727x^3-13829x^2-8150x+561. \qquad (6.30)$$

Here the functions $H_{\vec{a}}(x)$ denote the harmonic polylogarithms [30], which are defined by

$$H_{b,\vec{a}}(x) = \int_0^x dy f_b(y) H_{\vec{a}}(y), \quad H_\emptyset = 1, \quad a_i, b \in \{0, 1, -1\} \tag{6.31}$$

with

$$f_0(x) = \frac{1}{x}, \qquad f_1(x) = \frac{1}{1-x}, \qquad f_{-1}(x) = \frac{1}{1+x}. \tag{6.32}$$

In the case of the color–ζ terms the recurrences of which do not factorize to first order we can use the finite analytic representation given by the number of analytic moments $N_{\max} = 2000$ for numerical results and graphical illustrations given below, which already gives a good representation. In some cases they have a logarithmic divergence near $x = 0$, however.

7 The Results

In the following we present the results for the contribution $\propto n_h^k, k = 1, 2$, of the massive three–loop form factors in the vector-, axialvector-, scalar- and pseudoscalar case. The pole terms are given in terms of HPLs of the variable x. For the constant contributions, for a series of color–zeta terms also first order factorizing solutions are obtained, which are expressed using HPLs. Other contributions contain non–first order factorizing parts. For these we have derived the corresponding recurrences of their Taylor coefficients also using the method of arbitrarily high moments [15]. In the present paper we present these contributions in terms of analytic series expansions, for which we derived at least 2000 terms and in some cases 4000 or 8000 terms, cf. Section 6.

The renormalized form factors, which are lengthy expressions, except the functions $F_{C,i}^{(0)}, i = 1...3, C = V_1, V_2, A_1, A_2, P, S$ defined below, are given in an attachment to this paper, where we also present the other results in computer-readable form.

7.1 The Vector Form Factors

The vector form factor $F_{V,1}$ is given by

$$\begin{aligned}
F_{V,1} &= \frac{1}{\varepsilon^3}\Biggl\{n_h^2\Biggl[\frac{16}{27} - \frac{32(1+x^2)H_0}{27(1-x^2)}\Biggr] - n_h\Biggl[\frac{2(589+602x+589x^2)}{27(1+x)^2} - n_l\Biggl(\frac{16}{27} - \frac{32(1+x^2)H_0}{27(1-x^2)}\Biggr) \\
&\quad - \frac{16(41+190x+154x^2+190x^3+41x^4)H_0}{27(1-x)(1+x)^3} + \frac{128(1+x^2)^2}{27(1-x^2)^2}H_0^2\Biggr]\Biggr\} \\
&\quad + \frac{1}{\varepsilon^2}\Biggl\{n_h^2\Biggl[\frac{16}{27} - \Biggl(\frac{16(3+2x+3x^2)}{27(1-x^2)} - \frac{64(1+x^2)}{27(1-x^2)}H_{-1}\Biggr)H_0 - \frac{16(1+x^2)}{27(1-x^2)}H_0^2 \\
&\quad - \frac{64(1+x^2)}{27(1-x^2)}H_{0,-1} + \frac{32(1+x^2)}{27(1-x^2)}\zeta_2\Biggr] + n_h\Biggl[-\frac{4(263-66x+263x^2)}{27(1+x)^2} + n_l\Biggl(\frac{128}{81} \\
&\quad + \Biggl(-\frac{64(7+3x+7x^2)}{81(1-x^2)} + \frac{128(1+x^2)}{27(1-x^2)}H_{-1}\Biggr)H_0 - \frac{32(1+x^2)}{27(1-x^2)}H_0^2 - \frac{128(1+x^2)}{27(1-x^2)}H_{0,-1} \\
&\quad + \frac{64(1+x^2)}{27(1-x^2)}\zeta_2\Biggr] + \Biggl(\frac{8(371+1622x+1334x^2+1622x^3+371x^4)}{27(1-x)(1+x)^3} + \frac{64(1+x^2)}{3(1-x^2)}H_1
\end{aligned}$$

$$
-\frac{128\big(23+73x+64x^2+73x^3+23x^4\big)}{27(1-x)(1+x)^3}H_{-1}+\frac{64\big(1+x^2\big)^2}{3(1-x^2)^2}H_{0,1}-\frac{1088\big(1+x^2\big)^2}{27(1-x^2)^2}
$$

$$
\times H_{0,-1}\Bigg)H_0+\Bigg(\frac{16\big(13+51x-46x^2-10x^3-167x^4-97x^5\big)}{27(1-x)^2(1+x)^3}+\frac{512\big(1+x^2\big)^2}{27(1-x)^2(1+x)^2}
$$

$$
\times H_{-1}\big)H_0^2+\frac{64\big(-2+x^2\big)\big(1+x^2\big)}{27(1-x)^2(1+x)^2}H_0^3-\frac{64\big(1+x^2\big)}{3(1-x^2)}H_{0,1}+\frac{128}{27(1-x)(1+x)^3}
$$

$$
\times\big(23+73x+64x^2+73x^3+23x^4\big)H_{0,-1}-\frac{128\big(1+x^2\big)^2}{3(1-x)^2(1+x)^2}H_{0,0,1}+\frac{128\big(1+x^2\big)^2}{3(1-x)^2(1+x)^2}
$$

$$
\times H_{0,0,-1}+\big(-\frac{64}{27(1-x)(1+x)^3}\big(23+73x+46x^2+37x^3+5x^4\big)
$$

$$
+\frac{32\big(1+x^2\big)\big(-1+35x^2\big)}{27(1-x)^2(1+x)^2}H_0\big)\zeta_2+\frac{32\big(1+x^2\big)^2}{3(1-x)^2(1+x)^2}\zeta_3\big)\Bigg\}
$$

$$
+\frac{1}{\varepsilon}\Bigg\{n_h^2\Bigg[-\frac{16H_0P_4^{(1)}}{243(1-x)(1+x)^3}+\frac{8H_0^2P_9^{(1)}}{81(1-x)(1+x)^4}-\frac{16H_0^3P_{13}^{(1)}}{27(1-x)(1+x)^5}
$$

$$
-\frac{64\big(13-268x+13x^2\big)}{243(1+x)^2}-\frac{64\big(1+x^2\big)}{27(1-x^2)}H_{-1}^2H_0+\Bigg(\frac{32\big(3+2x+3x^2\big)}{27(1-x^2)}H_0+\frac{32\big(1+x^2\big)}{27(1-x^2)}H_0^2
$$

$$
+\frac{128\big(1+x^2\big)}{27(1-x^2)}H_{0,-1}\Bigg)H_{-1}-\frac{32\big(3+2x+3x^2\big)}{27(1-x^2)}H_{0,-1}-\frac{64\big(1+x^2\big)}{27(1-x^2)}H_{0,0,-1}
$$

$$
-\frac{128\big(1+x^2\big)}{27(1-x^2)}H_{0,-1,-1}+\Bigg(\frac{8P_{11}^{(1)}}{27(1-x)(1+x)^4}-\frac{16H_0P_{16}^{(1)}}{27(1-x)(1+x)^5}-\frac{64\big(1+x^2\big)}{27(1-x^2)}H_{-1}\Bigg)\zeta_2
$$

$$
+\frac{64\big(1+x^2\big)}{27(1-x^2)}\zeta_3\Bigg]+n_h\Bigg[-\frac{64H_{0,1}P_1^{(1)}}{27(1-x)(1+x)^3}+\frac{128H_{0,-1,-1}P_3^{(1)}}{27(1-x)(1+x)^3}-\frac{4P_6^{(1)}}{243(1+x)^4}
$$

$$
+\frac{128H_{0,0,1}P_{18}^{(1)}}{27(1-x)^2(1+x)^4}-\frac{64H_{0,0,-1}P_{21}^{(1)}}{27(1-x)^2(1+x)^4}+\frac{128H_{0,0,0,-1}P_{27}^{(1)}}{9(1-x)^2(1+x)^5}-\frac{128H_{0,0,0,1}P_{30}^{(1)}}{27(1-x)^2(1+x)^5}
$$

$$
-\frac{16\zeta_2^2P_{37}^{(1)}}{135(1-x)^2(1+x)^5}-\frac{8H_0^4P_{39}^{(1)}}{81(1-x)^2(1+x)^6}+n_l\Bigg(-\frac{32\big(1-194x+x^2\big)}{81(1+x)^2}
$$

$$
-\frac{32H_0P_2^{(1)}}{81(1-x)(1+x)^3}-\frac{16H_0^2P_7^{(1)}}{81(1-x)(1+x)^4}-\frac{32H_0^3P_{14}^{(1)}}{27(1-x)(1+x)^5}-\frac{256\big(1+x^2\big)}{27(1-x^2)}H_{-1}^2H_0
$$

$$
+\Bigg(\frac{256\big(7+3x+7x^2\big)}{81(1-x^2)}H_0+\frac{128\big(1+x^2\big)}{27(1-x^2)}H_0^2+\frac{512\big(1+x^2\big)}{27(1-x^2)}H_{0,-1}\Bigg)H_{-1}
$$

$$
-\frac{256\big(7+3x+7x^2\big)}{81(1-x^2)}H_{0,-1}-\frac{256\big(1+x^2\big)}{27(1-x^2)}H_{0,0,-1}+\frac{512\big(1+x^2\big)}{27(1-x^2)}H_{0,-1,-1}
$$

$$
+\Bigg(\frac{8P_{12}^{(1)}}{81(1-x)(1+x)^4}-\frac{16H_0P_{17}^{(1)}}{27(1-x)(1+x)^5}-\frac{256\big(1+x^2\big)}{27(1-x^2)}H_{-1}\Bigg)\zeta_2+\frac{256\big(1+x^2\big)}{27(1-x^2)}\zeta_3\Bigg)
$$

$$
+\Bigg(\frac{4P_{23}^{(1)}}{243(1-x)(1+x)^5}-\frac{128H_{0,1}P_{15}^{(1)}}{27(1-x)^2(1+x)^4}-\frac{64H_{0,-1}P_{19}^{(1)}}{27(1-x)^2(1+x)^4}
$$

$$
-\frac{512H_{0,0,1}P_{24}^{(1)}}{27(1-x)^2(1+x)^5}
+\frac{128H_{0,0,-1}P_{29}^{(1)}}{27(1-x)^2(1+x)^5}+\left(\frac{64P_1^{(1)}}{27(1-x)(1+x)^3}+\frac{256\left(1+x^2\right)^2}{27(1-x^2)^2}[H_{0,1}-H_{0,-1}]\right)
\times H_1+\frac{64\left(1+x^2\right)}{3(1-x^2)}H_1^2+\frac{640\left(1+x^2\right)^2}{27(1-x^2)^2}H_{0,1,1}-\frac{896\left(1+x^2\right)^2}{27(1-x^2)^2}H_{0,1,-1}
-\frac{896\left(1+x^2\right)^2}{27(1-x^2)^2}H_{0,-1,1}+\frac{128\left(1+x^2\right)^2}{27(1-x^2)^2}H_{0,-1,-1}\Bigg)H_0+\Bigg(\frac{64H_1P_8^{(1)}}{27(1-x)(1+x)^4}
+\frac{64H_{0,1}P_{31}^{(1)}}{27(1-x)^2(1+x)^5}-\frac{64H_{0,-1}P_{33}^{(1)}}{27(1-x)^2(1+x)^5}-\frac{4P_{41}^{(1)}}{243(1-x)^2(1+x)^6}\Bigg)H_0^2
+\left(-\frac{64H_1P_{28}^{(1)}}{81(1-x)^2(1+x)^5}-\frac{16P_{35}^{(1)}}{81(1-x)^2(1+x)^5}\right)H_0^3+\Bigg(-\frac{128\left(1+x^2\right)}{3(1-x^2)}H_{0,1}
+\frac{128\left(1+x^2\right)}{(1-x^2)}H_{0,-1}-\frac{512\left(1+x^2\right)^2}{27(1-x^2)^2}H_{0,0,1}+\frac{512\left(1+x^2\right)^2}{27(1-x^2)^2}H_{0,0,-1}\Bigg)H_1
+\Bigg(-\frac{128H_{0,-1}P_3^{(1)}}{27(1-x)(1+x)^3}+\frac{32H_0^2P_{22}^{(1)}}{27(1-x)^2(1+x)^4}-\Bigg(\frac{16P_5^{(1)}}{27(1-x)(1+x)^3}+\frac{128\left(1+x^2\right)}{(1-x^2)}
\times H_1+\frac{256\left(1+x^2\right)^2}{3(1-x^2)^2}H_{0,1}-\frac{4352\left(1+x^2\right)^2}{27(1-x^2)^2}H_{0,-1}\Bigg)H_0+\frac{64\left(1+x^2\right)\left(-1+5x^2\right)}{27(1-x^2)^2}H_0^3
+\frac{128\left(1+x^2\right)}{(1-x^2)}H_{0,1}+\frac{512\left(1+x^2\right)^2}{3(1-x^2)^2}H_{0,0,1}-\frac{512\left(1+x^2\right)^2}{3(1-x^2)^2}H_{0,0,-1}\Bigg)H_{-1}
+\left(\frac{64H_0P_3^{(1)}}{27(1-x)(1+x)^3}-\frac{1024\left(1+x^2\right)^2}{27(1-x^2)^2}H_0^2\right)H_{-1}^2-\frac{448\left(1+x^2\right)^2}{27(1-x^2)^2}H_{0,1}^2
+\left(\frac{16P_5^{(1)}}{27(1-x)(1+x)^3}+\frac{3200\left(1+x^2\right)^2}{27(1-x^2)^2}H_{0,1}\right)H_{0,-1}-\frac{2240\left(1+x^2\right)^2}{27(1-x^2)^2}H_{0,-1}^2
+\frac{128\left(1+x^2\right)}{3(1-x^2)}H_{0,1,1}-\frac{128\left(1+x^2\right)}{(1-x^2)}H_{0,1,-1}-\frac{128\left(1+x^2\right)}{(1-x^2)}H_{0,-1,1}+\frac{512\left(1+x^2\right)^2}{27(1-x^2)^2}
\times H_{0,0,1,1}-\frac{512\left(1+x^2\right)^2}{3(1-x^2)^2}H_{0,0,1,-1}-\frac{512\left(1+x^2\right)^2}{3(1-x^2)^2}H_{0,0,-1,1}+\frac{512\left(1+x^2\right)^2}{3(1-x^2)^2}H_{0,0,-1,-1}
-\frac{2048\left(1+x^2\right)^2}{27(1-x^2)^2}H_{0,-1,0,1}+\Bigg(-\frac{128\ln(2)\left(1+x^2\right)}{3(1+x)^2}-\frac{64H_{0,-1}P_{25}^{(1)}}{27(1-x)^2(1+x)^5}
+\frac{64H_{0,1}P_{32}^{(1)}}{27(1-x)^2(1+x)^5}-\frac{P_{38}^{(1)}}{27(1-x)(1+x)^6}-\frac{32H_0^2P_{40}^{(1)}}{27(1-x)^2(1+x)^6}
-\left(\frac{256H_1P_{26}^{(1)}}{27(1-x)^2(1+x)^5}-\frac{8P_{36}^{(1)}}{27(1-x)^2(1+x)^5}\right)H_0-\frac{64\left(1+x^2\right)}{3(1-x)(1+x)}H_1
+\left(\frac{64P_{10}^{(1)}}{27(1-x)(1+x)^4}-\frac{256\left(1+x^2\right)\left(13+4x^2\right)}{27(1-x)^2(1+x)^2}H_0\right)H_{-1}\Bigg)\zeta_2+\Bigg(\frac{32P_{20}^{(1)}}{27(1-x)^2(1+x)^4}
$$

$$+\frac{32H_0P_{34}^{(1)}}{27(1-x)^2(1+x)^5}+\frac{128\big(1+x^2\big)^2}{27(1-x^2)^2}H_1-\frac{128\big(1+x^2\big)^2}{3(1-x^2)^2}H_{-1}\Bigg)\zeta_3\Bigg]\Bigg\}+F_{V,1}^{(0)}, \qquad (7.1)$$

and the polynomials

$$P_1^{(1)} = 37x^4+104x^3+86x^2+104x+37 \qquad (7.2)$$
$$P_2^{(1)} = 103x^4+98x^3+702x^2+98x+103 \qquad (7.3)$$
$$P_3^{(1)} = 115x^4+284x^3+266x^2+284x+115 \qquad (7.4)$$
$$P_4^{(1)} = 337x^4+20x^3+3638x^2+20x+337 \qquad (7.5)$$
$$P_5^{(1)} = 913x^4+1462x^3+6506x^2+1462x+913 \qquad (7.6)$$
$$P_6^{(1)} = 14921x^4+130220x^3+280134x^2+130220x+14921 \qquad (7.7)$$
$$P_7^{(1)} = 9x^5+117x^4-60x^3+356x^2+75x+47 \qquad (7.8)$$
$$P_8^{(1)} = 14x^5-89x^4+761x^3-617x^2+197x+22 \qquad (7.9)$$
$$P_9^{(1)} = 29x^5-75x^4+362x^3-470x^2+9x-47 \qquad (7.10)$$
$$P_{10}^{(1)} = 55x^5+264x^4+277x^3+391x^2+210x+67 \qquad (7.11)$$
$$P_{11}^{(1)} = 87x^5+x^4+1502x^3-1430x^2+43x-75 \qquad (7.12)$$
$$P_{12}^{(1)} = 331x^5+249x^4+4942x^3-3758x^2+519x-107 \qquad (7.13)$$
$$P_{13}^{(1)} = x^6+4x^5+3x^4+48x^3+3x^2+4x+1 \qquad (7.14)$$
$$P_{14}^{(1)} = x^6+4x^5+5x^4+28x^3+5x^2+4x+1 \qquad (7.15)$$
$$P_{15}^{(1)} = 2x^6+123x^5-870x^4+1234x^3-870x^2+123x+2 \qquad (7.16)$$
$$P_{16}^{(1)} = 5x^6+20x^5+11x^4+280x^3+11x^2+20x+5 \qquad (7.17)$$
$$P_{17}^{(1)} = 7x^6+28x^5+25x^4+296x^3+25x^2+28x+7 \qquad (7.18)$$
$$P_{18}^{(1)} = 18x^6+143x^5-890x^4+1090x^3-926x^2+71x-18 \qquad (7.19)$$
$$P_{19}^{(1)} = 43x^6-685x^5+5403x^4-8498x^3+5403x^2-685x+43 \qquad (7.20)$$
$$P_{20}^{(1)} = 137x^6+1490x^5-6779x^4+12512x^3-7257x^2+606x-197 \qquad (7.21)$$
$$P_{21}^{(1)} = 142x^6+1359x^5-6912x^4+11114x^3-7268x^2+683x-142 \qquad (7.22)$$
$$P_{22}^{(1)} = 228x^6-11x^5+3894x^4-5882x^3+3538x^2-687x-56 \qquad (7.23)$$
$$P_{23}^{(1)} = 46991x^6+217814x^5+668705x^4+1193908x^3+668705x^2+217814x +46991 \qquad (7.24)$$
$$P_{24}^{(1)} = 5x^7-37x^6+421x^5-1247x^4+1268x^3-406x^2+46x-2 \qquad (7.25)$$
$$P_{25}^{(1)} = 7x^7+15x^6+17x^5-239x^4+x^3-187x^2-117x-41 \qquad (7.26)$$
$$P_{26}^{(1)} = 13x^7-14x^6+449x^5-1326x^4+1319x^3-454x^2+11x-14 \qquad (7.27)$$
$$P_{27}^{(1)} = 15x^7+201x^6-1135x^5+3933x^4-3709x^3+1295x^2-105x+17 \qquad (7.28)$$
$$P_{28}^{(1)} = 25x^7-31x^6+893x^5-2659x^4+2631x^3-913x^2+19x-29 \qquad (7.29)$$
$$P_{29}^{(1)} = 30x^7-171x^6+2127x^5-6392x^4+6448x^3-2087x^2+195x-22 \qquad (7.30)$$
$$P_{30}^{(1)} = 41x^7+382x^6-1814x^5+6397x^4-5949x^3+2134x^2-190x+23 \qquad (7.31)$$
$$P_{31}^{(1)} = 42x^7-31x^6+1377x^5-3742x^4+3966x^3-1217x^2+127x-10 \qquad (7.32)$$
$$P_{32}^{(1)} = 43x^7-83x^6+1751x^5-5367x^4+5213x^3-1861x^2+17x-65 \qquad (7.33)$$
$$P_{33}^{(1)} = 50x^7+45x^6+1011x^5-2442x^4+2750x^3-791x^2+87x-6 \qquad (7.34)$$

$$P_{34}^{(1)} = 71x^7 - 207x^6 + 3511x^5 - 10527x^4 + 10653x^3 - 3421x^2 + 261x - 53 \quad (7.35)$$
$$P_{35}^{(1)} = 229x^7 - 245x^6 + 6189x^5 - 4655x^4 - 731x^3 + 399x^2 - 167x + 5 \quad (7.36)$$
$$P_{36}^{(1)} = 281x^7 + 4495x^6 - 12289x^5 + 14449x^4 - 953x^3 + 2801x^2 - 671x + 79 \quad (7.37)$$
$$P_{37}^{(1)} = 809x^7 + 655x^6 + 16429x^5 - 41801x^4 + 43971x^3 - 14879x^2 + 275x - 499 \quad (7.38)$$
$$P_{38}^{(1)} = 18177x^7 + 16453x^6 + 426009x^5 + 662997x^4 - 333541x^3 - 273065x^2 + 23243x - 8305 \quad (7.39)$$
$$P_{39}^{(1)} = 8x^8 + 31x^7 + 42x^6 + 693x^5 + 212x^4 + 795x^3 + 214x^2 + 145x + 36 \quad (7.40)$$
$$P_{40}^{(1)} = 71x^8 + 232x^7 + 832x^6 - 74x^5 + 498x^4 + 1490x^3 - 256x^2 + 80x + 7 \quad (7.41)$$
$$P_{41}^{(1)} = 13421x^8 + 46436x^7 + 167926x^6 + 195276x^5 + 214672x^4 - 46932x^3 - 22982x^2 - 25564x - 5677, \quad (7.42)$$

and

$$\begin{aligned}
F_{V,2} = {} & \frac{x}{\varepsilon^3}\Bigg\{n_h 128\Bigg[-\frac{1}{3(1+x)^2} + \frac{(1+x^2)}{3(-1+x)(1+x)^3}H_0\Bigg]\Bigg\} \\
& + \frac{x}{\varepsilon^2}\Bigg\{-n_h^2\frac{64H_0}{27(1-x^2)} + n_h\Bigg[\frac{64}{27(1-x)(1+x)^3}\left(-73(1-x^2) + 18(1+x^2)\zeta_2\right) \\
& - n_l\frac{128H_0}{27(1-x^2)} - \left(\frac{128(18-73x+18x^2)}{27(1-x)(1+x)^3} - \frac{256(1+x^2)}{3(1-x)(1+x)^3}H_{-1}\right)H_0 \\
& + \frac{64(-17+x)(1+x^2)}{27(1-x)^2(1+x)^3}H_0^2 - \frac{256(1+x^2)}{3(1-x)(1+x)^3}H_{0,-1}\Bigg]\Bigg\} \\
& + \frac{x}{\varepsilon}\Bigg\{n_h^2\Bigg[-\frac{2176}{27(1+x)^2} + \left(-\frac{64(37-298x+37x^2)}{81(1-x)(1+x)^3} + \frac{512x^2\zeta_2}{3(1-x)(1+x)^5}\right)H_0 \\
& + \frac{128H_{-1}H_0}{27(1-x^2)} + \frac{32(-3+39x-45x^2+x^3)}{27(1-x)(1+x)^4}H_0^2 + \frac{256x^2H_0^3}{9(1-x)(1+x)^5} - \frac{128H_{0,-1}}{27(1-x^2)} \\
& - \frac{64(-7-213x+207x^2+5x^3)}{27(1-x)(1+x)^4}\zeta_2\Bigg] + n_h\Bigg[-\frac{512H_{0,0,1}P_4^{(2)}}{27(1-x)^2(1+x)^4} - \frac{256\zeta_3P_5^{(2)}}{27(1-x)^2(1+x)^4} \\
& + \frac{128H_{0,0,-1}P_9^{(2)}}{27(1-x)^2(1+x)^4} - \frac{64\zeta_2P_{18}^{(2)}}{27(1-x)^2(1+x)^6} + \frac{160(635+1654x+635x^2)}{81(1+x)^4} \\
& + n_l\Bigg(-\frac{2176}{27(1+x)^2} + \left(-\frac{128(37-112x+37x^2)}{81(1-x)(1+x)^3} + \frac{512x^2\zeta_2}{3(1-x)(1+x)^5}\right)H_0 + \frac{512H_{-1}H_0}{27(1-x^2)} \\
& - \frac{64(3-15x+27x^2+x^3)}{27(1-x)(1+x)^4}H_0^2 + \frac{256x^2H_0^3}{9(1-x)(1+x)^5} - \frac{512H_{0,-1}}{27(1-x^2)} \\
& - \frac{128(-5-111x+99x^2+x^3)}{27(1-x)(1+x)^4}\zeta_2\Bigg) + \Bigg(-\frac{128H_{0,-1}P_6^{(2)}}{9(1-x)^2(1+x)^4} - \frac{16P_{12}^{(2)}}{81(1-x)(1+x)^5} \\
& + \frac{64\zeta_2P_{16}^{(2)}}{27(1-x)^3(1+x)^5} + \left(\frac{256(5+22x+5x^2)}{27(1-x)(1+x)^3} - \frac{5120x(7-16x+7x^2)}{9(1-x)(1+x)^5}\zeta_2\right)H_1 \\
& + \frac{512(6-55x+6x^2)}{9(1+x)^4}H_{0,1} - \frac{26624x(8-17x+8x^2)}{27(1-x)(1+x)^5}H_{0,0,1}
\end{aligned}$$

$$
+\frac{512x\big(521-1142x+521x^2\big)}{27(1-x)(1+x)^5}H_{0,0,-1}+\frac{512x\big(209-479x+209x^2\big)}{27(1-x)(1+x)^5}\zeta_3\Bigg)H_0
$$

$$
-\frac{256\big(1+x^2\big)}{3(1-x)(1+x)^3}H_{-1}^2H_0+\Bigg(-\frac{256H_1P_3^{(2)}}{27(1-x)^2(1+x)^4}-\frac{64P_{17}^{(2)}}{81(1-x)^2(1+x)^6}
$$

$$
+\frac{256x\big(313-688x+313x^2\big)}{27(1-x)(1+x)^5}H_{0,1}-\frac{256xP_8^{(2)}}{27(1-x)^3(1+x)^5}H_{0,-1}-\frac{128xP_{14}^{(2)}}{27(1-x)^3(1+x)^6}\zeta_2\Bigg)
$$

$$
\times H_0^2+\Bigg(\frac{64P_{15}^{(2)}}{81(1-x)^3(1+x)^5}-\frac{2560x\big(7-16x+7x^2\big)}{27(1-x)(1+x)^5}H_1\Bigg)H_0^3-\frac{32xP_{13}^{(2)}}{81(1-x)^3(1+x)^6}H_0^4
$$

$$
+\Bigg(\frac{128\zeta_2P_2^{(2)}}{9(1-x)^2(1+x)^4}+\frac{64H_0^2P_7^{(2)}}{27(1-x)^2(1+x)^4}-\frac{128\big(143-318x+143x^2\big)}{27(1-x)(1+x)^3}H_0
$$

$$
+\frac{512\big(1+x^2\big)}{3(1-x)(1+x)^3}H_{0,-1}\Bigg)H_{-1}+\Bigg(-\frac{256\big(5+22x+5x^2\big)}{27(1-x)(1+x)^3}+\frac{5120x\big(7-16x+7x^2\big)}{9(1-x)(1+x)^5}\zeta_2\Bigg)
$$

$$
\times H_{0,1}+\Bigg(\frac{128\big(143-318x+143x^2\big)}{27(1-x)(1+x)^3}-\frac{512xP_1^{(2)}}{9(1-x)^3(1+x)^5}\zeta_2\Bigg)H_{0,-1}
$$

$$
-\frac{512\big(1+x^2\big)}{3(1-x)(1+x)^3}H_{0,-1,-1}+\frac{512x\big(173-356x+173x^2\big)}{9(1-x)(1+x)^5}H_{0,0,0,1}
$$

$$
-\frac{512xP_{10}^{(2)}}{9(1-x)^3(1+x)^5}H_{0,0,0,-1}-\frac{128xP_{11}^{(2)}}{45(1-x)^3(1+x)^5}\zeta_2^2\Bigg]\Bigg\}+F_{V,2}^{(0)}, \tag{7.43}
$$

with

$$
\begin{aligned}
P_1^{(2)} &= x^4-9x^3+10x^2-9x+1 && (7.44)\\
P_2^{(2)} &= 11x^4-12x^3+26x^2-12x-1 && (7.45)\\
P_3^{(2)} &= 14x^4-215x^3+370x^2-215x+14 && (7.46)\\
P_4^{(2)} &= 22x^4-187x^3+362x^2-187x+22 && (7.47)\\
P_5^{(2)} &= 58x^4-937x^3+1616x^2-937x+40 && (7.48)\\
P_6^{(2)} &= 65x^4-876x^3+1578x^2-876x+65 && (7.49)\\
P_7^{(2)} &= 125x^4-1772x^3+3262x^2-1772x+161 && (7.50)\\
P_8^{(2)} &= 209x^4-887x^3+1354x^2-887x+209 && (7.51)\\
P_9^{(2)} &= 265x^4-3484x^3+6206x^2-3484x+229 && (7.52)\\
P_{10}^{(2)} &= 312x^4-1297x^3+1972x^2-1297x+312 && (7.53)\\
P_{11}^{(2)} &= 590x^4-2447x^3+3760x^2-2447x+590 && (7.54)\\
P_{12}^{(2)} &= 609x^4+21136x^3+56414x^2+21136x+609 && (7.55)\\
P_{13}^{(2)} &= x^5+148x^4-143x^3+145x^2-140x+1 && (7.56)\\
P_{14}^{(2)} &= 104x^5-263x^4+173x^3+317x^2-407x+104 && (7.57)\\
P_{15}^{(2)} &= 86x^6-1756x^5+2685x^4-1008x^3-470x^2+76x-45 && (7.58)\\
P_{16}^{(2)} &= 157x^6-2540x^5+3393x^4-1728x^3+299x^2-532x+87 && (7.59)\\
P_{17}^{(2)} &= 671x^6+284x^5-322x^4-2888x^3+863x^2+188x+212 && (7.60)
\end{aligned}
$$

$$
\begin{aligned}
P_{18}^{(2)} &= 24\ln(2)(x-1)^2(x+1)^4-61x^6+7514x^5+2979x^4-17684x^3\\
&\quad +1601x^2+7194x+185.
\end{aligned}
\tag{7.61}
$$

The constant parts $F_{V,1(2)}^{(0)}$ are given by the genuine terms and more simple terms, $F_{V,1(2)}^{(0),r}$, resulting from the pole-terms in the ε-expansion. The latter terms are not displayed explicitly, because of space reasons. In an attachment we will present the complete renormalized form factors. Because of contributions due to non first order factorizing terms (elliptic and higher) in some color–zeta combinations higher functions contribute, for which we have calculated analytically at least 2000 moments, and in some cases 4000, 6000 and 8000 to determine their recurrence relations. They are used to define the corresponding numeric representations. The corresponding recurrences, which do not factorize in first order completely will be studied elsewhere.

$$
\begin{aligned}
F_{V,1}^{(0)} &= n_h^2\Biggl\{-\frac{64H_{0,-1,-1}P_3^{(3)}}{27(1-x)(1+x)^3}+\frac{256P_5^{(3)}}{27(1+x)^4}\ln(2)\zeta_2-\frac{128H_{0,0,1}P_8^{(3)}}{81(1+x)^4}-\frac{64H_{0,-1}P_{12}^{(3)}}{243(1-x)(1+x)^3}\\
&+\frac{8P_{14}^{(3)}}{729(1+x)^4}+\frac{32H_{0,0,-1}P_{17}^{(3)}}{81(1-x)(1+x)^4}+\frac{64\zeta_2^2P_{22}^{(3)}}{9(1-x)(1+x)^5}-\frac{64H_{0,0,0,-1}P_{25}^{(3)}}{27(1-x)(1+x)^5}\\
&-\frac{4H_0^4P_{31}^{(3)}}{27(1-x)(1+x)^5}+\frac{8H_0^3P_{52}^{(3)}}{243(1-x)(1+x)^7}+\Biggl(\frac{128H_{0,1}P_8^{(3)}}{81(1+x)^4}-\frac{16P_{46}^{(3)}}{729(1-x)(1+x)^5}\\
&+\frac{512\bigl(1+6x+x^2\bigr)P_1^{(3)}}{27(1-x)(1+x)^5}H_{0,0,1}-\frac{128(1+x^2)}{27(1-x^2)}H_{0,0,-1}\Biggr)H_0+\frac{128\bigl(1+x^2\bigr)}{81(1-x^2)}H_{-1}^3H_0\\
&+\Biggl(-\frac{64H_1P_8^{(3)}}{81(1+x)^4}+\frac{32P_{47}^{(3)}}{243(1-x)(1+x)^6}-\frac{128\bigl(1+6x+x^2\bigr)P_1^{(3)}}{27(1-x)(1+x)^5}H_{0,1}\\
&+\frac{64\bigl(1+x^2\bigr)}{27(1-x^2)}H_{0,-1}\Biggr)H_0^2+\Biggl(\frac{64H_{0,-1}P_3^{(3)}}{27(1-x)(1+x)^3}+\frac{64H_0P_{12}^{(3)}}{243(1-x)(1+x)^3}\\
&-\frac{16H_0^2P_{17}^{(3)}}{81(1-x)(1+x)^4}+\frac{32H_0^3P_{25}^{(3)}}{81(1-x)(1+x)^5}+\frac{128\bigl(1+x^2\bigr)}{27(1-x^2)}[H_{0,0,-1}+2H_{0,-1,-1}]\Biggr)\\
&\times H_{-1}+\Biggl(-\frac{32H_0P_3^{(3)}}{27(1-x)(1+x)^3}-\frac{32\bigl(1+x^2\bigr)}{27(1-x^2)}H_0^2-\frac{128\bigl(1+x^2\bigr)}{27(1-x^2)}H_{0,-1}\Biggr)H_{-1}^2\\
&-\frac{256\bigl(1+6x+x^2\bigr)P_1^{(3)}}{9(1-x)(1+x)^5}H_{0,0,0,1}-\frac{128\bigl(1+x^2\bigr)}{27(1-x^2)}H_{0,0,-1,-1}-\frac{256\bigl(1+x^2\bigr)}{27(1-x^2)}H_{0,-1,-1,-1}\\
&+\Biggl(-\frac{8H_0^2P_{26}^{(3)}}{3(1-x)(1+x)^5}+\frac{32H_{0,-1}P_{32}^{(3)}}{9(1-x)(1+x)^5}+\frac{32P_{50}^{(3)}}{1215(1-x)(1+x)^6}+\frac{8H_0P_{53}^{(3)}}{81(1-x)(1+x)^7}\\
&-\Biggl(\frac{32P_{18}^{(3)}}{27(1-x)(1+x)^4}-\frac{32H_0P_{24}^{(3)}}{27(1-x)(1+x)^5}\Biggr)H_{-1}+\frac{64\bigl(1+x^2\bigr)}{27(1-x^2)}H_{-1}^2\Biggr)\zeta_2\\
&+\Biggl(\frac{16P_{20}^{(3)}}{81(1-x)(1+x)^4}+\frac{32H_0P_{34}^{(3)}}{27(1-x)(1+x)^5}-\frac{128\bigl(1+x^2\bigr)}{27(1-x^2)}H_{-1}\Biggr)\zeta_3\Biggr\}\\
&+n_h\Biggl\{\Biggl(\frac{128P_9^{(3)}}{27(1+x)^4}+\Bigl(-\frac{2048P_{30}^{(3)}}{27(1-x)(1+x)^5}-\frac{1024x^2\bigl(5-2x+5x^2\bigr)}{(1-x)(1+x)^5}H_1\Bigr)H_0\\
&-\frac{256x^2\bigl(5-2x+5x^2\bigr)}{(1-x)(1+x)^5}H_0^2+\frac{1024x^2\bigl(5-2x+5x^2\bigr)}{(1-x)(1+x)^5}H_{0,1}\Biggr)\mathrm{Li}_4\left(\frac{1}{2}\right)
\end{aligned}
$$

$$
+\left(\frac{16P_9^{(3)}}{81(1+x)^4}+\left(-\frac{256P_{30}^{(3)}}{81(1-x)(1+x)^5}-\frac{128x^2(5-2x+5x^2)}{3(1-x)(1+x)^5}H_1\right)H_0
-\frac{32x^2(5-2x+5x^2)}{3(1-x)(1+x)^5}H_0^2+\frac{128x^2(5-2x+5x^2)}{3(1-x)(1+x)^5}H_{0,1}-\frac{128x^2(5-2x+5x^2)}{3(1-x)(1+x)^5}\zeta_2\right)
\times\ln^4(2)+\left[\frac{64\zeta_2^2P_{41}^{(3)}}{9(1-x)(1+x)^5}+\left(-\frac{128H_{0,1}P_{36}^{(3)}}{9(1-x)(1+x)^5}-\frac{128H_{0,-1}P_{40}^{(3)}}{9(1-x)(1+x)^5}
+\frac{64H_0^2P_{51}^{(3)}}{9(1-x)^2(1+x)^6}+\frac{16P_{54}^{(3)}}{81x(1+x)^6}+\left(\frac{128H_1P_{36}^{(3)}}{9(1-x)(1+x)^5}-\frac{32P_{43}^{(3)}}{27(1-x)(1+x)^5}\right)H_0
-\frac{64x^2(7-22x+7x^2)}{(1-x)(1+x)^5}H_0^3+\left(-\frac{128P_4^{(3)}}{3(1+x)^4}+\frac{128H_0P_{39}^{(3)}}{9(1-x)(1+x)^5}\right)H_{-1}\right)\zeta_2\right]\ln(2)
+\left[\left(\frac{32P_{11}^{(3)}}{27(1+x)^4}+\left(\frac{128P_{35}^{(3)}}{27(1-x)(1+x)^5}+\frac{256x^2(5-2x+5x^2)}{(1-x)(1+x)^5}H_1\right)H_0
+\frac{64x^2(5-2x+5x^2)}{(1-x)(1+x)^5}H_0^2-\frac{256x^2(5-2x+5x^2)}{(1-x)(1+x)^5}H_{0,1}\right)\zeta_2+\frac{256x^2(5-2x+5x^2)}{(1-x)(1+x)^5}
\times\zeta_2^2\right]\ln^2(2)+n_l\left[\frac{256P_5^{(3)}}{27(1+x)^4}\ln(2)\zeta_2-\frac{512H_{0,-1,-1}P_6^{(3)}}{81(1-x)(1+x)^3}-\frac{128H_{0,0,1}P_8^{(3)}}{81(1+x)^4}
-\frac{128H_{0,-1}P_{10}^{(3)}}{81(1-x)(1+x)^3}+\frac{64P_{13}^{(3)}}{729(1+x)^4}-\frac{128H_{0,0,-1}P_{16}^{(3)}}{81(1-x)(1+x)^4}-\frac{128H_{0,0,0,-1}P_{28}^{(3)}}{9(1-x)(1+x)^5}
-\frac{8H_0^4P_{33}^{(3)}}{27(1-x)(1+x)^5}+\frac{64\zeta_2^2P_{37}^{(3)}}{135(1-x)(1+x)^5}+\frac{16H_0^3P_{38}^{(3)}}{81(1-x)(1+x)^5}+\left(\frac{128H_{0,1}P_8^{(3)}}{81(1+x)^4}
-\frac{64P_{45}^{(3)}}{729(1-x)(1+x)^5}+\frac{(1+6x+x^2)P_1^{(3)}}{(1-x)(1+x)^5}\left[\frac{512}{27}H_{0,0,1}-\frac{128}{9}H_{0,0,-1}\right]\right)H_0
+\frac{1024(1+x^2)}{81(1-x^2)}H_{-1}^3H_0-\left(\frac{64H_1P_8^{(3)}}{81(1+x)^4}+\frac{32P_{48}^{(3)}}{81(1-x)(1+x)^6}+\frac{128(1+6x+x^2)P_1^{(3)}}{27(1-x)(1+x)^5}
\times H_{0,1}-\frac{64(1+6x+x^2)P_1^{(3)}}{9(1-x)(1+x)^5}H_{0,-1}\right)H_0^2+\left(\frac{512H_{0,-1}P_6^{(3)}}{81(1-x)(1+x)^3}+\frac{128H_0P_{10}^{(3)}}{81(1-x)(1+x)^3}
+\frac{64H_0^2P_{16}^{(3)}}{81(1-x)(1+x)^4}+\frac{64H_0^3P_{28}^{(3)}}{27(1-x)(1+x)^5}+\frac{1024(1+x^2)}{27(1-x^2)}H_{0,0,-1}+\frac{2048(1+x^2)}{27(1-x^2)}
\times H_{0,-1,-1}\right)H_{-1}-\left(\frac{256H_0P_6^{(3)}}{81(1-x)(1+x)^3}+\frac{256(1+x^2)}{27(1-x^2)}H_0^2+\frac{1024(1+x^2)}{27(1-x^2)}H_{0,-1}\right)
\times H_{-1}^2-\frac{256(1+6x+x^2)P_1^{(3)}}{9(1-x)(1+x)^5}H_{0,0,0,1}-\frac{1024(1+x^2)}{27(1-x^2)}H_{0,0,-1,-1}-\frac{2048(1+x^2)}{27(1-x^2)}
\times H_{0,-1,-1,-1}+\left(\frac{64H_{0,-1}P_{23}^{(3)}}{9(1-x)(1+x)^5}-\frac{16H_0^2P_{27}^{(3)}}{3(1-x)(1+x)^5}+\frac{16H_0P_{42}^{(3)}}{81(1-x)(1+x)^5}
+\frac{8P_{49}^{(3)}}{81(1-x)(1+x)^6}+\big(-\frac{128P_{19}^{(3)}}{81(1-x)(1+x)^4}+\frac{64H_0P_{29}^{(3)}}{27(1-x)(1+x)^5}\big)H_{-1}
$$

$$
+\frac{512\big(1+x^2\big)}{27(1-x^2)}H_{-1}^2\Bigg)\zeta_2+\Bigg(\frac{16P_{21}^{(3)}}{81(1-x)(1+x)^4}+\frac{32\big(1+x^2\big)}{3(1-x^2)}H_0-\frac{1024\big(1+x^2\big)}{27(1-x^2)}
$$

$$
\times H_{-1}\Bigg)\zeta_3\Bigg]+\Bigg(-\frac{1024x^2\big(5-2x+5x^2\big)}{(1-x)(1+x)^5}\text{Li}_4\left(\frac{1}{2}\right)+\Bigg(-\frac{32x^2\big(5-32x+5x^2\big)}{(1-x)(1+x)^5}H_0
$$

$$
-\frac{8P_7^{(3)}}{(1+x)^4}\Bigg)\zeta_3\Bigg)\zeta_2+\Bigg(-\frac{8P_{15}^{(3)}}{135(1+x)^4}+\Bigg(\frac{64x^2\big(43+98x+43x^2\big)}{5(1-x)(1+x)^5}H_1
$$

$$
-\frac{32P_{44}^{(3)}}{135(1-x)(1+x)^5}\Bigg)H_0+\frac{16x^2\big(43+98x+43x^2\big)}{5(1-x)(1+x)^5}H_0^2-\frac{64x^2\big(43+98x+43x^2\big)}{5(1-x)(1+x)^5}
$$

$$
\times H_{0,1}\Bigg)\zeta_2^2+\frac{64x^2\big(43+98x+43x^2\big)}{5(1-x)(1+x)^5}\zeta_2^3+\Bigg(\frac{80x^2\big(1-16x+x^2\big)}{(1-x)(1+x)^5}H_0
$$

$$
+\frac{20P_2^{(3)}}{(1+x)^4}\Bigg)\zeta_5\Bigg\}+n_hF_{V,1,1}^{(0)}(x)+n_h\zeta_2F_{V,1,2}^{(0)}(x)+n_h\zeta_3F_{V,1,3}^{(0)}(x)+F_{V,1}^{(0),r}(x), \tag{7.62}
$$

with the polynomials

$$P_1^{(3)} = x^4-2x^3+12x^2-2x+1, \tag{7.63}$$
$$P_2^{(3)} = x^4+8x^3+50x^2+8x+1, \tag{7.64}$$
$$P_3^{(3)} = x^4+12x^3+6x^2+12x+1, \tag{7.65}$$
$$P_4^{(3)} = 2x^4-7x^3+12x^2-7x+2, \tag{7.66}$$
$$P_5^{(3)} = 11x^4+14x^3+238x^2+14x+11, \tag{7.67}$$
$$P_6^{(3)} = 11x^4+40x^3+34x^2+40x+11, \tag{7.68}$$
$$P_7^{(3)} = 13x^4+56x^3+170x^2+56x+13, \tag{7.69}$$
$$P_8^{(3)} = 19x^4-2x^3+206x^2-2x+19, \tag{7.70}$$
$$P_9^{(3)} = 53x^4+184x^3+502x^2+184x+53, \tag{7.71}$$
$$P_{10}^{(3)} = 89x^4+108x^3+710x^2+108x+89, \tag{7.72}$$
$$P_{11}^{(3)} = 91x^4-94x^3-70x^2-94x+91, \tag{7.73}$$
$$P_{12}^{(3)} = 155x^4+10x^3+1846x^2+10x+155, \tag{7.74}$$
$$P_{13}^{(3)} = 377x^4+3452x^3+6006x^2+3452x+377, \tag{7.75}$$
$$P_{14}^{(3)} = 1367x^4+26372x^3+46362x^2+26372x+1367, \tag{7.76}$$
$$P_{15}^{(3)} = 5363x^4+15190x^3+15322x^2+15190x+5363, \tag{7.77}$$
$$P_{16}^{(3)} = 3x^5+123x^4-60x^3+356x^2+81x+41, \tag{7.78}$$
$$P_{17}^{(3)} = 35x^5-81x^4+362x^3-470x^2+3x-41, \tag{7.79}$$
$$P_{18}^{(3)} = 39x^5-29x^4+434x^3-398x^2+55x-37, \tag{7.80}$$
$$P_{19}^{(3)} = 79x^5+39x^4+772x^3-476x^2+165x-35, \tag{7.81}$$
$$P_{20}^{(3)} = 253x^5+435x^4+7450x^3-7234x^2-279x-241, \tag{7.82}$$
$$P_{21}^{(3)} = 463x^5+837x^4+7414x^3-5046x^2+795x-111, \tag{7.83}$$
$$P_{22}^{(3)} = x^6+4x^5+78x^3+4x+1, \tag{7.84}$$

$$P_{23}^{(3)} = x^6 + 4x^5 - 21x^4 + 288x^3 - 21x^2 + 4x + 1, \tag{7.85}$$
$$P_{24}^{(3)} = x^6 + 4x^5 - 17x^4 + 248x^3 - 17x^2 + 4x + 1, \tag{7.86}$$
$$P_{25}^{(3)} = x^6 + 4x^5 - 5x^4 + 128x^3 - 5x^2 + 4x + 1, \tag{7.87}$$
$$P_{26}^{(3)} = x^6 + 4x^5 - x^4 + 88x^3 - x^2 + 4x + 1, \tag{7.88}$$
$$P_{27}^{(3)} = x^6 + 4x^5 + 3x^4 + 48x^3 + 3x^2 + 4x + 1, \tag{7.89}$$
$$P_{28}^{(3)} = x^6 + 4x^5 + 9x^4 - 12x^3 + 9x^2 + 4x + 1, \tag{7.90}$$
$$P_{29}^{(3)} = x^6 + 4x^5 + 19x^4 - 112x^3 + 19x^2 + 4x + 1, \tag{7.91}$$
$$P_{30}^{(3)} = x^6 + 10x^5 + 36x^4 + 84x^3 + 36x^2 + 10x + 1, \tag{7.92}$$
$$P_{31}^{(3)} = 3x^6 + 12x^5 + x^4 + 224x^3 + x^2 + 12x + 3, \tag{7.93}$$
$$P_{32}^{(3)} = 3x^6 + 12x^5 + 5x^4 + 184x^3 + 5x^2 + 12x + 3, \tag{7.94}$$
$$P_{33}^{(3)} = 3x^6 + 12x^5 + 11x^4 + 124x^3 + 11x^2 + 12x + 3, \tag{7.95}$$
$$P_{34}^{(3)} = 11x^6 + 44x^5 + 5x^4 + 808x^3 + 5x^2 + 44x + 11, \tag{7.96}$$
$$P_{35}^{(3)} = 13x^6 + 67x^5 + 243x^4 + 228x^3 + 243x^2 + 67x + 13, \tag{7.97}$$
$$P_{36}^{(3)} = 21x^6 - 76x^5 - 193x^4 - 36x^3 - 193x^2 - 76x + 21, \tag{7.98}$$
$$P_{37}^{(3)} = 31x^6 + 124x^5 + 304x^4 - 622x^3 + 304x^2 + 124x + 31, \tag{7.99}$$
$$P_{38}^{(3)} = 35x^6 - 118x^5 + 325x^4 - 404x^3 - 423x^2 - 110x - 41, \tag{7.100}$$
$$P_{39}^{(3)} = 99x^6 + 70x^5 + 61x^4 + 24x^3 + 61x^2 + 70x + 99, \tag{7.101}$$
$$P_{40}^{(3)} = 105x^6 + 88x^5 + 127x^4 - 48x^3 + 127x^2 + 88x + 105, \tag{7.102}$$
$$P_{41}^{(3)} = 147x^6 - 64x^5 - 259x^4 - 120x^3 - 259x^2 - 64x + 147, \tag{7.103}$$
$$P_{42}^{(3)} = 189x^6 - 390x^5 + 1827x^4 - 1684x^3 - 1165x^2 - 358x - 115, \tag{7.104}$$
$$P_{43}^{(3)} = 381x^6 + 316x^5 - 2779x^4 + 13392x^3 - 3139x^2 + 568x + 309, \tag{7.105}$$
$$P_{44}^{(3)} = 806x^6 + 2501x^5 + 5574x^4 + 9924x^3 + 5574x^2 + 2501x + 806, \tag{7.106}$$
$$P_{45}^{(3)} = 1471x^6 + 5920x^5 + 19873x^4 + 30272x^3 + 19873x^2 + 5920x + 1471, \tag{7.107}$$
$$P_{46}^{(3)} = 2945x^6 + 9044x^5 + 56687x^4 + 91384x^3 + 56687x^2 + 9044x + 2945, \tag{7.108}$$
$$P_{47}^{(3)} = 19x^7 - 43x^6 + 476x^5 - 2038x^4 - 3695x^3 - 2817x^2 - 432x - 174, \tag{7.109}$$
$$P_{48}^{(3)} = 62x^7 + 269x^6 + 965x^5 + 2426x^4 + 2876x^3 + 1637x^2 + 481x + 116, \tag{7.110}$$
$$P_{49}^{(3)} = 1969x^7 + 7833x^6 + 31477x^5 + 37933x^4 + 4483x^3 - 10661x^2 - 1833x - 545, \tag{7.111}$$
$$P_{50}^{(3)} = 4452x^7 + 17001x^6 + 81141x^5 + 78032x^4 - 20702x^3 - 57731x^2 - 12251x - 2902, \tag{7.112}$$
$$P_{51}^{(3)} = 70x^8 + 173x^7 - 342x^6 + 245x^5 + 860x^4 + 241x^3 - 458x^2 + 45x - 2, \tag{7.113}$$
$$P_{52}^{(3)} = 181x^8 + 240x^7 + 1444x^6 + 1952x^5 - 3522x^4 - 4000x^3 - 1788x^2 - 336x - 123, \tag{7.114}$$
$$P_{53}^{(3)} = 213x^8 + 240x^7 + 1452x^6 + 1264x^5 - 6642x^4 - 4688x^3 - 1780x^2 - 336x - 91, \tag{7.115}$$
$$\begin{aligned} P_{54}^{(3)} &= 2768x^8 - 60837x^7 - 222044x^6 - 979155x^5 - 1790040x^4 - 979155x^3 - 222044x^2 \\ &\quad - 60837x + 2768. \end{aligned} \tag{7.116}$$

In the variable $y = 1 - x$ the first expansion coefficients of the functions $F_{V,1,i}^{(0)}(x), i = 1..3$ are given by

$$F_{V,1,1}^{(0)}(x) = -\frac{2222242}{243} + \frac{1047067y^2}{729} + \frac{1047067y^3}{729} + \frac{3436873681y^4}{3499200} + \frac{923912881y^5}{1749600} + O(y^6) \tag{7.117}$$

$$F_{V,1,2}^{(0)}(x) = \frac{2390434}{243} - \frac{8763197y^2}{4860} - \frac{8763197y^3}{4860} - \frac{4103868673y^4}{3402000} - \frac{345583241y^5}{567000} + O(y^6) \tag{7.118}$$

$$F_{V,1,3}^{(0)}(x) = \frac{311488}{81} - \frac{259276y^2}{243} - \frac{259276y^3}{243} - \frac{571282067y^4}{777600} - \frac{156440467y^5}{388800} + O(y^6) \tag{7.119}$$

and

$$\begin{aligned}
F_{V,2}^{(0)} = x\Bigg\{ & n_h^2\Bigg\{-\frac{128\big(451+830x+451x^2\big)}{243(1+x)^4} + \Bigg(\frac{128P_{11}^{(4)}}{243(x-1)(1+x)^5} - \frac{512\big(7-79x+7x^2\big)}{81(x-1)(1+x)^3}H_{-1} \\
& -\frac{128(x-1)H_{-1}^2}{27(1+x)^3}\Bigg)H_0 + \Bigg(-\frac{128P_{14}^{(4)}}{81(x-1)(1+x)^6} + \frac{64\big(1-40x+3x^2\big)}{27(1+x)^4}H_{-1}\Bigg)H_0^2 \\
& +\Bigg(-\frac{32P_{18}^{(4)}}{81(x-1)(1+x)^7} + \frac{512x^2H_{-1}}{9(x-1)(1+x)^5}\Bigg)H_0^3 - \frac{320x^2H_0^4}{9(x-1)(1+x)^5} \\
& -\frac{256\big(1-20x+x^2\big)H_0^2H_1}{27(1+x)^4} + \Bigg(\frac{512\big(1-20x+x^2\big)H_0}{27(1+x)^4} \\
& -\frac{1024x^2H_0^2}{3(x-1)(1+x)^5}\Bigg)H_{0,1} + \Bigg(\frac{512\big(7-79x+7x^2\big)}{81(x-1)(1+x)^3} + \frac{256(x-1)H_{-1}}{27(1+x)^3}\Bigg)H_{0,-1} \\
& +\Bigg(-\frac{512\big(1-20x+x^2\big)}{27(1+x)^4} + \frac{4096x^2H_0}{3(x-1)(1+x)^5}\Bigg)H_{0,0,1} - \frac{128\big(1-40x+3x^2\big)}{27(1+x)^4}H_{0,0,-1} \\
& -\frac{256(x-1)H_{0,-1,-1}}{27(1+x)^3} - \frac{2048x^2H_{0,0,0,1}}{(x-1)(1+x)^5} - \frac{1024x^2H_{0,0,0,-1}}{3(x-1)(1+x)^5} \\
& +\Bigg(-\frac{1024\big(1+22x+x^2\big)}{9(1+x)^4}\ln(2) - \frac{32P_{16}^{(4)}}{405(x-1)(1+x)^6} + \Bigg(-\frac{32P_{19}^{(4)}}{27(x-1)(1+x)^7} \\
& +\frac{1024x^2H_{-1}}{3(x-1)(1+x)^5}\Bigg)H_0 - \frac{256x^2H_0^2}{(x-1)(1+x)^5} + \frac{128\big(7-120x+5x^2\big)}{27(1+x)^4}H_{-1} \\
& +\frac{2048x^2H_{0,-1}}{3(x-1)(1+x)^5}\Bigg)\zeta_2 + \frac{1792x^2\zeta_2^2}{3(x-1)(1+x)^5} + \Bigg(\frac{128\big(27+364x+29x^2\big)}{27(1+x)^4} \\
& +\frac{1024x^2H_0}{(x-1)(1+x)^5}\Bigg)\zeta_3\Bigg\} \\
& +n_h\Bigg\{-\frac{160\zeta_5P_1^{(4)}}{(x-1)^2(1+x)^4} + \ln^4(2)\Bigg(\frac{128xH_0P_7^{(4)}}{27(x-1)^3(1+x)^5} - \frac{128P_8^{(4)}}{81(x-1)^2(1+x)^4}
\end{aligned}$$

$$-\frac{128x\big(1-x+x^2\big)}{3(x-1)(1+x)^5}H_0^2-\frac{512x\big(1-x+x^2\big)}{3(x-1)(1+x)^5}H_0H_1+\frac{512x\big(1-x+x^2\big)}{3(x-1)(1+x)^5}H_{0,1}\Bigg)$$

$$+\text{Li}_4\left(\frac{1}{2}\right)\Bigg(\frac{1024xH_0P_7^{(4)}}{9(x-1)^3(1+x)^5}-\frac{1024P_8^{(4)}}{27(x-1)^2(1+x)^4}-\frac{1024x\big(1-x+x^2\big)}{(x-1)(1+x)^5}H_0^2$$

$$-\frac{4096x\big(1-x+x^2\big)}{(x-1)(1+x)^5}H_0H_1+\frac{4096x\big(1-x+x^2\big)}{(x-1)(1+x)^5}H_{0,1}\Bigg)+n_l\Bigg[\Bigg(\frac{256P_9^{(4)}}{81(x-1)(1+x)^5}$$

$$-\frac{512\big(23-130x+23x^2\big)}{81(x-1)(1+x)^3}H_{-1}+\frac{2048xH_{-1}^2}{27(x-1)(1+x)^3}\Bigg)H_0-\frac{1024\big(14+27x+14x^2\big)}{81(1+x)^4}$$

$$+\Bigg(\frac{128P_{15}^{(4)}}{81(x-1)(1+x)^6}+\frac{256\big(-1+17x-25x^2+x^3\big)}{27(x-1)(1+x)^4}H_{-1}\Bigg)H_0^2$$

$$+\Bigg(-\frac{64P_4^{(4)}}{27(x-1)(1+x)^5}-\frac{512x^2H_{-1}}{9(x-1)(1+x)^5}\Bigg)H_0^3-\frac{320x^2H_0^4}{9(x-1)(1+x)^5}$$

$$-\frac{256\big(1-20x+x^2\big)H_0^2H_1}{27(1+x)^4}+\Bigg(\frac{512\big(1-20x+x^2\big)H_0}{27(1+x)^4}-\frac{1024x^2H_0^2}{3(x-1)(1+x)^5}\Bigg)H_{0,1}$$

$$+\Bigg(\frac{512\big(23-130x+23x^2\big)}{81(x-1)(1+x)^3}+\frac{512x^2H_0^2}{(x-1)(1+x)^5}-\frac{4096xH_{-1}}{27(x-1)(1+x)^3}\Bigg)H_{0,-1}$$

$$+\Bigg(-\frac{512\big(1-20x+x^2\big)}{27(1+x)^4}+\frac{4096x^2H_0}{3(x-1)(1+x)^5}\Bigg)H_{0,0,1}-\Bigg(\frac{1024x^2H_0}{(x-1)(1+x)^5}$$

$$+\frac{512\big(-1+17x-25x^2+x^3\big)}{27(x-1)(1+x)^4}\Bigg)H_{0,0,-1}+\frac{4096xH_{0,-1,-1}}{27(x-1)(1+x)^3}-\frac{2048x^2H_{0,0,0,1}}{(x-1)(1+x)^5}$$

$$+\frac{1024x^2H_{0,0,0,-1}}{3(x-1)(1+x)^5}+\Bigg(-\frac{1024\big(1+22x+x^2\big)}{9(1+x)^4}\ln(2)-\frac{32P_{17}^{(4)}}{81(x-1)(1+x)^6}$$

$$+\Bigg(-\frac{128P_6^{(4)}}{27(x-1)(1+x)^5}-\frac{1024x^2H_{-1}}{3(x-1)(1+x)^5}\Bigg)H_0-\frac{256x^2H_0^2}{(x-1)(1+x)^5}$$

$$+\frac{512\big(-3+67x-59x^2+3x^3\big)}{27(x-1)(1+x)^4}H_{-1}+\frac{7168x^2H_{0,-1}}{3(x-1)(1+x)^5}\Bigg)\zeta_2-\frac{7424x^2\zeta_2^2}{15(x-1)(1+x)^5}$$

$$+\frac{256\big(-7-155x+139x^2+7x^3\big)}{27(x-1)(1+x)^4}\zeta_3\Bigg]+\Bigg(-\frac{4096x\big(1-x+x^2\big)}{(x-1)(1+x)^5}\text{Li}_4\left(\frac{1}{2}\right)$$

$$-\frac{512x\big(1-x+x^2\big)}{3(x-1)(1+x)^5}\ln^4(2)+\ln(2)\Bigg(-\frac{256H_{-1}P_5^{(4)}}{3(x-1)^2(1+x)^4}+\frac{1024H_{0,-1}P_{20}^{(4)}}{9(x-1)^3(1+x)^5}$$

$$-\frac{512H_0H_1P_{23}^{(4)}}{9(x-1)^3(1+x)^5}+\frac{512H_{0,1}P_{23}^{(4)}}{9(x-1)^3(1+x)^5}+\frac{256xH_0^2P_{24}^{(4)}}{9(x-1)^3(1+x)^6}$$

$$+\frac{64P_{26}^{(4)}}{81(x-1)^2x(1+x)^6}+\Bigg(-\frac{1024H_{-1}P_{22}^{(4)}}{9(x-1)^3(1+x)^5}+\frac{256P_{25}^{(4)}}{27(x-1)^3(1+x)^5}\Bigg)H_0$$

$$
\begin{aligned}
&-\frac{256x\big(1-7x+x^2\big)}{(x-1)(1+x)^5}H_0^3\Bigg)+\ln^2(2)\Bigg(-\frac{256xH_0P_3^{(4)}}{9(x-1)^3(1+x)^5}+\frac{128P_{10}^{(4)}}{27(x-1)^2(1+x)^4}\\
&+\frac{256x\big(1-x+x^2\big)}{(x-1)(1+x)^5}H_0^2+\frac{1024x\big(1-x+x^2\big)}{(x-1)(1+x)^5}H_0H_1-\frac{1024x\big(1-x+x^2\big)}{(x-1)(1+x)^5}H_{0,1}\Bigg)\\
&+\Bigg(\frac{64P_2^{(4)}}{(x-1)^2(1+x)^4}+\frac{768x^2\big(2-3x+2x^2\big)}{(x-1)^3(1+x)^5}H_0\Bigg)\zeta_3\Bigg)\zeta_2+\Bigg(-\frac{64xH_0P_{12}^{(4)}}{45(x-1)^3(1+x)^5}\\
&-\frac{32P_{13}^{(4)}}{135(x-1)^2(1+x)^4}-\frac{1024P_{21}^{(4)}}{9(x-1)^3(1+x)^5}\ln(2)+\frac{1024x\big(1-x+x^2\big)}{(x-1)(1+x)^5}\ln^2(2)\\
&+\frac{64x\big(11+25x+11x^2\big)}{5(x-1)(1+x)^5}H_0^2+\frac{256x\big(11+25x+11x^2\big)}{5(x-1)(1+x)^5}H_0H_1\\
&-\frac{256x\big(11+25x+11x^2\big)}{5(x-1)(1+x)^5}H_{0,1}\Bigg)\zeta_2^2+\frac{256x\big(11+25x+11x^2\big)}{5(x-1)(1+x)^5}\zeta_2^3\\
&-\frac{640(-2+x)x^2(-1+2x)}{(x-1)^3(1+x)^5}H_0\zeta_5\Bigg\}\Bigg\}\\
&+n_hF_{V,2,1}^{(0)}(x)+n_hF_{V,2,2}^{(0)}(x)\zeta_2+n_hF_{V,2,3}^{(0)}(x)\zeta_3+F_{V,2}^{(0),r}(x),
\end{aligned}
\tag{7.120}
$$

with the polynomials

$$P_1^{(4)}=x^4+5x^3-10x^2+5x+1, \tag{7.121}$$
$$P_2^{(4)}=x^4+9x^3-26x^2+9x+1, \tag{7.122}$$
$$P_3^{(4)}=x^4+82x^3+62x^2+82x+1, \tag{7.123}$$
$$P_4^{(4)}=3x^4-64x^3-6x^2+16x-1, \tag{7.124}$$
$$P_5^{(4)}=5x^4-12x^3+26x^2-12x+5, \tag{7.125}$$
$$P_6^{(4)}=7x^4-147x^3-8x^2+13x-1, \tag{7.126}$$
$$P_7^{(4)}=7x^4+28x^3+122x^2+28x+7, \tag{7.127}$$
$$P_8^{(4)}=17x^4+102x^3+50x^2+102x+17, \tag{7.128}$$
$$P_9^{(4)}=67x^4-10x^3-138x^2-10x+67, \tag{7.129}$$
$$P_{10}^{(4)}=115x^4+96x^3+262x^2+96x+115, \tag{7.130}$$
$$P_{11}^{(4)}=187x^4-1300x^3-2590x^2-1300x+187, \tag{7.131}$$
$$P_{12}^{(4)}=671x^4+1042x^3-702x^2+1042x+671, \tag{7.132}$$
$$P_{13}^{(4)}=1145x^4-960x^3-8542x^2-960x+1145, \tag{7.133}$$
$$P_{14}^{(4)}=2x^5-61x^4+137x^3+281x^2+177x-16, \tag{7.134}$$
$$P_{15}^{(4)}=14x^5-35x^4-271x^3-325x^2-87x+32, \tag{7.135}$$
$$P_{16}^{(4)}=77x^5-19649x^4-22606x^3+5886x^2+15009x+483, \tag{7.136}$$
$$P_{17}^{(4)}=361x^5-3677x^4-5558x^3+790x^2+2701x+7, \tag{7.137}$$
$$P_{18}^{(4)}=13x^6-174x^5-381x^4+60x^3+243x^2+114x-3, \tag{7.138}$$
$$P_{19}^{(4)}=17x^6-206x^5-449x^4+124x^3+175x^2+82x+1, \tag{7.139}$$

$$P_{20}^{(4)} = 27x^6 - 31x^5 + 21x^4 - 10x^3 + 21x^2 - 31x + 27, \tag{7.140}$$
$$P_{21}^{(4)} = 27x^6 - 28x^5 - 24x^4 + 56x^3 - 24x^2 - 28x + 27, \tag{7.141}$$
$$P_{22}^{(4)} = 27x^6 - 28x^5 - 6x^4 + 20x^3 - 6x^2 - 28x + 27, \tag{7.142}$$
$$P_{23}^{(4)} = 27x^6 - 25x^5 - 69x^4 + 122x^3 - 69x^2 - 25x + 27, \tag{7.143}$$
$$P_{24}^{(4)} = 27x^6 + 136x^5 - 245x^4 - 221x^3 + 202x^2 + 223x - 134, \tag{7.144}$$
$$P_{25}^{(4)} = 165x^6 - 101x^5 - 2215x^4 + 5984x^3 - 2404x^2 + 7x + 120, \tag{7.145}$$
$$P_{26}^{(4)} = 1280x^8 - 7735x^7 + 155816x^6 + 33391x^5 - 401120x^4 + 33391x^3 + 155816x^2 - 7735x + 1280, \tag{7.146}$$

and

$$F_{V,2,1}^{(0)}(x) = \frac{1083812}{243} - \frac{32229191y^2}{18225} - \frac{32229191y^3}{18225} - \frac{488527686149y^4}{428652000} - \frac{109512399989y^5}{214326000} + O(y^6) \tag{7.147}$$
$$F_{V,2,2}^{(0)}(x) = -\frac{1126457}{243} + \frac{150475907y^2}{72900} + \frac{150475907y^3}{72900} + \frac{462427840529y^4}{357210000} + \frac{93761868379y^5}{178605000} + O(y^6) \tag{7.148}$$
$$F_{V,2,3}^{(0)}(x) = -\frac{334736}{81} + \frac{8944046y^2}{6075} + \frac{8944046y^3}{6075} + \frac{92909575471y^4}{95256000} + \frac{22788254831y^5}{47628000} + O(y^6). \tag{7.149}$$

7.2 The Scalar Form Factor

The scalar form factor is given by

$$\begin{aligned} F_S = & -\frac{1}{\varepsilon^3}\frac{1}{2(1+x)^2}\Bigg\{n_h^2\Bigg[-\frac{64}{27}(1+x)^2 + \frac{64(1+x)\big(1+x^2\big)}{27(1-x)}H_0\Bigg] + n_h\Bigg[\frac{4}{27}\big(997+1418x \\ & +997x^2\big) - \frac{32H_0P_8^{(5)}}{27(1-x^2)} - n_l\Bigg[\frac{32}{9}(1+x)^2 - \frac{64(1+x)\big(1+x^2\big)}{27(1-x)}H_0\Bigg] + \frac{256\big(1+x^2\big)^2}{27(1-x)^2}H_0^2\Bigg]\Bigg\} \\ & -\frac{1}{\varepsilon^2}\frac{1}{2(1+x)^2}\Bigg\{n_h^2\Bigg[-\frac{832}{81}(1+x)^2 - \frac{256x(1+x)H_0}{27(1-x)} - \frac{128(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}H_0 \\ & +\frac{32(1+x)\big(1+x^2\big)}{27(1-x)}H_0^2 + \frac{128(1+x)\big(1+x^2\big)}{27(1-x)}H_{0,-1} - \frac{64(1+x)\big(1+x^2\big)}{27(1-x)}\zeta_2\Bigg] \\ & +n_h\Bigg[\frac{16}{27}\big(897+1786x+897x^2\big) + n_l\Bigg[-\frac{64}{3}(1+x)^2 + \frac{64(1+x)\big(5-24x+5x^2\big)}{81(1-x)}H_0 \\ & -\frac{256(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}H_0 + \frac{64(1+x)\big(1+x^2\big)}{27(1-x)}H_0^2 + \frac{256(1+x)\big(1+x^2\big)}{27(1-x)}H_{0,-1} \\ & -\frac{128(1+x)\big(1+x^2\big)}{27(1-x)}\zeta_2\Bigg] + \Bigg(\frac{128H_{-1}P_7^{(5)}}{27(1-x^2)} - \frac{16P_{13}^{(5)}}{27(1-x^2)}\Bigg)H_0 + \Bigg(\frac{64P_{26}^{(5)}}{27(1-x)^2(1+x)} \end{aligned}$$

$$
-\frac{1024\big(1+x^2\big)^2}{27(1-x)^2}H_{-1}\bigg)H_0^2-\frac{128\big(-2+x^2\big)\big(1+x^2\big)}{27(1-x)^2}H_0^3-\frac{128(1+x)\big(1+x^2\big)}{3(1-x)}H_0H_1
$$

$$
+\bigg(\frac{128(1+x)\big(1+x^2\big)}{3(1-x)}-\frac{128\big(1+x^2\big)^2}{3(1-x)^2}H_0\bigg)H_{0,1}-\bigg(\frac{128P_7^{(5)}}{27(1-x^2)}-\frac{2176\big(1+x^2\big)^2}{27(1-x)^2}H_0\bigg)
$$

$$
\times H_{0,-1}+\frac{256\big(1+x^2\big)^2}{3(1-x)^2}H_{0,0,1}-\frac{256\big(1+x^2\big)^2}{3(1-x)^2}H_{0,0,-1}+\bigg(\frac{64P_5^{(5)}}{27(1-x^2)}
$$

$$
-\frac{64\big(1+x^2\big)\big(-1+35x^2\big)}{27(1-x)^2}H_0\bigg)\zeta_2-\frac{64\big(1+x^2\big)^2}{3(1-x)^2}\zeta_3\bigg]\bigg\}
$$

$$
-\frac{1}{\varepsilon}\frac{1}{2(1+x)^2}\bigg\{n_h^2\bigg[-\frac{32}{81}\big(107+294x+107x^2\big)+\bigg(\frac{64P_{10}^{(5)}}{243(1-x^2)}+\frac{512x(1+x)H_{-1}}{27(1-x)}
$$

$$
+\frac{128(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}^2\bigg)H_0-\bigg(\frac{64P_{17}^{(5)}}{81(1-x)(1+x)^2}+\frac{64(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\bigg)H_0^2
$$

$$
+\frac{32(1+x)\big(1+x^2\big)}{27(1-x)}H_0^3-\bigg(\frac{512x(1+x)}{27(1-x)}+\frac{256(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\bigg)H_{0,-1}
$$

$$
+\frac{128(1+x)\big(1+x^2\big)}{27(1-x)}H_{0,0,-1}+\frac{256(1+x)\big(1+x^2\big)}{27(1-x)}H_{0,-1,-1}+\bigg(-\frac{32P_{25}^{(5)}}{27(1-x)(1+x)^2}
$$

$$
+\frac{160(1+x)\big(1+x^2\big)}{27(1-x)}H_0+\frac{128(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\bigg)\zeta_2-\frac{128(1+x)\big(1+x^2\big)}{27(1-x)}\zeta_3\bigg]
$$

$$
+n_h\bigg[\frac{64P_{15}^{(5)}}{243(1+x)^2}+n_l\bigg[-\frac{640}{243}\big(37+86x+37x^2\big)+\bigg(\frac{512(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}^2
$$

$$
+\frac{128P_4^{(5)}}{81(1-x^2)}-\frac{256(1+x)\big(5-24x+5x^2\big)}{81(1-x)}H_{-1}\bigg)H_0+\bigg(-\frac{128P_2^{(5)}}{81(1-x)(1+x)^2}
$$

$$
-\frac{256(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\bigg)H_0^2+\frac{64(1+x)\big(1+x^2\big)}{27(1-x)}H_0^3+\bigg(\frac{256(1+x)\big(5-24x+5x^2\big)}{81(1-x)}
$$

$$
-\frac{1024(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\bigg)H_{0,-1}+\frac{512(1+x)\big(1+x^2\big)}{27(1-x)}[H_{0,0,-1}+2H_{0,-1,-1}]
$$

$$
+\bigg(-\frac{16P_{32}^{(5)}}{81(1-x)(1+x)^2}+\frac{224(1+x)\big(1+x^2\big)}{27(1-x)}H_0+\frac{512(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\bigg)\zeta_2
$$

$$
-\frac{512(1+x)\big(1+x^2\big)}{27(1-x)}\zeta_3\bigg]+\bigg(-\frac{256H_{-1}^2P_9^{(5)}}{27(1-x^2)}+\frac{32H_{-1}P_{14}^{(5)}}{27(1-x^2)}-\frac{32P_{37}^{(5)}}{243(1-x)(1+x)^3}\bigg)H_0
$$

$$
+\bigg(\frac{8P_{41}^{(5)}}{243(1-x)^2(1+x)^4}-\frac{32P_{34}^{(5)}}{27(1-x)^2(1+x)}H_{-1}+\frac{2048\big(1+x^2\big)^2}{27(1-x)^2}H_{-1}^2\bigg)H_0^2
$$

$$
+\bigg(\frac{32P_{39}^{(5)}}{81(1-x)^2(1+x)^3}-\frac{128\big(1+x^2\big)\big(-1+5x^2\big)}{27(1-x)^2}H_{-1}\bigg)H_0^3+\frac{16P_{19}^{(5)}}{81(1-x)^2(1+x)}H_0^4
$$

$$
+\Bigg(\frac{256H_0^2P_1^{(5)}}{27(1-x)^2}-\Bigg(\frac{256(1+x)\big(7-32x+7x^2\big)}{27(1-x)}-\frac{256(1+x)\big(1+x^2\big)}{1-x}H_{-1}\Bigg)H_0
$$

$$
+\frac{128P_{24}^{(5)}}{81(1-x)^2(1+x)}H_0^3\Bigg)H_1-\frac{128(1+x)\big(1+x^2\big)}{3(1-x)}H_0H_1^2+\Bigg(\frac{256(1+x)\big(7-32x+7x^2\big)}{27(1-x)}
$$

$$
+\Bigg(\frac{512P_3^{(5)}}{27(1-x)^2}+\frac{512\big(1+x^2\big)^2}{3(1-x)^2}H_{-1}\Bigg)H_0-\frac{256P_{23}^{(5)}H_0^2}{27(1-x)^2(1+x)}+\Bigg(\frac{256(1+x)\big(1+x^2\big)}{3(1-x)}
$$

$$
-\frac{512\big(1+x^2\big)^2}{27(1-x)^2}H_0\Bigg)H_1-\frac{256(1+x)\big(1+x^2\big)}{1-x}H_{-1}-\frac{6400\big(1+x^2\big)^2}{27(1-x)^2}H_{0,-1}\big)H_{0,1}
$$

$$
+\frac{896\big(1+x^2\big)^2}{27(1-x)^2}H_{0,1}^2+\Bigg(-\frac{512H_{-1}P_9^{(5)}}{27(-1+x)(1+x)}+\frac{32P_{14}^{(5)}}{27(-1+x)(1+x)}-\Bigg(\frac{64P_{12}^{(5)}}{27(1-x)^2}
$$

$$
+\frac{8704\big(1+x^2\big)^2}{27(1-x)^2}H_{-1}\Bigg)H_0+\frac{128P_{29}^{(5)}}{27(1-x)^2(1+x)}H_0^2+\Bigg(-\frac{256(1+x)\big(1+x^2\big)}{1-x}
$$

$$
+\frac{512\big(1+x^2\big)^2}{27(1-x)^2}H_0\Bigg)H_1\Bigg)H_{0,-1}+\frac{4480\big(1+x^2\big)^2}{27(1-x)^2}H_{0,-1}^2+\Bigg(-\frac{512P_6^{(5)}}{27(1-x)^2}
$$

$$
+\frac{1024P_{16}^{(5)}}{27(1-x)^2(1+x)}H_0+\frac{1024\big(1+x^2\big)^2}{27(1-x)^2}H_1-\frac{1024\big(1+x^2\big)^2}{3(1-x)^2}H_{-1}\Bigg)H_{0,0,1}
$$

$$
+\Bigg(\frac{64P_{35}^{(5)}}{27(1-x)^2(1+x)}-\frac{512P_{21}^{(5)}}{27(1-x)^2(1+x)}H_0-\frac{1024\big(1+x^2\big)^2}{27(1-x)^2}H_1+\frac{1024\big(1+x^2\big)^2}{3(1-x)^2}
$$

$$
\times H_{-1}\Bigg)H_{0,0,-1}+\Bigg(-\frac{256(1+x)\big(1+x^2\big)}{3(1-x)}-\frac{1280\big(1+x^2\big)^2}{27(1-x)^2}H_0\Bigg)H_{0,1,1}
$$

$$
+\Bigg(\frac{256(1+x)\big(1+x^2\big)}{1-x}+\frac{1792\big(1+x^2\big)^2}{27(1-x)^2}H_0\Bigg)[H_{0,1,-1}+H_{0,-1,1}]+\Bigg(\frac{512P_9^{(5)}}{27(-1+x)(1+x)}
$$

$$
-\frac{256\big(1+x^2\big)^2}{27(1-x)^2}H_0\Bigg)H_{0,-1,-1}+\frac{256P_{27}^{(5)}}{27(1-x)^2(1+x)}H_{0,0,0,1}-\frac{256P_{22}^{(5)}}{9(1-x)^2(1+x)}H_{0,0,0,-1}
$$

$$
+\frac{1024\big(1+x^2\big)^2}{3(1-x)^2}[-\frac{1}{9}H_{0,0,1,1}+H_{0,0,1,-1}+H_{0,0,-1,1}-H_{0,0,-1,-1}+\frac{4}{9}H_{0,-1,0,1}]
$$

$$
+\Bigg(\frac{256}{3}\ln(2)\big(1+x^2\big)-\frac{64H_{-1}P_{11}^{(5)}}{27(1-x^2)}+\frac{2P_{40}^{(5)}}{27(1-x)(1+x)^4}+\Bigg(-\frac{16P_{38}^{(5)}}{27(1-x)^2(1+x)^3}
$$

$$
+\frac{512\big(1+x^2\big)\big(13+4x^2\big)}{27(1-x)^2}H_{-1}\Bigg)H_0+\frac{64P_{31}^{(5)}}{27(1-x)^2(1+x)}H_0^2+\Bigg(\frac{128(1+x)\big(1+x^2\big)}{3(1-x)}
$$

$$
+\frac{512P_{20}^{(5)}}{27(1-x)^2(1+x)}H_0\Bigg)H_1-\frac{128P_{28}^{(5)}}{27(1-x)^2(1+x)}H_{0,1}+\frac{128P_{18}^{(5)}}{27(1-x)^2(1+x)}H_{0,-1}\Bigg)\zeta_2
$$

$$
+\frac{32P_{36}^{(5)}}{135(1-x)^2(1+x)}\zeta_2^2-\Bigg(\frac{64(P_{33}^{(5)}+P_{30}^{(5)}H_0)}{27(1-x)^2(1+x)}+\frac{256\big(1+x^2\big)^2}{27(1-x)^2}H_1-\frac{256\big(1+x^2\big)^2}{3(1-x)^2}
$$

$$\times H_{-1}\Big)\zeta_3\Big]\Big]\Big\} + F_S^{(0)}, \tag{7.150}$$

with the polynomials

$$P_1^{(5)} = x^4 - 32x^3 + 48x^2 - 32x - 17, \tag{7.151}$$
$$P_2^{(5)} = 3x^4 + 42x^3 + 10x^2 + 6x - 5, \tag{7.152}$$
$$P_3^{(5)} = 13x^4 + 47x^3 - 56x^2 + 47x + 13, \tag{7.153}$$
$$P_4^{(5)} = 17x^4 - 40x^3 - 34x^2 - 40x + 17, \tag{7.154}$$
$$P_5^{(5)} = 19x^4 + 92x^3 + 110x^2 + 164x + 55, \tag{7.155}$$
$$P_6^{(5)} = 27x^4 + 62x^3 - 64x^2 + 62x + 9, \tag{7.156}$$
$$P_7^{(5)} = 55x^4 + 164x^3 + 146x^2 + 164x + 55, \tag{7.157}$$
$$P_8^{(5)} = 59x^4 + 226x^3 + 190x^2 + 226x + 59, \tag{7.158}$$
$$P_9^{(5)} = 62x^4 + 151x^3 + 142x^2 + 151x + 62, \tag{7.159}$$
$$P_{10}^{(5)} = 65x^4 - 68x^3 + 214x^2 - 68x + 65, \tag{7.160}$$
$$P_{11}^{(5)} = 137x^4 + 436x^3 + 352x^2 + 340x + 143, \tag{7.161}$$
$$P_{12}^{(5)} = 163x^4 + 1050x^3 - 1146x^2 + 1050x + 163, \tag{7.162}$$
$$P_{13}^{(5)} = 255x^4 - 214x^3 - 1530x^2 - 214x + 255, \tag{7.163}$$
$$P_{14}^{(5)} = 499x^4 - 730x^3 - 3050x^2 - 730x + 499, \tag{7.164}$$
$$P_{15}^{(5)} = 11134x^4 + 45469x^3 + 67950x^2 + 45469x + 11134, \tag{7.165}$$
$$P_{16}^{(5)} = 5x^5 - 11x^4 + 89x^3 - 83x^2 + 14x - 2, \tag{7.166}$$
$$P_{17}^{(5)} = 5x^5 + 3x^4 + 50x^3 - 14x^2 + 9x - 5, \tag{7.167}$$
$$P_{18}^{(5)} = 7x^5 + 7x^4 - 52x^3 - 16x^2 - 41x - 41, \tag{7.168}$$
$$P_{19}^{(5)} = 8x^5 + 8x^4 + 35x^3 + 53x^2 + 36x + 36, \tag{7.169}$$
$$P_{20}^{(5)} = 13x^5 - 3x^4 + 79x^3 - 81x^2 + 2x - 14, \tag{7.170}$$
$$P_{21}^{(5)} = 15x^5 - 25x^4 + 216x^3 - 208x^2 + 29x - 11, \tag{7.171}$$
$$P_{22}^{(5)} = 15x^5 + 63x^4 - 225x^3 + 289x^2 - 31x + 17, \tag{7.172}$$
$$P_{23}^{(5)} = 21x^5 - 3x^4 + 142x^3 - 110x^2 + 19x - 5, \tag{7.173}$$
$$P_{24}^{(5)} = 25x^5 - 7x^4 + 156x^3 - 164x^2 + 3x - 29, \tag{7.174}$$
$$P_{25}^{(5)} = 33x^5 + 19x^4 + 130x^3 - 178x^2 - 35x - 33, \tag{7.175}$$
$$P_{26}^{(5)} = 41x^5 + 36x^4 - 59x^3 - 41x^2 - 82x - 23, \tag{7.176}$$
$$P_{27}^{(5)} = 41x^5 + 121x^4 - 372x^3 + 500x^2 - 57x + 23, \tag{7.177}$$
$$P_{28}^{(5)} = 43x^5 - 21x^4 + 298x^3 - 342x^2 - x - 65, \tag{7.178}$$
$$P_{29}^{(5)} = 50x^5 + 18x^4 + 211x^3 - 123x^2 + 26x - 6, \tag{7.179}$$
$$P_{30}^{(5)} = 71x^5 - 57x^4 + 682x^3 - 646x^2 + 75x - 53, \tag{7.180}$$
$$P_{31}^{(5)} = 71x^5 + 55x^4 + 157x^3 - x^2 + 23x + 7, \tag{7.181}$$

$$P_{32}^{(5)} = 133x^5 - 225x^4 + 298x^3 - 1130x^2 + 81x - 53, \quad (7.182)$$
$$P_{33}^{(5)} = 221x^5 + 799x^4 - 194x^3 - 122x^2 + 213x - 149, \quad (7.183)$$
$$P_{34}^{(5)} = 273x^5 - 425x^4 - 196x^3 - 124x^2 - 1245x - 331, \quad (7.184)$$
$$P_{35}^{(5)} = 599x^5 + 2001x^4 - 388x^3 - 316x^2 + 1181x - 5, \quad (7.185)$$
$$P_{36}^{(5)} = 809x^5 + 265x^4 + 3072x^3 - 2452x^2 + 45x - 499, \quad (7.186)$$
$$P_{37}^{(5)} = 5411x^6 + 5015x^5 - 11707x^4 - 28382x^3 - 11707x^2 + 5015x + 5411, \quad (7.187)$$
$$P_{38}^{(5)} = 155x^7 + 1277x^6 + 1073x^5 - 2993x^4 - 1563x^3 - 1661x^2 - 433x + 49, \quad (7.188)$$
$$P_{39}^{(5)} = 181x^7 + 191x^6 - 288x^5 - 210x^4 - 1103x^3 - 1097x^2 - 646x - 100, \quad (7.189)$$
$$P_{40}^{(5)} = 13209x^7 + 22973x^6 + 44113x^5 - 24083x^4 - 106013x^3 - 82753x^2 - 11469x - 7017, \quad (7.190)$$
$$P_{41}^{(5)} = 5069x^8 - 892x^7 - 75926x^6 - 69132x^5 + 50716x^4 + 90780x^3 + 17638x^2 - 5140x - 4921. \quad (7.191)$$

The constant part, $F_S^{(0)}$, reads

$$\begin{aligned}
F_S^{(0)} \;=\; & -\frac{1}{2(1+x)^2}\Bigg\{n_h^2\Bigg[\frac{256P_2^{(6)}H_0^2H_1}{81(1+x)^2} - \frac{16P_{14}^{(6)}}{243(1+x)^2} + \Bigg(\frac{32P_{26}^{(6)}}{729(1-x)(1+x)^3} - \frac{128P_{12}^{(6)}H_{-1}}{243(1-x^2)} \\
& -\frac{512x(1+x)}{27(1-x)}H_{-1}^2 - \frac{256(1+x)\big(1+x^2\big)}{81(1-x)}H_{-1}^3\Bigg)H_0 + \Bigg(\frac{64P_{27}^{(6)}}{243(1-x)(1+x)^4} \\
& +\frac{128P_{16}^{(6)}}{81(1-x)(1+x)^2}H_{-1} + \frac{64(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}^2\Bigg)H_0^2 + \Bigg(-\frac{64P_{30}^{(6)}}{243(1-x)(1+x)^5} \\
& -\frac{64(1+x)\big(1+x^2\big)}{81(1-x)}H_{-1}\Bigg)H_0^3 + \frac{8(1+x)\big(1+x^2\big)}{9(1-x)}H_0^4 + \Bigg(+\frac{256(1+x)\big(1+x^2\big)}{27(1-x)}H_0^2 \\
& -\frac{512H_0P_2^{(6)}}{81(1+x)^2}\Bigg)H_{0,1} + \Bigg(\frac{128P_{12}^{(6)}}{243(1-x)(1+x)} - \frac{128(1+x)\big(1+x^2\big)}{27(1-x)}H_0^2 + \frac{1024x(1+x)H_{-1}}{27(1-x)} \\
& +\frac{256(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}^2\Bigg)H_{0,-1} + \Bigg(\frac{512P_2^{(6)}}{81(1+x)^2} - \frac{1024(1+x)\big(1+x^2\big)}{27(1-x)}H_0\Bigg)H_{0,0,1} \\
& +\Bigg(-\frac{256P_{16}^{(6)}}{81(1-x)(1+x)^2} + \frac{256(1+x)\big(1+x^2\big)}{27(1-x)}H_0 - \frac{256(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\Bigg)H_{0,0,-1} \\
& +\Bigg(-\frac{1024x(1+x)}{27(1-x)} - \frac{512(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\Bigg)H_{0,-1,-1} + \frac{512(1+x)\big(1+x^2\big)}{9(1-x)}H_{0,0,0,1} \\
& +\frac{128(1+x)\big(1+x^2\big)}{27(1-x)}H_{0,0,0,-1} + \frac{256(1+x)\big(1+x^2\big)}{27(1-x)}H_{0,0,-1,-1} + \frac{512(1+x)\big(1+x^2\big)}{27(1-x)} \\
& \times H_{0,-1,-1,-1} + \Bigg(-\frac{512P_5^{(6)}}{27(1+x)^2}\ln(2) - \frac{32P_{28}^{(6)}}{1215(1-x)(1+x)^4} + \Bigg(-\frac{128P_{29}^{(6)}}{81(1-x)(1+x)^5} \\
& -\frac{64(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\Bigg)H_0 + \frac{16(1+x)\big(1+x^2\big)}{3(1-x)}H_0^2 + \frac{256P_{15}^{(6)}}{27(1-x)(1+x)^2}H_{-1}
\end{aligned}$$

$$
-\frac{128(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}^2-\frac{64(1+x)\big(1+x^2\big)}{3(1-x)}H_{0,-1}\Bigg)\zeta_2-\frac{128(1+x)\big(1+x^2\big)}{9(1-x)}\zeta_2^2
$$

$$
+\Bigg(-\frac{256P_{20}^{(6)}}{81(1-x)(1+x)^2}-\frac{704(1+x)\big(1+x^2\big)}{27(1-x)}H_0+\frac{256(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\Bigg)\zeta_3\Bigg]
$$

$$
+n_h\Bigg[\ln^4(2)\Bigg(-\frac{32}{81}\big(35+106x+35x^2\big)+\frac{64P_4^{(6)}}{81(1-x^2)}H_0\Bigg)+\text{Li}_4\left(\frac{1}{2}\right)\Bigg(-\frac{256}{27}\big(35+106x
$$

$$
+35x^2\big)+\frac{512P_4^{(6)}}{27(1-x)(1+x)}H_0\Bigg)+n_l\Bigg[-\frac{1664}{243}\big(77+142x+77x^2\big)+\frac{256H_0^2H_1P_2^{(6)}}{81(1+x)^2}
$$

$$
+\Bigg(\frac{2048P_6^{(6)}}{729(1-x)(1+x)}-\frac{512P_7^{(6)}}{81(1-x)(1+x)}H_{-1}+\frac{512(1+x)\big(5-24x+5x^2\big)}{81(1-x)}H_{-1}^2
$$

$$
-\frac{2048(1+x)\big(1+x^2\big)}{81(1-x)}H_{-1}^3\Bigg)H_0+\Bigg(\frac{128P_{19}^{(6)}}{81(1-x)(1+x)^2}+\frac{512P_1^{(6)}}{81(1-x)(1+x)^2}H_{-1}
$$

$$
+\frac{512(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}^2\Bigg)H_0^2+\Bigg(-\frac{128P_{17}^{(6)}}{81(1-x)(1+x)^2}-\frac{128(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\Bigg)
$$

$$
\times H_0^3+\frac{16(1+x)\big(1+x^2\big)}{9(1-x)}H_0^4+\Bigg(-\frac{512H_0P_2^{(6)}}{81(1+x)^2}+\frac{256(1+x)\big(1+x^2\big)}{27(1-x)}H_0^2\Bigg)H_{0,1}
$$

$$
+\Bigg(\frac{512P_7^{(6)}}{81(1-x)(1+x)}-\frac{128(1+x)\big(1+x^2\big)}{9(1-x)}H_0^2-\frac{1024(1+x)\big(5-24x+5x^2\big)}{81(1-x)}H_{-1}
$$

$$
+\frac{2048(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}^2\Bigg)H_{0,-1}+\Bigg(\frac{512P_2^{(6)}}{81(1+x)^2}-\frac{1024(1+x)\big(1+x^2\big)}{27(1-x)}H_0\Bigg)H_{0,0,1}
$$

$$
+\Bigg(-\frac{1024P_1^{(6)}}{81(1-x)(1+x)^2}+\frac{256(1+x)\big(1+x^2\big)}{9(1-x)}H_0-\frac{2048(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\Bigg)H_{0,0,-1}
$$

$$
+\Bigg(\frac{1024(1+x)\big(5-24x+5x^2\big)}{81(1-x)}-\frac{4096(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\Bigg)H_{0,-1,-1}
$$

$$
+\frac{512(1+x)\big(1+x^2\big)}{9(1-x)}H_{0,0,0,1}+\frac{256(1+x)\big(1+x^2\big)}{9(1-x)}H_{0,0,0,-1}+\frac{2048(1+x)\big(1+x^2\big)}{27(1-x)}
$$

$$
\times H_{0,0,-1,-1}+\frac{4096(1+x)\big(1+x^2\big)}{27(1-x)}H_{0,-1,-1,-1}+\Bigg(\Bigg(-\frac{512P_5^{(6)}}{27(1+x)^2}\ln(2)
$$

$$
-\frac{32P_{24}^{(6)}}{81(1-x)(1+x)^2}-\Bigg(\frac{32P_{22}^{(6)}}{81(1-x)(1+x)^2}+\frac{128(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\Bigg)H_0
$$

$$
+\frac{32(1+x)\big(1+x^2\big)}{3(1-x)}H_0^2+\frac{1024P_{18}^{(6)}}{81(1-x)(1+x)^2}H_{-1}-\frac{1024(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}^2
$$

$$
-\frac{128(1+x)\big(1+x^2\big)}{9(1-x)}H_{0,-1}\Bigg)\zeta_2-\frac{3968(1+x)\big(1+x^2\big)}{135(1-x)}\zeta_2^2+\Bigg(-\frac{32P_{23}^{(6)}}{81(1-x)(1+x)^2}
$$

$$
-\frac{64(1+x)\big(1+x^2\big)}{3(1-x)}H_0+\frac{2048(1+x)\big(1+x^2\big)}{27(1-x)}H_{-1}\Bigg)\zeta_3\Bigg]+\Bigg(\ln^2(2)\Bigg(-\frac{128}{27}\big(41-125x
$$

$$+41x^2\Big) - \frac{128P_8^{(6)}}{27(1-x)(1+x)}H_0\Bigg) + \ln(2)\Bigg(-\frac{128P_{31}^{(6)}}{81x(1+x)^4} + \frac{64P_{25}^{(6)}}{27(1-x)(1+x)^3}H_0$$

$$-\frac{128P_{21}^{(6)}}{9(1-x)^2(1+x)}H_0^2 - \frac{256P_3^{(6)}}{3(1-x)(1+x)}H_0H_1 + \Bigg(\frac{128}{3}\big(1-16x+x^2\big)$$

$$-\frac{256P_9^{(6)}}{3(1-x)(1+x)}H_0\Bigg)H_{-1} + \frac{256P_3^{(6)}}{3(1-x)(1+x)}H_{0,1} + \frac{256P_{10}^{(6)}}{3(1-x)(1+x)}H_{0,-1}\Bigg)$$

$$+\Bigg(16\big(13+20x+13x^2\big) + \frac{192x^2}{(1-x)(1+x)}H_0\Bigg)\zeta_3\Bigg)\zeta_2 + \frac{32}{135}\big(2146+3383x+2146x^2\big)$$

$$-\frac{128P_{11}^{(6)}}{3(1-x)(1+x)}\ln(2) + \frac{32P_{13}^{(6)}}{135(1-x)(1+x)}H_0\Bigg)\zeta_2^2 - 40\Bigg(1+4x+x^2\Bigg)\zeta_5$$

$$+\frac{160x^2}{(1-x)(1+x)}H_0\zeta_5\Bigg)\Bigg\} + n_hF_{S,1}^{(0)}(x) + n_h\zeta_2F_{S,2}^{(0)}(x) + n_h\zeta_3F_{S,3}^{(0)}(x), \tag{7.192}$$

with

$$P_1^{(6)} = 3x^4+42x^3+10x^2+6x-5, \tag{7.193}$$
$$P_2^{(6)} = 5x^4+2x^3+34x^2+2x+5, \tag{7.194}$$
$$P_3^{(6)} = 7x^4-13x^3+34x^2-13x+7, \tag{7.195}$$
$$P_4^{(6)} = 8x^4-11x^3-2x^2-11x+8, \tag{7.196}$$
$$P_5^{(6)} = 11x^4+26x^3+70x^2+26x+11, \tag{7.197}$$
$$P_6^{(6)} = 16x^4-193x^3-445x^2-193x+16, \tag{7.198}$$
$$P_7^{(6)} = 17x^4-40x^3-34x^2-40x+17, \tag{7.199}$$
$$P_8^{(6)} = 26x^4+25x^3-20x^2+25x+26, \tag{7.200}$$
$$P_9^{(6)} = 33x^4+12x^3+100x^2+12x+33, \tag{7.201}$$
$$P_{10}^{(6)} = 35x^4+16x^3+98x^2+16x+35, \tag{7.202}$$
$$P_{11}^{(6)} = 49x^4-10x^3+166x^2-10x+49, \tag{7.203}$$
$$P_{12}^{(6)} = 65x^4-68x^3+214x^2-68x+65, \tag{7.204}$$
$$P_{13}^{(6)} = 1612x^4+3737x^3+6068x^2+3737x+1612, \tag{7.205}$$
$$P_{14}^{(6)} = 3859x^4+15236x^3+22882x^2+15236x+3859, \tag{7.206}$$
$$P_{15}^{(6)} = 5x^5-5x^4+26x^3-38x^2+x-5, \tag{7.207}$$
$$P_{16}^{(6)} = 5x^5+3x^4+50x^3-14x^2+9x-5, \tag{7.208}$$
$$P_{17}^{(6)} = 5x^5+3x^4+86x^3+22x^2+9x-5, \tag{7.209}$$
$$P_{18}^{(6)} = 10x^5-9x^4+22x^3-74x^2-5, \tag{7.210}$$
$$P_{19}^{(6)} = 16x^5+13x^4-25x^3-123x^2-59x+18, \tag{7.211}$$
$$P_{20}^{(6)} = 35x^5+54x^4+107x^3-143x^2-66x-35, \tag{7.212}$$
$$P_{21}^{(6)} = 70x^5-11x^4+71x^3-70x^2-2x-2, \tag{7.213}$$
$$P_{22}^{(6)} = 105x^5+51x^4+1548x^3+524x^2+147x-55, \tag{7.214}$$

$$P_{23}^{(6)} = 325x^5 + 87x^4 + 274x^3 - 1938x^2 - 375x - 165, \tag{7.215}$$
$$P_{24}^{(6)} = 391x^5 - 979x^4 - 1930x^3 + 746x^2 + 611x - 119, \tag{7.216}$$
$$P_{25}^{(6)} = 261x^6 + 790x^5 - 1127x^4 + 1600x^3 - 569x^2 + 1042x + 243, \tag{7.217}$$
$$P_{26}^{(6)} = 599x^6 - 4456x^5 - 20887x^4 - 30512x^3 - 20887x^2 - 4456x + 599, \tag{7.218}$$
$$P_{27}^{(6)} = 11x^7 + 202x^6 + 445x^5 + 304x^4 + 131x^3 - 240x^2 - 75x + 54, \tag{7.219}$$
$$P_{28}^{(6)} = 7239x^7 + 16287x^6 + 11607x^5 + 7399x^4 + 1301x^3 - 7507x^2 - 13747x - 5939, \tag{7.220}$$
$$P_{29}^{(6)} = 15x^8 + 57x^7 + 264x^6 + 779x^5 + 1062x^4 + 539x^3 + 112x^2 + 9x - 5, \tag{7.221}$$
$$P_{30}^{(6)} = 25x^8 + 84x^7 + 358x^6 + 944x^5 + 1122x^4 + 464x^3 + 54x^2 - 12x - 15, \tag{7.222}$$
$$P_{31}^{(6)} = 536x^8 - 15135x^7 - 63923x^6 - 147609x^5 - 194394x^4 - 147609x^3 - 63923x^2 - 15135x + 536. \tag{7.223}$$

The first expansion coefficients of $F_{S,i}^{(0)}, i = 1..3$ are given by

$$F_{S,1}^{(0)}(x) = -\frac{96756433y^5}{218700} - \frac{316061833y^4}{437400} - \frac{731018y^3}{729} - \frac{731018y^2}{729} - \frac{874750}{243} + O(y^6) \tag{7.224}$$
$$F_{S,2}^{(0)}(x) = \frac{3932123y^5}{18225} + \frac{16041283y^4}{36450} + \frac{2421832y^3}{3645} + \frac{2421832y^2}{3645} + \frac{343864}{81} + O(y^6) \tag{7.225}$$
$$F_{S,3}^{(0)}(x) = -\frac{7752703y^5}{48600} - \frac{21262303y^4}{97200} - \frac{22516y^3}{81} - \frac{22516y^2}{81} + \frac{62968}{27} + O(y^6). \tag{7.226}$$

7.3 The Pseudoscalar Form Factor

The unrenormalized pseudoscalar form factor reads

$$\begin{aligned}
F_P \;=\; & \frac{1}{\varepsilon^3}\frac{1}{2(1-x)^2}\Bigg\{ n_h^2\Bigg[\frac{64}{27}(1-x)^2 - \frac{64(1-x)\big(1+x^2\big)H_0}{27(1+x)}\Bigg] + n_h\Bigg[-\frac{3988}{27}(1-x)^2 \\
& + n_l\Bigg(\frac{32}{9}(1-x)^2 - \frac{64(1-x)\big(1+x^2\big)H_0}{27(1+x)}\Bigg) + \frac{32(1-x)\big(59+36x+59x^2\big)}{27(1+x)}H_0 \\
& - \frac{256\big(1+x^2\big)^2H_0^2}{27(1+x)^2}\Bigg]\Bigg\} + \frac{1}{\varepsilon^2}\frac{1}{2(1-x)^2}\Bigg\{ n_h^2\Bigg[\frac{832}{81}(1-x)^2 + \frac{128(1-x)\big(1+x^2\big)H_{-1}H_0}{27(1+x)} \\
& - \frac{32(1-x)\big(1+x^2\big)H_0^2}{27(1+x)} - \frac{128(1-x)\big(1+x^2\big)H_{0,-1}}{27(1+x)} + \frac{64(1-x)\big(1+x^2\big)\zeta_2}{27(1+x)}\Bigg] \\
& + n_h\Bigg[-\frac{16(1-x)^2\big(299+694x+299x^2\big)}{9(1+x)^2} + n_l\Bigg[\frac{64}{3}(1-x)^2 - \frac{320(1-x)\big(1+x^2\big)H_0}{81(1+x)} \\
& + \frac{256(1-x)\big(1+x^2\big)H_{-1}H_0}{27(1+x)} - \frac{64(1-x)\big(1+x^2\big)H_0^2}{27(1+x)} - \frac{256(1-x)\big(1+x^2\big)H_{0,-1}}{27(1+x)} \\
& + \frac{128(1-x)\big(1+x^2\big)\zeta_2}{27(1+x)}\Bigg] + \Bigg(\frac{16(1-x)P_{12}^{(7)}}{27(1+x)^3} - \frac{128(1-x)\big(55+18x+55x^2\big)}{27(1+x)}H_{-1}\Bigg)H_0 \\
& + \Bigg(\frac{64(1-x)\big(23+9x+41x^2\big)}{27(1+x)} + \frac{1024\big(1+x^2\big)^2H_{-1}}{27(1+x)^2}\Bigg)H_0^2 - \frac{128\big(2-x^2\big)\big(1+x^2\big)}{27(1+x)^2}H_0^3
\end{aligned}$$

$$
+\frac{128(1-x)\big(1+x^2\big)H_0H_1}{3(1+x)}-\left(\frac{128(1-x)\big(1+x^2\big)}{3(1+x)}+\frac{128\big(1+x^2\big)^2H_0}{3(1+x)^2}\right)H_{0,1}
$$

$$
+\left(\frac{128(1-x)\big(55+18x+55x^2\big)}{27(1+x)}-\frac{2176\big(1+x^2\big)^2H_0}{27(1+x)^2}\right)H_{0,-1}-\frac{256\big(1+x^2\big)^2H_{0,0,1}}{3(1+x)^2}
$$

$$
+\frac{256\big(1+x^2\big)^2H_{0,0,-1}}{3(1+x)^2}-\left(\frac{64(1-x)\big(55+18x+19x^2\big)}{27(1+x)}+\frac{64\big(1+x^2\big)\big(1-35x^2\big)}{27(1+x)^2}H_0\right)\zeta_2
$$

$$
+\frac{64\big(1+x^2\big)^2\zeta_3}{3(1+x)^2}\Bigg)\Bigg\}+\frac{1}{\varepsilon}\frac{1}{2(1-x)^2}\Bigg\{n_h^2\Bigg[\frac{32(1-x)^2\big(107+198x+107x^2\big)}{81(1+x)^2}
$$

$$
-\left(\frac{64(1-x)P_8^{(7)}}{243(1+x)^3}+\frac{128(1-x)\big(1+x^2\big)H_{-1}^2}{27(1+x)}\right)H_0+\left(\frac{64(1-x)\big(1+x^2\big)H_{-1}}{27(1+x)}\right.
$$

$$
\left.-\frac{64(1-x)^2\big(1+x^2\big)\big(5+14x+5x^2\big)}{81(1+x)^4}\right)H_0^2-\frac{32(1-x)\big(1+x^2\big)H_0^3}{27(1+x)}
$$

$$
+\frac{256(1-x)\big(1+x^2\big)H_{-1}H_{0,-1}}{27(1+x)}-\frac{128(1-x)\big(1+x^2\big)H_{0,0,-1}}{27(1+x)}-\frac{256(1-x)\big(1+x^2\big)}{27(1+x)}
$$

$$
\times H_{0,-1,-1}-\left(\frac{32(1-x)^2P_6^{(7)}}{27(1+x)^4}+\frac{160(1-x)\big(1+x^2\big)H_0}{27(1+x)}+\frac{128(1-x)\big(1+x^2\big)H_{-1}}{27(1+x)}\right)\zeta_2
$$

$$
+\frac{128(1-x)\big(1+x^2\big)\zeta_3}{27(1+x)}\Bigg]+n_h\Bigg[-\frac{128(1-x)^2P_{16}^{(7)}}{243(1+x)^4}
$$

$$
+n_l\Bigg[\frac{128(1-x)^2\big(185+358x+185x^2\big)}{243(1+x)^2}-\left(\frac{128(1-x)P_3^{(7)}}{81(1+x)^3}-\frac{1280(1-x)\big(1+x^2\big)H_{-1}}{81(1+x)}\right.
$$

$$
\left.+\frac{512(1-x)\big(1+x^2\big)H_{-1}^2}{27(1+x)}\right)H_0+\left(-\frac{128(1-x)\big(1+x^2\big)\big(5+12x+3x^2\big)}{81(1+x)^4}\right.
$$

$$
\left.+\frac{256(1-x)\big(1+x^2\big)H_{-1}}{27(1+x)}\right)H_0^2-\frac{64(1-x)\big(1+x^2\big)H_0^3}{27(1+x)}-\left(\frac{1280(1-x)\big(1+x^2\big)}{81(1+x)}\right.
$$

$$
\left.-\frac{1024(1-x)\big(1+x^2\big)H_{-1}}{27(1+x)}\right)H_{0,-1}-\frac{512(1-x)\big(1+x^2\big)H_{0,0,-1}}{27(1+x)}
$$

$$
-\frac{1024(1-x)\big(1+x^2\big)H_{0,-1,-1}}{27(1+x)}+\left(\frac{16(1-x)P_{27}^{(7)}}{81(1+x)^4}-\frac{224(1-x)\big(1+x^2\big)H_0}{27(1+x)}\right.
$$

$$
\left.-\frac{512(1-x)\big(1+x^2\big)H_{-1}}{27(1+x)}\right)\zeta_2+\frac{512(1-x)\big(1+x^2\big)}{27(1+x)}\zeta_3\Bigg)-\left(\frac{32(1-x)H_{-1}P_{14}^{(7)}}{27(1+x)^3}\right.
$$

$$
\left.-\frac{32(1-x)P_{29}^{(7)}}{243(1+x)^5}-\frac{256(1-x)\big(62+9x+62x^2\big)}{27(1+x)}H_{-1}^2\right)H_0+\left(\frac{32H_{-1}P_{13}^{(7)}}{27(1+x)^2}-\frac{8P_{32}^{(7)}}{243(1+x)^6}\right.
$$

$$
\left.-\frac{2048\big(1+x^2\big)^2H_{-1}^2}{27(1+x)^2}\right)H_0^2+\left(\frac{32P_{31}^{(7)}}{81(1-x)(1+x)^5}+\frac{128\big(1+x^2\big)\big(-1+5x^2\big)}{27(1+x)^2}H_{-1}\right)H_0^3
$$

$$
+\frac{16P_{19}^{(7)}}{81(1-x)(1+x)^2}H_0^4+\Biggl(-\frac{128H_0^3P_4^{(7)}}{81(1+x)^2}+\Biggl(\frac{1792(1-x)\bigl(1+x^2\bigr)}{27(1+x)}
$$

$$
-\frac{256(1-x)\bigl(1+x^2\bigr)H_{-1}}{1+x}\Biggr)H_0-\frac{256\bigl(-17-6x-6x^3+x^4\bigr)}{27(1+x)^2}H_0^2\Biggr)H_1
$$

$$
+\frac{128(1-x)\bigl(1+x^2\bigr)H_0H_1^2}{3(1+x)}+\Biggl(-\frac{1792(1-x)\bigl(1+x^2\bigr)}{27(1+x)}-\Biggl(\frac{512P_2^{(7)}}{27(1+x)^2}
$$

$$
+\frac{512\bigl(1+x^2\bigr)^2H_{-1}}{3(1+x)^2}\Biggr)H_0-\frac{256P_{22}^{(7)}}{27(1-x)(1+x)^2}H_0^2-\Biggl(\frac{256(1-x)\bigl(1+x^2\bigr)}{3(1+x)}
$$

$$
-\frac{512\bigl(1+x^2\bigr)^2H_0}{27(1+x)^2}\Biggr)H_1+\frac{256(1-x)\bigl(1+x^2\bigr)H_{-1}}{1+x}+\frac{6400\bigl(1+x^2\bigr)^2H_{0,-1}}{27(1+x)^2}\Biggr)H_{0,1}
$$

$$
-\frac{896\bigl(1+x^2\bigr)^2H_{0,1}^2}{27(1+x)^2}+\Biggl(\frac{32(1-x)P_{14}^{(7)}}{27(1+x)^3}+\Biggl(\frac{64P_{10}^{(7)}}{27(1+x)^2}+\frac{8704\bigl(1+x^2\bigr)^2H_{-1}}{27(1+x)^2}\Biggr)H_0
$$

$$
+\frac{128P_{24}^{(7)}}{27(1-x)(1+x)^2}H_0^2+\Biggl(\frac{256(1-x)\bigl(1+x^2\bigr)}{1+x}-\frac{512\bigl(1+x^2\bigr)^2H_0}{27(1+x)^2}\Biggr)H_1
$$

$$
-\frac{512(1-x)\bigl(62+9x+62x^2\bigr)}{27(1+x)}H_{-1}\Biggr)H_{0,-1}-\frac{4480\bigl(1+x^2\bigr)^2H_{0,-1}^2}{27(1+x)^2}+\Biggl(\frac{512P_5^{(7)}}{27(1+x)^2}
$$

$$
+\frac{1024P_{17}^{(7)}}{27(1-x)(1+x)^2}H_0-\frac{1024\bigl(1+x^2\bigr)^2H_1}{27(1+x)^2}+\frac{1024\bigl(1+x^2\bigr)^2H_{-1}}{3(1+x)^2}\Biggr)H_{0,0,1}
$$

$$
-\Biggl(\frac{64P_{15}^{(7)}}{27(1+x)^2}+\frac{512P_{20}^{(7)}}{27(1-x)(1+x)^2}H_0-\frac{1024\bigl(1+x^2\bigr)^2H_1}{27(1+x)^2}+\frac{1024\bigl(1+x^2\bigr)^2H_{-1}}{3(1+x)^2}\Biggr)
$$

$$
\times H_{0,0,-1}+\Biggl(\frac{256(1-x)\bigl(1+x^2\bigr)}{3(1+x)}+\frac{1280\bigl(1+x^2\bigr)^2H_0}{27(1+x)^2}\Biggr)H_{0,1,1}-\Biggl(\frac{256(1-x)\bigl(1+x^2\bigr)}{1+x}
$$

$$
+\frac{1792\bigl(1+x^2\bigr)^2H_0}{27(1+x)^2}\Biggr)H_{0,1,-1}+\Biggl(-\frac{256(1-x)\bigl(1+x^2\bigr)}{1+x}-\frac{1792\bigl(1+x^2\bigr)^2H_0}{27(1+x)^2}\Biggr)H_{0,-1,1}
$$

$$
+\Biggl(\frac{512(1-x)\bigl(62+9x+62x^2\bigr)}{27(1+x)}+\frac{256\bigl(1+x^2\bigr)^2H_0}{27(1+x)^2}\Biggr)H_{0,-1,-1}+\frac{256P_{23}^{(7)}H_{0,0,0,1}}{27(1-x)(1+x)^2}
$$

$$
-\frac{256P_{21}^{(7)}}{9(1-x)(1+x)^2}H_{0,0,0,-1}+\frac{1024\bigl(1+x^2\bigr)^2H_{0,0,1,1}}{27(1+x)^2}-\frac{1024\bigl(1+x^2\bigr)^2H_{0,0,1,-1}}{3(1+x)^2}
$$

$$
-\frac{1024\bigl(1+x^2\bigr)^2H_{0,0,-1,1}}{3(1+x)^2}+\frac{1024\bigl(1+x^2\bigr)^2H_{0,0,-1,-1}}{3(1+x)^2}-\frac{4096\bigl(1+x^2\bigr)^2H_{0,-1,0,1}}{27(1+x)^2}
$$

$$
+\Biggl(\frac{128H_{0,1}P_7^{(7)}}{27(1+x)^2}-\frac{64H_{-1}P_9^{(7)}}{27(1+x)^2}+\frac{2P_{33}^{(7)}}{27(1+x)^6}-\frac{256(1-x)^2\bigl(3+4x+3x^2\bigr)}{9(1+x)^2}\ln(2)
$$

$$
+\Biggl(-\frac{16P_{30}^{(7)}}{27(1-x)(1+x)^5}-\frac{512\bigl(1+x^2\bigr)\bigl(13+4x^2\bigr)}{27(1+x)^2}H_{-1}\Biggr)H_0+\frac{64P_{26}^{(7)}}{27(1-x)(1+x)^2}H_0^2
$$

$$+\left(-\frac{128(1-x)\big(1+x^2\big)}{3(1+x)}-\frac{512H_0P_1^{(7)}}{27(1+x)^2}\right)H_1+\frac{128P_{18}^{(7)}}{27(1-x)(1+x)^2}H_{0,-1}\Bigg)\zeta_2$$

$$+\frac{32P_{28}^{(7)}}{135(1-x)(1+x)^2}\zeta_2^2+\Bigg(\frac{64P_{11}^{(7)}}{27(1+x)^2}-\frac{64P_{25}^{(7)}}{27(1-x)(1+x)^2}H_0+\frac{256\big(1+x^2\big)^2H_1}{27(1+x)^2}$$

$$-\frac{256\big(1+x^2\big)^2H_{-1}}{3(1+x)^2}\Bigg)\zeta_3\Bigg]\Bigg\}+F_P^{(0)}, \tag{7.227}$$

with

$$P_1^{(7)}=13x^4-16x^3-x^2+16x-14, \tag{7.228}$$

$$P_2^{(7)}=13x^4+5x^3-8x^2+5x+13, \tag{7.229}$$

$$P_3^{(7)}=17x^4+48x^3+46x^2+48x+17, \tag{7.230}$$

$$P_4^{(7)}=25x^4-32x^3-4x^2+32x-29, \tag{7.231}$$

$$P_5^{(7)}=27x^4+4x^3-16x^2+4x+9, \tag{7.232}$$

$$P_6^{(7)}=33x^4+108x^3+118x^2+108x+33, \tag{7.233}$$

$$P_7^{(7)}=43x^4-64x^3-22x^2+64x-65, \tag{7.234}$$

$$P_8^{(7)}=65x^4+196x^3+166x^2+196x+65, \tag{7.235}$$

$$P_9^{(7)}=137x^4+54x^3-54x^2-18x-143, \tag{7.236}$$

$$P_{10}^{(7)}=163x^4+174x^3+54x^2+174x+163, \tag{7.237}$$

$$P_{11}^{(7)}=221x^4+166x^3+164x^2+94x-149, \tag{7.238}$$

$$P_{12}^{(7)}=255x^4+370x^3+806x^2+370x+255, \tag{7.239}$$

$$P_{13}^{(7)}=273x^4-94x^3-170x^2-166x-331, \tag{7.240}$$

$$P_{14}^{(7)}=499x^4+998x^3+1574x^2+998x+499, \tag{7.241}$$

$$P_{15}^{(7)}=599x^4+254x^3-62x^2+182x-5, \tag{7.242}$$

$$P_{16}^{(7)}=5567x^4+21986x^3+33054x^2+21986x+5567, \tag{7.243}$$

$$P_{17}^{(7)}=5x^5-21x^4+21x^3+15x^2-18x+2, \tag{7.244}$$

$$P_{18}^{(7)}=7x^5-7x^4-28x^3+40x^2-41x+41, \tag{7.245}$$

$$P_{19}^{(7)}=8x^5-8x^4+39x^3-49x^2+36x-36, \tag{7.246}$$

$$P_{20}^{(7)}=15x^5-55x^4+48x^3+40x^2-51x+11, \tag{7.247}$$

$$P_{21}^{(7)}=15x^5+33x^4-21x^3-85x^2+65x-17, \tag{7.248}$$

$$P_{22}^{(7)}=21x^5-45x^4+42x^3+10x^2-29x+5, \tag{7.249}$$

$$P_{23}^{(7)}=41x^5+39x^4-28x^3-156x^2+103x-23, \tag{7.250}$$

$$P_{24}^{(7)}=50x^5-82x^4+79x^3-9x^2-38x+6, \tag{7.251}$$

$$P_{25}^{(7)}=71x^5-199x^4+154x^3+118x^2-181x+53, \tag{7.252}$$

$$P_{26}^{(7)}=71x^5-87x^4+89x^3-67x^2-9x-7, \tag{7.253}$$

$$P_{27}^{(7)}=133x^5+255x^4+10x^3+310x^2-15x-53, \tag{7.254}$$

$$P_{28}^{(7)}=809x^5-1353x^4+776x^3+156x^2-1043x+499, \tag{7.255}$$

$$P_{29}^{(7)} = 5411x^6 + 25439x^5 + 53201x^4 + 69802x^3 + 53201x^2 + 25439x + 5411, \quad (7.256)$$

$$P_{30}^{(7)} = 155x^8 + 794x^7 + 852x^6 - 778x^5 - 2634x^4 - 346x^3 + 396x^2 + 74x - 49, \quad (7.257)$$

$$P_{31}^{(7)} = 181x^8 + 266x^7 - 359x^6 - 726x^5 + 123x^4 + 278x^3 + 275x^2 + 246x + 100, \quad (7.258)$$

$$P_{32}^{(7)} = 5069x^8 + 25148x^7 + 56746x^6 + 50652x^5 + 21436x^4 - 10044x^3 - 13562x^2 - 14812x - 4921, \quad (7.259)$$

$$P_{33}^{(7)} = 13209x^8 + 44196x^7 + 34324x^6 + 5148x^5 - 48650x^4 - 38052x^3 - 12044x^2 + 19428x + 7017, \quad (7.260)$$

and

$$F_P^{(0)} = \frac{1}{4(1-x)^2}\Biggl\{ n_h^2 \Biggl[\frac{32(1-x)^2 P_{14}^{(8)}}{243(1+x)^4} + (1-x)\Biggl(\frac{256 H_{-1} P_9^{(8)}}{243(1+x)^3} + \frac{512\bigl(1+x^2\bigr) H_{-1}^3}{81(1+x)}$$
$$- \frac{64 P_{21}^{(8)}}{729(1+x)^5} \Biggr) H_0 + \Biggl(- \frac{128(1-x) P_{22}^{(8)}}{243(1+x)^6} + \frac{256(1-x)^2\bigl(1+x^2\bigr)\bigl(5+14x+5x^2\bigr)}{81(1+x)^4} H_{-1}$$
$$- \frac{128(1-x)\bigl(1+x^2\bigr) H_{-1}^2}{27(1+x)} \Biggr) H_0^2 + \Biggl(\frac{128(1-x)\bigl(1+x^2\bigr) P_{20}^{(8)}}{243(1+x)^7} + \frac{128(1-x)\bigl(1+x^2\bigr) H_{-1}}{81(1+x)} \Biggr)$$
$$\times H_0^3 - \frac{16(1-x)\bigl(1+x^2\bigr) H_0^4}{9(1+x)} - \frac{512(1-x)^2\bigl(1+x^2\bigr)\bigl(5+14x+5x^2\bigr)}{81(1+x)^4} H_0^2 H_1$$
$$+ \Biggl(\frac{1024(1-x)^2\bigl(1+x^2\bigr)\bigl(5+14x+5x^2\bigr)}{81(1+x)^4} H_0 - \frac{512(1-x)\bigl(1+x^2\bigr) H_0^2}{27(1+x)} \Biggr) H_{0,1}$$
$$- \Biggl(\frac{256(1-x) P_9^{(8)}}{243(1+x)^3} - \frac{256(1-x)\bigl(1+x^2\bigr) H_0^2}{27(1+x)} + \frac{512(1-x)\bigl(1+x^2\bigr) H_{-1}^2}{27(1+x)} \Biggr) H_{0,-1}$$
$$- \Biggl(\frac{1024(1-x)^2\bigl(1+x^2\bigr)\bigl(5+14x+5x^2\bigr)}{81(1+x)^4} - \frac{2048(1-x)\bigl(1+x^2\bigr) H_0}{27(1+x)} \Biggr) H_{0,0,1}$$
$$+ \Biggl(- \frac{512(1-x)^2\bigl(1+x^2\bigr)\bigl(5+14x+5x^2\bigr)}{81(1+x)^4} - \frac{512(1-x)\bigl(1+x^2\bigr) H_0}{27(1+x)}$$
$$+ \frac{512(1-x)\bigl(1+x^2\bigr) H_{-1}}{27(1+x)} \Biggr) H_{0,0,-1} + \frac{1024(1-x)\bigl(1+x^2\bigr)}{27(1+x)} [H_{-1} H_{0,-1,-1} - 3 H_{0,0,0,1}$$
$$- \frac{1}{4}(H_{0,0,0,-1} + 2 H_{0,0,-1,-1}) - H_{0,-1,-1,-1}] + \Biggl(\frac{1024(1-x)^2 P_3^{(8)}}{27(1+x)^4} \ln(2) + \frac{64(1-x) P_{23}^{(8)}}{1215(1+x)^6}$$
$$+ \Biggl(\frac{256(1-x)\bigl(1+x^2\bigr) P_{19}^{(8)}}{81(1+x)^7} + \frac{128(1-x)\bigl(1+x^2\bigr) H_{-1}}{27(1+x)} \Biggr) H_0 - \frac{32(1-x)\bigl(1+x^2\bigr) H_0^2}{3(1+x)}$$
$$+ \frac{512(1-x)^2\bigl(1+x^2\bigr)\bigl(5+14x+5x^2\bigr)}{27(1+x)^4} H_{-1} + \frac{256(1-x)\bigl(1+x^2\bigr) H_{-1}^2}{27(1+x)}$$
$$+ \frac{128(1-x)\bigl(1+x^2\bigr) H_{0,-1}}{3(1+x)} \Biggr) \zeta_2 + \frac{256(1-x)\bigl(1+x^2\bigr)}{9(1+x)} \zeta_2^2 + \Biggl(\frac{1408(1-x)\bigl(1+x^2\bigr) H_0}{27(1+x)}$$
$$- \frac{512(1-x)\bigl(1+x^2\bigr) H_{-1}}{27(1+x)} - \frac{2560(1-x)^2\bigl(1+x+x^2\bigr)\bigl(7+18x+7x^2\bigr)}{81(1+x)^4} \Biggr) \zeta_3 \Biggr]$$

$$
+n_h\Biggl[\ln^4(2)\Biggl(\frac{64}{81}\bigl(35-22x+35x^2\bigr)-\frac{128P_2^{(8)}}{81(1-x)(1+x)}H_0\Biggr)+\text{Li}_4\left(\frac{1}{2}\right)
$$

$$
\times\Biggl(\frac{512}{27}\bigl(35-22x+35x^2\bigr)-\frac{1024P_2^{(8)}}{27(1-x)(1+x)}H_0\Biggr)
$$

$$
+n_l\Biggl[-\frac{512(1-x)(1+x^2)}{9(1+x)}H_0+\frac{256(1-x)^2\bigl(1001+1966x+1001x^2\bigr)}{243(1+x)^2}
$$

$$
+\Biggl(\frac{1024(1-x)H_{-1}P_4^{(8)}}{81(1+x)^3}-\frac{1024(1-x)P_8^{(8)}}{729(1+x)^3}-\frac{5120(1-x)\bigl(1+x^2\bigr)H_{-1}^2}{81(1+x)}
$$

$$
+\frac{4096(1-x)\bigl(1+x^2\bigr)H_{-1}^3}{81(1+x)}\Biggr)H_0-\Biggl(-\frac{1024(1-x)\bigl(1+x^2\bigr)\bigl(5+12x+3x^2\bigr)}{81(1+x)^4}H_{-1}
$$

$$
+\frac{256(1-x)P_{15}^{(8)}}{81(1+x)^4}+\frac{1024(1-x)\bigl(1+x^2\bigr)H_{-1}^2}{27(1+x)}\Biggr)H_0^2+\Biggl(+\frac{256(1-x)\bigl(1+x^2\bigr)H_{-1}}{27(1+x)}
$$

$$
-\frac{256(1-x)^2\bigl(1+x^2\bigr)\bigl(5+14x+5x^2\bigr)}{81(1+x)^4}\Biggr)H_0^3-\frac{32(1-x)\bigl(1+x^2\bigr)H_0^4}{9(1+x)}
$$

$$
-\frac{512(1-x)^2\bigl(1+x^2\bigr)\bigl(5+14x+5x^2\bigr)}{81(1+x)^4}H_0^2H_1+\Biggl(\frac{1024(1-x)^2\bigl(1+x^2\bigr)\bigl(5+14x+5x^2\bigr)}{81(1+x)^4}
$$

$$
\times H_0+\frac{512(-1+x)\bigl(1+x^2\bigr)H_0^2}{27(1+x)}\Biggr)H_{0,1}+\Biggl(-\frac{1024(1-x)P_4^{(8)}}{81(1+x)^3}+\frac{256(1-x)\bigl(1+x^2\bigr)H_0^2}{9(1+x)}
$$

$$
+\frac{10240(1-x)\bigl(1+x^2\bigr)H_{-1}}{81(1+x)}-\frac{4096(1-x)\bigl(1+x^2\bigr)H_{-1}^2}{27(1+x)}\Biggr)H_{0,-1}+\Biggl(\frac{2048(1-x)\bigl(1+x^2\bigr)}{27(1+x)}
$$

$$
\times H_0-\frac{1024(1-x)^2\bigl(1+x^2\bigr)\bigl(5+14x+5x^2\bigr)}{81(1+x)^4}\Biggr)H_{0,0,1}+\frac{512(1-x)\bigl(1+x^2\bigr)H_0}{9(1+x)}
$$

$$
-\Biggl(\frac{2048(1-x)\bigl(1+x^2\bigr)\bigl(5+12x+3x^2\bigr)}{81(1+x)^4}-\frac{4096(1-x)\bigl(1+x^2\bigr)H_{-1}}{27(1+x)}\Biggr)H_{0,0,-1}
$$

$$
+\Biggl(-\frac{10240(1-x)\bigl(1+x^2\bigr)}{81(1+x)}+\frac{8192(1-x)\bigl(1+x^2\bigr)H_{-1}}{27(1+x)}\Biggr)H_{0,-1,-1}
$$

$$
-\frac{512(1-x)\bigl(1+x^2\bigr)}{9(1+x)}\Biggl[2H_{0,0,0,1}+H_{0,0,0,-1}+\frac{8}{3}H_{0,0,-1,-1}+\frac{16}{3}H_{0,-1,-1,-1}\Biggr]
$$

$$
+\Biggl(\frac{1024(1-x)^2P_3^{(8)}}{27(1+x)^4}\ln(2)+\frac{64(1-x)P_{18}^{(8)}}{81(1+x)^4}+\Biggl(+\frac{256(1-x)\bigl(1+x^2\bigr)H_{-1}}{27(1+x)}
$$

$$
+\frac{64(1-x)\bigl(1+x^2\bigr)\bigl(-55-69x+219x^2+105x^3\bigr)}{81(1+x)^4}\Biggr)H_0-\frac{64(1-x)\bigl(1+x^2\bigr)H_0^2}{3(1+x)}
$$

$$
-\frac{2048(1-x)\bigl(1+x^2\bigr)\bigl(-5-6x+21x^2+10x^3\bigr)}{81(1+x)^4}H_{-1}+\frac{2048(1-x)\bigl(1+x^2\bigr)H_{-1}^2}{27(1+x)}
$$

$$
\begin{aligned}
&+\frac{256(1-x)\big(1+x^2\big)H_{0,-1}}{9(1+x)}\Bigg)\zeta_2+\frac{7936(1-x)\big(1+x^2\big)}{135(1+x)}\zeta_2^2+\Bigg(\frac{64(1-x)P_{17}^{(8)}}{81(1+x)^4}\\
&+\frac{128(1-x)\big(1+x^2\big)H_0}{3(1+x)}-\frac{4096(1-x)\big(1+x^2\big)H_{-1}}{27(1+x)}\Bigg)\zeta_3\Bigg)+\Bigg(\ln^2(2)\Bigg(\frac{256P_7^{(8)}}{27(1+x)^2}\\
&+\frac{256P_6^{(8)}}{27(1-x)(1+x)}H_0\Bigg)+\ln(2)\Bigg(\frac{256P_{25}^{(8)}}{81x(1+x)^6}-\frac{128P_{24}^{(8)}}{27(1-x)(1+x)^5}H_0\\
&-\frac{256P_{16}^{(8)}}{9(1-x)(1+x)^2}H_0^2+\frac{512P_5^{(8)}}{9(1-x)(1+x)}H_0H_1-\Bigg(\frac{256P_1^{(8)}}{3(1+x)^2}-\frac{512P_{10}^{(8)}}{9(1-x)(1+x)}H_0\Bigg)\\
&\times H_{-1}-\frac{512P_5^{(8)}}{9(1-x)(1+x)}H_{0,1}-\frac{512P_{11}^{(8)}}{9(1-x)(1+x)}H_{0,-1}\Bigg)+\Bigg(-32\big(13-12x+13x^2\big)\\
&-\frac{896x^2}{(1-x)(1+x)}H_0\Bigg)\zeta_3\Bigg)\zeta_2-\Bigg(\frac{64}{135}\big(2146-3893x+2146x^2\big)-\frac{256P_{12}^{(8)}}{9(1-x)(1+x)}\ln(2)\\
&+\frac{64P_{13}^{(8)}}{135(1-x)(1+x)}H_0\Bigg)\zeta_2^2+80\big(1+4x+x^2\big)\zeta_5+\frac{960x^2}{(1-x)(1+x)}H_0\zeta_5\Bigg]\\
&-\frac{512(1-x)\big(1+x^2\big)}{9(1+x)}H_0H_{0,0,-1}\Bigg\}+n_hF_{P,1}^{(0)}(x)+n_h\zeta_2F_{P,2}^{(0)}(x)+n_h\zeta_3F_{P,3}^{(0)}(x),
\end{aligned}
\tag{7.261}
$$

with

$$P_1^{(8)}=x^4-6x^3+18x^2-6x+1, \tag{7.262}$$
$$P_2^{(8)}=8x^4-43x^3+22x^2-43x+8, \tag{7.263}$$
$$P_3^{(8)}=11x^4+38x^3+46x^2+38x+11, \tag{7.264}$$
$$P_4^{(8)}=17x^4+48x^3+46x^2+48x+17, \tag{7.265}$$
$$P_5^{(8)}=21x^4-69x^3+94x^2-69x+21, \tag{7.266}$$
$$P_6^{(8)}=26x^4-79x^3+76x^2-79x+26, \tag{7.267}$$
$$P_7^{(8)}=41x^4-87x^3+32x^2-87x+41, \tag{7.268}$$
$$P_8^{(8)}=64x^4+263x^3+218x^2+263x+64, \tag{7.269}$$
$$P_9^{(8)}=65x^4+196x^3+166x^2+196x+65, \tag{7.270}$$
$$P_{10}^{(8)}=99x^4-252x^3+308x^2-252x+99, \tag{7.271}$$
$$P_{11}^{(8)}=105x^4-264x^3+326x^2-264x+105, \tag{7.272}$$
$$P_{12}^{(8)}=147x^4-402x^3+514x^2-402x+147, \tag{7.273}$$
$$P_{13}^{(8)}=1612x^4-2711x^3+2996x^2-2711x+1612, \tag{7.274}$$
$$P_{14}^{(8)}=3859x^4+15092x^3+22402x^2+15092x+3859, \tag{7.275}$$
$$P_{15}^{(8)}=16x^5+57x^4+107x^3+81x^2+73x+18, \tag{7.276}$$
$$P_{16}^{(8)}=70x^5-97x^4+13x^3+12x^2-2x+2, \tag{7.277}$$
$$P_{17}^{(8)}=325x^5+807x^4+418x^3+222x^2-327x-165, \tag{7.278}$$

$$P_{18}^{(8)} = 391x^5 + 1357x^4 + 854x^3 + 650x^2 - 317x - 119, \quad (7.279)$$
$$P_{19}^{(8)} = 15x^6 + 78x^5 + 159x^4 + 152x^3 + 27x^2 - 18x - 5, \quad (7.280)$$
$$P_{20}^{(8)} = 25x^6 + 126x^5 + 225x^4 + 152x^3 - 39x^2 - 66x - 15, \quad (7.281)$$
$$P_{21}^{(8)} = 599x^6 + 3824x^5 + 7193x^4 + 7360x^3 + 7193x^2 + 3824x + 599, \quad (7.282)$$
$$P_{22}^{(8)} = 11x^7 + 70x^6 + 361x^5 + 712x^4 + 635x^3 + 588x^2 + 321x + 54, \quad (7.283)$$
$$P_{23}^{(8)} = 7239x^7 + 36399x^6 + 60903x^5 + 38023x^4 - 11083x^3 - 41923x^2 - 28579x - 5939, \quad (7.284)$$
$$P_{24}^{(8)} = 261x^8 + 1096x^7 - 702x^6 - 3888x^5 - 1232x^4 - 4392x^3 - 810x^2 + 1168x + 243, \quad (7.285)$$
$$P_{25}^{(8)} = 848x^{10} - 14847x^9 - 55535x^8 - 37906x^7 + 58575x^6 + 123650x^5 + 58575x^4 - 37906x^3 - 55535x^2 - 14847x + 848. \quad (7.286)$$

The first expansion coefficients of $F_{P,i}^{(0)}, i = 1...3$, are given by

$$F_{P,1}^{(0)}(x) = \frac{19068183229y^5}{85730400} + \frac{22625094013y^4}{171460800} + \frac{756146y^3}{18225} + \frac{756146y^2}{18225} - \frac{5529994}{729} + O(y^6) \quad (7.287)$$
$$F_{P,2}^{(0)}(x) = -\frac{524338481y^5}{1488375} - \frac{804767441y^4}{2976750} - \frac{381536y^3}{2025} - \frac{381536y^2}{2025} + \frac{4990072}{729} + O(y^6) \quad (7.288)$$
$$F_{P,3}^{(0)}(x) = -\frac{4050340711y^5}{19051200} - \frac{622908463y^4}{4233600} - \frac{496121y^3}{6075} - \frac{496121y^2}{6075} + \frac{426952}{243} + O(y^6). \quad (7.289)$$

7.4 The Axialvector Form Factors

The axialvector form factors are given by

$$\begin{aligned}
F_{A,1} = & \frac{1}{\varepsilon^3}\Bigg\{ n_h^2\Bigg[\frac{16}{27} - \frac{32(1+x^2)}{27(1-x)(1+x)}H_0\Bigg] + n_h\Bigg[-\frac{2(589+602x+589x^2)}{27(1+x)^2} + n_l\Bigg[\frac{16}{27} \\
& -\frac{32(1+x^2)}{27(1-x)(1+x)}H_0\Bigg] + \frac{16P_8^{(9)}}{27(1-x)(1+x)^3}H_0 - \frac{128(1+x^2)^2}{27(1-x^2)^2}H_0^2\Bigg]\Bigg\} \\
& +\frac{1}{\varepsilon^2}\Bigg\{ n_h^2\Bigg[\frac{16(1+x^2)}{27(1-x^2)} + \Bigg(-\frac{16(3-2x+3x^2)}{27(1-x^2)} + \frac{64(1+x^2)H_{-1}}{27(1-x^2)}\Bigg)H_0 - \frac{16(1+x^2)H_0^2}{27(1-x^2)} \\
& -\frac{64(1+x^2)H_{0,-1}}{27(1-x^2)} + \frac{32(1+x^2)}{27(1-x^2)}\zeta_2\Bigg] + n_h\Bigg[\frac{1}{9(1-x)^2(1+x)^3}\Bigg(-\frac{1052}{3} - \frac{988}{3}x \\
& +680x^2 + 680x^3 - \frac{988}{3}x^4 - \frac{1052}{3}x^5\Bigg) + n_l\Bigg[\frac{128(1+x^2)}{81(1-x^2)} + \Bigg(-\frac{64(7-3x+7x^2)}{81(1-x^2)} \\
& +\frac{128(1+x^2)H_{-1}}{27(1-x^2)}\Bigg)H_0 - \frac{32(1+x^2)H_0^2}{27(1-x^2)} - \frac{128(1+x^2)H_{0,-1}}{27(1-x^2)} + \frac{64(1+x^2)}{27(1-x^2)}\zeta_2\Bigg] \\
& +\frac{2968}{27(1-x)^2(1+x)^3}(1+2x-x^2+x^3-2x^4-x^5)H_0 - \frac{128P_7^{(9)}}{27(1-x)(1+x)^3}H_{-1}H_0 \\
& +\Bigg(-\frac{16P_{22}^{(9)}}{27(1-x)^2(1+x)^3} + \frac{512(1+x^2)^2}{27(1-x)^2(1+x)^2}H_{-1}\Bigg)H_0^2 + \frac{64(-2+x^2)(1+x^2)}{27(1-x^2)^2}H_0^3
\end{aligned}$$

$$
+\frac{64\left(1+x^2\right)H_0H_1}{3(1-x^2)}-\frac{64\left(1+x^2\right)H_{0,1}}{3(1-x^2)}+\frac{64\left(1+x^2\right)^2}{3(1-x^2)^2}H_0H_{0,1}+\frac{128P_7^{(9)}}{27(1-x)(1+x)^3}H_{0,-1}
$$

$$
-\frac{1088\left(1+x^2\right)^2}{27(1-x^2)^2}H_0H_{0,-1}-\frac{128\left(1+x^2\right)^2}{3(1-x^2)^2}H_{0,0,1}+\frac{128\left(1+x^2\right)^2}{3(1-x^2)^2}H_{0,0,-1}
$$

$$
+\frac{1}{(1-x)^2(1+x)^3}\left(-\frac{1472}{27}-\frac{3200}{27}x+64x^2+\frac{64}{3}x^3+\frac{2048}{27}x^4+\frac{320}{27}x^5\right)
$$

$$
-\left[\frac{32(1+x+x^2+x^3)H_0}{27(1-x)^2(1+x)^3}-\frac{1120}{27(1-x)^2(1+x)^3}x^2(1+x+x^2+x^3)H_0\right]\zeta_2+\frac{32}{3(1-x^2)}
$$

$$
\times\frac{1}{(1+x)^3}(1+x+2x^2+2x^3+x^4+x^5)\zeta_3\Bigg\}+\frac{1}{\varepsilon}\Bigg\{n_h^2\Bigg[-\frac{64\left(13-34x+13x^2\right)}{243(1+x)^2}
$$

$$
-\frac{16P_{12}^{(9)}}{243(1-x)(1+x)^3}H_0+\frac{32\left(3-2x+3x^2\right)}{27(1-x^2)}H_{-1}H_0-\frac{64\left(1+x^2\right)H_{-1}^2H_0}{27(1-x^2)}
$$

$$
+\frac{8P_{18}^{(9)}}{81(1-x)(1+x)^4}H_0^2+\frac{32\left(1+x^2\right)H_{-1}H_0^2}{27(1-x^2)}-\frac{16P_1^{(9)}}{27(1-x)(1+x)^3}H_0^3
$$

$$
-\frac{32\left(3-2x+3x^2\right)}{27(1-x^2)}H_{0,-1}+\frac{128\left(1+x^2\right)H_{-1}H_{0,-1}}{27(1-x^2)}-\frac{64\left(1+x^2\right)H_{0,0,-1}}{27(1-x^2)}
$$

$$
-\frac{128\left(1+x^2\right)H_{0,-1,-1}}{27(1-x^2)}+\Bigg(\frac{8P_{21}^{(9)}}{27(1-x)(1+x)^4}-\frac{16P_3^{(9)}}{27(1-x)(1+x)^3}H_0
$$

$$
-\frac{64\left(1+x^2\right)H_{-1}}{27(1-x^2)}\Bigg)\zeta_2+\frac{64\left(1+x^2\right)}{27(1-x^2)}\zeta_3\Bigg]+n_h\Bigg[-\frac{128H_0H_{0,1}P_2^{(9)}}{27(1-x^2)^2}-\frac{64H_0^2H_1P_5^{(9)}}{27(1-x^2)^2}
$$

$$
+\frac{128H_{0,0,1}P_6^{(9)}}{27(1-x^2)^2}-\frac{64H_0H_{0,-1}P_9^{(9)}}{27(1-x^2)^2}-\frac{4P_{14}^{(9)}}{243(1+x)^4}-\frac{64H_{0,0,-1}P_{24}^{(9)}}{27(1-x)^2(1+x)^3}
$$

$$
+\frac{32H_{-1}H_0^2P_{25}^{(9)}}{27(1-x)^2(1+x)^3}+\frac{512H_0H_{0,0,1}P_{27}^{(9)}}{27(1-x^2)^3}-\frac{128H_{0,0,0,-1}P_{29}^{(9)}}{9(1-x^2)^3}-\frac{128H_0H_{0,0,-1}P_{30}^{(9)}}{27(1-x^2)^3}
$$

$$
+\frac{128H_{0,0,0,1}P_{31}^{(9)}}{27(1-x^2)^3}-\frac{64H_0^2H_{0,1}P_{32}^{(9)}}{27(1-x^2)^3}+\frac{64H_0^2H_{0,-1}P_{33}^{(9)}}{27(1-x^2)^3}+\frac{16P_{168}^{(9)}}{135(1-x^2)^3}\zeta_2^2
$$

$$
+\frac{8H_0^4P_{37}^{(9)}}{81(1-x)^3(1+x)^4}-\frac{4H_0^2P_{41}^{(9)}}{243(1-x)^2(1+x)^6}+n_l\Bigg[-\frac{32\left(1-38x+x^2\right)}{81(1+x)^2}
$$

$$
-\frac{32P_{10}^{(9)}}{81(1-x)(1+x)^3}H_0+\frac{256\left(7-3x+7x^2\right)}{81(1-x^2)}H_{-1}H_0-\frac{256\left(1+x^2\right)H_{-1}^2H_0}{27(1-x^2)}
$$

$$
-\frac{16P_{15}^{(9)}}{81(1-x)(1+x)^4}H_0^2+\frac{128\left(1+x^2\right)H_{-1}H_0^2}{27(1-x^2)}-\frac{32\left(1+2x+2x^3+x^4\right)}{27(1-x)(1+x)^3}H_0^3
$$

$$
-\frac{256\left(7-3x+7x^2\right)}{81(1-x^2)}H_{0,-1}+\frac{512\left(1+x^2\right)H_{-1}H_{0,-1}}{27(1-x^2)}-\frac{256\left(1+x^2\right)H_{0,0,-1}}{27(1-x^2)}
$$

$$
-\frac{512\left(1+x^2\right)H_{0,-1,-1}}{27(1-x^2)}+\Bigg(\frac{8P_{26}^{(9)}}{81(1-x)(1+x)^4}-\frac{16P_4^{(9)}}{27(1-x)(1+x)^3}H_0
$$

$$
-\frac{256\left(1+x^2\right)H_{-1}}{27(1-x^2)}\Bigg)\zeta_2+\frac{256\left(1+x^2\right)\zeta_3}{27(1-x^2)}\Bigg]+\frac{4P_{36}^{(9)}}{243(1-x)(1+x)^5}H_0
$$

$$
\begin{aligned}
&-\frac{16P_{13}^{(9)}}{27(1-x)(1+x)^3}H_{-1}H_0+\Bigg(-\frac{64H_1P_{17}^{(9)}}{81(1-x)^2(1+x)^3}+\frac{16P_{39}^{(9)}}{81(1-x)^3(1+x)^5}\Bigg)H_0^3\\
&+\frac{64\big(1+x^2\big)\big(-1+5x^2\big)}{27(1-x^2)^2}H_{-1}H_0^3+\frac{64\big(37-14x+37x^2\big)}{27(1-x^2)}H_0H_1\\
&-\frac{128\big(1+x^2\big)H_{-1}H_0H_1}{(1-x^2)}+\frac{64\big(1+x^2\big)H_0H_1^2}{3(1-x^2)}+\Bigg(\frac{64P_{11}^{(9)}}{27(1-x)(1+x)^3}H_0\\
&-\frac{1024\big(1+x^2\big)^2}{27(1-x^2)^2}H_0^2\Bigg)H_{-1}^2-\frac{64\big(37-14x+37x^2\big)}{27(1-x^2)}H_{0,1}-\frac{256\big(1+x^2\big)^2}{3(1-x^2)^2}H_{-1}H_0H_{0,1}\\
&-\frac{128\big(1+x^2\big)H_1H_{0,1}}{3(1-x^2)}+\frac{256\big(1+x^2\big)^2}{27(1-x^2)^2}H_0H_1H_{0,1}+\frac{128\big(1+x^2\big)H_{-1}H_{0,1}}{(1-x^2)}\\
&+\frac{3200\big(1+x^2\big)^2}{27(1-x^2)^2}H_{0,-1}H_{0,1}-\frac{448\big(1+x^2\big)^2}{27(1-x^2)^2}H_{0,1}^2+\frac{16P_{13}^{(9)}}{27(1-x)(1+x)^3}H_{0,-1}\\
&+\frac{4352\big(1+x^2\big)^2}{27(1-x^2)^2}H_{-1}H_0H_{0,-1}+\frac{128\big(1+x^2\big)H_1H_{0,-1}}{1-x^2}-\frac{256\big(1+x^2\big)^2}{27(1-x^2)^2}H_0H_1H_{0,-1}\\
&-\frac{128P_{11}^{(9)}}{27(1-x)(1+x)^3}H_{-1}H_{0,-1}-\frac{2240\big(1+x^2\big)^2}{27(1-x^2)^2}H_{0,-1}^2-\frac{512\big(1+x^2\big)^2}{27(1-x^2)^2}H_1H_{0,0,1}\\
&+\frac{512\big(1+x^2\big)^2}{3(1-x^2)^2}H_{-1}H_{0,0,1}+\frac{512\big(1+x^2\big)^2}{27(1-x^2)^2}H_1H_{0,0,-1}-\frac{512\big(1+x^2\big)^2}{3(1-x^2)^2}H_{-1}H_{0,0,-1}\\
&+\frac{128\big(1+x^2\big)H_{0,1,1}}{3(1-x^2)}+\frac{640\big(1+x^2\big)^2}{27(1-x^2)^2}H_0H_{0,1,1}-\frac{128\big(1+x^2\big)H_{0,1,-1}}{(1-x^2)}\\
&-\frac{896\big(1+x^2\big)^2}{27(1-x^2)^2}H_0H_{0,1,-1}-\frac{128\big(1+x^2\big)H_{0,-1,1}}{1-x^2}-\frac{896\big(1+x^2\big)^2}{27(1-x^2)^2}H_0H_{0,-1,1}\\
&+\frac{128P_{11}^{(9)}}{27(1-x)(1+x)^3}H_{0,-1,-1}+\frac{128\big(1+x^2\big)^2}{27(1-x^2)^2}H_0H_{0,-1,-1}+\frac{512\big(1+x^2\big)^2}{27(1-x^2)^2}H_{0,0,1,1}\\
&-\frac{512\big(1+x^2\big)^2}{3(1-x^2)^2}H_{0,0,1,-1}-\frac{512\big(1+x^2\big)^2}{3(1-x^2)^2}H_{0,0,-1,1}+\frac{512\big(1+x^2\big)^2}{3(1-x^2)^2}H_{0,0,-1,-1}\\
&-\frac{2048\big(1+x^2\big)^2}{27(1-x^2)^2}H_{0,-1,0,1}+\Bigg[-\frac{256H_0H_1P_{16}^{(9)}}{27(1-x)^2(1+x)^3}+\frac{64H_{0,1}P_{19}^{(9)}}{27(1-x)^2(1+x)^3}\\
&-\frac{64H_{-1}P_{20}^{(9)}}{27(1-x)^2(1+x)^3}+\frac{64H_{0,-1}P_{28}^{(9)}}{27(1-x^2)^3}+\frac{32H_0^2P_{38}^{(9)}}{27(1-x)^3(1+x)^4}-\frac{8H_0P_{40}^{(9)}}{27(1-x)^3(1+x)^5}\\
&+\frac{P_{42}^{(9)}}{27(1-x)^2(1+x)^6}-\frac{256\big(1+x^2\big)\big(13+4x^2\big)}{27(1-x^2)^2}H_{-1}H_0-\frac{64\big(1+x^2\big)H_1}{3(1-x^2)}\Bigg]\zeta_2\\
&+\Bigg[\frac{32P_{23}^{(9)}}{27(1-x)^2(1+x)^3}-\frac{32H_0P_{34}^{(9)}}{27(1-x^2)^3}+\frac{128\big(1+x^2\big)^2}{27(1-x^2)^2}H_1\\
&-\frac{128\big(1+x^2\big)^2}{3(1-x^2)^2}H_{-1}\Bigg]\zeta_3\Bigg\}+F_{A,1}^{(0)},
\end{aligned}
\tag{7.290}
$$

with

$$P_1^{(9)} = x^4 + 2x^3 - 2x^2 + 2x + 1, \tag{7.291}$$
$$P_2^{(9)} = 2x^4 + 125x^3 - 346x^2 + 125x + 2, \tag{7.292}$$
$$P_3^{(9)} = 5x^4 + 10x^3 - 14x^2 + 10x + 5, \tag{7.293}$$
$$P_4^{(9)} = 7x^4 + 14x^3 - 10x^2 + 14x + 7, \tag{7.294}$$
$$P_5^{(9)} = 14x^4 - 121x^3 + 310x^2 - 121x - 22, \tag{7.295}$$
$$P_6^{(9)} = 18x^4 + 129x^3 - 382x^2 + 129x - 18, \tag{7.296}$$
$$P_7^{(9)} = 23x^4 + 73x^3 + 64x^2 + 73x + 23, \tag{7.297}$$
$$P_8^{(9)} = 41x^4 + 190x^3 + 154x^2 + 190x + 41, \tag{7.298}$$
$$P_9^{(9)} = 43x^4 - 777x^3 + 1970x^2 - 777x + 43, \tag{7.299}$$
$$P_{10}^{(9)} = 103x^4 + 70x^3 + 94x^2 + 70x + 103, \tag{7.300}$$
$$P_{11}^{(9)} = 115x^4 + 284x^3 + 266x^2 + 284x + 115, \tag{7.301}$$
$$P_{12}^{(9)} = 337x^4 - 64x^3 + 158x^2 - 64x + 337, \tag{7.302}$$
$$P_{13}^{(9)} = 913x^4 + 1014x^3 - 390x^2 + 1014x + 913, \tag{7.303}$$
$$P_{14}^{(9)} = 14921x^4 + 59084x^3 + 82566x^2 + 59084x + 14921, \tag{7.304}$$
$$P_{15}^{(9)} = 9x^5 + 105x^4 + 48x^3 + 104x^2 + 39x + 47, \tag{7.305}$$
$$P_{16}^{(9)} = 13x^5 - 38x^4 + 226x^3 - 228x^2 + 37x - 14, \tag{7.306}$$
$$P_{17}^{(9)} = 25x^5 - 77x^4 + 450x^3 - 458x^2 + 73x - 29, \tag{7.307}$$
$$P_{18}^{(9)} = 29x^5 - 87x^4 + 38x^3 - 74x^2 + 45x - 47, \tag{7.308}$$
$$P_{19}^{(9)} = 43x^5 - 161x^4 + 886x^3 - 930x^2 + 139x - 65, \tag{7.309}$$
$$P_{20}^{(9)} = 55x^5 + 136x^4 - 27x^3 + 9x^2 - 94x - 67, \tag{7.310}$$
$$P_{21}^{(9)} = 87x^5 - 151x^4 + 86x^3 - 62x^2 + 179x - 75, \tag{7.311}$$
$$P_{22}^{(9)} = 97x^5 + 135x^4 - 22x^3 + 14x^2 - 83x - 13, \tag{7.312}$$
$$P_{23}^{(9)} = 137x^5 + 1241x^4 - 1412x^3 - 1340x^2 + 691x - 197, \tag{7.313}$$
$$P_{24}^{(9)} = 142x^5 + 1207x^4 - 1655x^3 - 1619x^2 + 815x - 142, \tag{7.314}$$
$$P_{25}^{(9)} = 228x^5 - 261x^4 + 731x^3 + 767x^2 - 653x - 56, \tag{7.315}$$
$$P_{26}^{(9)} = 331x^5 - 279x^4 + 478x^3 + 130x^2 + 855x - 107, \tag{7.316}$$
$$P_{27}^{(9)} = 5x^6 - 52x^5 + 277x^4 - 458x^3 + 274x^2 - 52x + 2, \tag{7.317}$$
$$P_{28}^{(9)} = 7x^6 + 6x^5 - 65x^4 + 24x^3 - 31x^2 + 6x + 41, \tag{7.318}$$
$$P_{29}^{(9)} = 15x^6 + 156x^5 - 823x^4 + 1378x^3 - 855x^2 + 156x - 17, \tag{7.319}$$
$$P_{30}^{(9)} = 30x^6 - 259x^5 + 1378x^4 - 2298x^3 + 1370x^2 - 259x + 22, \tag{7.320}$$
$$P_{31}^{(9)} = 41x^6 + 261x^5 - 1367x^4 + 2282x^3 - 1431x^2 + 261x - 23, \tag{7.321}$$
$$P_{32}^{(9)} = 42x^6 - 155x^5 + 828x^4 - 1374x^3 + 796x^2 - 155x + 10, \tag{7.322}$$
$$P_{33}^{(9)} = 50x^6 - 103x^5 + 554x^4 - 920x^3 + 510x^2 - 103x + 6, \tag{7.323}$$

$$P_{34}^{(9)} = 71x^6 - 412x^5 + 2199x^4 - 3696x^3 + 2181x^2 - 412x + 53, \quad (7.324)$$
$$P_{35}^{(9)} = 809x^6 - 1764x^5 + 8941x^4 - 15196x^3 + 8631x^2 - 1764x + 499, \quad (7.325)$$
$$P_{36}^{(9)} = 46991x^6 + 160202x^5 + 253361x^4 + 257260x^3 + 253361x^2 + 160202x + 46991, \quad (7.326)$$
$$P_{37}^{(9)} = 8x^7 + 9x^6 - 21x^5 + 102x^4 - 38x^3 + 31x^2 - 35x - 36, \quad (7.327)$$
$$P_{38}^{(9)} = 71x^7 + 19x^6 + 211x^5 - 135x^4 - 261x^3 + 181x^2 - 59x - 7, \quad (7.328)$$
$$P_{39}^{(9)} = 229x^8 - 318x^7 + 98x^6 + 1828x^5 - 1004x^4 - 1266x^3 - 54x^2 + 300x - 5, \quad (7.329)$$
$$P_{40}^{(9)} = 281x^8 + 3062x^7 + 48x^6 - 6750x^5 + 2006x^4 + 1130x^3 + 688x^2 + 382x - 79, \quad (7.330)$$
$$P_{41}^{(9)} = 13421x^8 + 46988x^7 + 29734x^6 + 3876x^5 + 15184x^4 + 26916x^3 - 11702x^2 - 13780x - 5677, \quad (7.331)$$
$$P_{42}^{(9)} = -1152\ln(2)(1-x)^2(x+1)^4\left(x^2+1\right) + 18177x^8 - 28796x^7 - 86188x^6 + 6780x^5 + 175430x^4 + 42236x^3 - 91148x^2 - 57084x + 8305, \quad (7.332)$$

and

$$\begin{aligned}
F_{A,1}^{(0)} = \; & n_h^2 \Biggl\{ \frac{64H_{0,0,0,-1}P_2^{(10)}}{27(x-1)(1+x)^3} - \frac{64\zeta_2^2 P_3^{(10)}}{9(x-1)(1+x)^3} + \frac{256H_{0,0,0,1}P_4^{(10)}}{9(x-1)(1+x)^3} \\
& + \frac{4H_0^4 P_9^{(10)}}{27(x-1)(1+x)^3} - \frac{64H_0^2 H_1 P_{14}^{(10)}}{81(1+x)^4} + \frac{8P_{24}^{(10)}}{729(1+x)^4} \\
& + \Biggl(-\frac{320H_{-1}P_{15}^{(10)}}{243(x-1)(1+x)^3} + \frac{16P_{40}^{(10)}}{729(x-1)(1+x)^5} + \frac{32(x-1)H_{-1}^2}{27(1+x)} \\
& - \frac{128\bigl(1+x^2\bigr)}{81(x-1)(1+x)}H_{-1}^3 \Biggr) H_0 + \Biggl(\frac{16H_{-1}P_{17}^{(10)}}{81(1+x)^4} - \frac{32P_{41}^{(10)}}{243(x-1)(1+x)^6} \\
& + \frac{32\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_{-1}^2 \Biggr) H_0^2 + \Biggl(-\frac{32H_{-1}P_2^{(10)}}{81(x-1)(1+x)^3} - \frac{8P_{44}^{(10)}}{243(x-1)(1+x)^7} \Biggr) H_0^3 \\
& + \Biggl(\frac{128H_0^2 P_4^{(10)}}{27(x-1)(1+x)^3} + \frac{128H_0 P_{14}^{(10)}}{81(1+x)^4} \Biggr) H_{0,1} + \Biggl(\frac{320P_{15}^{(10)}}{243(x-1)(1+x)^3} \\
& - \frac{64\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_0^2 - \frac{64(x-1)H_{-1}}{27(1+x)} + \frac{128\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_{-1}^2 \Biggr) H_{0,-1} \\
& + \Biggl(-\frac{512H_0 P_4^{(10)}}{27(x-1)(1+x)^3} - \frac{128P_{14}^{(10)}}{81(1+x)^4} \Biggr) H_{0,0,1} + \Biggl(-\frac{32P_{17}^{(10)}}{81(1+x)^4} \\
& + \frac{128\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_0 - \frac{128\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_{-1} \Biggr) H_{0,0,-1} + \Biggl(\frac{64(x-1)}{27(1+x)} \\
& - \frac{256\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_{-1} \Biggr) H_{0,-1,-1} + \frac{128\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_{0,0,-1,-1} \\
& + \frac{256\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_{0,-1,-1,-1} + \frac{256(x-1)^2\bigl(11+12x+11x^2\bigr)}{27(1+x)^4}\ln(2)\zeta_2 \\
& + \Biggl(-\frac{32H_{0,-1}P_{10}^{(10)}}{9(x-1)(1+x)^3} + \frac{32H_{-1}P_{18}^{(10)}}{27(1+x)^4} - \frac{64P_{43}^{(10)}}{1215(x-1)(1+x)^6}
\end{aligned}$$

$$+\left(-\frac{8P_{45}^{(10)}}{81(x-1)(1+x)^7}-\frac{32\big(1-4x+x^2\big)\big(1+6x+x^2\big)}{27(x-1)(1+x)^3}H_{-1}\right)H_0$$

$$+\frac{8(x-1)\big(1+4x+x^2\big)}{3(1+x)^3}H_0^2-\frac{64\big(1+x^2\big)}{27(x-1)(1+x)}H_{-1}^2\Bigg)\zeta_2+\Bigg(-\frac{32H_0P_{12}^{(10)}}{27(x-1)(1+x)^3}$$

$$-\frac{16P_{23}^{(10)}}{81(1+x)^4}+\frac{128\big(1+x^2\big)}{27(x-1)(1+x)}H_{-1}\Bigg)\zeta_3\Bigg\}+n_h\Bigg\{\mathrm{Li}_4\left(\frac{1}{2}\right)\Bigg(\frac{128P_{19}^{(10)}}{27(x-1)^2(1+x)^2}$$

$$+\frac{1024H_0P_{34}^{(10)}}{27(x-1)^3(1+x)^3}+\frac{256x^2H_0^2}{(x-1)(1+x)^3}+\frac{1024x^2H_0H_1}{(x-1)(1+x)^3}-\frac{1024x^2H_{0,1}}{(x-1)(1+x)^3}\Bigg)$$

$$+\ln^4(2)\Bigg(\frac{16P_{19}^{(10)}}{81(x-1)^2(1+x)^2}+\frac{128H_0P_{34}^{(10)}}{81(x-1)^3(1+x)^3}+\frac{32x^2H_0^2}{3(x-1)(1+x)^3}$$

$$+\frac{128x^2H_0H_1}{3(x-1)(1+x)^3}-\frac{128x^2H_{0,1}}{3(x-1)(1+x)^3}+\frac{128x^2\zeta_2}{3(x-1)(1+x)^3}\Bigg)$$

$$+\ln(2)\Bigg(-\frac{64\zeta_2^2P_{38}^{(10)}}{9(x-1)^3(1+x)^3}+\Bigg(-\frac{128H_{-1}P_8^{(10)}}{3(x-1)^2(1+x)^2}-\frac{128H_0H_1P_{36}^{(10)}}{9(x-1)^3(1+x)^3}$$

$$+\frac{128H_{0,1}P_{36}^{(10)}}{9(x-1)^3(1+x)^3}+\frac{64H_0^2P_{42}^{(10)}}{9(x-1)^3(1+x)^4}+\frac{16P_{47}^{(10)}}{81(x-1)^2x(1+x)^6}$$

$$+\Bigg(-\frac{128H_{-1}P_{37}^{(10)}}{9(x-1)^3(1+x)^3}+\frac{32P_{46}^{(10)}}{27(x-1)^3(1+x)^5}\Bigg)H_0+\frac{192x^2H_0^3}{(x-1)(1+x)^3}$$

$$+\frac{128\big(1+x^2\big)P_{22}^{(10)}}{9(x-1)^3(1+x)^3}H_{0,-1}\Bigg)\zeta_2\Bigg)+\ln^2(2)\Bigg(\Bigg(\frac{32P_{21}^{(10)}}{27(x-1)^2(1+x)^2}-\frac{128H_0P_{35}^{(10)}}{27(x-1)^3(1+x)^3}$$

$$-\frac{64x^2H_0^2}{(x-1)(1+x)^3}-\frac{256x^2H_0H_1}{(x-1)(1+x)^3}+\frac{256x^2H_{0,1}}{(x-1)(1+x)^3}\Bigg)\zeta_2-\frac{256x^2\zeta_2^2}{(x-1)(1+x)^3}\Bigg)$$

$$+n_l\Bigg[\frac{256H_{0,0,0,1}P_4^{(10)}}{9(x-1)(1+x)^3}+\frac{128H_{0,0,0,-1}P_6^{(10)}}{9(x-1)(1+x)^3}+\frac{8H_0^4P_{11}^{(10)}}{27(x-1)(1+x)^3}$$

$$-\frac{64H_0^2H_1P_{14}^{(10)}}{81(1+x)^4}-\frac{64\zeta_2^2P_{16}^{(10)}}{135(x-1)(1+x)^3}+\frac{64\big(377+106x+377x^2\big)}{729(1+x)^2}$$

$$+\Bigg(-\frac{128H_{-1}P_{20}^{(10)}}{81(x-1)(1+x)^3}+\frac{64P_{25}^{(10)}}{729(x-1)(1+x)^3}+\frac{256\big(11-6x+11x^2\big)}{81(x-1)(1+x)}H_{-1}^2$$

$$-\frac{1024\big(1+x^2\big)}{81(x-1)(1+x)}H_{-1}^3\Bigg)H_0+\Bigg(-\frac{64H_{-1}P_{27}^{(10)}}{81(x-1)(1+x)^4}+\frac{32P_{29}^{(10)}}{81(x-1)(1+x)^4}$$

$$+\frac{256\big(1+x^2\big)}{27(x-1)(1+x)}H_{-1}^2\Bigg)H_0^2+\Bigg(-\frac{64H_{-1}P_6^{(10)}}{27(x-1)(1+x)^3}-\frac{16P_{28}^{(10)}}{81(x-1)(1+x)^4}\Bigg)H_0^3$$

$$+\Bigg(\frac{128H_0^2P_4^{(10)}}{27(x-1)(1+x)^3}+\frac{128H_0P_{14}^{(10)}}{81(1+x)^4}\Bigg)H_{0,1}+\Bigg(-\frac{64H_0^2P_4^{(10)}}{9(x-1)(1+x)^3}$$

$$+\frac{128P_{20}^{(10)}}{81(x-1)(1+x)^3}-\frac{512\big(11-6x+11x^2\big)}{81(x-1)(1+x)}H_{-1}+\frac{1024\big(1+x^2\big)}{27(x-1)(1+x)}H_{-1}^2\Bigg)H_{0,-1}$$

$$
\begin{aligned}
&+\Biggl(-\frac{512H_0P_4^{(10)}}{27(x-1)(1+x)^3}-\frac{128P_{14}^{(10)}}{81(1+x)^4}\Biggr)H_{0,0,1}+\Biggl(\frac{128H_0P_4^{(10)}}{9(x-1)(1+x)^3}\\
&+\frac{128P_{27}^{(10)}}{81(x-1)(1+x)^4}-\frac{1024\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_{-1}\Biggr)H_{0,0,-1}\\
&+\Biggl(\frac{512\bigl(11-6x+11x^2\bigr)}{81(x-1)(1+x)}-\frac{2048\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_{-1}\Biggr)H_{0,-1,-1}\\
&+\frac{1024\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_{0,0,-1,-1}+\frac{2048\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_{0,-1,-1,-1}\\
&+\frac{256(x-1)^2\bigl(11+12x+11x^2\bigr)}{27(1+x)^4}\ln(2)\zeta_2+\Biggl(-\frac{64H_{0,-1}P_1^{(10)}}{9(x-1)(1+x)^3}\\
&+\frac{16H_0^2P_5^{(10)}}{3(x-1)(1+x)^3}+\frac{128H_{-1}P_{30}^{(10)}}{81(x-1)(1+x)^4}-\frac{8P_{33}^{(10)}}{81(x-1)(1+x)^4}\\
&+\Bigl(-\frac{64H_{-1}P_7^{(10)}}{27(x-1)(1+x)^3}-\frac{16P_{31}^{(10)}}{81(x-1)(1+x)^4}\Bigr)H_0-\frac{512\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_{-1}^2\Biggr)\zeta_2\\
&+\Biggl(-\frac{16P_{32}^{(10)}}{81(x-1)(1+x)^4}-\frac{32\bigl(1+x^2\bigr)}{3(x-1)(1+x)}H_0+\frac{1024\bigl(1+x^2\bigr)}{27(x-1)(1+x)}H_{-1}\Biggr)\zeta_3\Biggr]\\
&+\Biggl(\frac{1024x^2}{(x-1)(1+x)^3}\mathrm{Li}_4\left(\frac{1}{2}\right)+\Biggl(-\frac{8P_{13}^{(10)}}{(x-1)^2(1+x)^2}+\frac{32x^2\bigl(5-24x+5x^2\bigr)}{(x-1)^3(1+x)^3}H_0\Biggr)\zeta_3\Biggr)\zeta_2\\
&+\Biggl(-\frac{8P_{26}^{(10)}}{135(x-1)^2(1+x)^2}+\frac{32H_0P_{39}^{(10)}}{135(x-1)^3(1+x)^3}+\frac{16x^2H_0^2}{5(x-1)(1+x)^3}\\
&+\frac{64x^2H_0H_1}{5(x-1)(1+x)^3}-\frac{64x^2H_{0,1}}{5(x-1)(1+x)^3}\Biggr)\zeta_2^2+\frac{64x^2\zeta_2^3}{5(x-1)(1+x)^3}\\
&+\Biggl(\frac{20\bigl(1+x^2\bigr)\bigl(1-8x+x^2\bigr)}{(x-1)^2(1+x)^2}-\frac{80x^2\bigl(1-8x+x^2\bigr)}{(x-1)^3(1+x)^3}H_0\Biggr)\zeta_5\Biggr\}\\
&+F_{A,1,1}^{(0)}+F_{A,1,2}^{(0)}\zeta_2+F_{A,1,3}^{(0)}\zeta_3+F_{A,1}^{(0),r},
\end{aligned}
\tag{7.333}
$$

with the polynomials

$$P_1^{(10)}=x^4+2x^3-26x^2+2x+1, \tag{7.334}$$
$$P_2^{(10)}=x^4+2x^3-10x^2+2x+1, \tag{7.335}$$
$$P_3^{(10)}=x^4+2x^3-5x^2+2x+1, \tag{7.336}$$
$$P_4^{(10)}=x^4+2x^3-4x^2+2x+1, \tag{7.337}$$
$$P_5^{(10)}=x^4+2x^3-2x^2+2x+1, \tag{7.338}$$
$$P_6^{(10)}=x^4+2x^3+4x^2+2x+1, \tag{7.339}$$
$$P_7^{(10)}=x^4+2x^3+14x^2+2x+1, \tag{7.340}$$
$$P_8^{(10)}=2x^4-9x^3+12x^2-9x+2, \tag{7.341}$$
$$P_9^{(10)}=3x^4+6x^3-14x^2+6x+3, \tag{7.342}$$

$$P_{10}^{(10)} = 3x^4 + 6x^3 - 10x^2 + 6x + 3, \tag{7.343}$$
$$P_{11}^{(10)} = 3x^4 + 6x^3 - 4x^2 + 6x + 3, \tag{7.344}$$
$$P_{12}^{(10)} = 11x^4 + 22x^3 - 50x^2 + 22x + 11, \tag{7.345}$$
$$P_{13}^{(10)} = 13x^4 - 24x^3 - 6x^2 - 24x + 13, \tag{7.346}$$
$$P_{14}^{(10)} = 19x^4 - 14x^3 + 14x^2 - 14x + 19, \tag{7.347}$$
$$P_{15}^{(10)} = 31x^4 - 10x^3 + 14x^2 - 10x + 31, \tag{7.348}$$
$$P_{16}^{(10)} = 31x^4 + 62x^3 + 149x^2 + 62x + 31, \tag{7.349}$$
$$P_{17}^{(10)} = 35x^4 - 34x^3 + 28x^2 - 22x + 41, \tag{7.350}$$
$$P_{18}^{(10)} = 39x^4 - 26x^3 + 28x^2 - 30x + 37, \tag{7.351}$$
$$P_{19}^{(10)} = 53x^4 - 188x^3 + 174x^2 - 188x + 53, \tag{7.352}$$
$$P_{20}^{(10)} = 89x^4 + 48x^3 + 78x^2 + 48x + 89, \tag{7.353}$$
$$P_{21}^{(10)} = 91x^4 - 190x^3 + 258x^2 - 190x + 91, \tag{7.354}$$
$$P_{22}^{(10)} = 105x^4 - 208x^3 + 198x^2 - 208x + 105, \tag{7.355}$$
$$P_{23}^{(10)} = 253x^4 - 296x^3 - 310x^2 - 320x + 241, \tag{7.356}$$
$$P_{24}^{(10)} = 1367x^4 + 3428x^3 + 4506x^2 + 3428x + 1367, \tag{7.357}$$
$$P_{25}^{(10)} = 1471x^4 + 2042x^3 - 10x^2 + 2042x + 1471, \tag{7.358}$$
$$P_{26}^{(10)} = 5363x^4 - 4754x^3 - 2814x^2 - 4754x + 5363, \tag{7.359}$$
$$P_{27}^{(10)} = 3x^5 + 87x^4 + 24x^3 + 80x^2 + 21x + 41, \tag{7.360}$$
$$P_{28}^{(10)} = 35x^5 - 141x^4 + 34x^3 - 78x^2 - 9x - 41, \tag{7.361}$$
$$P_{29}^{(10)} = 62x^5 + 125x^4 + 205x^3 + 47x^2 + 149x + 116, \tag{7.362}$$
$$P_{30}^{(10)} = 79x^5 - 45x^4 + 136x^3 - 32x^2 + 153x - 35, \tag{7.363}$$
$$P_{31}^{(10)} = 189x^5 - 579x^4 + 222x^3 - 226x^2 - 51x - 115, \tag{7.364}$$
$$P_{32}^{(10)} = 463x^5 - 219x^4 + 598x^3 + 234x^2 + 1083x - 111, \tag{7.365}$$
$$P_{33}^{(10)} = 1969x^5 + 1147x^4 - 1734x^3 + 3750x^2 + 1045x - 545, \tag{7.366}$$
$$P_{34}^{(10)} = 2x^6 + 15x^5 - 37x^4 + 64x^3 - 37x^2 + 15x + 2, \tag{7.367}$$
$$P_{35}^{(10)} = 13x^6 + 39x^5 - 119x^4 + 164x^3 - 119x^2 + 39x + 13, \tag{7.368}$$
$$P_{36}^{(10)} = 21x^6 - 110x^5 + 195x^4 - 208x^3 + 195x^2 - 110x + 21, \tag{7.369}$$
$$P_{37}^{(10)} = 99x^6 - 214x^5 + 333x^4 - 440x^3 + 333x^2 - 214x + 99, \tag{7.370}$$
$$P_{38}^{(10)} = 147x^6 - 428x^5 + 693x^4 - 832x^3 + 693x^2 - 428x + 147, \tag{7.371}$$
$$P_{39}^{(10)} = 806x^6 - 951x^5 + 1172x^4 - 2852x^3 + 1172x^2 - 951x + 806, \tag{7.372}$$
$$P_{40}^{(10)} = 2945x^6 + 6836x^5 + 287x^4 - 6056x^3 + 287x^2 + 6836x + 2945, \tag{7.373}$$
$$P_{41}^{(10)} = 19x^7 - 133x^6 - 466x^5 - 238x^4 + 73x^3 + 81x^2 - 282x - 174, \tag{7.374}$$
$$P_{42}^{(10)} = 70x^7 + 15x^6 - 69x^5 - 94x^4 + 6x^3 + 127x^2 - 53x + 2, \tag{7.375}$$
$$P_{43}^{(10)} = 2226x^7 + 4218x^6 - 1317x^5 - 2729x^4 + 3554x^3 + 3242x^2 - 2143x - 1451, \tag{7.376}$$

$$P_{44}^{(10)} = 181x^8 + 228x^7 - 212x^6 + 44x^5 + 942x^4 + 428x^3 + 12x^2 - 156x - 123, \quad (7.377)$$
$$P_{45}^{(10)} = 213x^8 + 276x^7 - 300x^6 + 268x^5 + 1854x^4 + 652x^3 - 76x^2 - 108x - 91, \quad (7.378)$$
$$P_{46}^{(10)} = 381x^8 + 214x^7 - 7742x^6 + 5550x^5 + 7394x^4 + 5154x^3 - 7526x^2 + 394x + 309, \quad (7.379)$$
$$P_{47}^{(10)} = 2768x^{10} - 63429x^9 - 44774x^8 + 54896x^7 + 46326x^6 - 43414x^5 + 46326x^4 + 54896x^3 - 44774x^2 - 63429x + 2768. \quad (7.380)$$

The first expansion coefficients of $F_{A,1,i}^{(0)}, i = 1...3$ are given by

$$F_{A,1,1}^{(0)}(x) = -\frac{1105690}{729} - \frac{19976198y^2}{18225} - \frac{19976198y^3}{18225} - \frac{647051207603y^4}{857304000} - \frac{177211030643y^5}{428652000} + O(y^6) \quad (7.381)$$
$$F_{A,1,2}^{(0)}(x) = \frac{1979131}{729} + \frac{17033692y^2}{18225} + \frac{17033692y^3}{18225} + \frac{27237088943y^4}{44651250} + \frac{6370816243y^5}{22325625} + O(y^6) \quad (7.382)$$
$$F_{A,1,3}^{(0)}(x) = \frac{24544}{243} + \frac{1061573y^2}{6075} + \frac{1061573y^3}{6075} + \frac{255928217y^4}{2352000} + \frac{4084720937y^5}{95256000} + O(y^6). \quad (7.383)$$

The form factor $F_{A,2}$ is given by

$$F_{A,2} = x\Biggl\{-\frac{128n_h}{\varepsilon^3}\Biggl\{\frac{1}{3(1+x)^2} + \frac{\bigl(1+x^2\bigr)}{3(1-x)(1+x)^3}H_0\Biggr\} + \frac{1}{\varepsilon^2}\Biggl\{n_h^2\Biggl[-\frac{128}{27(1-x)^2} - \frac{64\bigl(3-2x+3x^2\bigr)}{27(1-x)^3(1+x)}H_0\Biggr] + n_h\Biggl[\frac{4672}{27(1-x)^2(1+x)^3} + \frac{23488x(1+x)}{27(1-x)^2(1+x)^3} + \frac{4672x^3}{27(1-x)^2(1+x)^3} - \frac{64H_0^2P_{24}^{(11)}}{27(1-x)^4(1+x)^3} - n_l\Biggl[\frac{256}{27(1-x)^2} + \frac{128\bigl(3-2x+3x^2\bigr)}{27(1-x)^3(1+x)}H_0\Biggr] + \Biggl(\frac{128P_4^{(11)}}{27(1-x)^3(1+x)^3} + \frac{256\bigl(1+x^2\bigr)}{3(1-x)(1+x)^3}H_{-1}\Biggr)H_0 - \frac{256\bigl(1+x^2\bigr)}{3(1-x)(1+x)^3}H_{0,-1} + \frac{128(1+x^2)}{3(1-x)(1+x)^3}\zeta_2\Biggr\} + \frac{1}{\varepsilon}\Biggl\{n_h^2\Biggl[-\frac{64H_0P_7^{(11)}}{81(1-x^2)^3} + \frac{32H_0^2P_{20}^{(11)}}{27(1-x)^3(1+x)^4} + \frac{256\bigl(1+26x+x^2\bigr)}{81(1-x)^2(1+x)^2} + \frac{128\bigl(3-2x+3x^2\bigr)}{27(1-x)^3(1+x)}H_{-1}H_0 + \frac{256x^2H_0^3}{27(1-x^2)^3} - \frac{128\bigl(3-2x+3x^2\bigr)}{27(1-x)^3(1+x)} \times H_{0,-1} + \Biggl(\frac{64P_{26}^{(11)}}{27(1-x)^3(1+x)^4} + \frac{512x^2H_0}{9(1-x^2)^3}\Biggr)\zeta_2\Biggr] + n_h\Biggl[-\frac{256H_0^2H_1P_2^{(11)}}{27(1-x)^4(1+x)^2} + \frac{512H_0H_{0,1}P_3^{(11)}}{27(1-x)^4(1+x)^2} - \frac{512H_{0,0,1}P_6^{(11)}}{27(1-x)^4(1+x)^2} + \frac{128P_{13}^{(11)}}{81(10x)^2(1+x)^4} + \frac{128H_{-1}H_0P_{16}^{(11)}}{27(1-x)^3(1+x)^3} + \frac{128H_{0,-1}P_{16}^{(11)}}{27(1-x^2)^3} - \frac{128H_0H_{0,-1}P_{17}^{(11)}}{27(1-x)^4(1+x)^2} + \frac{64H_{-1}H_0^2P_{28}^{(11)}}{27(1-x)^4(1+x)^3} + \frac{128H_{0,0,-1}P_{29}^{(11)}}{27(1-x)^4(1+x)^3} + \frac{16H_0P_{30}^{(11)}}{81(1-x)^3(1+x)^5} - \frac{64H_0^2P_{33}^{(11)}}{81(1-x)^4(1+x)^6} + n_l\Biggl[-\frac{128H_0P_8^{(11)}}{81(1-x^2)^3} - \frac{64H_0^2P_{22}^{(11)}}{27(1-x)^3(1+x)^4}$$

$$
-\frac{128\big(23-2x+23x^2\big)}{81(1-x^2)^2}+\frac{512\big(3-2x+3x^2\big)}{27(1-x)^3(1+x)}H_{-1}H_0+\frac{256x^2H_0^3}{27(1-x^2)^3}
$$

$$
-\frac{512\big(3-2x+3x^2\big)}{27(1-x)^3(1+x)}H_{0,-1}+\left(\frac{128P_{25}^{(11)}}{27(1-x)^3(1+x)^4}+\frac{512x^2H_0}{9(1-x^2)^3}\right)\zeta_2\Bigg]
$$

$$
-\frac{256\big(1+x^2\big)}{3(1-x)(1+x)^3}H_{-1}^2H_0+\left(+\frac{64P_{31}^{(11)}}{81(1-x^2)^5}-\frac{512x\big(35-208x+35x^2\big)}{81(1-x^2)^3}H_1\right)H_0^3
$$

$$
+\frac{32xP_{19}^{(11)}}{81(1-x)^5(1+x)^4}H_0^4+\frac{256\big(23-14x+23x^2\big)}{27(1-x)^3(1+x)}H_0H_1-\frac{256\big(23-14x+23x^2\big)}{27(1-x)^3(1+x)}H_{0,1}
$$

$$
+\frac{256xP_{11}^{(11)}}{27(1-x)^5(1+x)^3}H_0^2H_{0,1}-\frac{256xP_{10}^{(11)}}{27(1-x)^5(1+x)^3}H_0^2H_{0,-1}+\frac{512\big(1+x^2\big)}{3(1-x)(1+x)^3}H_{-1}
$$

$$
\times H_{0,-1}-\frac{2048xP_5^{(11)}}{27(1-x)^5(1+x)^3}H_0H_{0,0,1}+\frac{512xP_{14}^{(11)}}{27(1-x)^5(1+x)^3}H_0H_{0,0,-1}
$$

$$
-\frac{512\big(1+x^2\big)}{3(1-x)(1+x)^3}H_{0,-1,-1}+\frac{512xP_{15}^{(11)}}{27(1-x)^5(1+x)^3}H_{0,0,0,1}-\frac{512x^2P_{12}^{(11)}}{9(1-x)^5(1+x)^3}H_{0,0,0,-1}
$$

$$
+\Bigg(\frac{128H_{-1}P_{21}^{(11)}}{3(1-x)^4(1+x)^3}-\frac{64H_0P_{32}^{(11)}}{27(1-x)^5(1+x)^5}-\frac{64P_{34}^{(11)}}{27(1-x)^4(1+x)^6}-\frac{128xP_{23}^{(11)}H_0^2}{9(1-x)^5(1+x)^4}
$$

$$
-\frac{1024x\big(35-208x+35x^2\big)}{27(1-x^2)^3}H_0H_1+\frac{1024x\big(35-208x+35x^2\big)}{27(1-x^2)^3}H_{0,1}
$$

$$
+\frac{512xP_1^{(11)}}{9(1-x)^5(1+x)^3}H_{0,-1}\Bigg)\zeta_2-\frac{128xP_{18}^{(11)}}{135(1-x)^5(1+x)^3}\zeta_2^2+\Bigg(-\frac{256P_{27}^{(11)}}{27(1-x)^4(1+x)^3}
$$

$$
+\frac{512xP_9^{(11)}}{27(1-x)^5(1+x)^3}H_0\Bigg)\zeta_3\Bigg]\Bigg\}+F_{A,2}^{(0)} \tag{7.384}
$$

with the polynomials

$$P_1^{(11)} = x^4-5x^3+2x^2-5x+1, \tag{7.385}$$
$$P_2^{(11)} = 12x^4-109x^3+310x^2-109x+12, \tag{7.386}$$
$$P_3^{(11)} = 24x^4-115x^3+330x^2-115x+24, \tag{7.387}$$
$$P_4^{(11)} = 29x^4+167x^3-16x^2+167x+29, \tag{7.388}$$
$$P_5^{(11)} = 36x^4-277x^3+494x^2-277x+36, \tag{7.389}$$
$$P_6^{(11)} = 36x^4-121x^3+350x^2-121x+36, \tag{7.390}$$
$$P_7^{(11)} = 69x^4-152x^3-58x^2-152x+69, \tag{7.391}$$
$$P_8^{(11)} = 69x^4-26x^3+2x^2-26x+69, \tag{7.392}$$
$$P_9^{(11)} = 71x^4-561x^3+992x^2-561x+71, \tag{7.393}$$
$$P_{10}^{(11)} = 71x^4-557x^3+990x^2-557x+71, \tag{7.394}$$
$$P_{11}^{(11)} = 107x^4-834x^3+1478x^2-834x+107, \tag{7.395}$$
$$P_{12}^{(11)} = 108x^4-835x^3+1484x^2-835x+108, \tag{7.396}$$
$$P_{13}^{(11)} = 158x^4+469x^3+1006x^2+469x+158, \tag{7.397}$$

$$P_{14}^{(11)} = 179x^4 - 1392x^3 + 2474x^2 - 1392x + 179, \quad (7.398)$$
$$P_{15}^{(11)} = 181x^4 - 1378x^3 + 2466x^2 - 1378x + 181, \quad (7.399)$$
$$P_{16}^{(11)} = 207x^4 + 8x^3 - 982x^2 + 8x + 207, \quad (7.400)$$
$$P_{17}^{(11)} = 249x^4 - 1380x^3 + 3994x^2 - 1380x + 249, \quad (7.401)$$
$$P_{18}^{(11)} = 610x^4 - 4759x^3 + 8064x^2 - 4759x + 610, \quad (7.402)$$
$$P_{19}^{(11)} = x^5 - 52x^4 + 81x^3 - 15x^2 + 44x + 1, \quad (7.403)$$
$$P_{20}^{(11)} = 3x^5 - 41x^4 + 18x^3 - 30x^2 + 27x - 9, \quad (7.404)$$
$$P_{21}^{(11)} = 3x^5 - 9x^4 + 6x^3 - 10x^2 + 3x - 1, \quad (7.405)$$
$$P_{22}^{(11)} = 3x^5 + 31x^4 + 24x^2 - 3x + 9, \quad (7.406)$$
$$P_{23}^{(11)} = 12x^5 - 75x^4 + 53x^3 + 69x^2 - 91x + 12, \quad (7.407)$$
$$P_{24}^{(11)} = 15x^5 + 35x^4 - 4x^3 + 68x^2 - 19x + 33, \quad (7.408)$$
$$P_{25}^{(11)} = 21x^5 - 31x^4 + 48x^3 - 24x^2 + 59x - 9, \quad (7.409)$$
$$P_{26}^{(11)} = 33x^5 - 83x^4 + 78x^3 - 66x^2 + 97x - 27, \quad (7.410)$$
$$P_{27}^{(11)} = 48x^5 - 421x^4 + 859x^3 + 787x^2 - 367x + 30, \quad (7.411)$$
$$P_{28}^{(11)} = 183x^5 - 701x^4 + 1726x^3 + 1870x^2 - 809x + 219, \quad (7.412)$$
$$P_{29}^{(11)} = 315x^5 - 1561x^4 + 3502x^3 + 3358x^2 - 1453x + 279, \quad (7.413)$$
$$P_{30}^{(11)} = 8449x^6 + 19482x^5 + 13519x^4 - 7316x^3 + 13519x^2 + 19482x + 8449, \quad (7.414)$$
$$P_{31}^{(11)} = 48x^8 - 584x^7 + 457x^6 + 2554x^5 - 1127x^4 - 1544x^3 - 329x^2 + 54x - 105, \quad (7.415)$$
$$P_{32}^{(11)} = 63x^8 + 1134x^7 - 402x^6 - 2986x^5 + 2320x^4 + 738x^3 + 146x^2 + 154x - 15, \quad (7.416)$$
$$P_{33}^{(11)} = 696x^8 + 1820x^7 - 2251x^6 - 3898x^5 - 521x^4 + 3080x^3 + 155x^2 + 86x - 63, \quad (7.417)$$
$$\begin{aligned} P_{34}^{(11)} &= 24\ln(2)\left(x^2-1\right)^4 - 843x^8 + 2432x^7 + 5402x^6 + 2028x^5 - 8680x^4 - 2888x^3 \\ &+ 2814x^2 + 2652x - 613 \end{aligned} \quad (7.418)$$

and

$$\begin{aligned} F_{A,2}^{(0)} \quad = \quad & x\Biggl\{ n_h^2 \Biggl\{ -\frac{256H_0^2H_1P_2^{(12)}}{27(x-1)^2(1+x)^4} - \frac{128H_{0,0,-1}P_3^{(12)}}{27(x-1)^2(1+x)^4} + \frac{512P_5^{(12)}}{243(x-1)^2(1+x)^4} \\ & + \left(-\frac{512H_{-1}P_6^{(12)}}{81(x-1)^3(1+x)^3} + \frac{128P_{28}^{(12)}}{243(x-1)^3(1+x)^5} + \frac{128H_{-1}^2}{27(x-1)(1+x)} \right) H_0 \\ & + \left(\frac{64H_{-1}P_3^{(12)}}{27(x-1)^2(1+x)^4} - \frac{128P_{29}^{(12)}}{81(x-1)^3(1+x)^6} \right) H_0^2 + \left(-\frac{32P_{31}^{(12)}}{81(x-1)^3(1+x)^7} \right. \\ & \left. + \frac{512x^2H_{-1}}{27(x-1)^3(1+x)^3} \right) H_0^3 - \frac{320x^2H_0^4}{27(x-1)^3(1+x)^3} + \left(\frac{512H_0P_2^{(12)}}{27(x-1)^2(1+x)^4} \right. \\ & \left. - \frac{1024x^2H_0^2}{9(x-1)^3(1+x)^3} \right) H_{0,1} + \left(\frac{512P_6^{(12)}}{81(x-1)^3(1+x)^3} - \frac{256H_{-1}}{27(x-1)(1+x)} \right) H_{0,-1} \\ & + \left(-\frac{512P_2^{(12)}}{27(x-1)^2(1+x)^4} + \frac{4096x^2H_0}{9(x-1)^3(1+x)^3} \right) H_{0,0,1} + \frac{256H_{0,-1,-1}}{27(x-1)(1+x)} \end{aligned}$$

$$
-\frac{2048x^2H_{0,0,0,1}}{3(x-1)^3(1+x)^3}-\frac{1024x^2H_{0,0,0,-1}}{9(x-1)^3(1+x)^3}+\Biggl(\frac{128H_{-1}P_8^{(12)}}{27(x-1)^2(1+x)^4}
$$

$$
-\frac{32P_{30}^{(12)}}{405(x-1)^3(1+x)^6}+\frac{1024\bigl(3+2x+3x^2\bigr)}{9(1+x)^4}\ln(2)+\Biggl(-\frac{32P_{32}^{(12)}}{27(x-1)^3(1+x)^7}
$$

$$
+\frac{1024x^2H_{-1}}{9(x-1)^3(1+x)^3}\Biggr)H_0-\frac{256x^2H_0^2}{3(x-1)^3(1+x)^3}+\frac{2048x^2H_{0,-1}}{9(x-1)^3(1+x)^3}\Biggr)\zeta_2
$$

$$
+\frac{1792x^2\zeta_2^2}{9(x-1)^3(1+x)^3}+\Biggl(-\frac{128P_{11}^{(12)}}{27(x-1)^2(1+x)^4}+\frac{1024x^2H_0}{3(x-1)^3(1+x)^3}\Biggr)\zeta_3\Biggr\}
$$

$$
+n_h\Biggl\{\frac{160\zeta_5P_1^{(12)}}{(x-1)^4(1+x)^2}+\text{Li}_4\left(\frac{1}{2}\right)\Biggl(\frac{1024P_7^{(12)}}{27(x-1)^4(1+x)^2}+\frac{1024xH_0P_{17}^{(12)}}{27(x-1)^5(1+x)^3}
$$

$$
+\frac{1024x^2H_0^2}{(x-1)^3(1+x)^3}+\frac{4096x^2H_0H_1}{(x-1)^3(1+x)^3}-\frac{4096x^2H_{0,1}}{(x-1)^3(1+x)^3}\Biggr)
$$

$$
+\ln^4(2)\Biggl(\frac{128P_7^{(12)}}{81(x-1)^4(1+x)^2}+\frac{128xH_0P_{17}^{(12)}}{81(x-1)^5(1+x)^3}+\frac{128x^2H_0^2}{3(x-1)^3(1+x)^3}
$$

$$
+\frac{512x^2H_0H_1}{3(x-1)^3(1+x)^3}-\frac{512x^2H_{0,1}}{3(x-1)^3(1+x)^3}\Biggr)+n_l\Biggl[-\frac{256H_0^2H_1P_2^{(12)}}{27(x-1)^2(1+x)^4}
$$

$$
-\frac{256\zeta_3P_{27}^{(12)}}{27(x-1)^3(1+x)^4}-\frac{512\bigl(11+32x+11x^2\bigr)}{27(x-1)^2(1+x)^2}+\Biggl(\frac{256P_{19}^{(12)}}{81(x-1)^3(1+x)^3}
$$

$$
-\frac{512\bigl(55+62x+55x^2\bigr)}{81(x-1)(1+x)^3}H_{-1}+\frac{2048\bigl(1-x+x^2\bigr)}{27(x-1)^3(1+x)}H_{-1}^2\Biggr)H_0+\Biggl(\frac{128P_{10}^{(12)}}{81(x-1)^2(1+x)^4}
$$

$$
-\frac{256H_{-1}P_{22}^{(12)}}{27(x-1)^3(1+x)^4}\Biggr)H_0^2+\Biggl(-\frac{64P_{25}^{(12)}}{81(x-1)^3(1+x)^4}-\frac{512x^2H_{-1}}{27(x-1)^3(1+x)^3}\Biggr)H_0^3
$$

$$
-\frac{320x^2H_0^4}{27(x-1)^3(1+x)^3}+\Biggl(\frac{512H_0P_2^{(12)}}{27(x-1)^2(1+x)^4}-\frac{1024x^2H_0^2}{9(x-1)^3(1+x)^3}\Biggr)H_{0,1}
$$

$$
+\Biggl(\frac{512\bigl(55+62x+55x^2\bigr)}{81(x-1)(1+x)^3}+\frac{512x^2H_0^2}{3(x-1)^3(1+x)^3}-\frac{4096\bigl(1-x+x^2\bigr)}{27(x-1)^3(1+x)}H_{-1}\Biggr)H_{0,-1}
$$

$$
+\Biggl(-\frac{512P_2^{(12)}}{27(x-1)^2(1+x)^4}+\frac{4096x^2H_0}{9(x-1)^3(1+x)^3}\Biggr)H_{0,0,1}+\Biggl(\frac{512P_{22}^{(12)}}{27(x-1)^3(1+x)^4}
$$

$$
-\frac{1024x^2H_0}{3(x-1)^3(1+x)^3}\Biggr)H_{0,0,-1}+\frac{4096\bigl(1-x+x^2\bigr)}{27(x-1)^3(1+x)}H_{0,-1,-1}-\frac{2048x^2H_{0,0,0,1}}{3(x-1)^3(1+x)^3}
$$

$$
+\frac{1024x^2H_{0,0,0,-1}}{9(x-1)^3(1+x)^3}+\Biggl(-\frac{32P_{20}^{(12)}}{81(x-1)^2(1+x)^4}+\frac{512H_{-1}P_{23}^{(12)}}{27(x-1)^3(1+x)^4}
$$

$$
+\frac{1024\bigl(3+2x+3x^2\bigr)}{9(1+x)^4}\ln(2)+\Biggl(-\frac{128P_{24}^{(12)}}{27(x-1)^3(1+x)^4}-\frac{1024x^2H_{-1}}{9(x-1)^3(1+x)^3}\Biggr)H_0
$$

$$
-\frac{256x^2H_0^2}{3(x-1)^3(1+x)^3}+\frac{7168x^2H_{0,-1}}{9(x-1)^3(1+x)^3}\Bigg)\zeta_2-\frac{7424x^2\zeta_2^2}{45(x-1)^3(1+x)^3}\Bigg]
$$

$$
+\Bigg(\frac{4096x^2}{(x-1)^3(1+x)^3}\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{512x^2}{3(x-1)^3(1+x)^3}\ln^4(2)+\ln(2)
$$

$$
\times\Bigg(-\frac{1024xH_{0,-1}P_{13}^{(12)}}{9(x-1)^5(1+x)^3}+\frac{512xH_0H_1P_{16}^{(12)}}{9(x-1)^5(1+x)^3}-\frac{512xH_{0,1}P_{16}^{(12)}}{9(x-1)^5(1+x)^3}
$$

$$
-\frac{64P_{34}^{(12)}}{27(x-1)^4x(1+x)^6}+\left(\frac{1024xH_{-1}P_{14}^{(12)}}{9(x-1)^5(1+x)^3}+\frac{256P_{33}^{(12)}}{27(x-1)^5(1+x)^5}\right)H_0
$$

$$
-\frac{256xH_0^2P_{26}^{(12)}}{9(x-1)^5(1+x)^4}+\frac{768x^2H_0^3}{(x-1)^3(1+x)^3}-\frac{256\left(1+x^2\right)\left(1-4x+x^2\right)}{(x-1)^4(1+x)^2}H_{-1}\Bigg)
$$

$$
+\ln^2(2)\Bigg(\frac{128P_9^{(12)}}{27(x-1)^4(1+x)^2}-\frac{256xH_0P_{18}^{(12)}}{27(x-1)^5(1+x)^3}-\frac{256x^2H_0^2}{(x-1)^3(1+x)^3}
$$

$$
-\frac{1024x^2H_0H_1}{(x-1)^3(1+x)^3}+\frac{1024x^2H_{0,1}}{(x-1)^3(1+x)^3}\Bigg)+\Bigg(-\frac{64P_4^{(12)}}{(x-1)^4(1+x)^2}
$$

$$
-\frac{256x^2\left(1+19x+x^2\right)}{(x-1)^5(1+x)^3}H_0\Bigg)\zeta_3\Bigg)\zeta_2+\Bigg(\frac{32P_{12}^{(12)}}{135(x-1)^4(1+x)^2}-\frac{64xH_0P_{21}^{(12)}}{135(x-1)^5(1+x)^3}
$$

$$
+\frac{1024xP_{15}^{(12)}}{9(x-1)^5(1+x)^3}\ln(2)-\frac{1024x^2}{(x-1)^3(1+x)^3}\ln^2(2)+\frac{64x^2H_0^2}{5(x-1)^3(1+x)^3}
$$

$$
+\frac{256x^2H_0H_1}{5(x-1)^3(1+x)^3}-\frac{256x^2H_{0,1}}{5(x-1)^3(1+x)^3}\Bigg)\zeta_2^2+\frac{256x^2\zeta_2^3}{5(x-1)^3(1+x)^3}
$$

$$
+\frac{640x^2\left(1+7x+x^2\right)}{(x-1)^5(1+x)^3}H_0\zeta_5\Bigg\}\Bigg\}+F_{A,2,1}^{(0)}+F_{A,2,2}^{(0)}\zeta_2+F_{A,2,3}^{(0)}\zeta_3+F_{A,2}^{(0),r}, \tag{7.419}
$$

with the polynomials

$$P_1^{(12)}=2x^4-7x^3-8x^2-7x+2, \tag{7.420}$$
$$P_2^{(12)}=3x^4-14x^3-2x^2-14x+3, \tag{7.421}$$
$$P_3^{(12)}=5x^4-30x^3-4x^2-26x+7, \tag{7.422}$$
$$P_4^{(12)}=6x^4-19x^3-16x^2-19x+6, \tag{7.423}$$
$$P_5^{(12)}=14x^4+35x^3+54x^2+35x+14, \tag{7.424}$$
$$P_6^{(12)}=15x^4-41x^3-16x^2-41x+15, \tag{7.425}$$
$$P_7^{(12)}=17x^4-118x^3+58x^2-118x+17, \tag{7.426}$$
$$P_8^{(12)}=19x^4-82x^3-12x^2-86x+17, \tag{7.427}$$
$$P_9^{(12)}=29x^4-16x^3+154x^2-16x+29, \tag{7.428}$$
$$P_{10}^{(12)}=30x^4+41x^3+32x^2-83x-80, \tag{7.429}$$
$$P_{11}^{(12)}=49x^4-2x^3+56x^2-6x+47, \tag{7.430}$$
$$P_{12}^{(12)}=65x^4+5552x^3-6446x^2+5552x+65, \tag{7.431}$$

$$P_{13}^{(12)} = 77x^4 - 200x^3 + 270x^2 - 200x + 77, \tag{7.432}$$
$$P_{14}^{(12)} = 80x^4 - 215x^3 + 276x^2 - 215x + 80, \tag{7.433}$$
$$P_{15}^{(12)} = 80x^4 - 209x^3 + 264x^2 - 209x + 80, \tag{7.434}$$
$$P_{16}^{(12)} = 83x^4 - 218x^3 + 258x^2 - 218x + 83, \tag{7.435}$$
$$P_{17}^{(12)} = 87x^4 - 92x^3 + 298x^2 - 92x + 87, \tag{7.436}$$
$$P_{18}^{(12)} = 105x^4 - 182x^3 + 334x^2 - 182x + 105, \tag{7.437}$$
$$P_{19}^{(12)} = 135x^4 + 110x^3 - 98x^2 + 110x + 135, \tag{7.438}$$
$$P_{20}^{(12)} = 1709x^4 + 1708x^3 - 690x^2 + 716x + 829, \tag{7.439}$$
$$P_{21}^{(12)} = 2415x^4 - 3158x^3 + 6274x^2 - 3158x + 2415, \tag{7.440}$$
$$P_{22}^{(12)} = x^5 + 25x^4 - 8x^3 + 16x^2 - 9x + 7, \tag{7.441}$$
$$P_{23}^{(12)} = 13x^5 - 43x^4 + 40x^3 - 32x^2 + 59x - 5, \tag{7.442}$$
$$P_{24}^{(12)} = 14x^5 - 133x^4 + 31x^3 - 65x^2 + 3x - 10, \tag{7.443}$$
$$P_{25}^{(12)} = 15x^5 - 177x^4 + 50x^3 - 94x^2 + 27x - 21, \tag{7.444}$$
$$P_{26}^{(12)} = 28x^5 - 42x^4 + 35x^3 + 29x^2 - 117x + 55, \tag{7.445}$$
$$P_{27}^{(12)} = 29x^5 - 33x^4 + 50x^3 - 34x^2 + 65x - 13, \tag{7.446}$$
$$P_{28}^{(12)} = 391x^6 + 502x^5 - 1151x^4 - 2236x^3 - 1151x^2 + 502x + 391, \tag{7.447}$$
$$P_{29}^{(12)} = 10x^7 - 21x^6 - 35x^5 + 158x^4 + 236x^3 + 223x^2 + 13x - 40, \tag{7.448}$$
$$P_{30}^{(12)} = 4647x^7 + 4667x^6 - 12281x^5 - 14941x^4 - 819x^3 + 4761x^2 - 4347x - 3447, \tag{7.449}$$
$$P_{31}^{(12)} = 27x^8 - 92x^7 - 404x^6 - 356x^5 + 66x^4 + 28x^3 + 76x^2 + 36x - 21, \tag{7.450}$$
$$P_{32}^{(12)} = 31x^8 - 116x^7 - 564x^6 - 524x^5 + 122x^4 - 140x^3 - 84x^2 + 12x - 17, \tag{7.451}$$
$$P_{33}^{(12)} = 60x^8 - 441x^7 - 3520x^6 + 4719x^5 + 4313x^4 + 4773x^3 - 3358x^2 - 387x + 33, \tag{7.452}$$
$$\begin{aligned} P_{34}^{(12)} &= 208x^{10} + 10715x^9 - 21650x^8 - 31368x^7 + 25186x^6 + 85658x^5 + 25186x^4 \\ &- 31368x^3 - 21650x^2 + 10715x + 208. \end{aligned} \tag{7.453}$$

The first expansion coefficients of $F_{A,2,i}^{(0)}, i = 1...3$ are given by

$$\begin{aligned} F_{A,2,1}^{(0)}(x) &= -\frac{82929376}{18225} + \frac{264976}{81y^2} - \frac{264976}{81y} + \frac{53768196023y^2}{53581500} \\ &+ \frac{53768196023y^3}{53581500} + \frac{42082197145871y^4}{54010152000} + \frac{14983026350279y^5}{27005076000} + O(y^6) \end{aligned} \tag{7.454}$$

$$\begin{aligned} F_{A,2,2}^{(0)}(x) &= \frac{81870064}{18225} + \frac{11132}{27y^2} - \frac{11132}{27y} - \frac{7224542428y^2}{7441875} \\ &- \frac{7224542428y^3}{7441875} - \frac{164328135367y^4}{218791125} - \frac{581273616754y^5}{1093955625} + O(y^6) \end{aligned} \tag{7.455}$$

$$\begin{aligned} F_{A,2,3}^{(0)}(x) &= \frac{6230776}{6075} - \frac{1024}{27y^2} + \frac{1024}{27y} - \frac{22054357y^2}{11907000} \\ &- \frac{22054357y^3}{11907000} - \frac{12191383321y^4}{266716800} - \frac{19907290871y^5}{222264000} + O(y^6). \end{aligned} \tag{7.456}$$

The $1/y^k$ behaviour of these expressions is cancelled by corresponding terms of other contributions.

The renormalization of the three–loop massive form factors is performed in the same way as in earlier calculations, cf. [9, 12]. We use a mixed scheme. The heavy quark mass and wave function have been renormalized in the on-shell (OS) renormalization scheme, while the strong coupling constant is renormalized in the $\overline{\text{MS}}$ scheme, where we set the universal factor $S_\varepsilon = \exp(-\varepsilon(\gamma_E - \ln(4\pi))$ for each loop order to one at the end of the calculation. The required renormalization constants are available and are denoted by $Z_{m,\text{OS}}$ [38, 41, 42, 112, 113], $Z_{2,\text{OS}}$ [38, 42, 112, 114] and Z_{a_s} [115–121] for the heavy quark mass, wave function and strong coupling constant, respectively. The renormalization of the heavy-quark wave function and the strong coupling constant are multiplicative, while the renormalization of massive fermion lines has been taken care of by properly considering the counter terms. Pseudoscalar and axialvector form factors are related by a Ward identity. We explicitly verified our results fulfill this relation.

7.5 Numerical Results

We present now numerical results for the ε^0 parts of the different unrenormalized form factors. For comparison the functions $F_{C,i}^{(0)}, i = 1...3$ are shown using their first 20, 50, 100. 200 and 500 expansion coefficients. In Figures 1–7 we show the results in the Euclidean region $0 < x < 1$. In Figures 8–13 the results in the region below threshold $(0 < z < 4)$ are shown, expanding around $z = 0$. At small values of x the form factor F_1 has logarithmic singularities both in its n_h (Figure 1, left panel) and n_h^2 parts (Figure 1, right panel). Here and in the following we also illustrate taking into account a rising number of terms $n = 20$ to 500 from the non–first order factorizing contributions to illustrate the degree of convergence.

The vector form factor F_2, cf. Figure 2 is proportional to x, damping out further $\ln^k(x)$ contributions. Despite taking only 500 expansion coefficients, one obtains the correct representation in the whole x range. In Figure 3 we show the behaviour of the axialvector form factor $F_{A,1}$ under the same conditions as in Figure 1, and for the form factor $F_{A,2}$ in Figure 4 similar to those in Figure 2. In Figure 5 we illustrate for the ratio of the vector form factor F_1 evaluated of n terms of the non–first order factorizing contributions for $n = 20, 50, 100, 200$ and 500 to the case of $n = 2000$ to see the relative convergence both for the n_h^2 and n_h contribution, which approves towards $x \to 0$. However, the complete logarithmic behaviour cannot be resembled by this representation.

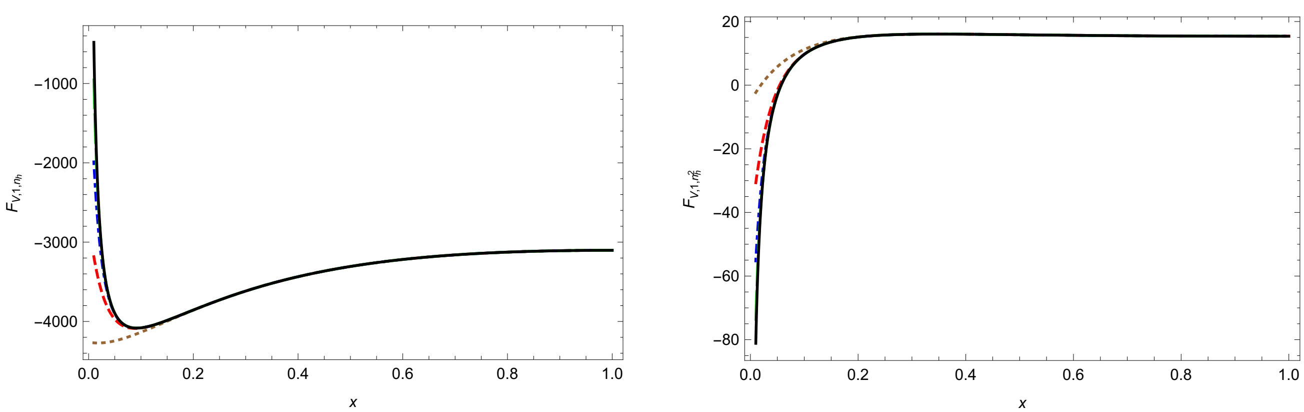

Figure 1: Vector form factor $F_{V,1}$: left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$, the approximation with 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

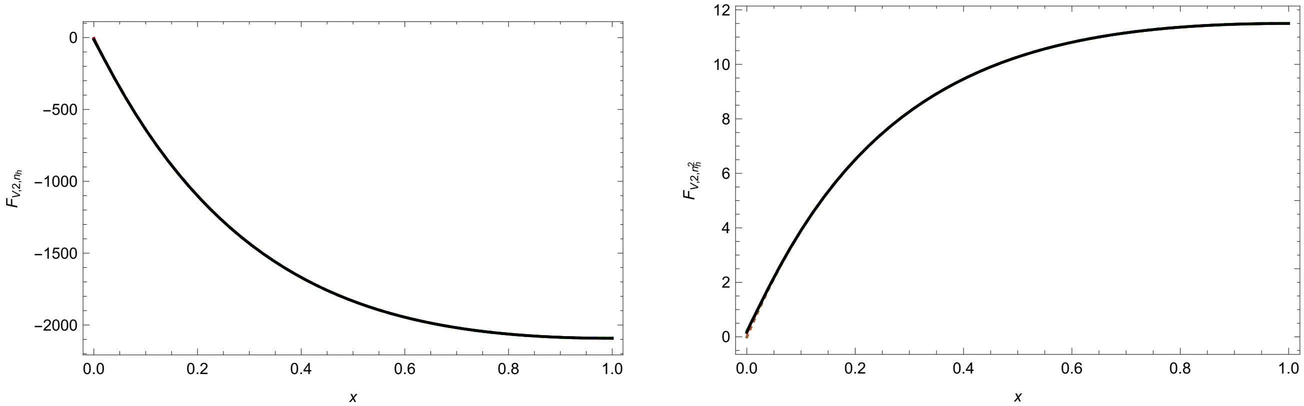

Figure 2: Vector form factor $F_{V,2}$: left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$, the approximation with 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

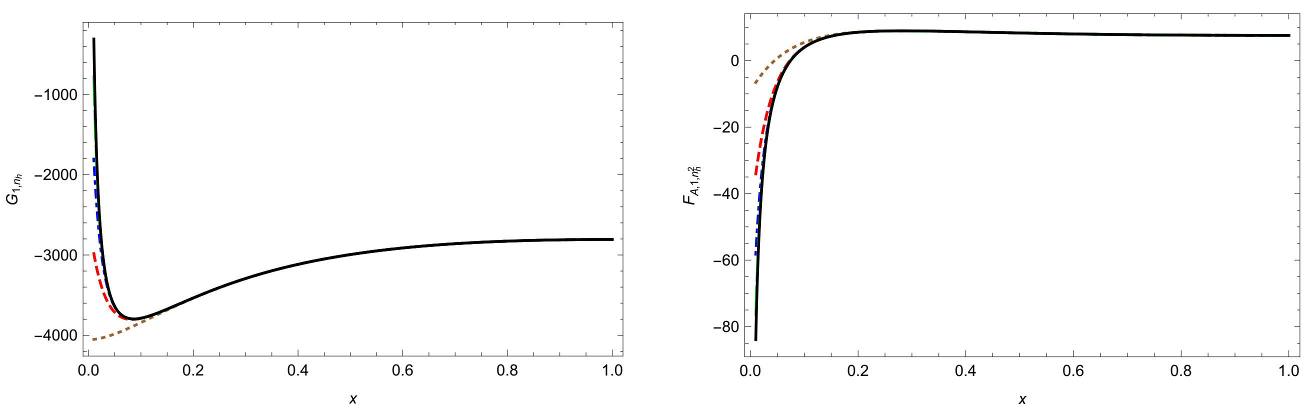

Figure 3: Axial vector form factor $F_{A,1}$: left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$, the approximation with 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

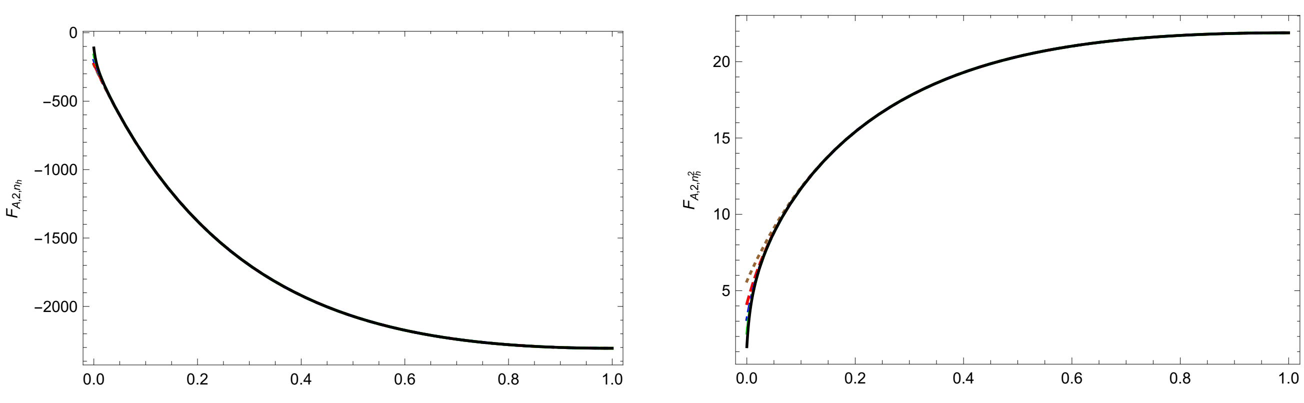

Figure 4: Axial vector form factor $F_{A,2}$: left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$, the approximation with 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

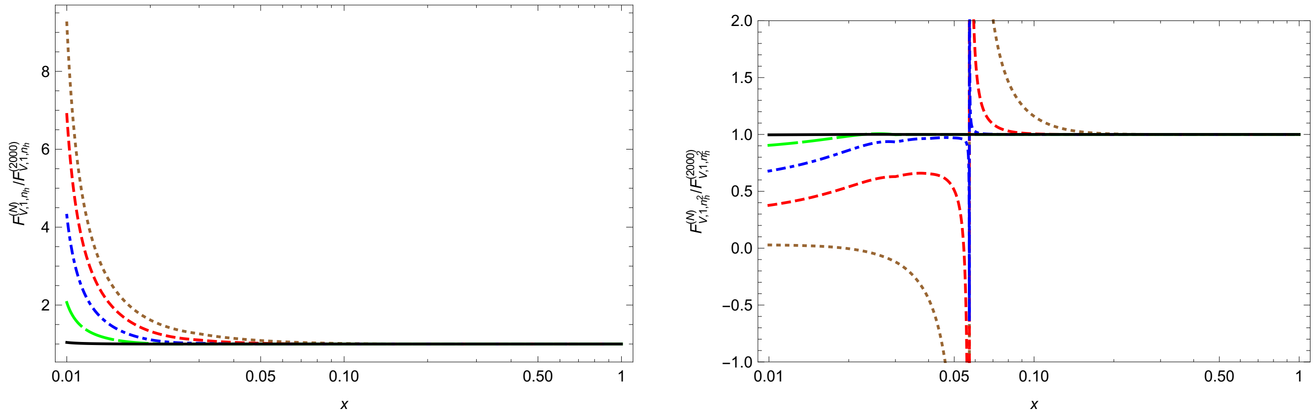

Figure 5: Ratios of the approximations with 20, 50, 100, 200, 500 terms and our best approximation using 2000 terms for the vector form factor $F_{V,1}$. Left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$. The ratio using 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

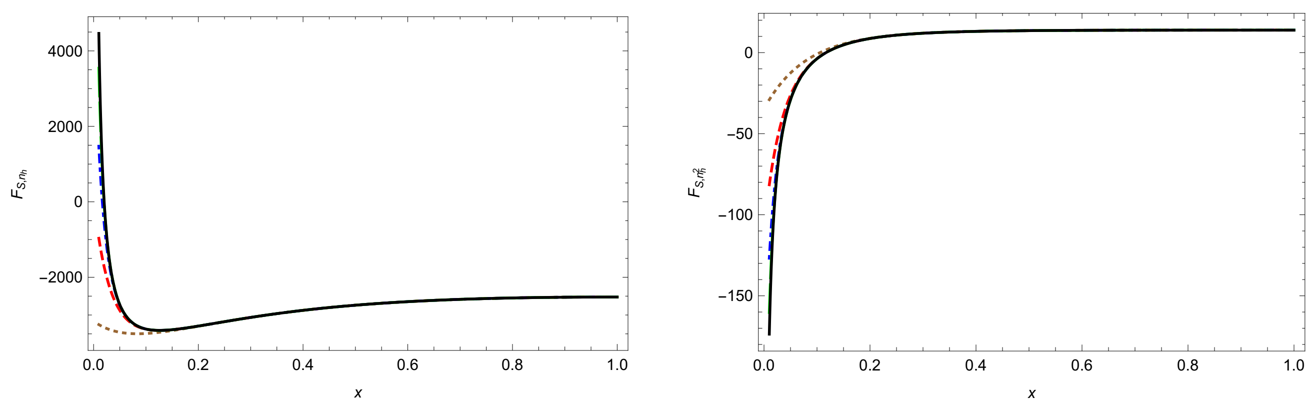

Figure 6: small Scalar form factor F_S: left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$, the approximation with 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

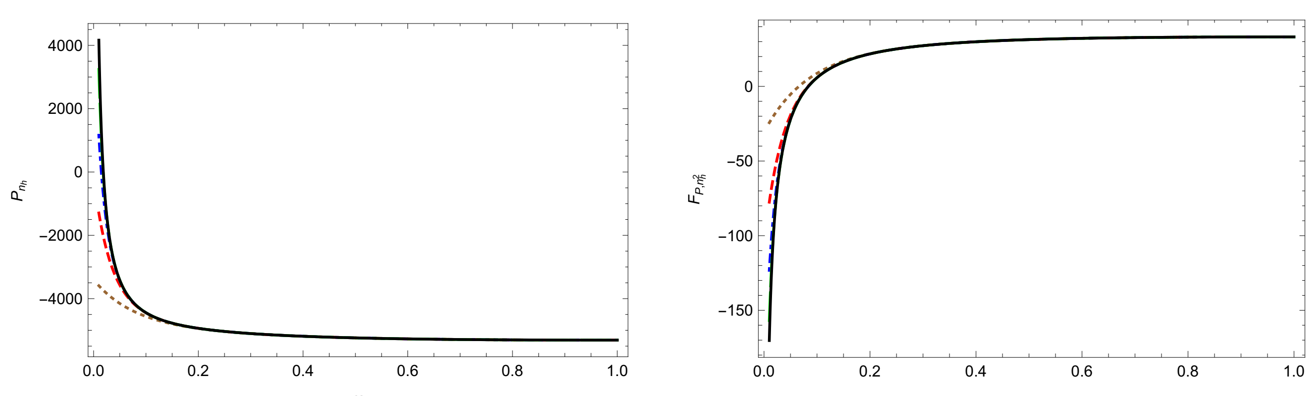

Figure 7: The pseudoscalar form factor F_P: left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$, the approximation with 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

The contributions to the scalar form Factor F_S are shown in Figure 6. Its x–dependence is similar to the one of the vector form factor. The pseudoscalar form factor is illustrated in Figure 7. It has similar behaviour as the axialvector form factor.

Now we turn to the illustration of the threshold expansion of the form factors. All form factors are multiplied by the factor $(4-z)^{3/2}$ for the n_h terms and by $(4-z)$ for the n_h^2 terms for convenience, with $z = q^2/m^2$ for the vector- (Figure 7, 8), axialvector- (Figures 9, 10), scalar- (Figure 11) and pseudoscalar form factor (Figure 12). In large z region differences due to the number of terms used in the expansion of the non–first order factorizing contributions are seen.

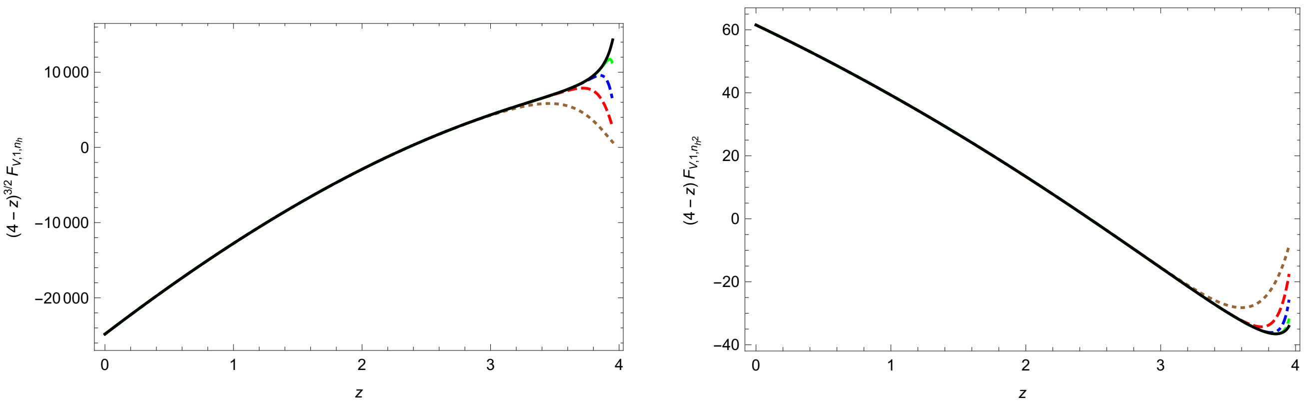

Figure 8: The vector form factor $F_{V,1}$ in the threshold region as a function of z: left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$, the approximation with 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

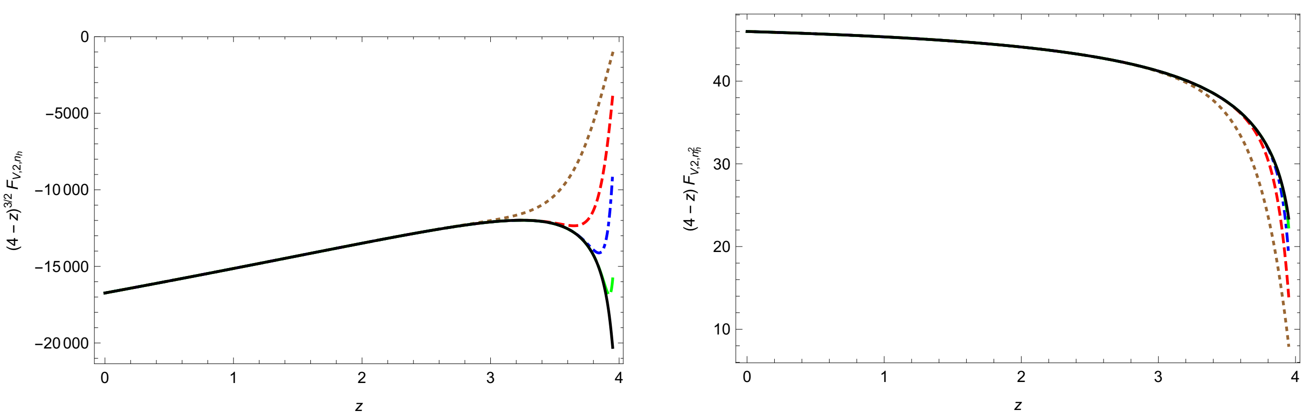

Figure 9: The vector form factor $F_{V,2}$ in the threshold region as a function of z: left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$, the approximation with 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

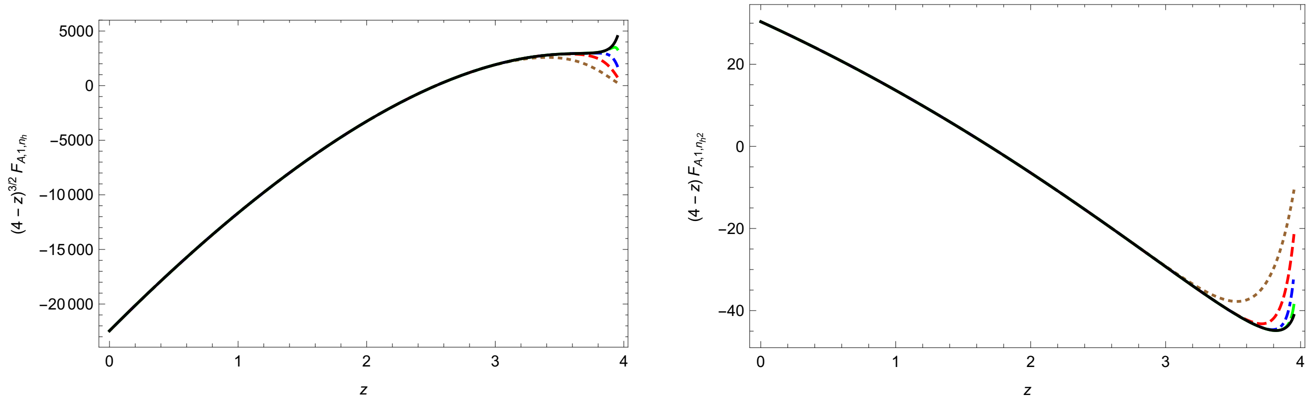

Figure 10: small The axialvector form factor $F_{A,1}$ in the threshold region as a function of z: left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$, the approximation with 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

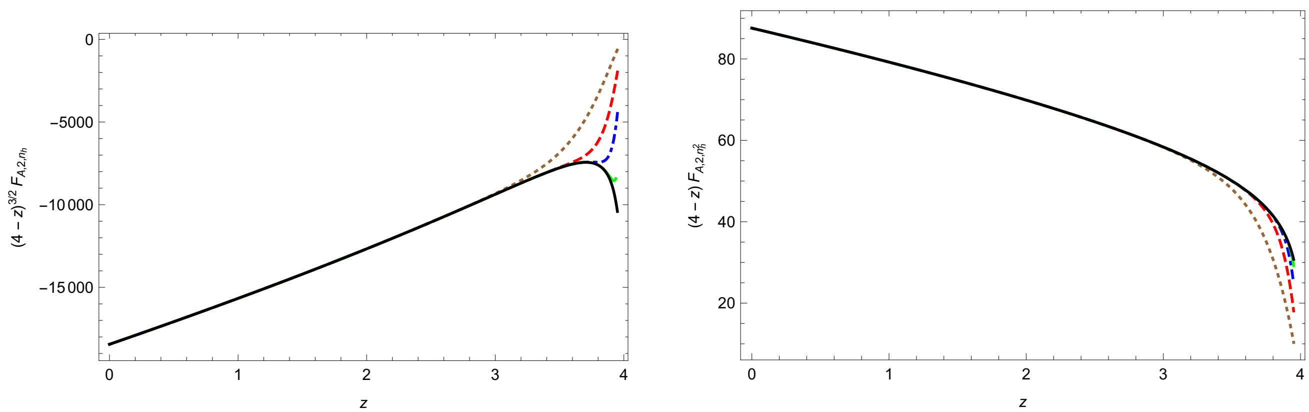

Figure 11: The axialvector form factor $F_{A,2}$ in the threshold region as a function of z: left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$, the approximation with 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

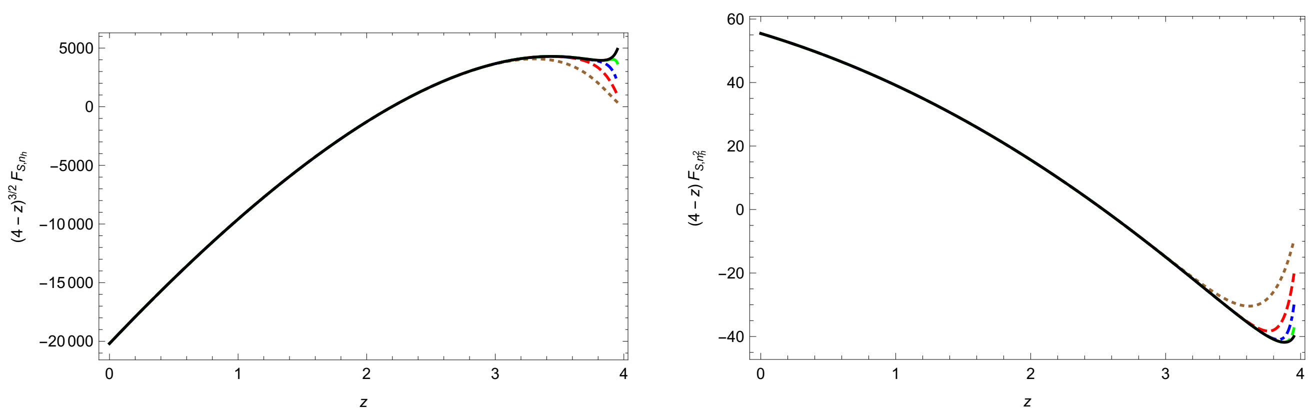

Figure 12: The scalar form factor F_S in the threshold region as a function of z: left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$, the approximation with 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

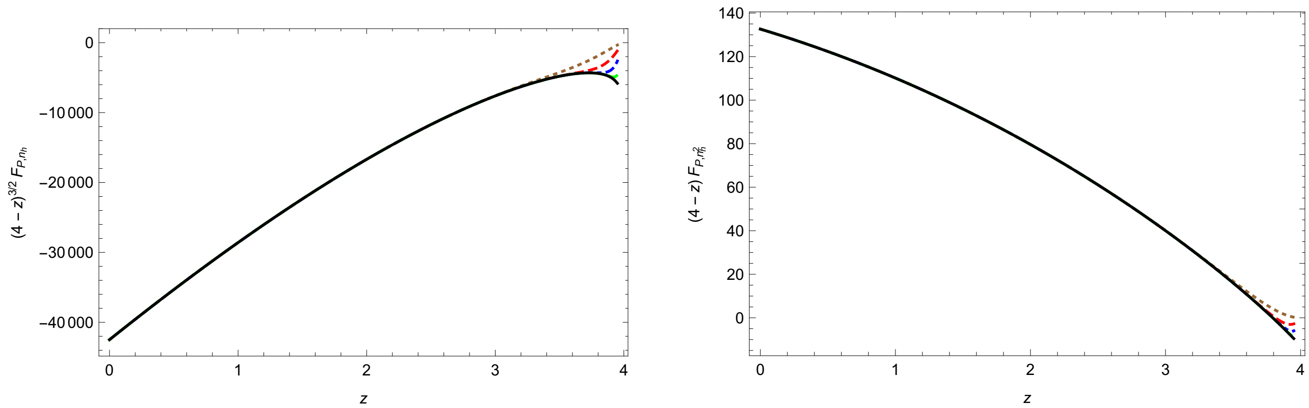

Figure 13: The pseudoscalar form factor F_P in the threshold region as a function of z: left $\varepsilon^0 n_h^1$, right $\varepsilon^0 n_h^2$, the approximation with 20, 50, 100, 200, 500 terms is shown in brown, red, blue, green and black, respectively.

8 Conclusions

We have calculated the n_h contributions to the massive three–loop form factors for vector-, axialvector-, scalar- and pseudoscalar currents. The contributing Feynman integrals were reduced to master integrals using the package `Crusher` [23]. The calculation of the analytic expansion coefficients of the master integrals in x required a new way of decoupling of the differential equations provided by the IBP–relations w.r.t. their optimal expansion in the dimensional parameter ε. Otherwise, it would have been very demanding to provide the required initial values. Here we used the method of arbitrary high moments [15], partly requiring to calculate 8000 moments. The recursions were derived using the guessing method [26]. The recursions for the pole terms and a part of the contributions of $O(\varepsilon^0)$ are first order factorizing and one obtains representations in terms of harmonic polylogarithms in the variable x. The recursions were solved using the packages `Sigma` [24,28] and `EvaluateMultisums` and `SumProduction` [125,126]. The resulting infinite sums were then converted into harmonic polylogarithms using the package `HarmonicSums` [59–67]. For some color-ζ contributions at $O(\varepsilon^0)$ non–first order factorizing parts are contained in the recurrences. Here the largest of the remaining recurrences are of order $\mathsf{o} = 15$, resulting from of original recurrences of order up to $\mathsf{o} = 55$ and degree $\mathsf{d} = 1324$. For those we have have obtained analytic polynomial expansions in x of degree $\mathsf{d} = 2000$, which can be extended in case needed. They already allow for precise numerical representations in a wide range of x. However, in the case of logarithmic divergences for $x \to 0$ these representations diverge. The analytic solution of the non–first order factorizing recurrences in terms of special higher functions still needs to be performed in the future. We also considered the leading color contributions for the scalar form factor as an example to see whether here simplifications can be obtained. This is indeed the case since the most involved of the remaining non–first order factorizing recurrences is only of order $\mathsf{o} = 5$, stemming from an original recurrence of of order $\mathsf{o} = 46$ and degree $\mathsf{d} = 901$. Complete first order factorization can, however, not be obtained in this case.

A Initial Values to High Order in ε

Boundary values for the master integrals appearing in the present calculation can be obtained using three loop propagator integrals as have been dealt with in Ref. [38] to orders in the dimensional parameter up to $O(\varepsilon^3)$ and lower. Depending on the method of uncoupling of the associated differential equations terms up to $O(\varepsilon^9)$, i.e. in total 13 orders in ε, may be necessary. For some of the integrals all-order in ε representations exist, [25, 112, 122–124]. For the integrals $I_9, I_{11}, I_{12}, I_{15}$ and I_{16} of [38,123,124] we have derived the corresponding representations. They are thoroughly given in terms of multiple zeta values, cf. [25]. The calculation of ε-expansions at such depth is very time consuming, despite the fact that the techniques to be used are rather standard ones available in the packages `Sigma` [24, 28], `EvaluateMultisums` and `SumProduction` [125, 126] together with the asymptotic expansions of harmonic sums of the package `HarmonicSums`. In the final results all constants of higher transcendentality will be cancelled. Given the associated calculational effort, it is of advantage having decoupling methods, in which only initial values requiring an expansion in ε to a far lower order in ε is needed. This has been possible by the algorithm described in Section 5.

It is interesting to see that all constants spanning the MZVs [25] are contributing to this integral.

weight	# basis elements
1	1
2	1
3	1
4	1
5	2
6	2
7	3
8	5
9	8
10	11
11	18

Table 3: Number of basis elements by weight.

These are (together with their approximate numerical values)

$$\ln(2) \approx 0.69314718055994530942 \tag{A.1}$$

$$\zeta_2 \approx 1.64493406684822643647 \tag{A.2}$$

$$\zeta_3 \approx 1.20205690315959428540 \tag{A.3}$$

$$\mathrm{Li}_4\left(\frac{1}{2}\right) \approx 0.51747906167389938633 \tag{A.4}$$

$$\zeta_5 \approx 1.03692775514336992633 \tag{A.5}$$

$$\mathrm{Li}_5\left(\frac{1}{2}\right) \approx 0.50840057924226870746 \tag{A.6}$$

$$s_6 = S_{-5,-1}(\infty) \approx 0.98744142640329971377 \tag{A.7}$$

$$\mathrm{Li}_6\left(\frac{1}{2}\right) \approx 0.50409539780398855069 \tag{A.8}$$

$$s_{7a} = S_{-5,1,1}(\infty) \approx -0.95296007575629860341 \quad (A.9)$$
$$s_{7b} = S_{5,-1,-1}(\infty) \approx 1.02912126296432453422 \quad (A.10)$$
$$\zeta_7 \approx 1.00834927738192282684 \quad (A.11)$$
$$\mathrm{Li}_7\left(\frac{1}{2}\right) \approx 0.50201456332470849457 \quad (A.12)$$
$$s_{8a} = S_{5,3}(\infty) \approx 1.04178502918279188339 \quad (A.13)$$
$$s_{8b} = S_{-7,-1}(\infty) \approx 0.99644774839783766598 \quad (A.14)$$
$$s_{8c} = S_{-5,-1,-1,-1}(\infty) \approx 0.98396667382173367092 \quad (A.15)$$
$$s_{8d} = S_{-5,-1,1,1}(\infty) \approx 0.99996261346268344770 \quad (A.16)$$
$$\mathrm{Li}_8\left(\frac{1}{2}\right) \approx 0.50099665909705191056 \quad (A.17)$$
$$\zeta_9 \approx 1.00200839282608221442 \quad (A.18)$$
$$\mathrm{Li}_9\left(\frac{1}{2}\right) \approx 0.50049488810595361004 \quad (A.19)$$
$$s_{9a} = S_{7,-1,-1}(\infty) \approx 1.00640196269235635901 \quad (A.20)$$
$$s_{9b} = S_{-7,-1,1}(\infty) \approx 0.99842952512288855440 \quad (A.21)$$
$$s_{9c} = S_{-6,-2,-1}(\infty) \approx -0.98747515763691535588 \quad (A.22)$$
$$s_{9d} = S_{-5,-1,1,1,1}(\infty) \approx 1.00219817413397743630 \quad (A.23)$$
$$s_{9e} = S_{-5,-1,-1,-1,1}(\infty) \approx 0.98591171955244547262 \quad (A.24)$$
$$s_{9f} = S_{-5,-1,-1,1,-1}(\infty) \approx 0.97848117128116624248 \quad (A.25)$$
$$(A.26)$$
$$\mathrm{Li}_{10}\left(\frac{1}{2}\right) \approx 0.50024632060600677501 \quad (A.27)$$
$$h_{313111} \approx 0.00000059060152818192 \quad (A.28)$$
$$h_{331111} \approx 0.00000040503762995428 \quad (A.29)$$
$$h_{3331} \approx 0.00003761590257651408 \quad (A.30)$$
$$h_{511111} \approx 0.00000030113797559813 \quad (A.31)$$
$$h_{5113} \approx 0.00008812461415599555 \quad (A.32)$$
$$h_{5131} \approx 0.00002330610214579640 \quad (A.33)$$
$$h_{5311} \approx 0.00001097851878750415 \quad (A.34)$$
$$h_{7111} \approx 0.00000654709718938270 \quad (A.35)$$
$$h_{73} \approx 0.00735713658369574885 \quad (A.36)$$
$$h_{91} \approx 0.00188302419181393899 \quad (A.37)$$
$$\zeta_{11} \approx 1.00049418860411946456 \quad (A.38)$$
$$\mathrm{Li}_{11}\left(\frac{1}{2}\right) \approx 0.50012279152986795519 \quad (A.39)$$
$$h_{3113111} \approx 0.00000004259335538990 \quad (A.40)$$
$$h_{3131111} \approx 0.00000002903877500717 \quad (A.41)$$
$$h_{3311111} \approx 0.00000002146685479915 \quad (A.42)$$
$$h_{33131} \approx 0.00000203422625738433 \quad (A.43)$$
$$h_{33311} \approx 0.00000095250833508071 \quad (A.44)$$

$$h_{5111111} \approx 0.0000001677568330189 \quad (A.45)$$
$$h_{51113} \approx 0.0000053599147863424 6 \quad (A.46)$$
$$h_{51131} \approx 0.0000014145611151887 1 \quad (A.47)$$
$$h_{51311} \approx 0.0000006648923825555 3 \quad (A.48)$$
$$h_{53111} \approx 0.0000003956433293178 5 \quad (A.49)$$
$$h_{533} \approx 0.0003936864988739547 1 \quad (A.50)$$
$$h_{551} \approx 0.0001010676637705645 2 \quad (A.51)$$
$$h_{71111} \approx 0.0000002678006366586 0 \quad (A.52)$$
$$h_{731} \approx 0.0000482128148540712 0 \quad (A.53)$$
$$h_{713} \approx 0.0001864966014630654 0 \quad (A.54)$$
$$h_{911} \approx 0.0000221903642577010 3, \quad (A.55)$$

with

$$h_{abcd} \equiv H_{-a,-b,-c,-d}(1) \quad (A.56)$$

in the (collected) notation as e.g. $H_{-3,-1}(1) \equiv H_{0,0,-1,-1,}(1)$. Up to weight $\mathtt{w} = 9$ we follow the representation of `summer.h`. Earlier a similar representation has been available for the weights $\mathtt{w} = 10$–12. These files do not exist anymore [127]. Therefore we change to the basis which is used in the package `HarmonicSums.m`, where also the the respective constant files are now available[7] The MZV data mine used another basis for the HPLs at argument one. This basis is of course equivalent. We used the values ζ_{2k+1} and $\mathrm{Li}_k\left(\frac{1}{2}\right)$ as basis elements, through which one of the high weight HPLs is replaced.

$$\begin{aligned}
\mathrm{Li}_{10}\left(\frac{1}{2}\right) &= \frac{1}{2}h_{31111111} - \frac{1}{4}h_{511111} + \frac{1}{2}h_{7111} - \frac{17}{8}h_{91} - \mathrm{Li}_9\left(\frac{1}{2}\right)\ln(2) - \frac{1}{2}\mathrm{Li}_8\left(\frac{1}{2}\right)\ln^2(2) \\
&\quad -\frac{1}{6}\mathrm{Li}_7\left(\frac{1}{2}\right)\ln^3(2) - \frac{1}{24}\mathrm{Li}_6\left(\frac{1}{2}\right)\ln^4(2) - \frac{1}{120}\mathrm{Li}_5\left(\frac{1}{2}\right)\ln^5(2) - \frac{1}{120}\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^6(2) \\
&\quad -\frac{1}{43200}\ln^{10}(2) + \frac{1}{11520}\ln^8(2)\zeta_2 - \frac{789}{8800}\zeta_2^5 - \frac{1}{5760}\ln^7(2)\zeta_3 + \frac{27}{160}\zeta_2^2\zeta_3^2 + \frac{3}{8}\zeta_2\zeta_3\zeta_5 \\
&\quad +\frac{3}{16}\zeta_5^2 + \frac{3}{8}\zeta_3\zeta_7 \qquad (A.57)\\
\mathrm{Li}_{11}\left(\frac{1}{2}\right) &= \frac{1}{2}h_{311111111} - \frac{1}{4}h_{5111111} + \frac{1}{2}h_{71111} - \frac{17}{8}h_{911} - \mathrm{Li}_1\left(\frac{1}{2}\right)\ln(2) - \frac{1}{2}\mathrm{Li}_9\left(\frac{1}{2}\right)\ln^2(2) \\
&\quad -\frac{1}{6}\mathrm{Li}_8\left(\frac{1}{2}\right)\ln^3(2) - \frac{1}{24}\mathrm{Li}_7\left(\frac{1}{2}\right)\ln^4(2) - \frac{1}{120}\mathrm{Li}_6\left(\frac{1}{2}\right)\ln^5(2) - \frac{1}{720}\mathrm{Li}_5\left(\frac{1}{2}\right)\ln^6(2) \\
&\quad -\frac{1}{5040}\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^7(2) - \frac{1}{332640}\ln^{11}(2) + \frac{1}{90720}\ln^9(2)\zeta_2 - \frac{1}{46080}\ln^8(2)\zeta_3 \\
&\quad -\frac{2533}{5600}\zeta_2^4\zeta_3 + \frac{5}{48}\zeta_2\zeta_3^3 - \frac{99}{112}\zeta_2^3\zeta_5 + \frac{3}{16}\zeta_3^2\zeta_5 - \frac{57}{40}\zeta_2^2\zeta_7 - \frac{5}{3}\zeta_2\zeta_9 + \frac{30945}{2048}\zeta_{11} \qquad (A.58)
\end{aligned}$$

The integrals I_9 and I_{11} are given by

$$I_9 = -\frac{2}{3\varepsilon^3} - \frac{10}{3\varepsilon^2} + \left(-\frac{26}{3} - 2\zeta_2\right)\frac{1}{\varepsilon} - 2 - \frac{16}{3}\zeta_3 - 22\zeta_2 + \varepsilon\Bigg(\frac{398}{3} - \frac{52}{5}\zeta_2^2 - \frac{248}{3}\zeta_3 + 96\ln(2)\zeta_2$$

[7] We thank J. Ablinger for making these files available.

$$-146\zeta_2\Bigg) + \varepsilon^2\Bigg(1038 - 96\zeta_5 - 16\zeta_2\zeta_3 - 512\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{64\ln^4(2)}{3} - 256\ln^2(2)\zeta_2 + \frac{108}{5}\zeta_2^2$$

$$-\frac{1888}{3}\zeta_3 + 960\ln(2)\zeta_2 - 774\zeta_2\Bigg) + \varepsilon^3\Bigg(\frac{17470}{3} - \frac{2944}{35}\zeta_2^3 - \frac{64}{3}\zeta_3^2 + 2496\zeta_5 - 4096\mathrm{Li}_5\left(\frac{1}{2}\right)$$

$$+\frac{512\ln^5(2)}{15} + \frac{2048}{3}\ln^3(2)\zeta_2 - \frac{2944}{5}\ln(2)\zeta_2^2 + 16\zeta_2\zeta_3 - 5120\mathrm{Li}_4\left(\frac{1}{2}\right) - 2560\ln^2(2)\zeta_2$$

$$-\frac{640\ln^4(2)}{3} + \frac{3004}{5}\zeta_2^2 - 3600\zeta_3 + 6144\ln(2)\zeta_2 - 3634\zeta_2\Bigg) + \varepsilon^4\Bigg(\frac{85562}{3} - 1328\zeta_7$$

$$-288\zeta_2\zeta_5 - \frac{416}{5}\zeta_2^2\zeta_3 - 12288s_6 - 32768\mathrm{Li}_6\left(\frac{1}{2}\right) - 1536\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2 - \frac{2048\ln^6(2)}{45}$$

$$-\frac{4288}{3}\ln^4(2)\zeta_2 + \frac{13696}{5}\ln^2(2)\zeta_2^2 - 2112\ln(2)\zeta_2\zeta_3 + \frac{19728}{5}\zeta_2^3 + \frac{14680}{3}\zeta_3^2 - 40960\mathrm{Li}_5\left(\frac{1}{2}\right)$$

$$+\frac{1024\ln^5(2)}{3} + \frac{20480}{3}\ln^3(2)\zeta_2 - 5888\ln(2)\zeta_2^2 + 28512\zeta_5 + 752\zeta_2\zeta_3 - 32768\mathrm{Li}_4\left(\frac{1}{2}\right)$$

$$-\frac{4096\ln^4(2)}{3} - 16384\ln^2(2)\zeta_2 + \frac{23396}{5}\zeta_2^2 - \frac{53264}{3}\zeta_3 + 32256\ln(2)\zeta_2 - 15894\zeta_2\Bigg)$$

$$+\varepsilon^5\Bigg(\frac{389806}{3} - \frac{133968}{175}\zeta_2^4 - 64\zeta_2\zeta_3^2 - 768\zeta_3\zeta_5 - \frac{325632s_{7a}}{7} + \frac{362496s_{7b}}{7} - 262144\mathrm{Li}_7\left(\frac{1}{2}\right)$$

$$-12288\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_2 - 4096\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_3 + \frac{325632\ln(2)s_6}{7} + \frac{16384\ln^7(2)}{315} + \frac{34304}{15}\ln^5(2)\zeta_2$$

$$-\frac{512}{3}\ln^4(2)\zeta_3 - \frac{109568}{15}\ln^3(2)\zeta_2^2 + 9472\ln^2(2)\zeta_2\zeta_3 - 95232\ln^2(2)\zeta_5 - \frac{905344}{35}\ln(2)\zeta_2^3$$

$$-\frac{407040}{7}\ln(2)\zeta_3^2 + \frac{3311468}{7}\zeta_7 - \frac{421296}{35}\zeta_2^2\zeta_3 - \frac{1016280}{7}\zeta_2\zeta_5 - 122880s_6 - 327680\mathrm{Li}_6\left(\frac{1}{2}\right)$$

$$-15360\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2 - \frac{4096\ln^6(2)}{9} - \frac{42880}{3}\ln^4(2)\zeta_2 + 27392\ln^2(2)\zeta_2^2 - 21120\ln(2)\zeta_2\zeta_3$$

$$+\frac{1489888}{35}\zeta_2^3 + \frac{149168}{3}\zeta_3^2 - 262144\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{32768\ln^5(2)}{15} + \frac{131072}{3}\ln^3(2)\zeta_2$$

$$-\frac{188416}{5}\ln(2)\zeta_2^2 + 190176\zeta_5 + 6096\zeta_2\zeta_3 - 172032\mathrm{Li}_4\left(\frac{1}{2}\right) - 7168\ln^4(2) - 86016\ln^2(2)\zeta_2$$

$$+\frac{133996}{5}\zeta_2^2 - 80400\zeta_3 + 152064\ln(2)\zeta_2 - 66626\zeta_2\Bigg) + \varepsilon^6\Bigg(566958 - \frac{160960}{9}\zeta_9 - 3984\zeta_2\zeta_7$$

$$-\frac{512}{9}\zeta_3^3 - \frac{23552}{35}\zeta_2^3\zeta_3 - \frac{7488}{5}\zeta_2^2\zeta_5 - \frac{21695376s_{8a}}{35} - \frac{28157952s_{8b}}{7} - \frac{2899968s_{8c}}{7}$$

$$-\frac{2605056s_{8d}}{7} + \frac{1449984\ln^2(2)s_6}{7} + \frac{7704576}{7}s_6\zeta_2 - 2097152\mathrm{Li}_8\left(\frac{1}{2}\right) - 98304\mathrm{Li}_6\left(\frac{1}{2}\right)\zeta_2$$

$$+\frac{4980736}{7}\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_3 - \frac{2884608}{35}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2^2 - \frac{16384\ln^8(2)}{315} - \frac{137216}{45}\ln^6(2)\zeta_2$$

$$-\frac{622592}{105}\ln^5(2)\zeta_3 + \frac{1173376}{105}\ln^4(2)\zeta_2^2 + \frac{772096}{21}\ln^3(2)\zeta_2\zeta_3 + \frac{468736}{35}\ln^2(2)\zeta_2^3$$

$$+\frac{407808}{7}\ln^2(2)\zeta_3^2+\frac{16493824}{35}\ln(2)\zeta_2^2\zeta_3+\frac{8558784}{7}\ln(2)\zeta_2\zeta_5+\frac{942502872\zeta_2^4}{6125}+\frac{3316480}{7}$$
$$\times\zeta_2\zeta_3^2-\frac{3194496}{7}\zeta_3\zeta_5-\frac{3256320s_{7a}}{7}+\frac{3624960s_{7b}}{7}+\frac{3256320\ln(2)s_6}{7}-2621440\text{Li}_7\left(\frac{1}{2}\right)$$
$$-122880\text{Li}_5\left(\frac{1}{2}\right)\zeta_2-40960\text{Li}_4\left(\frac{1}{2}\right)\zeta_3+\frac{32768\ln^7(2)}{63}+\frac{68608}{3}\ln^5(2)\zeta_2-\frac{5120}{3}\ln^4(2)\zeta_3$$
$$-\frac{219136}{3}\ln^3(2)\zeta_2^2+94720\ln^2(2)\zeta_2\zeta_3-952320\ln^2(2)\zeta_5-\frac{1810688}{7}\ln(2)\zeta_2^3-\frac{4070400}{7}$$
$$\times\ln(2)\zeta_3^2+\frac{33458632}{7}\zeta_7-\frac{10088208}{7}\zeta_2\zeta_5-\frac{4105216}{35}\zeta_2^2\zeta_3-786432s_6-2097152\text{Li}_6\left(\frac{1}{2}\right)$$
$$-98304\text{Li}_4\left(\frac{1}{2}\right)\zeta_2-\frac{131072\ln^6(2)}{45}-\frac{274432}{3}\ln^4(2)\zeta_2+\frac{876544}{5}\ln^2(2)\zeta_2^2-135168$$
$$\times\ln(2)\zeta_2\zeta_3+\frac{9771392}{35}\zeta_2^3+319936\zeta_3^2-1376256\text{Li}_5\left(\frac{1}{2}\right)+\frac{57344\ln^5(2)}{5}+229376$$
$$\times\ln^3(2)\zeta_2-\frac{989184}{5}\ln(2)\zeta_2^2+35440\zeta_2\zeta_3+1019040\zeta_5-811008\text{Li}_4\left(\frac{1}{2}\right)-33792\ln^4(2)$$
$$-405504\ln^2(2)\zeta_2+\frac{663876}{5}\zeta_2^2-\frac{1037584}{3}\zeta_3+672768\ln(2)\zeta_2-272326\zeta_2\Bigg)$$
$$+\varepsilon^7\Bigg(\frac{7221278}{3}-6912\zeta_5^2-2304\zeta_2\zeta_3\zeta_5-\frac{1956032}{275}\zeta_2^5-\frac{1664}{5}\zeta_2^2\zeta_3^2-10624\zeta_3\zeta_7$$
$$-\frac{114294784s_{9a}}{7}+6750208s_{9b}+\frac{90902528s_{9c}}{21}-\frac{9240576s_{9d}}{7}-\frac{11599872s_{9e}}{7}$$
$$+\frac{143271936}{49}s_{7a}\zeta_2-\frac{206985216}{49}s_{7b}\zeta_2+\frac{144875904\ln(2)s_{8a}}{35}+\frac{181805056\ln(2)s_{8b}}{7}$$
$$-\frac{1933312\ln^3(2)s_6}{7}-\frac{183871488}{49}\ln(2)s_6\zeta_2+227696640s_6\zeta_3-16777216\text{Li}_9\left(\frac{1}{2}\right)$$
$$-786432\text{Li}_7\left(\frac{1}{2}\right)\zeta_2+\frac{39845888}{7}\text{Li}_6\left(\frac{1}{2}\right)\zeta_3-\frac{81076224}{35}\text{Li}_5\left(\frac{1}{2}\right)\zeta_2^2$$
$$-\frac{283149312}{7}\text{Li}_4\left(\frac{1}{2}\right)\zeta_5+\frac{46116864}{7}\text{Li}_4\left(\frac{1}{2}\right)\zeta_2\zeta_3+\frac{131072\ln^9(2)}{2835}+\frac{1097728}{315}\ln^7(2)\zeta_2$$
$$+\frac{2490368}{315}\ln^6(2)\zeta_3-\frac{2137088}{525}\ln^5(2)\zeta_2^2+\frac{4220416}{21}\ln^4(2)\zeta_2\zeta_3-\frac{11797888}{7}\ln^4(2)\zeta_5$$
$$-\frac{3332096}{15}\ln^3(2)\zeta_2^3-\frac{543744}{7}\ln^3(2)\zeta_3^2-\frac{93715456}{35}\ln^2(2)\zeta_2^2\zeta_3+\frac{102425088}{7}\ln^2(2)\zeta_2\zeta_5$$
$$+606208\ln^2(2)\zeta_2\zeta_3+\frac{113401856}{7}\ln^2(2)\zeta_7-\frac{70934369088\ln(2)\zeta_2^4}{6125}$$
$$+\frac{168809664}{49}\ln(2)\zeta_2\zeta_3^2-\frac{224415616}{7}\ln(2)\zeta_3\zeta_5-\frac{7007214274}{63}\zeta_9+\frac{1046207136}{245}\zeta_2^3\zeta_3$$
$$-\frac{48494416}{9}\zeta_3^3+\frac{2042564800}{49}\zeta_2^2\zeta_5+\frac{1637594676}{49}\zeta_2\zeta_7-\frac{43390752s_{8a}}{7}-\frac{281579520s_{8b}}{7}$$
$$-\frac{28999680s_{8c}}{7}-\frac{26050560s_{8d}}{7}+\frac{14499840\ln(2)^2s_6}{7}+\frac{77045760}{7}s_6\zeta_2-20971520\text{Li}_8\left(\frac{1}{2}\right)$$
$$-983040\text{Li}_6\left(\frac{1}{2}\right)\zeta_2+\frac{49807360}{7}\text{Li}_5\left(\frac{1}{2}\right)\zeta_3-\frac{5769216}{7}\text{Li}_4\left(\frac{1}{2}\right)\zeta_2^2-\frac{32768\ln^8(2)}{63}$$

$$
\begin{aligned}
&-\frac{274432}{9}\ln^6(2)\zeta_2-\frac{1245184}{21}\ln^5(2)\zeta_3+\frac{2346752}{21}\ln^4(2)\zeta_2^2+\frac{7720960}{21}\ln^3(2)\zeta_2\zeta_3\\
&+\frac{937472}{7}\ln^2(2)\zeta_2^3+\frac{4078080}{7}\ln^2(2)\zeta_3^2+\frac{85587840}{7}\ln(2)\zeta_2\zeta_5+\frac{1919703456\zeta_2^4}{1225}\\
&+\frac{33181376}{7}\zeta_2\zeta_3^2-\frac{31746048}{7}\zeta_3\zeta_5-\frac{20840448s_{7a}}{7}+\frac{23199744s_{7b}}{7}\\
&+\frac{20840448\ln(2)s_6}{7}-16777216\mathrm{Li}_7\left(\frac{1}{2}\right)-786432\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_2-262144\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_3\\
&+\frac{1048576\ln^7(2)}{315}+\frac{2195456}{15}\ln^5(2)\zeta_2-\frac{32768}{3}\ln^4(2)\zeta_3-\frac{7012352}{15}\ln^3(2)\zeta_2^2\\
&+\frac{214880784}{7}\zeta_7-\frac{5207968}{7}\zeta_2^2\zeta_3-\frac{57942016}{35}\ln(2)\zeta_2^3-\frac{64402848}{7}\zeta_2\zeta_5\\
&-4128768s_6-6094848\ln^2(2)\zeta_5+\frac{32987648}{7}\ln(2)\zeta_2^2\zeta_3-\frac{26050560}{7}\ln(2)\zeta_3^2\\
&-11010048\mathrm{Li}_6\left(\frac{1}{2}\right)-516096\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2-\frac{229376\ln^6(2)}{15}-480256\ln^4(2)\zeta_2\\
&+\frac{4601856}{5}\ln^2(2)\zeta_2^2-709632\ln(2)\zeta_2\zeta_3+\frac{51932032}{35}\zeta_2^3+\frac{5052736}{3}\zeta_3^2\\
&-6488064\mathrm{Li}_5\left(\frac{1}{2}\right)+\frac{270336\ln^5(2)}{5}+1081344\ln^3(2)\zeta_2-\frac{4663296}{5}\ln(2)\zeta_2^2+4863456\zeta_5\\
&+176976\zeta_2\zeta_3-3588096\mathrm{Li}_4\left(\frac{1}{2}\right)-149504\ln^4(2)-1794048\ln^2(2)\zeta_2+\frac{3033164}{5}\zeta_2^2\\
&-\frac{4325936}{3}\zeta_3+2863104\ln(2)\zeta_2-1097042\zeta_2\Bigg)+\varepsilon^8\Bigg(10037278+\frac{2261065728}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\\
&\times\ln(2)\zeta_5-243216\zeta_{11}-\frac{160960}{3}\zeta_2\zeta_9-\frac{103584}{5}\zeta_2^2\zeta_7-3072\zeta_3^2\zeta_5-\frac{423936}{35}\zeta_2^3\zeta_5\\
&-\frac{512}{3}\zeta_2\zeta_3^3-\frac{1071744}{175}\zeta_2^4\zeta_3+\frac{1202564511072256h_{91}}{780419}+\frac{19687710418944h_{73}}{780419}\\
&-\frac{5326405632h_{7111}}{29}-\frac{257949696h_{511111}}{17}-\frac{24978358272h_{5113}}{493}+\frac{15639478272h_{5131}}{493}\\
&+\frac{8004796416h_{5311}}{493}-\frac{2606579712h_{3331}}{493}+\frac{185597952h_{331111}}{17}+\frac{914358272\ln(2)s_{9a}}{7}\\
&-54001664\ln(2)s_{9b}-\frac{727220224\ln(2)s_{9c}}{21}+\frac{73924608\ln(2)s_{9d}}{7}+\frac{92798976\ln(2)s_{9e}}{7}\\
&-13278720\ln^2(2)s_{8a}-\frac{553385984\ln^2(2)s_{8b}}{7}+\frac{92798976\ln^2(2)s_{8c}}{7}+\frac{83361792\ln^2(2)s_{8d}}{7}\\
&-\frac{673097882736}{17255}s_{8a}\zeta_2-\frac{84473856}{7}s_{8b}\zeta_2-\frac{8699904}{7}s_{8c}\zeta_2-\frac{7815168}{7}s_{8d}\zeta_2\\
&+\frac{27787264\ln^3(2)s_{7a}}{7}-\frac{30932992\ln^3(2)s_{7b}}{7}-\frac{1200881664}{49}\ln(2)s_{7a}\zeta_2+\frac{1716781056}{49}\\
&\times\ln(2)s_{7b}\zeta_2-\frac{132420673536s_{7a}\zeta_3}{3451}+\frac{2899968s_{7b}\zeta_3}{7}-\frac{1278001152}{7}\ln(2)s_6\zeta_3\\
&+\frac{164856262656s_6\zeta_2^2}{17255}-6291456\ln^4(2)s_6-\frac{227500032}{49}\ln^2(2)s_6\zeta_2-134217728\mathrm{Li}_{10}\left(\frac{1}{2}\right)\\
&-6291456\mathrm{Li}_8\left(\frac{1}{2}\right)\zeta_2-2097152\mathrm{Li}_7\left(\frac{1}{2}\right)\zeta_3-\frac{333447168}{7}\mathrm{Li}_6\left(\frac{1}{2}\right)\ln(2)\zeta_3
\end{aligned}
$$

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\begin{aligned}
&-\frac{2555904}{5}\mathrm{Li}_6\left(\frac{1}{2}\right)\zeta_2^2-\frac{166723584}{7}\mathrm{Li}_5\left(\frac{1}{2}\right)\ln^2(2)\zeta_3+\frac{630718464}{35}\mathrm{Li}_5\left(\frac{1}{2}\right)\ln(2)\zeta_2^2\\
&-589824\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_5+\frac{14942208}{7}\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_2\zeta_3+\frac{83361792}{35}\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2)^2\zeta_2^2\\
&-\frac{369623040}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2)\zeta_2\zeta_3-\frac{2015232}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2^3-16384\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_3^2\\
&-\frac{524288\ln^{10}(2)}{14175}-\frac{1097728}{315}\ln^8(2)\zeta_2+\frac{41811968}{315}\ln^7(2)\zeta_3-\frac{32247808}{1575}\ln^6(2)\zeta_2^2\\
&+\frac{727211008}{35}\ln^5(2)\zeta_5-\frac{345571328}{105}\ln^5(2)\zeta_2\zeta_3+\frac{297220096}{105}\ln^4(2)\zeta_2^3\\
&+\frac{8366080}{3}\ln^4(2)\zeta_3^2-\frac{1193759744}{7}\ln^3(2)\zeta_7-145342464\ln^3(2)\zeta_2\zeta_5+\frac{22380544}{3}\ln^3(2)\zeta_2^2\zeta_3\\
&+270697472\ln^2(2)\zeta_3\zeta_5+\frac{21425512704}{245}\ln^2(2)\zeta_2^4-\frac{2129968640}{49}\ln^2(2)\zeta_2\zeta_3^2\\
&-\frac{12544430784}{49}\ln(2)\zeta_2\zeta_7+\frac{56012645392}{63}\ln(2)\zeta_9-\frac{41723390470912\ln(2)\zeta_2^2\zeta_5}{120785}\\
&+\frac{300811136}{7}\ln(2)\zeta_3^3-\frac{8089162496}{245}\ln(2)\zeta_2^3\zeta_3-\frac{2721028640820472\zeta_3\zeta_7}{5462933}\\
&-\frac{116853114149 28\zeta_5^2}{26911}-\frac{1088749068720\zeta_2\zeta_3\zeta_5}{3451}+\frac{877301874261743216\zeta_2^5}{4780066375}\\
&-\frac{1430424876432\zeta_2^2\zeta_3^2}{17255}-\frac{1142947840s_{9a}}{7}+67502080s_{9b}+\frac{909025280s_{9c}}{21}-\frac{92405760s_{9d}}{7}\\
&-\frac{115998720s_{9e}}{7}+\frac{289751808\ln(2)s_{8a}}{7}+\frac{1818050560\ln(2)s_{8b}}{7}-\frac{19333120\ln^3(2)s_6}{7}\\
&-\frac{1838714880}{49}\ln(2)s_6\zeta_2-167772160\mathrm{Li}_9\left(\frac{1}{2}\right)-7864320\mathrm{Li}_7\left(\frac{1}{2}\right)\zeta_2+\frac{398458880}{7}\\
&\times\mathrm{Li}_6\left(\frac{1}{2}\right)\zeta_3-\frac{162152448}{7}\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_2^2-\frac{2831493120}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_5+\frac{461168640}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2\zeta_3\\
&+\frac{262144}{567}\ln^9(2)+\frac{2195456}{63}\ln^7(2)\zeta_2+\frac{4980736}{63}\ln^6(2)\zeta_3-\frac{4274176}{105}\ln^5(2)\zeta_2^2\\
&-\frac{117978880}{7}\ln^4(2)\zeta_5+\frac{42204160}{21}\ln^4(2)\zeta_2\zeta_3-\frac{6664192}{3}\ln^3(2)\zeta_2^3-\frac{5437440}{7}\ln^3(2)\zeta_3^2\\
&+\frac{1134018560}{7}\ln^2(2)\zeta_7+\frac{1024250880}{7}\ln^2(2)\zeta_2\zeta_5-\frac{187430912}{7}\ln^2(2)\zeta_2^2\zeta_3\\
&-\frac{2244156160}{7}\ln(2)\zeta_3\zeta_5-\frac{141868738176\ln(2)\zeta_2^4}{1225}+\frac{1688096640}{49}\ln(2)\zeta_2\zeta_3^2\\
&-\frac{23343484700}{21}\zeta_9+\frac{16383169752}{49}\zeta_2\zeta_7+\frac{102141815744}{245}\zeta_2^2\zeta_5-\frac{484925216}{9}\zeta_3^3\\
&+\frac{10468171328}{245}\zeta_2^3\zeta_3-\frac{1388504064s_{8a}}{35}-\frac{1802108928s_{8b}}{7}-\frac{185597952s_{8c}}{7}-\frac{166723584s_{8d}}{7}\\
&+\frac{1432719360}{49}s_{7a}\zeta_2-\frac{2069852160}{49}s_{7b}\zeta_2+\frac{92798976\ln^2(2)s_6}{7}+\frac{493092864}{7}s_6\zeta_2\\
&-134217728\mathrm{Li}_8\left(\frac{1}{2}\right)-6291456\mathrm{Li}_6\left(\frac{1}{2}\right)\zeta_2+\frac{318767104}{7}\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_3-\frac{184614912}{35}\\
&\times\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2^2-\frac{1048576\ln^8(2)}{315}-\frac{8781824}{45}\ln^6(2)\zeta_2-\frac{39845888}{105}\ln^5(2)\zeta_3+\frac{75096064}{105}
\end{aligned}
$$

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\begin{aligned}
&\times \ln^4(2)\zeta_2^2 + \frac{49414144}{21}\ln^3(2)\zeta_2\zeta_3 + \frac{29999104}{35}\ln^2(2)\zeta_2^3 + \frac{26099712}{7}\ln^2(2)\zeta_3^2 + \frac{547762176}{7} \\
&\times \ln(2)\zeta_2\zeta_5 + \frac{1055604736}{35}\ln(2)\zeta_2^2\zeta_3 - \frac{202743552}{7}\zeta_3\zeta_5 + \frac{61806558768\zeta_2^4}{6125} + \frac{212396736}{7}\zeta_2\zeta_3^2 \\
&-15630336 s_{7a} + 17399808 s_{7b} + 15630336\ln(2)s_6 - 88080384\mathrm{Li}_7\left(\frac{1}{2}\right) \\
&-4128768\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_2 - 1376256\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_3 + \frac{262144\ln^7(2)}{15} + \frac{3842048}{5}\ln^5(2)\zeta_2 \\
&-57344\ln^4(2)\zeta_3 - \frac{12271616}{5}\ln^3(2)\zeta_2^2 - 31997952\ln^2(2)\zeta_5 + 3182592\ln^2(2)\zeta_2\zeta_3 \\
&-\frac{43456512}{5}\ln(2)\zeta_2^3 - 19537920\ln(2)\zeta_3^2 + 161445776\zeta_7 - 48240288\zeta_2\zeta_5 - \frac{19440544}{5}\zeta_2^2\zeta_3 \\
&-19464192 s_6 - 51904512\mathrm{Li}_6\left(\frac{1}{2}\right) - 2433024\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2 - \frac{360448\ln^6(2)}{5} \\
&-2264064\ln^4(2)\zeta_2 + \frac{21694464}{5}\ln^2(2)\zeta_2^2 + \frac{22610657280}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_5\ln(2) - 3345408\ln(2)\zeta_2\zeta_3 \\
&+\frac{246644352}{35}\zeta_2^3 + 7953216\zeta_3^2 - 28704768\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{1196032\ln^5(2)}{5} + 4784128\ln(2)^3\zeta_2 \\
&-\frac{20631552}{5}\ln(2)\zeta_2^2 + 21694368\zeta_5 + 812528\zeta_2\zeta_3 - 15269888\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{1908736\ln^4(2)}{3} \\
&-7634944\ln^2(2)\zeta_2 + \frac{13199844}{5}\zeta_2^2 - \frac{17697424}{3}\zeta_3 + 11894784\ln(2)\zeta_2 \\
&\left.-4384758\zeta_2 + 227696640 s_6\zeta_3\right) + \varepsilon^9\left(\frac{124045582}{3} - \frac{1287680}{9}\zeta_3\zeta_9\right. \\
&-191232\zeta_5\zeta_7 - 31872\zeta_2\zeta_3\zeta_7 - 20736\zeta_2\zeta_5^2 - \frac{59904}{5}\zeta_2^2\zeta_3\zeta_5 - \frac{1024}{9}\zeta_3^4 - \frac{94208}{35}\zeta_2^3\zeta_3^2 \\
&-\frac{2321418752}{35035}\zeta_2^6 - \frac{1130454496528082960384 h_{911}}{394189048955} + \frac{378811595020225434192 h_{731}}{8277970028055} \\
&+\frac{1193251487848937005056 h_{713}}{2759323342685} - \frac{6057099264 h_{71111}}{731} - \frac{226872518626348089344 h_{551}}{1655594005611} \\
&-\frac{3233842372976071696384 h_{533}}{74501730252495} + \frac{23750770688 h_{53111}}{731} - \frac{2464531349504 h_{51113}}{47515} \\
&+\frac{72970403840 h_{51311}}{731} - \frac{2063597568 h_{5111111}}{17} + \frac{1484783616 h_{3311111}}{17} - \frac{1201267933184 h_{33311}}{47515} \\
&-\frac{9620516088578048 h_{91}\ln(2)}{780419} - \frac{157501683351552 h_{73}\ln(2)}{780419} + \frac{42611245056 h_{7111}\ln(2)}{29} \\
&+\frac{199826866176 h_{5113}\ln(2)}{493} - \frac{125115826176 h_{5131}\ln(2)}{493} - \frac{64038371328 h_{5311}\ln(2)}{493} \\
&+\frac{2063597568 h_{511111}\ln(2)}{17} + \frac{20852637696 h_{3331}\ln(2)}{493} - \frac{1484783616 h_{331111}\ln(2)}{17} \\
&-\frac{3657433088\ln^2(2)s_{9a}}{7} + 216006656\ln^2(2)s_{9b} + \frac{2908880896\ln^2(2)s_{9c}}{21} \\
&-\frac{295698432\ln^2(2)s_{9d}}{7} - \frac{371195904\ln^2(2)s_{9e}}{7} - \frac{20605079846912}{332605}s_{9a}\zeta_2 + \frac{346075561984}{47515} \\
&\times s_{9b}\zeta_2 + \frac{90902528}{7}s_{9c}\zeta_2 - \frac{27721728}{7}s_{9d}\zeta_2 - \frac{34799616}{7}s_{9e}\zeta_2 + \frac{933351424\ln^3(2)s_{8a}}{35}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1012137984\ln^3(2)s_{8b}}{7}-\frac{494927872\ln^3(2)s_{8c}}{7}-\frac{444596224\ln^3(2)s_{8d}}{7}\\
&+\frac{3468068558355783751 36s_{8a}\zeta_3}{1970945244775}-\frac{4200071353238881144832s_{8b}\zeta_3}{8277970028055}-\frac{23199744}{7}s_{8c}\zeta_3\\
&-\frac{13929023471616}{332605}s_{8d}\zeta_3+\frac{5342354835072}{17255}\ln(2)s_{8a}\zeta_2+\frac{545415168}{7}\ln(2)s_{8b}\zeta_2\\
&+\frac{34803127545856}{2328235}s_{7a}\zeta_2^2-\frac{3021041664}{245}s_{7b}\zeta_2^2+\frac{184549376\ln^5(2)s_6}{5}\\
&+\frac{250412578603008\ln^2(2)s_6\zeta_3}{332605}-\frac{166723584\ln^4(2)s_{7a}}{7}+\frac{185597952\ln^4(2)s_{7b}}{7}\\
&+\frac{365283511369728\ln(2)s_{7a}\zeta_3}{1377935}+\frac{4803526656}{49}\ln^2(2)s_{7a}\zeta_2-\frac{6867124224}{49}\ln^2(2)s_{7b}\zeta_2\\
&-\frac{7442343481344}{120785}\ln(2)s_6\zeta_2^2-\frac{2356066299904s_6\zeta_5}{47515}+\frac{3369369600}{49}\ln^3(2)s_6\zeta_2\\
&+\frac{61328336277504s_6\zeta_2\zeta_3}{332605}-1073741824\mathrm{Li}_{11}\left(\frac{1}{2}\right)-50331648\mathrm{Li}_9\left(\frac{1}{2}\right)\zeta_2\\
&-16777216\mathrm{Li}_8\left(\frac{1}{2}\right)\zeta_3-\frac{20447232}{5}\mathrm{Li}_7\left(\frac{1}{2}\right)\zeta_2^2-4718592\mathrm{Li}_6\left(\frac{1}{2}\right)\zeta_5\\
&+\frac{119537664}{7}\mathrm{Li}_6\left(\frac{1}{2}\right)\zeta_2\zeta_3+\frac{1333788672}{7}\mathrm{Li}_6\left(\frac{1}{2}\right)\ln^2(2)\zeta_3+\frac{27814451740672}{332605}\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_3^2\\
&-\frac{2522873856}{35}\mathrm{Li}_5\left(\frac{1}{2}\right)\ln^2(2)\zeta_2^2-7274496\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_2^3+\frac{889192448}{7}\mathrm{Li}_5\left(\frac{1}{2}\right)\ln^3(2)\zeta_3\\
&-\frac{444596224}{35}\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^3(2)\zeta_2^2-\frac{31735658955776}{332605}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2\zeta_5\\
&+\frac{1478492160}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^2(2)\zeta_2\zeta_3-1019904\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_7-\frac{9044262912}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^2(2)\zeta_5\\
&+\frac{18894660870144}{1663025}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2^2\zeta_3+\frac{4194304\ln^{11}(2)}{155925}+\frac{8781824\ln^9(2)\zeta_2}{2835}\\
&-\frac{250216448}{315}\ln^8(2)\zeta_3+\frac{57638912}{1575}\ln^7(2)\zeta_2^2-\frac{10204172288}{105}\ln^6(2)\zeta_5\\
&+\frac{699662336}{45}\ln^6(2)\zeta_2\zeta_3-\frac{85119722061824\ln^5(2)\zeta_3^2}{4989075}-\frac{201367552}{15}\ln^5(2)\zeta_2^3\\
&+\frac{5347831296}{7}\ln^4(2)\zeta_7+\frac{602089461569408\ln^4(2)\zeta_2\zeta_5}{997815}-\frac{53577818647552\ln^4(2)\zeta_2^2\zeta_3}{4989075}\\
&-\frac{376156375212032\ln^3(2)\zeta_3\zeta_5}{332605}+\frac{1403329698221056\ln^3(2)\zeta_2\zeta_3^2}{6984705}-\frac{224050581568}{63}\ln^2(2)\zeta_9\\
&-\frac{301110728704}{875}\ln^3(2)\zeta_2^4+\frac{2527896877478656\ln^2(2)\zeta_2\zeta_7}{2328235}+\frac{1433322075232256\ln^2(2)\zeta_2^3\zeta_3}{11641175}\\
&+2706974720\ln^2(2)\zeta_3\zeta_5+\frac{100081795637701376\ln^2(2)\zeta_2^2\zeta_5}{67518815}-\frac{13144557640192\ln^2(2)\zeta_3^3}{66521}\\
&+\frac{2502185878462204194255541 76\ln(2)\zeta_3\zeta_7}{54288110011131555}+\frac{268488970046321792\ln(2)\zeta_5^2}{75216245}\\
&+\frac{1370557129830144\ln(2)\zeta_2\zeta_3\zeta_5}{567385}+\frac{2847832763676416\ln(2)\zeta_2^2\zeta_3^2}{3971695}-\frac{16178324992}{49}\ln(2)\zeta_2^3\zeta_3
\end{aligned}
$$

$$
-\frac{4402072179795111099788\ln(2)\zeta_2^5}{29392628139875}+\frac{27629818338556398552313\zeta_{11}}{24833910084165}
+\frac{2551544671187332995658\zeta_2\zeta_9}{14900346050499}-\frac{641168745323279315967\zeta_2^2\zeta_7}{17042879469525}
+\frac{395838313918814227897\zeta_3^2\zeta_5}{2759323342685}-\frac{4238652572500683985395\zeta_2^3\zeta_5}{51128638408575}-\frac{1629519807808\zeta_2\zeta_3^3}{66521}
-\frac{62710643279150007707220\zeta_2^4\zeta_3}{7243223774548125}+\frac{12025645110722560h_{91}}{780419}+\frac{196877104189440h_{73}}{780419}
-\frac{53264056320h_{7111}}{29}+\frac{80047964160h_{5311}}{493}-\frac{249783582720h_{5113}}{493}+\frac{156394782720h_{5131}}{493}
-\frac{2579496960h_{511111}}{17}+\frac{1855979520h_{331111}}{17}-\frac{26065797120h_{3331}}{493}+\frac{9143582720\ln(2)s_{9a}}{7}
-540016640\ln(2)s_{9b}-\frac{7272202240\ln(2)s_{9c}}{21}+\frac{739246080\ln(2)s_{9d}}{7}+\frac{927989760\ln(2)s_{9e}}{7}
-132787200\ln^2(2)s_{8a}-\frac{5533859840\ln^2(2)s_{8b}}{7}+\frac{927989760\ln^2(2)s_{8c}}{7}
+\frac{833617920\ln^2(2)s_{8d}}{7}-\frac{1346195765472s_{8a}\zeta_2}{3451}-\frac{844738560}{7}s_{8b}\zeta_2-\frac{86999040}{7}s_{8c}\zeta_2
-\frac{78151680}{7}s_{8d}\zeta_2+\frac{277872640\ln^3(2)s_{7a}}{7}-\frac{309329920\ln^3(2)s_{7b}}{7}-\frac{1324206735360s_{7a}\zeta_3}{3451}
+\frac{28999680}{7}s_{7b}\zeta_3-\frac{12008816640}{49}\ln(2)s_{7a}\zeta_2+\frac{17167810560}{49}\ln(2)s_{7b}\zeta_2
+\frac{329712525312}{3451}s_6\zeta_2^2-62914560\ln^4(2)s_6-\frac{2275000320}{49}\ln^2(2)s_6\zeta_2
-\frac{12780011520}{7}\ln(2)s_6\zeta_3-1342177280\mathrm{Li}_{10}\left(\frac{1}{2}\right)-62914560\mathrm{Li}_8\left(\frac{1}{2}\right)\zeta_2
-20971520\mathrm{Li}_7\left(\frac{1}{2}\right)\zeta_3-\frac{3334471680}{7}\mathrm{Li}_6\left(\frac{1}{2}\right)\ln(2)\zeta_3-5111808\mathrm{Li}_6\left(\frac{1}{2}\right)\zeta_2^2
+\frac{1261436928}{7}\mathrm{Li}_5\left(\frac{1}{2}\right)\ln(2)\zeta_2^2+\frac{149422080}{7}\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_2\zeta_3-5898240\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_5
-\frac{1667235840}{7}\mathrm{Li}_5\left(\frac{1}{2}\right)\ln^2(2)\zeta_3+\frac{166723584}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^2(2)\zeta_2^2-163840\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_3^2
-\frac{20152320}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2^3-\frac{3696230400}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2)\zeta_2\zeta_3-\frac{1048576\ln^{10}(2)}{2835}
-\frac{2195456}{63}\ln^8(2)\zeta_2+\frac{83623936}{63}\ln^7(2)\zeta_3-\frac{64495616}{315}\ln^6(2)\zeta_2^2+\frac{1454422016}{7}\ln^5(2)\zeta_5
-\frac{691142656}{21}\ln^5(2)\zeta_2\zeta_3+\frac{83660800}{3}\ln^4(2)\zeta_3^2+\frac{594440192}{21}\ln^4(2)\zeta_2^3
-\frac{11937597440}{7}\ln^3(2)\zeta_7-1453424640\ln^3(2)\zeta_2\zeta_5+\frac{223805440}{3}\ln^3(2)\zeta_2^2\zeta_3
-\frac{21299686400}{49}\ln^2(2)\zeta_2\zeta_3^2+\frac{42851025408}{49}\ln^2(2)\zeta_2^4+\frac{560126453920}{63}\ln(2)\zeta_9
-\frac{125444307840}{49}\ln(2)\zeta_2\zeta_7-\frac{83446780941824\ln(2)\zeta_2^2\zeta_5}{24157}+\frac{3008111360}{7}\ln(2)\zeta_3^3
+\frac{10803818496}{49}\ln(2)\zeta_2\zeta_3^2-\frac{27208138994797616\zeta_3\zeta_7}{5462933}-\frac{116846231822496\zeta_5^2}{26911}
$$

$$
\begin{aligned}
&-\frac{10887196496352\zeta_2\zeta_3\zeta_5}{3451}-\frac{14304036293152\zeta_2^2\zeta_3^2}{17255}+\frac{1930340882274467145 6\zeta_2^5}{10516146025}\\
&-\frac{7314866176s_{9a}}{7}+432013312s_{9b}+\frac{5817761792s_{9c}}{21}-\frac{591396864s_{9d}}{7}-\frac{742391808s_{9e}}{7}\\
&+\frac{9272057856\ln(2)s_{8a}}{35}+\frac{11635523584\ln(2)s_{8b}}{7}+\frac{9169403904}{49}s_{7a}\zeta_2-\frac{13247053824}{49}s_{7b}\zeta_2\\
&-\frac{11767775232}{49}\ln(2)s_6\zeta_2-\frac{123731968\ln^3(2)s_6}{7}+1457258496s_6\zeta_3-1073741824\mathrm{Li}_9\left(\frac{1}{2}\right)\\
&-50331648\mathrm{Li}_7\left(\frac{1}{2}\right)\zeta_2+\frac{2550136832}{7}\mathrm{Li}_6\left(\frac{1}{2}\right)\zeta_3-\frac{5188878336}{35}\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_2^2\\
&-\frac{18121555968}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_5+\frac{2951479296}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2\zeta_3+\frac{8388608\ln^9(2)}{2835}\\
&+\frac{70254592}{315}\ln^7(2)\zeta_2+\frac{159383552}{315}\ln^6(2)\zeta_3-\frac{136773632}{525}\ln^5(2)\zeta_2^2-\frac{755064832}{7}\ln^4(2)\zeta_5\\
&+\frac{270106624}{21}\ln^4(2)\zeta_2\zeta_3-\frac{34799616}{7}\ln^3(2)\zeta_3^2-\frac{213254144}{15}\ln^3(2)\zeta_2^3+\frac{7257718784}{7}\ln^2(2)\zeta_7\\
&+\frac{6555205632}{7}\ln^2(2)\zeta_2\zeta_5-\frac{5997789184}{35}\ln^2(2)\zeta_2^2\zeta_3-\frac{4539799621632\ln(2)\zeta_2^4}{6125}\\
&-\frac{14362599424}{7}\ln(2)\zeta_3\zeta_5-\frac{448104543296}{63}\zeta_9+\frac{104867942736}{49}\zeta_2\zeta_7+\frac{653737047104}{245}\zeta_2^2\zeta_5\\
&-\frac{1034493440}{3}\zeta_3^3+\frac{67009518592}{245}\zeta_2^3\zeta_3-\frac{1041378048s_{8a}}{5}-1351581696s_{8b}-139198464s_{8c}\\
&-125042688s_{8d}+69599232\ln^2(2)s_6+369819648s_6\zeta_2-704643072\mathrm{Li}_8\left(\frac{1}{2}\right)\\
&-33030144\mathrm{Li}_6\left(\frac{1}{2}\right)\zeta_2+239075328\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_3-\frac{138461184}{5}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2^2-\frac{262144\ln^8(2)}{15}\\
&-\frac{15368192}{15}\ln^6(2)\zeta_2-\frac{9961472}{5}\ln^5(2)\zeta_3+\frac{18774016}{5}\ln^4(2)\zeta_2^2+12353536\ln^3(2)\zeta_2\zeta_3\\
&+19574784\ln^2(2)\zeta_3^2+\frac{22499328}{5}\ln^2(2)\zeta_2^3+410821632\ln(2)\zeta_2\zeta_5+\frac{791703552}{5}\ln(2)\zeta_2^2\zeta_3\\
&-151892736\zeta_3\zeta_5+159311296\zeta_2\zeta_3^2+\frac{46498767216}{875}\zeta_2^4-\frac{515801088s_{7a}}{7}+\frac{574193664s_{7b}}{7}\\
&+\frac{515801088\ln(2)s_6}{7}-415236096\mathrm{Li}_7\left(\frac{1}{2}\right)-19464192\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_2-6488064\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_3\\
&+\frac{2883584\ln^7(2)}{35}+\frac{18112512}{5}\ln^5(2)\zeta_2\\
&-270336\ln^4(2)\zeta_3-\frac{57851904}{5}\ln^3(2)\zeta_2^2-150847488\ln(2)^2\zeta_5+15003648\ln^2(2)\zeta_2\zeta_3\\
&-\frac{644751360}{7}\ln(2)\zeta_3^2-\frac{1434064896}{35}\ln(2)\zeta_2^3+\frac{5333463504}{7}\zeta_7-\frac{1590681888}{7}\zeta_2\zeta_5\\
&-\frac{127947168}{7}\zeta_2^2\zeta_3-86114304s_6-229638144\mathrm{Li}_6\left(\frac{1}{2}\right)-10764288\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2\\
&-\frac{4784128\ln^6(2)}{15}-10016768\ln^4(2)\zeta_2+\frac{61079552}{3}\ln^3(2)\zeta_2+\frac{95981568}{5}\ln^2(2)\zeta_2^2\\
&-14800896\ln(2)\zeta_2\zeta_3+\frac{105679040}{3}\zeta_3^2+\frac{1096650368}{35}\zeta_2^3-122159104\mathrm{Li}_5\left(\frac{1}{2}\right)
\end{aligned}
$$

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+\frac{15269888\ln^5(2)}{15} - \frac{87801856}{5}\ln(2)\zeta_2^2 + 92863200\zeta_5 + 3547600\zeta_2\zeta_3 - 63438848\mathrm{Li}_4\left(\frac{1}{2}\right)
-\frac{7929856\ln^4(2)}{3} - 31719424\ln^2(2)\zeta_2 + \frac{55731628}{5}\zeta_2^2 - 23863952\zeta_3 + 48685056\ln(2)\zeta_2
-17459170\zeta_2 + \frac{22610657280}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_5\ln(2)\Bigg) \tag{A.59}
$$

$$
\begin{aligned}
I_{11} = {} & \frac{1}{\varepsilon^3} + \frac{7}{2\varepsilon^2} + \frac{253}{36\varepsilon} + \frac{2501}{216} + \varepsilon\left(\frac{59437}{1296} - \frac{128\zeta_2}{3}\right) + \varepsilon^2\Bigg(\frac{2831381}{7776} - \frac{1792}{9}\zeta_3 + 512\ln(2)\zeta_2 \\
& -\frac{4544}{9}\zeta_2\Bigg) + \varepsilon^3\Bigg(\frac{117529021}{46656} - \frac{8192}{3}\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{1024}{9}\ln^4(2) - \frac{7168}{3}\ln^2(2)\zeta_2 + \frac{11008}{15}\zeta_2^2 \\
& -\frac{63616}{27}\zeta_3 + \frac{18176}{3}\ln(2)\zeta_2 - \frac{99680}{27}\zeta_2\Bigg) + \varepsilon^4\Bigg(\frac{4081770917}{279936} - 32768\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{87296}{3}\zeta_5 \\
& +\frac{4096}{15}\ln^5(2) + \frac{28672}{3}\ln^3(2)\zeta_2 - \frac{44032}{5}\ln(2)\zeta_2^2 + \frac{3584}{3}\zeta_2\zeta_3 - \frac{290816}{9}\mathrm{Li}_4\left(\frac{1}{2}\right) \\
& -\frac{36352}{27}\ln^4(2) - \frac{254464}{9}\ln^2(2)\zeta_2 + \frac{390784}{45}\zeta_2^2 - \frac{1395520}{81}\zeta_3 + \frac{398720}{9}\ln(2)\zeta_2 \\
& -\frac{1750448}{81}\zeta_2\Bigg) + \varepsilon^5\Bigg(\frac{125873914573}{1679616} - 180224s_6 - 393216\mathrm{Li}_6\left(\frac{1}{2}\right) - \frac{8192}{15}\ln^6(2) \\
& +\frac{633344}{9}\zeta_3^2 - 28672\ln^4(2)\zeta_2 + \frac{264192}{5}\ln^2(2)\zeta_2^2 - 14336\ln(2)\zeta_2\zeta_3 + \frac{745472}{15}\zeta_2^3 \\
& -\frac{1163264}{3}\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{3099008}{9}\zeta_5 + \frac{145408}{45}\ln^5(2) + \frac{1017856}{9}\ln^3(2)\zeta_2 - \frac{1563136}{15}\ln(2)\zeta_2^2 \\
& +\frac{127232}{9}\zeta_2\zeta_3 - \frac{6379520}{27}\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{797440}{81}\ln^4(2) - \frac{5582080}{27}\ln^2(2)\zeta_2 + \frac{1714496}{27}\zeta_2^2 \\
& -\frac{24506272}{243}\zeta_3 + \frac{7001792}{27}\ln(2)\zeta_2 - \frac{27091736}{243}\zeta_2\Bigg) + \varepsilon^6\Bigg(\frac{3593750577461}{10077696} - \frac{19922944}{21}s_{7a} \\
& +\frac{25493504}{21}s_{7b} - 4718592\mathrm{Li}_7\left(\frac{1}{2}\right) + \frac{19922944}{21}\ln(2)s_6 + \frac{72259840}{7}\zeta_7 + \frac{32768}{35}\ln^7(2) \\
& +\frac{344064}{5}\ln^5(2)\zeta_2 - \frac{1056768}{5}\ln^3(2)\zeta_2^2 + 86016\ln^2(2)\zeta_2\zeta_3 - 2095104\ln^2(2)\zeta_5 \\
& -\frac{5758976}{15}\ln(2)\zeta_2^3 - \frac{24903680}{21}\ln(2)\zeta_3^2 - \frac{7158784}{21}\zeta_2^2\zeta_3 - \frac{76080640}{21}\zeta_2\zeta_5 - \frac{6397952}{3}s_6 \\
& -4653056\mathrm{Li}_6\left(\frac{1}{2}\right) - \frac{290816}{45}\ln^6(2) - \frac{1017856}{3}\ln^4(2)\zeta_2 + \frac{3126272}{5}\ln^2(2)\zeta_2^2 \\
& -\frac{508928}{3}\ln(2)\zeta_2\zeta_3 + \frac{26464256}{45}\zeta_2^3 + \frac{22483712}{27}\zeta_3^2 - \frac{25518080}{9}\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{67981760}{27}\zeta_5 \\
& +\frac{637952}{27}\ln^5(2) + \frac{22328320}{27}\ln^3(2)\zeta_2 - \frac{6857984}{9}\ln(2)\zeta_2^2 + \frac{2791040}{27}\zeta_2\zeta_3 \\
& -\frac{112028672}{81}\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{14003584}{243}\ln^4(2) - \frac{98025088}{81}\ln^2(2)\zeta_2 + \frac{150538528}{405}\zeta_2^2
\end{aligned}
$$

$$
+\frac{108366944}{81}\ln(2)\zeta_2-\frac{379284304}{729}\zeta_3-\frac{387541868}{729}\zeta_2\Bigg)+\varepsilon^7\Bigg(\frac{97480790072029}{60466176}
-\frac{2183312384}{105}s_{8a}-\frac{897384448}{7}s_{8b}-\frac{101974016}{7}s_{8c}-\frac{79691776}{7}s_{8d}+\frac{50987008}{7}\ln^2(2)s_6
+\frac{250216448}{7}\zeta_2 s_6-56623104\mathrm{Li}_8\left(\frac{1}{2}\right)+\frac{159383552}{7}\zeta_3\mathrm{Li}_5\left(\frac{1}{2}\right)-\frac{79691776}{35}\zeta_2^2\mathrm{Li}_4\left(\frac{1}{2}\right)
-\frac{49152}{35}\ln^8(2)-\frac{688128}{5}\ln^6(2)\zeta_2-\frac{19922944}{105}\ln^5(2)\zeta_3+\frac{56614912}{105}\ln^4(2)\zeta_2^2
+\frac{32620544}{21}\ln^3(2)\zeta_2\zeta_3-\frac{4751360}{7}\ln^2(2)\zeta_2^3+\frac{14340096}{7}\ln^2(2)\zeta_3^2+\frac{572674048}{35}\ln(2)\zeta_2^2\zeta_3
+\frac{305018880}{7}\ln(2)\zeta_2\zeta_5-\frac{378638336}{21}\zeta_3\zeta_5+\frac{22032280576}{6125}\zeta_2^4+\frac{331879424}{21}\zeta_2\zeta_3^2
-\frac{707264512}{63}s_{7a}+\frac{905019392}{63}s_{7b}+\frac{707264512}{63}\ln(2)s_6-55836672\mathrm{Li}_7\left(\frac{1}{2}\right)
+\frac{2565224320}{21}\zeta_7+\frac{1163264}{105}\ln^7(2)+\frac{4071424}{5}\ln^5(2)\zeta_2-\frac{12505088}{5}\ln^3(2)\zeta_2^2
-24792064\ln^2(2)\zeta_5+1017856\ln^2(2)\zeta_2\zeta_3-\frac{204443648}{45}\ln(2)\zeta_2^3-\frac{884080640}{63}\ln(2)\zeta_3^2
-\frac{2700862720}{63}\zeta_2\zeta_5-\frac{254136832}{63}\zeta_2^2\zeta_3-\frac{140349440}{9}s_6-\frac{102072320}{3}\mathrm{Li}_6\left(\frac{1}{2}\right)
-\frac{1275904}{27}\ln^6(2)-\frac{22328320}{9}\ln^4(2)\zeta_2+\frac{13715968}{3}\ln^2(2)\zeta_2^2-\frac{11164160}{9}\ln(2)\zeta_2\zeta_3
+\frac{493216640}{81}\zeta_3^2+\frac{116107264}{27}\zeta_2^3-\frac{448114688}{27}\mathrm{Li}_5\left(\frac{1}{2}\right)+\frac{56014336}{405}\ln^5(2)
+\frac{392100352}{81}\ln^3(2)\zeta_2-\frac{602154112}{135}\ln(2)\zeta_2^2+\frac{1193805536}{81}\zeta_5+\frac{49012544}{81}\zeta_2\zeta_3
-\frac{1733871104}{243}\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{216733888}{729}\ln^4(2)-\frac{1517137216}{243}\ln^2(2)\zeta_2+\frac{2329889296}{1215}\zeta_2^2
-\frac{5425586152}{2187}\zeta_3+\frac{1550167472}{243}\ln(2)\zeta_2-\frac{5263826150}{2187}\zeta_2\Bigg)+\varepsilon^8\Bigg(\frac{2553476823634373}{362797056}
-\frac{16011624448}{21}s_{9a}+\frac{1157496832}{3}s_{9b}+\frac{13187219456}{63}s_{9c}-\frac{344457216}{7}s_{9d}
-\frac{611844096}{7}s_{9e}+\frac{7005710336}{35}\ln(2)s_{8a}+\frac{26374438912}{21}\ln(2)s_{8b}+\frac{5416157184}{49}s_{7a}\zeta_2
-\frac{11319115776}{49}s_{7b}\zeta_2+1117716480s_6\zeta_3-\frac{101974016}{7}s_6\ln^3(2)-\frac{7557611520}{49}s_6\ln(2)\zeta_2
-679477248\mathrm{Li}_9\left(\frac{1}{2}\right)+\frac{1912602624}{7}\mathrm{Li}_6\left(\frac{1}{2}\right)\zeta_3-\frac{4015521792}{35}\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_2^2
+\frac{1722286080}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2\zeta_3-\frac{13673512960}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_5-\frac{380773366144}{63}\zeta_9+\frac{65536}{35}\ln^9(2)
+\frac{1179648}{5}\ln^7(2)\zeta_2+\frac{39845888}{105}\ln^6(2)\zeta_3-\frac{98992128}{175}\ln^5(2)\zeta_2^2+\frac{39141376}{7}\ln^4(2)\zeta_2\zeta_3
-\frac{1709189120}{21}\ln^4(2)\zeta_5-\frac{249135104}{35}\ln^3(2)\zeta_2^3-\frac{28680192}{7}\ln^3(2)\zeta_3^2
-\frac{4011491328}{35}\ln^2(2)\zeta_2^2\zeta_3+\frac{4822073344}{7}\ln^2(2)\zeta_2\zeta_5+\frac{15886533632}{21}\ln^2(2)\zeta_7
$$

$$
\begin{aligned}
&-\frac{3435955044352}{6125}\ln(2)\zeta_2^4+\frac{6177746944}{49}\ln(2)\zeta_2\zeta_3^2-\frac{32555948032}{21}\ln(2)\zeta_3\zeta_5\\
&+\frac{61344280576}{245}\zeta_2^3\zeta_3-\frac{7109786624}{27}\zeta_3^3+\frac{1581235817984}{735}\zeta_2^2\zeta_5+\frac{78334057728}{49}\zeta_2\zeta_7\\
&-\frac{77507589632}{315}s_{8a}-\frac{31857147904}{21}s_{8b}-\frac{3620077568}{21}s_{8c}-\frac{2829058048}{21}s_{8d}\\
&-670040064\mathrm{Li}_8\left(\frac{1}{2}\right)+\frac{1810038784}{21}s_6\ln^2(2)+\frac{8882683904}{21}s_6\zeta_2+\frac{5658116096}{21}\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_3\\
&-\frac{2829058048}{105}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2^2-\frac{581632}{35}\ln^8(2)-\frac{392100352}{27}\ln^4(2)\zeta_2-\frac{8142848}{5}\ln^6(2)\zeta_2\\
&-\frac{707264512}{315}\ln^5(2)\zeta_3+\frac{2009829376}{315}\ln^4(2)\zeta_2^2+\frac{1158029312}{63}\ln^3(2)\zeta_2\zeta_3\\
&-\frac{168673280}{21}\ln^2(2)\zeta_2^3+\frac{169691136}{7}\ln^2(2)\zeta_3^2+\frac{20329928704}{105}\ln(2)\zeta_2^2\zeta_3\\
&+\frac{3609390080}{7}\ln(2)\zeta_2\zeta_5+\frac{782145960448}{18375}\zeta_2^4+\frac{11781719552}{63}\zeta_2\zeta_3^2-\frac{13441660928}{63}\zeta_3\zeta_5\\
&-\frac{2216427520}{27}s_{7a}+\frac{2836152320}{27}s_{7b}-408289280\mathrm{Li}_7\left(\frac{1}{2}\right)+\frac{2216427520}{27}\ln(2)s_6\\
&+\frac{8038907200}{9}\zeta_7+\frac{729088}{9}\ln^7(2)+\frac{17862656}{3}\ln^5(2)\zeta_2-\frac{54863872}{3}\ln^3(2)\zeta_2^2\\
&+\frac{22328320}{3}\ln^2(2)\zeta_2\zeta_3-\frac{543854080}{3}\ln^2(2)\zeta_5-\frac{896960512}{27}\ln(2)\zeta_2^3-\frac{2770534400}{27}\ln(2)\zeta_3^2\\
&-\frac{796414720}{27}\zeta_2^2\zeta_3-\frac{8463971200}{27}\zeta_2\zeta_5-\frac{1792458752}{9}\mathrm{Li}_6\left(\frac{1}{2}\right)-\frac{2464630784}{27}s_6\\
&-\frac{112028672}{405}\ln^6(2)+\frac{1204308224}{45}\ln^2(2)\zeta_2^2-\frac{196050176}{27}\ln(2)\zeta_2\zeta_3+\frac{10194609152}{405}\zeta_2^3\\
&+\frac{8661216704}{243}\zeta_3^2-\frac{6935484416}{81}\mathrm{Li}_5\left(\frac{1}{2}\right)+\frac{866935552}{1215}\ln^5(2)+\frac{6068548864}{243}\ln^3(2)\zeta_2\\
&-\frac{9319557184}{405}\ln(2)\zeta_2^2+\frac{758568608}{243}\zeta_2\zeta_3+\frac{18476563952}{243}\zeta_5-\frac{24802679552}{729}\mathrm{Li}_4\left(\frac{1}{2}\right)\\
&-\frac{3100334944}{2187}\ln^4(2)-\frac{21702344608}{729}\ln^2(2)\zeta_2+\frac{33328600648}{3645}\zeta_2^2-\frac{73693566100}{6561}\zeta_3\\
&+\frac{21055304600}{729}\ln(2)\zeta_2-\frac{69020223371}{6561}\zeta_2\Bigg)+\varepsilon^9\Bigg(\frac{652827188634337709}{2176782336}\\
&+\frac{14684258304}{17}h_{331111}-\frac{19013284397056}{54723}h_{3331}-\frac{24574427136}{17}h_{511111}\\
&-\frac{407096785895424}{127687}h_{5113}+\frac{38026568794112}{18241}h_{5131}+\frac{128886518054912}{127687}h_{5311}\\
&-\frac{77744179249152}{7511}h_{7111}+\frac{982846081352335360}{606385563}h_{73}+\frac{498389163900928000}{5462933}h_{91}\\
&-8153726976\mathrm{Li}_{10}\left(\frac{1}{2}\right)-8040480768\mathrm{Li}_9\left(\frac{1}{2}\right)-4899471360\mathrm{Li}_8\left(\frac{1}{2}\right)\\
&-\frac{7169835008}{3}\mathrm{Li}_7\left(\frac{1}{2}\right)-\frac{27741937664}{27}\mathrm{Li}_6\left(\frac{1}{2}\right)-\frac{99210718208}{243}\mathrm{Li}_5\left(\frac{1}{2}\right)\\
&-\frac{336884873600}{2187}\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{393216}{175}\ln^{10}(2)+\frac{2326528}{105}\ln^9(2)-\frac{364544}{3}\ln^8(2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{64016384}{135}\ln^7(2)-\frac{1733871104}{1215}\ln^6(2)+\frac{12401339776}{3645}\ln^5(2)-\frac{42110609200}{6561}\ln^4(2)\\
&-467140608s_6\ln^4(2)+\frac{1912602624}{7}s_{7a}\ln^3(2)-\frac{38921961472}{81}s_{7a}-\frac{2447376384}{7}s_{7b}\ln^3(2)\\
&+\frac{49804746752}{81}s_{7b}-\frac{3620077568}{21}s_6\ln^3(2)+\frac{5672304640}{9}s_6\ln^2(2)\\
&+\frac{38921961472}{81}s_6\ln(2)-\frac{38145164288}{81}s_6-\frac{4524146688}{5}s_{8a}\ln^2(2)\\
&+\frac{248702716928}{105}s_{8a}\ln(2)-\frac{48578700544}{27}s_{8a}-\frac{99834019840}{9}s_{8b}\\
&+\frac{936292581376}{63}s_{8b}\ln(2)-\frac{40886075392}{7}s_{8b}\ln^2(2)-\frac{11344609280}{9}s_{8c}\\
&+\frac{7342129152}{7}s_{8c}\ln^2(2)-\frac{8865710080}{9}s_{8d}+\frac{5737807872}{7}s_{8d}\ln^2(2)-\frac{568412667904}{63}s_{9a}\\
&+\frac{64046497792}{7}s_{9a}\ln(2)+\frac{41091137536}{9}s_{9b}-4629987328s_{9b}\ln(2)\\
&+\frac{468146290688}{189}s_{9c}-\frac{52748877824}{21}s_{9c}\ln(2)-\frac{4076077056}{7}s_{9d}+\frac{4133486592}{7}s_{9d}\ln(2)\\
&-\frac{7240155136}{7}s_{9e}+\frac{7342129152}{7}s_{9e}\ln(2)-\frac{1766128887523}{39366}\zeta_2\\
&+\frac{276080893484}{2187}\ln(2)\zeta_2-\frac{294774264400}{2187}\ln^2(2)\zeta_2+\frac{86809378432}{729}\ln^3(2)\zeta_2\\
&-\frac{6068548864}{81}\ln^4(2)\zeta_2+\frac{1568401408}{45}\ln^5(2)\zeta_2-\frac{35725312}{3}\ln^6(2)\zeta_2+\frac{13959168}{5}\ln^7(2)\zeta_2\\
&-\frac{1769472}{5}\ln^8(2)\zeta_2+\frac{27836579840}{9}s_6\zeta_2-\frac{89431736320}{49}s_6\ln(2)\zeta_2-\frac{35417751552}{49}s_6\ln^2(2)\zeta_2\\
&+\frac{64091193344}{49}s_{7a}\zeta_2-\frac{64993886208}{49}s_{7a}\ln(2)\zeta_2-\frac{133942870016}{49}s_{7b}\zeta_2\\
&+\frac{135829389312}{49}\ln(2)s_{7b}\zeta_2-\frac{42561416368128}{18241}s_{8a}\zeta_2+\frac{90537809780}{2187}\zeta_2^2\\
&-\frac{47517007872}{35}\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_2^2-\frac{1773142016}{9}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2^2-\frac{133314402592}{1215}\ln(2)\zeta_2^2\\
&+\frac{48186261504}{35}\mathrm{Li}_5\left(\frac{1}{2}\right)\ln(2)\zeta_2^2+\frac{18639114368}{135}\ln^2(2)\zeta_2^2+\frac{5737807872}{35}\mathrm{Li}_4\left(\frac{1}{2}\right)\ln^2(2)\zeta_2^2\\
&-\frac{4817232896}{45}\ln^3(2)\zeta_2^2+\frac{1259681792}{27}\ln^4(2)\zeta_2^2-\frac{1171406848}{175}\ln^5(2)\zeta_2^2\\
&-\frac{279773184}{175}\ln^6(2)\zeta_2^2+\frac{50887098236928}{127687}s_6\zeta_2^2+\frac{157782270464}{1215}\zeta_2^3-\frac{78756156416}{405}\ln(2)\zeta_2^3\\
&-\frac{528588800}{9}\ln^2(2)\zeta_2^3-\frac{8844296192}{105}\ln^3(2)\zeta_2^3+\frac{7067271168}{35}\ln^4(2)\zeta_2^3+\frac{490218242816}{1575}\zeta_2^4\\
&-\frac{121976404074496}{18375}\ln(2)\zeta_2^4+\frac{39645136330752}{6125}\ln^2(2)\zeta_2^4+\frac{262144173134771720192}{23822289975}\zeta_2^5\\
&-\frac{966283127194}{19683}\zeta_3+\frac{17731420160}{9}\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_3+\frac{22632464384}{7}\mathrm{Li}_6\left(\frac{1}{2}\right)\zeta_3\\
&-\frac{22951231488}{7}\mathrm{Li}_6\left(\frac{1}{2}\right)\ln(2)\zeta_3-\frac{11475615744}{7}\mathrm{Li}_5\left(\frac{1}{2}\right)\ln^2(2)\zeta_3-\frac{443285504}{27}\ln^5(2)\zeta_3\\
&+\frac{1414529024}{315}\ln^6(2)\zeta_3+\frac{318767104}{35}\ln^7(2)\zeta_3+13226311680s_6\zeta_3
\end{aligned}
$$

$$
\begin{aligned}
&-13412597760 s_6 \ln(2)\zeta_3 - \frac{305322589421568}{127687} s_{7a}\zeta_3 + \frac{10851172304}{729}\zeta_2\zeta_3 \\
&+\frac{20380385280}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_2\zeta_3 - \frac{3034274432}{81}\ln(2)\zeta_2\zeta_3 - \frac{20667432960}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2)\zeta_2\zeta_3 \\
&+\frac{392100352}{9}\ln^2(2)\zeta_2\zeta_3 + \frac{3629035520}{27}\ln^3(2)\zeta_2\zeta_3 + \frac{1389518848}{21}\ln^4(2)\zeta_2\zeta_3 \\
&-\frac{6783172608}{35}\ln^5(2)\zeta_2\zeta_3 - \frac{13985579392}{81}\zeta_2^2\zeta_3 + \frac{12741997568}{9}\ln(2)\zeta_2^2\zeta_3 \\
&-\frac{47469314048}{35}\ln^2(2)\zeta_2^2\zeta_3 + \frac{2068316160}{7}\ln^3(2)\zeta_2^2\zeta_3 + \frac{2177721960448}{735}\zeta_2^3\zeta_3 \\
&-\frac{736131366912}{245}\ln(2)\zeta_2^3\zeta_3 + \frac{134049909728}{729}\zeta_3^2 - \frac{48652451840}{81}\ln(2)\zeta_3^2 \\
&+177259520\ln^2(2)\zeta_3^2 - \frac{339382272}{7}\ln^3(2)\zeta_3^2 + 182403072\ln^4(2)\zeta_3^2 + \frac{36921585920}{27}\zeta_2\zeta_3^2 \\
&+\frac{219310016512}{147}\ln(2)\zeta_2\zeta_3^2 - \frac{129888706560}{49}\ln^2(2)\zeta_2\zeta_3^2 - \frac{9358852028348416}{1915305}\zeta_2^2\zeta_3^2 \\
&-\frac{252397425152}{81}\zeta_3^3 + \frac{28439146496}{9}\ln(2)\zeta_3^3 + \frac{264303553976}{729}\zeta_5 \\
&-\frac{485409710080}{21}\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta_5 + \frac{164082155520}{7}\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2)\zeta_5 - \frac{9550444288}{9}\ln^2(2)\zeta_5 \\
&-\frac{60676213760}{63}\ln^4(2)\zeta_5 + \frac{53190565888}{35}\ln^5(2)\zeta_5 - \frac{148633040320}{81}\zeta_2\zeta_5 \\
&+\frac{11311116800}{3}\ln(2)\zeta_2\zeta_5 + \frac{171183603712}{21}\ln^2(2)\zeta_2\zeta_5 - 10360356864\ln^3(2)\zeta_2\zeta_5 \\
&+\frac{56133871538432}{2205}\zeta_2^2\zeta_5 - \frac{118824071572576256}{4469045}\ln(2)\zeta_2^2\zeta_5 - \frac{42123514880}{27}\zeta_3\zeta_5 \\
&-\frac{1155736155136}{63}\ln(2)\zeta_3\zeta_5 + 19901587456\ln^2(2)\zeta_3\zeta_5 - \frac{2399840466176000}{127687}\zeta_2\zeta_3\zeta_5 \\
&-\frac{186051331138705408}{6969949}\zeta_5^2 + \frac{141168629920}{27}\zeta_7 + \frac{563971943936}{63}\ln^2(2)\zeta_7 \\
&-\frac{84356968448}{7}\ln^3(2)\zeta_7 + \frac{926953016448}{49}\zeta_2\zeta_7 - \frac{940008692736}{49}\ln(2)\zeta_2\zeta_7 \\
&-\frac{18265279696831990784}{606385563}\zeta_3\zeta_7 - \frac{13517454498112}{189}\zeta_9 + \frac{1523093464576}{21}\ln(2)\zeta_9\Bigg) \qquad \text{(A.60)}
\end{aligned}
$$

$$
\begin{aligned}
I_{12} = &\frac{2}{\varepsilon^3} + \frac{23}{3\varepsilon^2} + \frac{35}{2\varepsilon} + \frac{275}{12} + \frac{7}{24}\varepsilon\big(-81 + 128\zeta_3\big) + \varepsilon^2\Bigg(-\frac{14917}{48} - \frac{136\pi^4}{45} - \frac{32}{3}\pi^2\ln^2(2) \\
&+\frac{32\ln^4(2)}{3} + 256\mathrm{Li}_4\left(\frac{1}{2}\right) + 280\zeta_3\Bigg) + \varepsilon^3\Bigg(-\frac{48005}{32} - \frac{68\pi^4}{3} + \frac{272\pi^4\ln(2)}{15} - 80\pi^2\ln^2(2) \\
&+\frac{64}{3}\pi^2\ln^3(2) + 80\ln^4(2) - \frac{64\ln^5(2)}{5} + 1920\mathrm{Li}_4\left(\frac{1}{2}\right) + 1536\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{4060\zeta_3}{3} - 1240\zeta_5\Big) \\
&+\varepsilon^4\Bigg(-\frac{1108525}{192} - \frac{986\pi^4}{9} - \frac{32\pi^6}{5} + 136\pi^4\ln(2) - \frac{1160}{3}\pi^2\ln^2(2) - \frac{272}{5}\pi^4\ln^2(2) \\
&+160\pi^2\ln^3(2) + \frac{1160\ln^4(2)}{3} - 32\pi^2\ln^4(2) - 96\ln^5(2) + \frac{64\ln^6(2)}{5} + 9280\mathrm{Li}_4\left(\frac{1}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ 11520\mathrm{Li}_5\left(\frac{1}{2}\right) + 9216\mathrm{Li}_6\left(\frac{1}{2}\right) + 3840 s_6 + 5390\zeta_3 - \frac{4880\zeta_3^2}{3} - 9300\zeta_5\Bigg) \\
&+ \varepsilon^5\Bigg(-\frac{2570029}{128} - \frac{1309\pi^4}{3} - 48\pi^6 + \frac{1972\pi^4\ln(2)}{3} + \frac{3824\pi^6\ln(2)}{135} - 1540\pi^2\ln^2(2) \\
&- 408\pi^4\ln^2(2) + \frac{2320}{3}\pi^2\ln^3(2) + \frac{544}{5}\pi^4\ln^3(2) + 1540\ln^4(2) - 240\pi^2\ln^4(2) - 464\ln^5(2) \\
&+ \frac{192}{5}\pi^2\ln^5(2) + 96\ln^6(2) - \frac{384\ln^7(2)}{35} + 36960\mathrm{Li}_4\left(\frac{1}{2}\right) + 55680\mathrm{Li}_5\left(\frac{1}{2}\right) + 69120\mathrm{Li}_6\left(\frac{1}{2}\right) \\
&+ 55296\mathrm{Li}_7\left(\frac{1}{2}\right) + 28800 s_6 - \frac{74240}{7}\ln(2) s_6 + \frac{74240 s_{7\mathrm{a}}}{7} - \frac{87040 s_{7\mathrm{b}}}{7} + \frac{57967\zeta_3}{3} + \frac{720\pi^4\zeta_3}{7} \\
&- 12200\zeta_3^2 + \frac{92800}{7}\ln(2)\zeta_3^2 - 44950\zeta_5 + \frac{130360\pi^2\zeta_5}{21} + 22320\ln^2(2)\zeta_5 - \frac{772868\zeta_7}{7}\Bigg) \\
&+ \varepsilon^6\Bigg(-\frac{50743957}{768} - \frac{140777\pi^4}{90} - 232\pi^6 - \frac{593716\pi^8}{33075} + 2618\pi^4\ln(2) + \frac{1912\pi^6\ln(2)}{9} \\
&- \frac{16562}{3}\pi^2\ln^2(2) - 1972\pi^4\ln^2(2) - \frac{3328}{315}\pi^6\ln^2(2) + 3080\pi^2\ln^3(2) + 816\pi^4\ln^3(2) \\
&+ \frac{16562\ln^4(2)}{3} - 1160\pi^2\ln^4(2) - \frac{46768}{315}\pi^4\ln^4(2) - 1848\ln^5(2) + 288\pi^2\ln^5(2) + 464\ln^6(2) \\
&- \frac{192}{5}\pi^2\ln^6(2) - \frac{576\ln^7(2)}{7} + \frac{288\ln^8(2)}{35} + 132496\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7424\pi^4\mathrm{Li}_4\left(\frac{1}{2}\right)}{21} \\
&+ 221760\mathrm{Li}_5\left(\frac{1}{2}\right) + 334080\mathrm{Li}_6\left(\frac{1}{2}\right) + 414720\mathrm{Li}_7\left(\frac{1}{2}\right) + 331776\mathrm{Li}_8\left(\frac{1}{2}\right) + 139200 s_6 \\
&- \frac{229120}{7}\pi^2 s_6 - \frac{556800}{7}\ln(2) s_6 - \frac{261120}{7}\ln^2(2) s_6 + \frac{556800 s_{7\mathrm{a}}}{7} - \frac{652800 s_{7\mathrm{b}}}{7} \\
&+ \frac{767504 s_{8\mathrm{a}}}{7} + \frac{4862976 s_{8\mathrm{b}}}{7} + \frac{522240 s_{8\mathrm{c}}}{7} + \frac{445440 s_{8\mathrm{d}}}{7} + \frac{130095\zeta_3}{2} + \frac{5400\pi^4\zeta_3}{7} \\
&- \frac{51952}{21}\pi^4\ln(2)\zeta_3 - \frac{37120}{21}\pi^2\ln^3(2)\zeta_3 + \frac{7424}{7}\ln^5(2)\zeta_3 - \frac{890880}{7}\mathrm{Li}_5\left(\frac{1}{2}\right)\zeta_3 - \frac{176900\zeta_3^2}{3} \\
&- \frac{96080}{7}\pi^2\zeta_3^2 + \frac{696000}{7}\ln(2)\zeta_3^2 - \frac{73440}{7}\ln^2(2)\zeta_3^2 - 179025\zeta_5 + \frac{325900\pi^2\zeta_5}{7} \\
&- \frac{261120}{7}\pi^2\ln(2)\zeta_5 + 167400\ln^2(2)\zeta_5 + \frac{636208\zeta_3\zeta_5}{7} - \frac{5796510\zeta_7}{7}\Bigg) \\
&+ \varepsilon^7\Bigg(-\frac{107716245}{512} - \frac{21063\pi^4}{4} - 924\pi^6 - \frac{296858\pi^8}{2205} + \frac{140777\pi^4\ln(2)}{15} + \frac{27724\pi^6\ln(2)}{27} \\
&+ \frac{38646953\pi^8\ln(2)}{33075} - 18585\pi^2\ln^2(2) - 7854\pi^4\ln^2(2) - \frac{1664}{21}\pi^6\ln^2(2) + \frac{33124}{3}\pi^2\ln^3(2) \\
&+ 3944\pi^4\ln^3(2) + \frac{43376}{315}\pi^6\ln^3(2) + 18585\ln^4(2) - 4620\pi^2\ln^4(2) - \frac{23384}{21}\pi^4\ln^4(2) \\
&- \frac{33124\ln^5(2)}{5} + 1392\pi^2\ln^5(2) + \frac{22112}{175}\pi^4\ln^5(2) + 1848\ln^6(2) - 288\pi^2\ln^6(2) \\
&- \frac{2784\ln^7(2)}{7} + \frac{1152}{35}\pi^2\ln^7(2) + \frac{432\ln^8(2)}{7} - \frac{192\ln^9(2)}{35} + 446040\mathrm{Li}_4\left(\frac{1}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{18560\pi^4\text{Li}_4\left(\frac{1}{2}\right)}{7}+794976\text{Li}_5\left(\frac{1}{2}\right)+\frac{58368\pi^4}{7}\text{Li}_5\left(\frac{1}{2}\right)+1330560\text{Li}_6\left(\frac{1}{2}\right)\\
&+2004480\text{Li}_7\left(\frac{1}{2}\right)+2488320\text{Li}_8\left(\frac{1}{2}\right)+1990656\text{Li}_9\left(\frac{1}{2}\right)+554400s_6-\frac{1718400}{7}\pi^2 s_6\\
&-\frac{2691200}{7}\ln(2)s_6+\frac{3878400}{49}\pi^2\ln(2)s_6-\frac{1958400}{7}\ln^2(2)s_6+\frac{261120}{7}\ln^3(2)s_6\\
&+\frac{2691200s_{7\text{a}}}{7}-\frac{2964480}{49}\pi^2 s_{7\text{a}}-\frac{3155200s_{7\text{b}}}{7}+\frac{4830720\pi^2 s_{7\text{b}}}{49}+\frac{5756280s_{8\text{a}}}{7}\\
&-\frac{3787464}{7}\ln(2)s_{8\text{a}}+\frac{36472320s_{8\text{b}}}{7}-\frac{23764480}{7}\ln(2)s_{8\text{b}}+\frac{3916800s_{8\text{c}}}{7}+\frac{3340800s_{8\text{d}}}{7}\\
&+\frac{14821888s_{9\text{a}}}{7}-951808s_{9\text{b}}-\frac{11882240s_{9\text{c}}}{21}+\frac{1105920s_{9\text{d}}}{7}+\frac{1566720s_{9\text{e}}}{7}+\frac{2526055\zeta_3}{12}\\
&+\frac{26100\pi^4\zeta_3}{7}-\frac{18482398\pi^6\zeta_3}{6615}-\frac{129880}{7}\pi^4\ln(2)\zeta_3+\frac{67912}{7}\pi^4\ln^2(2)\zeta_3-\frac{92800}{7}\pi^2\ln^3(2)\zeta_3\\
&-\frac{19840}{7}\pi^2\ln^4(2)\zeta_3+\frac{55680}{7}\ln^5(2)\zeta_3-\frac{7424}{7}\ln^6(2)\zeta_3-\frac{921600}{7}\pi^2\text{Li}_4\left(\frac{1}{2}\right)\zeta_3\\
&-\frac{6681600}{7}\text{Li}_5\left(\frac{1}{2}\right)\zeta_3-\frac{5345280}{7}\text{Li}_6\left(\frac{1}{2}\right)\zeta_3-3011328s_6\zeta_3-234850\zeta_3^2-\frac{720600}{7}\pi^2\zeta_3^2\\
&+\frac{3364000}{7}\ln(2)\zeta_3^2-\frac{3448560}{49}\pi^2\ln(2)\zeta_3^2-\frac{550800}{7}\ln^2(2)\zeta_3^2+\frac{73440}{7}\ln^3(2)\zeta_3^2\\
&+\frac{6408524\zeta_3^3}{9}-\frac{1283555\zeta_5}{2}+\frac{4725550\pi^2\zeta_5}{21}-\frac{690281971\pi^4\zeta_5}{4410}-\frac{1958400}{7}\pi^2\ln(2)\zeta_5\\
&+809100\ln^2(2)\zeta_5-\frac{2232728}{7}\pi^2\ln^2(2)\zeta_5+\frac{1545608}{7}\ln^4(2)\zeta_5+\frac{37094592\text{Li}_4\left(\frac{1}{2}\right)\zeta_5}{7}\\
&+\frac{4771560\zeta_3\zeta_5}{7}+\frac{29334280}{7}\ln(2)\zeta_3\zeta_5-\frac{28016465\zeta_7}{7}-\frac{70452569}{98}\pi^2\zeta_7\\
&-\left.\frac{14706092}{7}\ln^2(2)\zeta_7+\frac{216126121\zeta_9}{14}\right)
\end{aligned}
\tag{A.61}
$$

$$
\begin{aligned}
I_{15} \;=\;& \frac{1}{2\varepsilon^3}+\frac{7}{4\varepsilon^2}+\frac{1}{24\varepsilon}\left(75+8\pi^2\right)-\frac{5}{16}+\frac{7\pi^2}{6}+4\zeta_3+\varepsilon\left(-\frac{959}{32}+\frac{25\pi^2}{12}+\frac{16\pi^4}{45}+14\zeta_3\right)\\
&+\varepsilon^2\left(-\frac{10493}{64}+\frac{56\pi^4}{45}+\pi^2\left(-\frac{5}{24}+\frac{8\zeta_3}{3}\right)+25\zeta_3+72\zeta_5\right)+\varepsilon^3\left(-\frac{85175}{128}+\frac{20\pi^4}{9}\right.\\
&\left.+\frac{458\pi^6}{945}+\pi^2\left(-\frac{959}{48}+\frac{28\zeta_3}{3}\right)-\frac{5}{2}\zeta_3+16\zeta_3^2+252\zeta_5\right)+\varepsilon^4\left(-\frac{610085}{256}+\frac{229\pi^6}{135}\right.\\
&+\pi^2\left(-\frac{10493}{96}+\frac{50\zeta_3}{3}+48\zeta_5\right)+\pi^4\left(-\frac{2}{9}+\frac{128\zeta_3}{45}\right)-\frac{959}{4}\zeta_3+56\zeta_3^2+450\zeta_5\\
&\left.+996\zeta_7\right)+\varepsilon^5\left(-\frac{4087919}{512}+\frac{1145\pi^6}{378}+\frac{3337\pi^8}{4725}+\pi^2\left(-\frac{85175}{192}-\frac{5}{3}\zeta_3+\frac{32}{3}\zeta_3^2\right.\right.\\
&\left.\left.+168\zeta_5\right)+\pi^4\left(-\frac{959}{45}+\frac{448\zeta_3}{45}\right)+\left(-\frac{10493}{8}+576\zeta_5\right)\zeta_3+100\zeta_3^2-45\zeta_5+3486\zeta_7\right)
\end{aligned}
$$

$$+\varepsilon^6\Bigg(-\frac{26332493}{1024}+\frac{3337\pi^8}{1350}+\pi^4\Bigg(-\frac{10493}{90}+\frac{160\zeta_3}{9}+\frac{256\zeta_5}{5}\Bigg)+\pi^2\Bigg(-\frac{610085}{384}$$
$$-\frac{959}{6}\zeta_3+\frac{112}{3}\zeta_3^2+300\zeta_5+664\zeta_7\Bigg)+\pi^6\Bigg(-\frac{229}{756}+\frac{3664\zeta_3}{945}\Bigg)-\Bigg(\frac{85175}{16}-2016\zeta_5\Bigg)\zeta_3$$
$$-10\zeta_3^2+\frac{128}{3}\zeta_3^3-\frac{8631}{2}\zeta_5+6225\zeta_7+\frac{40240}{3}\zeta_9\Bigg)+\varepsilon^7\Bigg(-\frac{165503975}{2048}+\frac{3337\pi^8}{756}$$
$$+\frac{70336\pi^{10}}{66825}+\pi^4\Bigg(-\frac{17035}{36}-\frac{16}{9}\zeta_3+\frac{512}{45}\zeta_3^2+\frac{896}{5}\zeta_5\Bigg)+\pi^2\Bigg(-\frac{4087919}{768}+\Bigg(-\frac{10493}{12}$$
$$+384\zeta_5\Bigg)\zeta_3+\frac{200}{3}\zeta_3^2-30\zeta_5+2324\zeta_7\Bigg)+\pi^6\Bigg(-\frac{31373}{1080}+\frac{1832\zeta_3}{135}\Bigg)+\Bigg(-\frac{610085}{32}$$
$$+3600\zeta_5+7968\zeta_7\Bigg)\zeta_3-959\zeta_3^2+\frac{448}{3}\zeta_3^3-\frac{94437}{4}\zeta_5+5184\zeta_5^2-\frac{1245}{2}\zeta_7+\frac{140840}{3}\zeta_9\Bigg)$$
$$+\varepsilon^8\Bigg(-\frac{1023933365}{4096}+\frac{246176\pi^{10}}{66825}+\pi^6\Bigg(-\frac{343271}{2160}+\frac{4580\zeta_3}{189}+\frac{7328\zeta_5}{105}\Bigg)$$
$$+\pi^4\Bigg(-\frac{122017}{72}-\frac{7672}{45}\zeta_3+\frac{1792}{45}\zeta_3^2+320\zeta_5+\frac{10624}{15}\zeta_7\Bigg)+\pi^2\Bigg(-\frac{26332493}{1536}$$
$$+\Bigg(-\frac{85175}{24}+1344\zeta_5\Bigg)\zeta_3-\frac{20}{3}\zeta_3^2+\frac{256}{9}\zeta_3^3-2877\zeta_5+4150\zeta_7+\frac{80480}{9}\zeta_9\Bigg)$$
$$+\pi^8\Bigg(-\frac{3337}{7560}+\frac{26696\zeta_3}{4725}\Bigg)+\Bigg(-\frac{4087919}{64}-360\zeta_5+27888\zeta_7\Bigg)\zeta_3+\Bigg(-\frac{10493}{2}$$
$$+2304\zeta_5\Bigg)\zeta_3^2+\frac{800}{3}\zeta_3^3-\frac{766575}{8}\zeta_5+18144\zeta_5^2-\frac{238791}{4}\zeta_7+\frac{251500}{3}\zeta_9+182412\zeta_{11}\Bigg) \tag{A.62}$$

$$I_{16} = -\frac{1}{6\varepsilon^2}-\frac{35}{36\varepsilon}-\frac{559}{216}-\frac{\pi^2}{3}+\varepsilon\Bigg(\frac{2737}{1296}-\frac{35\pi^2}{18}-\frac{16}{3}\zeta_3\Bigg)+\varepsilon^2\Bigg(\frac{552041}{7776}-\frac{559\pi^2}{108}-\frac{37\pi^4}{45}$$
$$-\frac{280}{9}\zeta_3\Bigg)+\varepsilon^3\Bigg(\frac{25027345}{46656}-\frac{259\pi^4}{54}+\pi^2\Bigg(\frac{2737}{648}-\frac{32\zeta_3}{3}\Bigg)-\frac{2236}{27}\zeta_3-208\zeta_5\Bigg)$$
$$+\varepsilon^4\Bigg(\frac{855963737}{279936}-\frac{20683\pi^4}{1620}-\frac{2318\pi^6}{945}+\pi^2\Bigg(\frac{552041}{3888}-\frac{560\zeta_3}{9}\Bigg)+\frac{5474}{81}\zeta_3-\frac{256}{3}\zeta_3^2$$
$$-\frac{3640}{3}\zeta_5\Bigg)+\varepsilon^5\Bigg(\frac{25728647329}{1679616}-\frac{1159\pi^6}{81}+\pi^2\Bigg(\frac{25027345}{23328}-\frac{4472\zeta_3}{27}-416\zeta_5\Bigg)$$
$$+\pi^4\Bigg(\frac{101269}{9720}-\frac{1184\zeta_3}{45}\Bigg)+\frac{552041}{243}\zeta_3-\frac{4480}{9}\zeta_3^2-\frac{29068}{9}\zeta_5-6168\zeta_7\Bigg)$$
$$+\varepsilon^6\Bigg(\frac{718599153833}{10077696}-\frac{647881\pi^6}{17010}-\frac{37189\pi^8}{4725}+\pi^2\Bigg(\frac{855963737}{139968}+\frac{10948}{81}\zeta_3-\frac{512}{3}\zeta_3^2$$

$$
-\frac{7280}{3}\zeta_5\Bigg) + \pi^4\left(\frac{20425517}{58320} - \frac{4144\zeta_3}{27}\right) + \left(\frac{25027345}{1458} - 6656\zeta_5\right)\zeta_3 - \frac{35776}{27}\zeta_3^2
$$

$$
+\frac{71162}{27}\zeta_5 - 35980\zeta_7\Bigg) + \varepsilon^7\Bigg(\frac{19166358676465}{60466176} - \frac{37189\pi^8}{810} + \pi^4\Bigg(\frac{185202353}{69984}
$$

$$
-\frac{165464\zeta_3}{405} - \frac{15392\zeta_5}{15}\Bigg) + \pi^2\Bigg(\frac{25728647329}{839808} + \frac{1104082}{243}\zeta_3 - \frac{8960}{9}\zeta_3^2 - \frac{58136}{9}\zeta_5
$$

$$
-12336\zeta_7\Bigg) + \pi^6\left(\frac{453169}{14580} - \frac{74176\zeta_3}{945}\right) + \left(\frac{855963737}{8748} - \frac{116480\zeta_5}{3}\right)\zeta_3 + \frac{87584}{81}\zeta_3^2
$$

$$
-\frac{8192}{9}\zeta_3^3 + \frac{7176533}{81}\zeta_5 - \frac{287326}{3}\zeta_7 - \frac{1629280}{9}\zeta_9\Bigg) + \varepsilon^8\Bigg(\frac{495815946702713}{362797056}
$$

$$
-\frac{20788651\pi^8}{170100} - \frac{1738426\pi^{10}}{66825} + \pi^4\left(\frac{31670658269}{2099520} + \frac{405076\zeta_3}{1215} - \frac{18944}{45}\zeta_3^2 - \frac{53872}{9}\zeta_5\right)
$$

$$
+\pi^2\left(\frac{718599153833}{5038848} + \left(\frac{25027345}{729} - 13312\zeta_5\right)\zeta_3 - \frac{71552}{27}\zeta_3^2 + \frac{142324}{27}\zeta_5 - 71960\zeta_7\right)
$$

$$
+\pi^6\left(\frac{91402217}{87480} - \frac{37088\zeta_3}{81}\right) + \left(\frac{25728647329}{52488} - \frac{930176\zeta_5}{9} - 197376\zeta_7\right)\zeta_3
$$

$$
+\frac{8832656}{243}\zeta_3^2 - \frac{143360}{27}\zeta_3^3 + \frac{325355485}{486}\zeta_5 - 129792\zeta_5^2 + \frac{703409}{9}\zeta_7 - \frac{28512400}{27}\zeta_9\Bigg)
\tag{A.63}
$$

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References

[1] W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, T. Leineweber, P. Mastrolia, and E. Remiddi, Nucl. Phys. **B706** (2005) 245–324 [hep-ph/0406046].

[2] W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, T. Leineweber, P. Mastrolia, and E. Remiddi, Nucl. Phys. **B712** (2005) 229–286 [hep-ph/0412259].

[3] W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, T. Leineweber, and E. Remiddi, Nucl. Phys. **B723** (2005) 91–116 [hep-ph/0504190].

[4] W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, P. Mastrolia, and E. Remiddi, Phys. Rev. **D72** (2005) 096002 [hep-ph/0508254].

[5] J. Gluza, A. Mitov, S. Moch, and T. Riemann, JHEP **07** (2009) 001 [arXiv:0905.1137 [hep-ph]].

[6] J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, N. Rana and C. Schneider, Phys. Rev. D **97** (2018) no.9, 094022 [arXiv:1712.09889 [hep-ph]].

[7] J. Henn, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, JHEP **1701** (2017) 074 [arXiv:1611.07535 [hep-ph]].

[8] J.M. Henn, A.V. Smirnov, and V.A. Smirnov, JHEP **12** (2016) 144 [arXiv:1611.06523 [hep-ph]].

[9] J. Ablinger, J. Blümlein, P. Marquard, N. Rana and C. Schneider, Phys. Lett. B **782** (2018) 528–532 [arXiv:1804.07313 [hep-ph]].

[10] R.N. Lee, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, JHEP **03** (2018) 136 [arXiv:1801.08151 [hep-ph]].

[11] R.N. Lee, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, JHEP **05** (2018) 187 [arXiv:1804.07310 [hep-ph]].

[12] J. Ablinger, J. Blümlein, P. Marquard, N. Rana and C. Schneider, Nucl. Phys. B **939** (2019) 253–291. [arXiv:1810.12261 [hep-ph]].

[13] T. Ahmed, J.M. Henn and M. Steinhauser, JHEP **06** (2017) 125 [arXiv:1704.07846 [hep-ph]].

[14] J. Blümlein, P. Marquard and N. Rana, Phys. Rev. D **99** (2019) no.1, 016013 [arXiv:1810.08943 [hep-ph]].

[15] J. Blümlein and C. Schneider, Phys. Lett. B **771** (2017) 31–36, [arXiv:1701.04614 [hep-ph]].

[16] J. Lagrange, *Nouvelles recherches sur la nature et la propagation du son*, Miscellanea Taurinensis t. **II** (1760-61) 263.

[17] C.F. Gauß, *Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo novo tractate*, Commentationes societas scientiarum Gottingensis recentiores **III** (1813) 5–7.

[18] G. Green, *Essay on the mathematical theory of electricity and magnetism*, (Nottingham, 1828), printed for the author, by T. Wheelhouse, Green Papers, pp. 1–115.

[19] M. Ostrogradski, Mem. Ac. Sci. St. Peters. **6** (1831) 129–133.

[20] K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B **192** (1981) 159–204.

[21] S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087–5159 [hep-ph/0102033].

[22] C. Studerus, Comput. Phys. Commun. **181** (2010) 1293–1300, [arXiv:0912.2546 [physics.comp-ph]];
A. von Manteuffel and C. Studerus, *Reduze 2 - Distributed Feynman Integral Reduction*, arXiv:1201.4330 [hep-ph].

[23] P. Marquard and D. Seidel, *The* `Crusher` *algorithm*, (unpublished).

[24] C. Schneider, in: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, Texts and Monographs in Symbolic Computation eds. C. Schneider and J. Blümlein (Springer, Wien, 2013), 325–360 [arXiv:1304.4134 [cs.SC]].

[25] J. Blümlein, D.J. Broadhurst and J.A.M. Vermaseren, Comput. Phys. Commun. **181** (2010) 582–625, [arXiv:0907.2557 [math-ph]].

[26] M. Kauers, M. Jaroschek, and F. Johansson, in: Computer Algebra and Polynomials, Editors: J. Gutierrez, J. Schicho, Josef, M. Weimann, Lecture Notes in Computer Science **8942** (Springer, Berlin, 2015) 105–125, [arXiv:1306.4263 [cs.SC]].

[27] J. Blümlein, M. Kauers, S. Klein, and C. Schneider, Comput. Phys. Commun. **180** (2009) 2143–2165,

[28] C. Schneider, Sém. Lothar. Combin. **56** (2007) 1–36.

[29] J. Blümlein and C. Schneider, Int. J. Mod. Phys. A **33** (2018) no.17, 1830015 [arXiv:1809.02889 [hep-ph]].

[30] E. Remiddi and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) 725–754 [hep-ph/9905237].

[31] P. Nogueira, J. Comput. Phys. **105** (1993) 279–289.

[32] T. van Ritbergen, A.N. Schellekens and J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 41–96 [hep-ph/9802376].

[33] R. Harlander, T. Seidensticker, and M. Steinhauser, Phys. Lett. **B426** (1998) 125–132, [hep-ph/9712228].

[34] T. Seidensticker, in: Proc. of the 6th International Workshop on New Computing Techniques in Physics Research (AIHENP 99) Heraklion, Crete, Greece, April 12-16, 1999, hep-ph/9905298.

[35] J.A.M. Vermaseren, *New features of FORM*, math-ph/0010025.

[36] M. Tentyukov and J.A.M. Vermaseren, Comput. Phys. Commun. **181** (2010) 1419–1427 [hep-ph/0702279].

[37] C. Schneider, in preparation.

[38] K. Melnikov and T. van Ritbergen, Nucl. Phys. B **591** (2000) 515–546 [hep-ph/0005131].

[39] P. Marquard, J.H. Piclum, D. Seidel and M. Steinhauser, Nucl. Phys. B **758** (2006) 144–160 [hep-ph/0607168].

[40] A. Kurz, T. Liu, P. Marquard, A. Smirnov, V. Smirnov and M. Steinhauser, Phys. Rev. D **93** (2016) no.5, 053017 doi:10.1103/PhysRevD.93.053017 [arXiv:1602.02785 [hep-ph]].

[41] P. Marquard, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *Phys. Rev. Lett.* **114** (2015) no.14, 142002 [arXiv:1502.01030 [hep-ph]].

[42] P. Marquard, L. Mihaila, J.H. Piclum, and M. Steinhauser, Nucl. Phys. **B773** (2007) 1–18 [arXiv:hep-ph/0702185 [hep-ph]].

[43] I. Bierenbaum, J. Blumlein and S. Klein, Nucl. Phys. B **820** (2009) 417–482 [arXiv:0904.3563 [hep-ph]].

[44] J. Blümlein, J. Ablinger, A. Behring, A. De Freitas, A. von Manteuffel, C. Schneider and C. Schneider, PoS (QCDEV) (2017) 031 [arXiv:1711.07957 [hep-ph]].

[45] Sage, http://www.sagemath.org/.

[46] M. Karr, J. ACM **28** (1981) 305–350.

[47] M. Bronstein, J. Symbolic Comput. **29** (2000), no. 6 841–877.

[48] C. Schneider, Symbolic Summation in Difference Fields, Ph.D. Thesis RISC, Johannes Kepler University, Linz technical report 01–17 (2001).

[49] C. Schneider, An. Univ. Timisoara Ser. Mat.-Inform. **42** (2004) 163–179.

[50] C. Schneider, J. Differ. Equations Appl. **11** (2005) 799–821.

[51] C. Schneider, Appl. Algebra Engrg. Comm. Comput. **16** (2005) 1–32.

[52] C. Schneider, J. Algebra Appl. **6** (2007) 415–441.

[53] C. Schneider, Motives, Quantum Field Theory, and Pseudodifferential Operators, Clay Mathematics Proceedings Vol. **12**, eds. A. Carey, D. Ellwood, S. Paycha and S. Rosenberg, (Amer. Math. Soc) (2010), 285–308, [arXiv:0904.2323].

[54] C. Schneider, Ann. Comb. **14** (2010) 533–552, [arXiv:0808.2596].

[55] C. Schneider, in: Computer Algebra and Polynomials, Applications of Algebra and Number Theory, J. Gutierrez, J. Schicho, M. Weimann (ed.), Lecture Notes in Computer Science (LNCS) 8942 (2015), 157–191, [arXiv:1307.7887 [cs.SC]].

[56] C. Schneider, J. Symbolic Comput. **43** (2008) 611–644 [arXiv:0808.2543].

[57] C. Schneider, J. Symb. Comput. **72** (2016) 82–127 [arXiv:1408.2776 [cs.SC]].

[58] C. Schneider, J. Symb. Comput. **80** (2017) 616–664 [arXiv:1603.04285[cs.SC]].

[59] J. Vermaseren, Int. J. Mod. Phys. **A14** (1999) 2037–2076 [hep-ph/9806280].

[60] J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [hep-ph/9810241].

[61] J. Ablinger, J. Blümlein, and C. Schneider, J. Math. Phys. **54** (2013) 082301 [arXiv: 1302.0378 [math-ph]].

[62] J. Ablinger, J. Blümlein, and C. Schneider, J. Math. Phys. **52** (2011) 102301 [arXiv: 1105.6063 [math-ph]].

[63] J. Ablinger, J. Blümlein, C.G. Raab, and C. Schneider, J. Math. Phys. **55** (2014) 112301 [arXiv: 1407.1822 [hep-th]].

[64] J. Ablinger, PoS (LL2014) 019 [arXiv:1407.6180[cs.SC]].

[65] J. Ablinger, *A Computer Algebra Toolbox for Harmonic Sums Related to Particle Physics*, Diploma Thesis, JKU Linz, 2009, arXiv:1011.1176[math-ph].

[66] J. Ablinger, *Computer Algebra Algorithms for Special Functions in Particle Physics*, Ph.D. Thesis, Linz U. (2012) arXiv:1305.0687[math-ph].

[67] J. Ablinger, PoS (RADCOR2017) 001 [arXiv:1801.01039 [cs.SC]].

[68] E.E. Kummer, J. Reine Angew. Math. (Crelle) **21** (1840) 74–90; 193–225; 328–371.

[69] H. Poincaré, Acta Math. **4** (1884) 201–312.

[70] S. Moch, P. Uwer, and S. Weinzierl, J. Math. Phys. **43** (2002) 3363–3386 [hep-ph/0110083].

[71] J.C. Collins, *Algorithm to Compute Corrections to the Sudakov Form-factor*, Phys. Rev. D **22** (1980) 1478–1489; *Sudakov form-factors*, Adv. Ser. Direct. High Energy Phys. **5** (1989) 573–614, Ed. A.H. Mueller (World Scientific, Singapore, 1989). [hep-ph/0312336].

[72] S. Catani, Phys. Lett. B **427** (1998) 161–171 [hep-ph/9802439].

[73] G.F. Sterman and M.E. Tejeda-Yeomans, Phys. Lett. B **552** (2003) 48–56 [hep-ph/0210130].

[74] T. Becher and M. Neubert, Phys. Rev. Lett. **102** (2009) 162001 Erratum: [Phys. Rev. Lett. **111** (2013) no.19, 199905] [arXiv:0901.0722 [hep-ph]].

[75] E. Gardi and L. Magnea, JHEP **0903** (2009) 079 [arXiv:0901.1091 [hep-ph]].

[76] V. Ravindran, J. Smith and W.L. van Neerven, Nucl. Phys. B **704** (2005) 332–348 [hep-ph/0408315].

[77] S. Moch, J.A.M. Vermaseren and A. Vogt, Phys. Lett. B **625** (2005) 245–252 [hep-ph/0508055].

[78] A. Mitov and S. Moch, JHEP **0705** (2007) 001 [hep-ph/0612149].

[79] A.A. Penin, Nucl. Phys. B **734** (2006) 185–202 [hep-ph/0508127].

[80] T. Becher and K. Melnikov, JHEP **0706** (2007) 084 [arXiv:0704.3582 [hep-ph]].

[81] V. Ravindran, Nucl. Phys. B **746** (2006) 58–76 [hep-ph/0512249].

[82] V. Ravindran, Nucl. Phys. B **752** (2006) 173–196 [hep-ph/0603041].

[83] T. Ahmed, M. Mahakhud, N. Rana and V. Ravindran, Phys. Rev. Lett. **113** (2014) no.11, 112002 [arXiv:1404.0366 [hep-ph]].

[84] T. Ahmed, N. Rana and V. Ravindran, JHEP **1410** (2014) 139 [arXiv:1408.0787 [hep-ph]].

[85] T. Becher and M. Neubert, Phys. Rev. D **79** (2009) 125004 Erratum: [Phys. Rev. D **80** (2009) 109901] [arXiv:0904.1021 [hep-ph]].

[86] S. Weinberg, Phys. Lett. **91B** (1980) 51–55.

[87] B.A. Ovrut and H.J. Schnitzer, Phys. Lett. **100B** (1981) 403–406.

[88] W. Wetzel, Nucl. Phys. B **196** (1982) 259–272.

[89] W. Bernreuther and W. Wetzel, Nucl. Phys. B **197** (1982) 228–236 Erratum: [Nucl. Phys. B **513** (1998) 758].

[90] W. Bernreuther, Annals Phys. **151** (1983) 127–162.

[91] K.G. Chetyrkin, B.A. Kniehl and M. Steinhauser, Nucl. Phys. B **510** (1998) 61–87 [hep-ph/9708255].

[92] G.P. Korchemsky and A.V. Radyushkin, Nucl. Phys. B **283** (1987) 342–364.

[93] G.P. Korchemsky and A.V. Radyushkin, Phys. Lett. B **279** (1992) 359–366. [hep-ph/9203222].

[94] A. Grozin, J.M. Henn, G.P. Korchemsky and P. Marquard, Phys. Rev. Lett. **114** (2015) no.6, 062006 [arXiv:1409.0023 [hep-ph]].

[95] A. Grozin, J.M. Henn, G.P. Korchemsky and P. Marquard, JHEP **1601** (2016) 140 [arXiv:1510.07803 [hep-ph]].

[96] A.G. Grozin, P. Marquard, J.H. Piclum and M. Steinhauser, Nucl. Phys. B **789** (2008) 277–293 [arXiv:0707.1388 [hep-ph]].

[97] A.G. Grozin, M. Höschele, J. Hoff, and M. Steinhauser, JHEP **1109** (2011) 066 [arXiv:1107.5970 [hep-ph]].

[98] S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren, *The FORM version of MINCER* NIKHEF-H-91-18.

[99] M. Steinhauser, Comput. Phys. Commun. **134** (2001) 335–364 [hep-ph/0009029].

[100] R. Boughezal, M. Czakon and T. Schutzmeier, Nucl. Phys. Proc. Suppl. **160** (2006) 160–164 [hep-ph/0607141].

[101] A. Maier, P. Maierhofer and P. Marquard, Nucl. Phys. B **797** (2008) 218–242 [arXiv:0711.2636 [hep-ph]].

[102] A. Maier and P. Marquard, Nucl. Phys. B **859** (2012) 1–12 [arXiv:1110.5581 [hep-ph]].

[103] A. Maier and P. Marquard, Nucl. Phys. B **899** (2015) 451–462 [arXiv:1506.00900 [hep-ph]].

[104] P. Maierhöfer and P. Marquard, Phys. Lett. B **721** (2013) 131–135 [arXiv:1212.6233 [hep-ph]].

[105] R.N. Lee, A.V. Smirnov and V.A. Smirnov, JHEP **1803** (2018) 008 [arXiv:1709.07525 [hep-ph]].

[106] K. Kudashkin, K. Melnikov and C. Wever, JHEP **1802** (2018) 135 [arXiv:1712.06549 [hep-ph]].

[107] J. Ablinger, A. Behring, J. Blümlein A. De Freitas, A. von Manteuffel and C. Schneider, Comput. Phys. Commun. 202 (2016) 33–112, [arXiv:1509.08324[hep-ph]]

[108] S. Gerhold, Uncoupling systems of linear Ore operator equations, Master's thesis, RISC, J. Kepler University, Linz, 2002.

[109] B. Zürcher, Rationale Normalformen von pseudo-linearen Abbildungen, Master's thesis, Mathematik, ETH Zürich (1994).

[110] J. Blümlein, S. Klein, C. Schneider and F. Stan, J. Symbolic Comput. **47** (2012) 1267–1289 [arXiv:1011.2656 [cs.SC]].

[111] J. Ablinger, J. Blümlein, A. De Freitas, M. van Hoeij, E. Imamoglu, C.G. Raab, C. S. Radu and C. Schneider, J. Math. Phys. **59** (2018) no.6, 062305 [arXiv:1706.01299 [hep-th]].

[112] D.J. Broadhurst, N. Gray, and K. Schilcher, Z. Phys. **C52** (1991) 111–122.

[113] P. Marquard, A.V. Smirnov, V.A. Smirnov, M. Steinhauser and D. Wellmann, Phys. Rev. D **94** (2016) no.7, 074025 [arXiv:1606.06754 [hep-ph]].

[114] P. Marquard, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, Phys. Rev. D **97** (2018) no.5, 054032 [arXiv:1801.08292 [hep-ph]].

[115] O.V. Tarasov, A.A. Vladimirov and A.Y. Zharkov, Phys. Lett. **93B** (1980) 429–432

[116] S.A. Larin and J.A.M. Vermaseren, Phys. Lett. B **303** (1993) 334–336 [hep-ph/9302208].

[117] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, Phys. Lett. B **400** (1997) 379–384 [hep-ph/9701390].

[118] M. Czakon, Nucl. Phys. B **710** (2005) 485–498 [hep-ph/0411261].

[119] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. **118** (2017) no.8, 082002 [arXiv:1606.08659 [hep-ph]].

[120] F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, JHEP **02** (2017) 090 [arXiv:1701.01404 [hep-ph]].

[121] T. Luthe, A. Maier, P. Marquard and Y. Schröder, JHEP **10** (2017) 166 [arXiv:1709.07718 [hep-ph]].

[122] A.I. Davydychev and M.Y. Kalmykov, Nucl. Phys. B **699** (2004) 3–64 [hep-th/0303162].

[123] D.J. Broadhurst, Z. Phys. C **54** (1992) 599–606.

[124] D.J. Broadhurst, J. Fleischer and O.V. Tarasov, Z. Phys. C **60** (1993) 287–302 [hep-ph/9304303].

[125] J. Blümlein, A. Hasselhuhn and C. Schneider, PoS (RADCOR2011) 032 [arXiv:1202.4303 [math-ph]].

[126] C. Schneider, J. Phys. Conf. Ser. **523** (2014) 012037 [arXiv:1310.0160 [cs.SC]].

[127] J. Vermaseren, private communication, July 2018.