# Top quark mass effects in $g g \rightarrow Z Z$ at two loops and off-shell Higgs interference 

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#### Abstract

We consider top-quark mass effects in the Higgs-interference contribution to $Z$-boson pair production in gluon fusion. While this production mechanism is formally of next-to-next-to leading order, its contribution is numerically important above the top threshold $M_{Z Z}^{2}=4 m_{t}^{2}$. This region is essential to constrain the width of the Higgs boson and good control over the top-quark mass dependence is crucial. We determine the form factors that are relevant for the interference contribution at two-loop order using a method based on a conformal mapping and Padé approximants constructed from the expansions of the amplitude for large top mass and around the top threshold.


## I. INTRODUCTION

A direct measurement of the Higgs boson width $\Gamma_{H}$ is not possible at the LHC or even the envisioned next generation of collider experiments. However, indirect constraints can be obtained at the LHC by studying the process $p p \rightarrow H \rightarrow Z Z(\rightarrow 4 l)$ on the Higgs boson peak where the cross section depends on the combination $g_{H g g}^{2} g_{H Z Z}^{2} / \Gamma_{H}$ and off the peak where the measurement of the cross section constrains the product $g_{H g g}^{2} g_{H Z Z}^{2}$ of the effective Higgs boson-gluon coupling $g_{H g g}$ and the Higgs boson- $Z$ boson coupling $g_{H Z Z}$, as proposed in [1[3]. ${ }^{1}$ The same strategy can be employed with $W W$ final states [6]. The latest studies from the LHC experiments give an upper limit of 14.4 MeV at $95 \%$ C.L. from the $Z Z$ final state at ATLAS [7] and the value $3.2_{-2.2}^{+2.8} \mathrm{MeV}$ from the combination of $V V$ final states in CMS [8], close to the SM prediction $\Gamma_{H}^{S M}=4.10 \pm 0.06 \mathrm{MeV}$ [9]. Measurements of the Higgs boson signal at large invariant mass can also be used to directly constrain physics beyond the Standard Model in the Higgs sector [10 13].

Here, we focus on the loop-induced continuum gluon fusion process $g g \rightarrow Z Z$ and in particular its interference with the off-shell Higgs contribution $g g \rightarrow H^{*} \rightarrow Z Z$. Despite the narrow width of the Higgs boson these interference effects are sizable with $10 \%$ of the Higgs signal stemming from the off-shell region where the invariant mass of the two decay products is greater than $2 m_{Z}$ [1] and higher-order corrections are required to control the uncertainties. The Higgs-mediated amplitude only de-

[^0]pends on two scales, the mass $m_{q}$ of the quark in the loop and the invariant mass $M_{Z Z}$ of the final state. Next-to-leading order (NLO) corrections with the full quarkmass dependence have been known for some time [14-17, and the top-quark mass dependence at next-to-next-to leading order (NNLO) has been reconstructed very recently [18] (see also [19]). On the other hand the continuum amplitude depends on four scales $m_{q}, m_{Z}, M_{Z Z}$ and the transverse momentum $p_{T}$ of one of the $Z$ bosons, and the exact result is only known at leading order (LO) [20] while an analytic NLO calculation appears extremely challenging. In the massless limit $m_{q}=0$ the two-loop amplitude has been determined in $21+23$ and the NLO cross section in 24]. Recently, also the quark-gluon channel has been included 25].

The contribution from top quarks at two-loop order has been computed in a large-mass expansion (LME) [24, [26, 27] and is known up to $1 / m_{t}^{12}$. While the contribution from massless quarks dominates the interference correction at small invariant masses, the top-quark contribution is of the same size near the top threshold $M_{Z Z}=4 m_{t}^{2}$ and dominates in the large invariant-mass regime. Since the LME ceases to provide a reliable description above the top threshold, the authors of 27 have improved their prediction by a conformal mapping and the construction of Padé approximants based on the available number of LME coefficients. In [28] we have extended this method by considering the expansion around the top threshold in addition to the LME and demonstrated that the top-mass effects can be reproduced correctly by comparing results for the two-loop amplitude for $g g \rightarrow H H$ with the numerical calculation from [29] [31. ${ }^{2}$

[^1]In this work we consider the form factors of the continuum $g g \rightarrow Z Z$ amplitude that are relevant for the interference contribution at one and two loops. The nonanalytic terms in the expansion around the top threshold are computed up to at least order $(1-z)^{4}$, where $z=M_{Z Z}^{2} /\left(4 m_{t}^{2}\right)+i 0$, and used to construct Padé approximants. Together with the exactly known real NLO top quark [27, 38] and the massless quark corrections [21, 25] this is sufficient to determine the full NLO interference contribution with realistic top-quark mass dependence.

## II. FORM FACTORS FOR INTERFERENCE

Up to the two loop level, the amplitude for the topmediated non-resonant continuum production process $g\left(\mu, A, p_{1}\right)+g\left(\nu, B, p_{2}\right) \rightarrow Z\left(\alpha, p_{3}\right)+Z\left(\beta, p_{4}\right)$ receives contributions from both box and double-triangle diagrams, see figure 1 The latter are known for arbitrary quark masses [27, 39] and will not be discussed in the following.


FIG. 1: Examples for box (left) and double-triangle (right) top-mediated contributions to $g g \rightarrow Z Z$.

The box amplitude $\left|B_{\mu \nu \alpha \beta}^{A B}\right\rangle$ has a complicated tensor structure [20 22, 40]. However, the interference with the Higgs-mediated amplitude is described by a single form factor. Adopting the conventions of [27] it takes the form

$$
\begin{equation*}
|\mathcal{B}\rangle=\frac{\delta^{A B}}{N_{A}}\left(p_{1} \cdot p_{2} g^{\mu \nu}-p_{1}^{\nu} p_{2}^{\mu}\right) P_{Z}^{\alpha \rho}\left(p_{3}\right) P_{Z, \rho}^{\beta}\left(p_{4}\right)\left|B_{\mu \nu \alpha \beta}^{A B}\right\rangle, \tag{1}
\end{equation*}
$$

with $N_{A}=N_{c}^{2}-1$ and $P_{Z}^{\alpha \rho}(p)=-g^{\alpha \rho}+p^{\alpha} p^{\rho} / m_{Z}^{2}$. The form factor can be decomposed into a vector and axialvector part

$$
\begin{equation*}
|\mathcal{B}\rangle=\frac{i g_{W}^{2}}{4 \cos ^{2} \theta_{W}}\left(v_{t}^{2}\left|\widetilde{\mathcal{B}}_{V V}\right\rangle+a_{t}^{2}\left|\widetilde{\mathcal{B}}_{A A}\right\rangle\right) \tag{2}
\end{equation*}
$$

where $a_{t}=1 / 2$ and $v_{t}=1 / 2-4 / 3 \sin ^{2} \theta_{W}$ denote the axial-vector and vector couplings for an up-type quark. Mixed $v_{t} a_{t}$ terms are forbidden by charge conjugation symmetry. The order in the strong coupling constant $\alpha_{s}$ is indicated as follows

$$
\begin{equation*}
\left|\widetilde{\mathcal{B}}_{i}\right\rangle=\frac{\alpha_{s}}{4 \pi}\left|\widetilde{\mathcal{B}}_{i}^{(1)}\right\rangle+\left(\frac{\alpha_{s}}{4 \pi}\right)^{2}\left|\widetilde{\mathcal{B}}_{i}^{(2)}\right\rangle+\ldots, \tag{3}
\end{equation*}
$$

[^2]with $i=V V, A A$. At order $\alpha_{s}^{2}$ the renormalized form factors contain IR divergences, which cancel in the combination with real corrections, and we define the finite remainder by applying the subtraction $41{ }^{3}$
\[

$$
\begin{equation*}
\left|\widetilde{\mathcal{F}}_{i}^{(2)}\right\rangle=\left|\widetilde{\mathcal{B}}_{i}^{(2)}\right\rangle+\frac{e^{\epsilon \gamma_{E}}}{\Gamma(1-\epsilon)}\left[\frac{2 C_{A}}{\epsilon^{2}}\left(\frac{\mu^{2}}{-s}\right)^{\epsilon}+\frac{\beta_{0}}{\epsilon}\right]\left|\widetilde{\mathcal{B}}_{i}^{(1)}\right\rangle \tag{4}
\end{equation*}
$$

\]

where $\beta_{0}=\frac{11}{3} C_{A}-\frac{4}{3} T_{f} n_{l}, C_{A}=3, T_{f}=\frac{1}{2}, n_{l}=5$, and the form factors $\left|\widetilde{\mathcal{B}}_{i}^{(1,2)}\right\rangle$ are defined in $d=4-2 \epsilon$ dimensions. The one-loop form-factors $\left|\widetilde{\mathcal{B}}_{i}^{(1)}\right\rangle$ are already finite; we define $\left|\widetilde{\mathcal{F}}_{i}^{(1)}\right\rangle=\left|\widetilde{\mathcal{B}}_{i}^{(1)}\right\rangle$ for the sake of a consistent notation.

## A. The amplitude near threshold

Above the top threshold at $z=1$ the top quarks in the loop can go on shell which manifests as non-analytic terms in the expansion of the form factors in $\bar{z} \equiv 1-z$, generating a sizable imaginary part. As shown in [28] the knowledge of these terms alone provides very valuable information for the determination of top-quark mass effects in our approach. The calculation of the non-analytic terms is significantly simpler than that of the analytic contributions and was described in detail in [28] for the three leading non-analytic expansion terms of the one and two-loop form factors for $g g \rightarrow H H$. For $g g \rightarrow Z Z$ we expand the amplitude up to high orders in $\bar{z} \equiv 1-z$ and therefore use the expansion by regions 42, 43] to expand the full-theory diagrams instead of an EFT approach where a large number of effective vertices is required due to the deep expansion. We use QGRAF [44] to generate the Feynman diagrams which are processed and expanded using private FORM [45] code. The IBP reduction [46] is performed with FIRE 47] which is based on the Laporta algorithm [48].

Our results are given in Appendix A and an ancillary Mathematica file. They are of the form

$$
\begin{align*}
& \left|\widetilde{\mathcal{F}}_{i}^{(1)}\right\rangle \stackrel{z \overbrace{}^{1}}{ } \sum_{n=3}^{\infty} a_{i}^{(n, 0)} \bar{z}^{\frac{n}{2}}  \tag{5}\\
& \left|\widetilde{\mathcal{F}}_{i}^{(2)}\right\rangle^{z \rightarrow 1} \overbrace{n=2}^{\infty} \sum_{m=\bar{n}_{2}}^{1}\left[b_{i}^{(n, m)}+b_{i, \ln }^{(n, m)} \ln (-4 z)\right] \bar{z}^{\frac{n}{2}} \ln ^{m} \bar{z}
\end{align*}
$$

where $\bar{n}_{2}$ is $n$ modulo 2 , the coefficients are functions of the dimensionless variables $r_{Z}=m_{Z}^{2} / M_{Z Z}^{2}$ and $\tilde{x}=$ $\left(p_{T}^{2}+m_{Z}^{2}\right) / M_{Z Z}^{2}$. We use the symbol $\asymp$ to indicate that terms which are analytic in $\bar{z}$ and currently unknown have been dropped on the right-hand side.

[^3]Threshold logarithms $\ln \bar{z}$ and logarithms $\ln (-4 z)$ related to massless cuts in the amplitude first appear at two-loop order. While we generally compute the expansion coefficients up to $n=8$, i.e. expand up to $\bar{z}^{4}$, we find that for the massless-cut contribution proportional to $\ln (-4 z)$ more input is required to achieve a reliable Padé approximation. We therefore compute the corresponding coefficients $b_{i, \ln }^{(n, m)}$ up to $n=9$.

As in Higgs pair production there is no S-wave contribution to the form factors relevant for the interference and the leading non-analytic terms involve the $\bar{z}$ suppressed P -wave Green function [49.

## B. Behavior for $z \rightarrow \infty$

In addition to the LME and threshold expansions we can exploit scaling information in the small-mass limit $m_{t} \rightarrow 0$ which corresponds to $z \rightarrow \infty$. This does not require an additional calculation in this region but relies solely on the symmetries of QCD. The absence of infrared $1 / m_{t}$ power divergences as $m_{t} \rightarrow 0$ implies that the form factors can only show logarithmic behavior as $z \rightarrow \infty$. Below we show that the difference

$$
\begin{equation*}
\left|\widetilde{\mathcal{B}}_{A A-V V}\right\rangle \equiv\left|\widetilde{\mathcal{B}}_{A A}\right\rangle-\left|\widetilde{\mathcal{B}}_{V V}\right\rangle \tag{6}
\end{equation*}
$$

vanishes as $z \rightarrow \infty$. To prove this we note that chirality is conserved in massless QCD and hence the fourpoint correlator of two vector currents, a left-handed and a right-handed current, which we denote in short by $[\mathrm{V}, \mathrm{V}, \mathrm{V}-\mathrm{A}, \mathrm{V}+\mathrm{A}]$, vanishes in the limit of zero quark masses. ${ }^{4}$ Using that the correlator $[\mathrm{V}, \mathrm{V}, \mathrm{V}, \mathrm{A}]$ vanishes due to charge conjugation we immediately conclude that $[\mathrm{V}, \mathrm{V}, \mathrm{V}, \mathrm{V}]-[\mathrm{V}, \mathrm{V}, \mathrm{A}, \mathrm{A}] \rightarrow 0$ as $z \rightarrow \infty$. We exploit this below and reconstruct the top-mass dependence of $\left|\widetilde{\mathcal{F}}_{V V}^{(i)}\right\rangle$ and $\left|\widetilde{\mathcal{F}}_{A A}^{(i)}\right\rangle-\left|\widetilde{\mathcal{F}}_{V V}^{(i)}\right\rangle$ where we have one additional condition for the latter.

## III. THE METHOD

We approximate the box form factors (2) using our approach from [28]. First, we introduce subtraction functions $s_{V V}^{(2)}, s_{A A}^{(2)}$ in such a way that the combinations $\left|\widetilde{\mathcal{F}}_{i}^{(2)}\right\rangle-s_{i}^{(2)}$ retain their analytic structure for $|z|<1$ but have threshold expansions which are free of logarithms $\ln (\bar{z})$ up to the highest known order, i.e. up to $\bar{z}^{4}$. The construction of such subtraction functions is detailed in [28] and we give the ones we explicitly need

[^4]in Appendix B. Note that even after this subtraction the threshold and large mass expansions of the two-loop form factors still receive contributions proportional to a single logarithm $L_{s} \equiv \ln (-4 z)$ from diagrams with massless cuts. We therefore split the subtracted two-loop form factors into a constant and a logarithmic part and construct separate approximants for each part.

The top mass dependence is contained in the variable $z$ and the conformal transformation 50 ]

$$
\begin{equation*}
z=\frac{4 \omega}{(1+\omega)^{2}} \tag{7}
\end{equation*}
$$

is used to map the entire complex $z$ plane onto the unit disc $|\omega| \leq 1$ with the branch cut for $z \geq 1$ corresponding to the perimeter. Thus, the top-mass dependence is encoded by a function that is analytic in the region $|\omega|<1$ and can be reconstructed using Padé approximants

$$
\begin{equation*}
[n / m](\omega)=\frac{\sum_{i=0}^{n} a_{i} \omega^{i}}{1+\sum_{j=1}^{m} b_{j} \omega^{j}} \tag{8}
\end{equation*}
$$

where the $n+m+1$ coefficients $a_{i}, b_{j}$ can be fixed by imposing the condition that the expansion of eq. (8) in the LME and threshold region must reproduce the known coefficients for given, fixed values of $r_{Z}$ and $\tilde{x}$. The smallmass behavior discussed in sec. IIB is not used to further constrain the Padé coefficients, but is taken into account by a rescaling of the Padé ansatz. Hence, we use approximation functions of the form

$$
\begin{align*}
P_{A A-V V}^{(1)}(\omega)= & \frac{[n / m](\omega)}{1+a_{R, 0} z(\omega)} \\
P_{A A-V V}^{(2)}(\omega)= & \frac{[n / m](\omega)}{1+a_{R, 0} z(\omega)}+\frac{[k / l](\omega)}{1+a_{R, 1} z(\omega)} L_{s} \\
& +s_{A A}^{(2)}(z(w))-s_{V V}^{(2)}(z(w)) \\
P_{V V}^{(1)}(\omega)= & \frac{z(\omega)[n / m](\omega)}{1+a_{R, 0} z(\omega)}, \\
P_{V V}^{(2)}(\omega)= & \frac{z(\omega)[n / m](\omega)}{1+a_{R, 0} z(\omega)}+\frac{z(\omega)[k / l](\omega)}{1+a_{R, 1} z(\omega)} L_{s} \\
& +s_{V V}^{(2)}(z(w)) \tag{9}
\end{align*}
$$

where $P_{A A-V V}^{(j)}$ is used to approximate the difference between the axial-vector and vector form factors, whereas the vector form factors in isolation are approximated using $P_{V V}^{(j)}$. The limit $z \rightarrow \infty$ corresponds to $\omega \rightarrow-1$ where the approximants in eq. (8) approach a constant value. Thus, the rescaling eq. (9) enforces the correct asymptotic behavior for $z \rightarrow \infty$ discussed in sec. IIB and provides us with free parameters $a_{R, i}$ that can be varied in addition to the polynomial degrees $n, m, k$ and $l$ to assess the stability of the approximation. We note that these variations are performed independently for all the terms in eq. (9). Our final ansätze for the form factor


FIG. 2: The form factors $\left|\widetilde{F}_{V V}^{(1)}\right\rangle$ (upper row) and $\left|\widetilde{F}_{A A}^{(1)}\right\rangle$ (lower row) at LO for $\tilde{x}=0.09$ (left side) and $\tilde{x}=0.25$ (right side) as a function of the invariant mass of the $Z$-boson pair. $\tilde{x}=0.25$ corresponds to the maximum possible transverse momentum for a given invariant mass. The dark blue and light blue points correspond to the real and imaginary parts of the Padé approximants from eq. 10 and eq. (11), the solid lines are the full result and the shaded regions are Padé approximants that were constructed using only the information from the LME (cf. text for details).
approximation are then

$$
\begin{align*}
& \left|\widetilde{F}_{A A}^{(j)}(z(\omega))\right\rangle \simeq P_{A A-V V}^{(j)}(\omega)+P_{V V}^{(j)}(\omega),  \tag{10}\\
& \left|\widetilde{F}_{V V}^{(j)}(z(\omega))\right\rangle \simeq P_{V V}^{(j)}(\omega) \tag{11}
\end{align*}
$$

## IV. RESULTS

Before showing our results at NLO for the form factors, we can compare the LO form factors constructed as discussed in the previous sections with the full analytic result. We choose as input for the on-shell $Z$-boson and top quark masses

$$
\begin{equation*}
m_{Z}=91.1876 \mathrm{GeV}, \quad m_{t}=173 \mathrm{GeV} \tag{12}
\end{equation*}
$$

and show results for two different values of $\tilde{x}$ in fig. 2. The plots contain the maximum information we have available from the LME at LO (see [27]) and our threshold expansion. By construction the Padé ansatz in eq. (8) contains poles in the complex $\omega$ plane whereas the functions it approximates are analytic in $z$ implying the absence of
poles in the unit disc $|\omega| \leq 1$. Furthermore poles in the vicinity of the unit disc can cause unphysical behavior in the reconstructed form factors. We were not able to construct Padé approximants without poles inside a larger disc $|\omega| \leq 1.2$. Therefore we focus on the time-like region of the form factors and construct only Padé approximants which do not contain poles for

$$
\begin{equation*}
\operatorname{Re}(z(\omega))>0 \quad \text { and } \quad|\omega|<1.2 \tag{13}
\end{equation*}
$$

We obtain an uncertainty estimate for our results in the following way. For every phase space point, we calculate the mean and standard deviation for each contributing Padé approximant in eq. (9). To this end, we vary the rescaling parameters $a_{R, i}$ in the region

$$
\begin{equation*}
a_{R, i} \in[0.1,10] \tag{14}
\end{equation*}
$$

and vary $[n / m]$ within $|n-m| \leq 3$, where $n+m+1$ is the number of available constraints. We construct 100 variants for each Padé approximant. Our final prediction then follows from the sum of the mean values of the Padé approximants, with an uncertainty obtained by adding the individual errors in quadrature.


FIG. 3: The NLO form factors $\left|\widetilde{F}_{V V}^{(2)}\right\rangle$ (upper row) and $\left|\widetilde{F}_{A A}^{(2)}\right\rangle$ (lower row) for $\tilde{x}=0.09$ (left side) and $\tilde{x}=0.25$ (right side) as a function of the invariant mass of the $Z$-boson pair. The conventions are the same as in fig. 2 with the points and shaded regions corresponding to the Padé approximation constructed from the LME only.

Fig. 2 shows the Padé approximants from eq. (10) and eq. (11) for the LO form factors $\left|\widetilde{F}_{V V}^{(1)}\right\rangle$ and $\left|\widetilde{F}_{A A}^{(1)}\right\rangle$ including our uncertainty estimate as points with error bars. We observe good agreement with the full results, which are indicated by the solid lines, up to large values of the invariant mass $M_{Z Z}$ of the $Z$-boson pair. The error remains small throughout the whole invariant mass range, increasing somewhat towards large $M_{Z Z}$. The behavior for different values of $\tilde{x}$ is similar. To demonstrate the importance of including the threshold expansion we also show an approximation based solely on the LME as shaded regions. For this we adopt the prescription given in ref. [27] and show the envelope of the [2/2], [2/3], [3/2] and $[3 / 3]$ Padé approximants which we have constructed without applying the rescaling of eq. (9) or the pole criterion eq. 13). We note that the resonant structure near $z=1$ in the upper right plot showing the vector form factor for maximal transverse momentum is caused by a pole near $w=1$ in the $[3 / 3]$ Padé approximant. In our full results from eq. (10) and eq. (11) we apply the criterion eq. (13) to exclude approximants which feature such resonances in the time-like region $z \geq 0$. We conclude that the threshold expansion is essential for the reconstruction of the full top mass dependence above the top
quark threshold.
We now turn to the NLO form factors. In fig. 3 we show the results for the virtual corrections to the form factors $\left|\widetilde{F}_{V V}^{(2)}\right\rangle$ (upper panel) and $\left|\widetilde{F}_{A A}^{(2)}\right\rangle$ (lower panel) for two values of $\tilde{x}$. Note that we do not include the double-triangle contribution to the form factors, as they have been computed analytically in [27]. As at LO, we include only the top quark contributions. The uncertainty associated with the Padé construction increases with $M_{Z Z}$. Since we input information mainly at low $M_{Z Z}$ this behavior is expected. With the exception of the vector form factor $\left|\widetilde{F}_{V V}^{(2)}\right\rangle$ for small transverse momenta (upper left panel in fig. 3) we find that the Padé approximation based on the LME alone does not yield a realistic reconstruction of the top-quark mass effects of the form factors. In particular, the important axialvector form factor suffers from very large uncertainties. We remark though that in 27 for the NLO cross section the Padé prediction was improved by a reweighting with the full LO cross section.

We note that $\left|\widetilde{F}_{V V}^{(2)}\right\rangle$ shows a small oscillation in the region of large $M_{Z Z}$ when the transverse momentum of the $Z$ bosons is small as is evident from the upper left


FIG. 4: The NLO form factors $\left|\widetilde{F}_{V V}^{(2)}\right\rangle$ (upper row) and $\left|\widetilde{F}_{A A}^{(2)}\right\rangle$ (lower row) for $\tilde{x}=0.09$ as a function of the invariant mass of the $Z$-boson pair. In dark/light blue we show the same points as in fig. 3 while in pink/rose we show the real/imaginary part of the Padé approximants expanded up to $\mathcal{O}\left(\bar{z}^{2}\right)$ (left side) and $\mathcal{O}\left(\bar{z}^{3}\right)$ (right side).
plot in fig. 3. We trace the appearance of the second peak back to the contribution proportional to $L_{s}$ stemming from diagrams with massless cuts. In general, we find that this contribution shows worse convergence behavior than the non-logarithmic terms when including more and more terms in the LME and the threshold expansion. This is shown in fig. 4 where we compare our results from fig. 3 to the Padé approximants obtained with the same procedure but only using threshold input up to the order $\bar{z}^{2}$ and $\bar{z}^{3}$. We observe good convergence in the case of the axial-vector form factor. On the other hand, the $\mathcal{O}\left(\bar{z}^{2}\right)$ approximation for the vector form factor does not feature the oscillatory behavior described above and there is no overlap with the full approximation in a significant part of the phase space. However, the $\mathcal{O}\left(\bar{z}^{3}\right)$ and $\mathcal{O}\left(\bar{z}^{4}\right)$ results are in good agreement with the full approximation where we have also included the $\mathcal{O}\left(\bar{z}^{5}\right)$ term in the coefficient of the logarithm $L_{s}$ to verify that this stabilization persists with the addition of higher orders in the threshold expansion. We conclude
from this discussion, that the Padé approximation can be improved systematically when including higher orders in the various expansions. Nevertheless, we believe that the prediction for $\left|\widetilde{F}_{V V}^{(2)}\right\rangle$ should be taken with a grain of salt above $M_{Z Z} \geq 500 \mathrm{GeV}$ because of the slower convergence.

In fig. 5 we show the virtual corrections to the form factor $v_{f}^{2}\left|\widetilde{F}_{V V}^{(2)}\right\rangle+a_{f}^{2}\left|\widetilde{F}_{V V}^{(2)}\right\rangle$ as it enters in the interference term with the Higgs boson exchange. The dashed lines show the form factor $v_{f}^{2}\left|\widetilde{F}_{V V}^{(2)}\right\rangle$ increased by a factor of 300. This clearly demonstrates that the interference term will be dominated by $\left|\widetilde{F}_{A A}^{(2)}\right\rangle$ and we therefore choose not to modify the uncertainty estimate for the vector form factor. The fact that $\left|\widetilde{F}_{V V}^{(2)}\right\rangle$ is negligible compared to $\left|\widetilde{F}_{A A}^{(2)}\right\rangle$ allows us to make trustworthy predictions for the interference with the Higgs production with subsequent decay to $Z$ bosons up to $M_{Z Z} \rightarrow \infty$, even though


FIG. 5: The interference form factor $v_{f}^{2}\left|\widetilde{F}_{V V}^{(2)}\right\rangle+a_{f}^{2}\left|\widetilde{F}_{V V}^{(2)}\right\rangle$ for $\tilde{x}=0.09$ (left side) and $\tilde{x}=0.25$ (right side) as a function of the invariant mass of the $Z$-boson pair. The dashed lines show a rescaled form factor $\left|\widetilde{F}_{V V}^{(2)}\right\rangle$ to demonstrate that it is negligible compared to $\left|\widetilde{F}_{A A}^{(2)}\right\rangle$.
as stated above we trust our results for $\left|\widetilde{F}_{V V}^{(2)}\right\rangle$ only for $M_{Z Z} \leq 500 \mathrm{GeV}$.

The numerical implementation of the form factors is available as a FORTRAN routine on request and can be combined with existing computations of the massless loop contributions and the real corrections for the interference of the Higgs exchange with decay to $Z Z$ with the continuum background.

## V. CONCLUSIONS AND OUTLOOK

We have considered top-quark mass effects in the continuum process $g g \rightarrow Z Z$, focusing on the form factors relevant for the NLO interference with the production of a Higgs boson and its subsequent decay into two $Z$ bosons. We have presented a Padé-based approximation using information from an expansion around a large top quark mass and an expansion around the top quark pair production threshold.

At LO, we have shown that our Padé construction approximates very well the full top mass dependence of the form factors for the whole range of the invariant mass $M_{Z Z}$ of the $Z$ bosons. At NLO, we provide a new prediction with very small uncertainties at small and moderate $M_{Z Z}$, with an increased uncertainty towards large $M_{Z Z}$. We expect that adding more information into the Padé construction at large $M_{Z Z}$ would improve the description also in this region.

Our results can be combined both with virtual corrections mediated by massless loops and the real corrections. The latter constitute a one-loop process and can therefore be computed with well-established techniques. The Padé construction can also be applied to the remaining form
factors contributing to $g g \rightarrow Z Z$, which do not interfere with the Higgs signal.

We note also that while in this work we have applied our method to the production of on-shell $Z$ bosons, there is no obstruction for applying it also to off-shell $Z$ boson production. Indeed, the LME for off-shell $Z$ boson production is already known up to the order $z^{4}$ [24]. While a calculation of the full top mass dependence for onshell $Z$ bosons with numerical methods seems to be feasible with current techniques in a reasonable time-frame (see [51, 52]) a computation of the off-shell form factors appears to be beyond the current state-of-the-art.

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## Appendix A: Threshold expansion of form factors

In the following we give explicit expressions for the coefficients in the threshold expansions of the form factors. For convenience, we quote the definition already given in
eq. (5):

$$
\begin{align*}
& \left|\widetilde{\mathcal{F}}_{i}^{(1)}\right\rangle \stackrel{z \sqsupset 1}{\asymp} \sum_{n=3}^{\infty} a_{i}^{(n, 0)} \bar{z}^{\frac{n}{2}},  \tag{A1}\\
& \left.\left|\widetilde{\mathcal{F}}_{i}^{(2)}\right\rangle\right\rangle^{z \rightarrow 1} \sum_{n=2}^{\infty} \sum_{m=\bar{n}_{2}}^{1}\left[b_{i}^{(n, m)}+b_{i, \ln }^{(n, m)} \ln (-4 z)\right] \bar{z}^{\frac{n}{2}} \ln ^{m} \bar{z}, \tag{A2}
\end{align*}
$$

where $i \in\{V V, A A\}$ and $\bar{n}_{2}$ is $n$ modulo 2 . The coefficients $a, b$ are most conveniently written in terms of the two dimensionless ratios $r_{Z}=\frac{m_{Z}^{2}}{M_{Z Z}^{2}}$ and $r_{p_{T}}=\frac{p_{T}^{2}}{M_{Z Z}^{2}}=$ $\tilde{x}-r_{Z}$. We define the loop integral measure as

$$
\begin{equation*}
[d l]=\frac{d^{d} l}{i \pi^{\frac{d}{2}}} e^{\epsilon \gamma_{E}} \tag{A3}
\end{equation*}
$$

and use the short-hand notation

$$
\begin{equation*}
C_{0}=\int[d l] \frac{1}{l^{2}\left[(l+q)^{2}-1\right]\left[\left(l+q-p_{Z}\right)^{2}-1\right]} \tag{A4}
\end{equation*}
$$

with $q^{2}=1, p_{Z}^{2}=4 r_{Z}^{2}, q \cdot p_{Z}=1$. The coefficients $b_{i, \ln }^{(n, 1)}$ and $b_{i, \ln }^{(2 n, m)}$ vanish. Furthermore, coefficients with $m=0$ and even $n$ do not contribute to the imaginary part and are therefore not listed here. We have calculated the remaining coefficients $a_{i}^{n, 0}, b_{i}^{n, m}$ up to $n=8$ and the coefficients $b_{i, \ln }^{(n, 0)}$ up to $n=9$, obtaining the following results:

$$
\begin{align*}
& a_{A A}^{(3,0)}=\frac{4 \pi}{3\left(1-2 r_{Z}\right)^{2} r_{Z}^{2}}\left(-1+6 r_{Z}-18 r_{Z}^{2}+16 r_{Z}^{3}\right) \text {, }  \tag{A5}\\
& a_{A A}^{(5,0)}=\frac{2 \pi}{15\left(1-2 r_{Z}\right)^{4} r_{Z}^{2}}\left[-21+210 r_{Z}-958 r_{Z}^{2}+2336 r_{Z}^{3}-2968 r_{Z}^{4}+1472 r_{Z}^{5}+8 r_{p_{T}}\left(1-2 r_{Z}\right)^{2}\left(1-2 r_{Z}+4 r_{Z}^{2}\right)\right] \text {, }  \tag{A6}\\
& a_{A A}^{(7,0)}=\frac{\pi}{210\left(1-2 r_{Z}\right)^{6} r_{Z}^{2}}\left[-905+12670 r_{Z}-80954 r_{Z}^{2}+301104 r_{Z}^{3}-695264 r_{Z}^{4}+985120 r_{Z}^{5}\right. \\
& \left.-788896 r_{Z}^{6}+269568 r_{Z}^{7}+16 r_{p_{T}}\left(1-2 r_{Z}\right)^{2}\left(39-234 r_{Z}+672 r_{Z}^{2}-920 r_{Z}^{3}+592 r_{Z}^{4}\right)\right] \text {, }  \tag{A7}\\
& b_{A A}^{(2,1)}=\frac{32 \pi^{2}}{9\left(1-2 r_{Z}\right)^{2} r_{Z}^{2}}\left(-1+6 r_{Z}-18 r_{Z}^{2}+16 r_{Z}^{3}\right) \text {, }  \tag{A8}\\
& b_{A A}^{(3,0)}=-\frac{\pi}{9\left(1-2 r_{Z}\right)^{3}\left(1-4 r_{Z}\right)^{2} r_{Z}^{2}}\left[-2\left(1-2 r_{Z}\right)\left(1-4 r_{Z}\right)^{2}\left[-136+3 \pi^{2}+168 \ln (2)\right] r_{p_{T}}\left(1-2 r_{Z}+4 r_{Z}^{2}\right)\right. \\
& -64 C_{0}\left(1-2 r_{Z}\right)^{2}\left(1-4 r_{Z}\right)^{2}\left(-1-7 r_{Z}+34 r_{Z}^{2}-44 r_{Z}^{3}+8 r_{Z}^{4}\right) \\
& +64 r_{Z}\left(1-4 r_{Z}\right)^{2} \sqrt{\frac{1-r_{Z}}{r_{Z}}} \arctan \left(\frac{2 \sqrt{\left(1-r_{Z}\right) r_{Z}}}{1-2 r_{Z}}\right)\left(9-45 r_{Z}+70 r_{Z}^{2}-56 r_{Z}^{3}+32 r_{Z}^{4}\right) \\
& -\left(1-2 r_{Z}\right)\left(1-4 r_{Z}\right)\left[192+9 \pi^{2}-56 \ln (2)-1728 r_{Z}-90 \pi^{2} r_{Z}+560 \ln (2) r_{Z}+6256 r_{Z}^{2}+384 \pi^{2} r_{Z}^{2}\right. \\
& \left.-2016 \ln (2) r_{Z}^{2}-12480 r_{Z}^{3}-816 \pi^{2} r_{Z}^{3}+3584 \ln (2) r_{Z}^{3}+12032 r_{Z}^{4}+576 \pi^{2} r_{Z}^{4}-3584 \ln (2) r_{Z}^{4}-2048 r_{Z}^{5}\right] \\
& \left.+128\left(1-2 r_{Z}\right) \ln \left(2-4 r_{Z}\right)\left(2-23 r_{Z}+98 r_{Z}^{2}-184 r_{Z}^{3}+152 r_{Z}^{4}-112 r_{Z}^{5}+96 r_{Z}^{6}\right)\right],  \tag{A9}\\
& b_{A A, \ln }^{(3,0)}=0,  \tag{A10}\\
& b_{A A}^{(3,1)}=0,  \tag{A11}\\
& b_{A A}^{(4,1)}=\frac{32 \pi^{2}}{45\left(1-2 r_{Z}\right)^{4} r_{Z}^{2}}\left[-3+30 r_{Z}-134 r_{Z}^{2}+328 r_{Z}^{3}-464 r_{Z}^{4}+256 r_{Z}^{5}+4\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(1-2 r_{Z}+4 r_{Z}^{2}\right)\right],  \tag{A12}\\
& b_{A A}^{(5,0)}=\frac{\pi}{4050\left(1-2 r_{Z}\right)^{5}\left(1-4 r_{Z}\right)^{3} r_{Z}^{2}}\left[-\left(1-2 r_{Z}\right)\left(1-4 r_{Z}\right)\left[21472-6075 \pi^{2}-104520 \ln (2)-643776 r_{Z}\right.\right. \\
& +109350 \pi^{2} r_{Z}+1881360 \ln (2) r_{Z}+7528432 r_{Z}^{2}-855900 \pi^{2} r_{Z}^{2}-14848320 \ln (2) r_{Z}^{2}-44282176 r_{Z}^{3} \\
& +3817800 \pi^{2} r_{Z}^{3}+67172160 \ln (2) r_{Z}^{3}+141881152 r_{Z}^{4}-10558080 \pi^{2} r_{Z}^{4}-187708800 \ln (2) r_{Z}^{4} \\
& -245787136 r_{Z}^{5}+18135360 \pi^{2} r_{Z}^{5}+319941120 \ln (2) r_{Z}^{5}+202149888 r_{Z}^{6}-17763840 \pi^{2} r_{Z}^{6} \\
& \left.-302008320 \ln (2) r_{Z}^{6}-35688448 r_{Z}^{7}+7326720 \pi^{2} r_{Z}^{7}+117350400 \ln (2) r_{Z}^{7}-25067520 r_{Z}^{8}\right] \\
& -18\left(1-2 r_{Z}\right)\left(1-4 r_{Z}\right) r_{p_{T}}\left[2136+375 \pi^{2}-360 \ln (2)-37584 r_{Z}-5250 \pi^{2} r_{Z}+5040 \ln (2) r_{Z}\right.
\end{align*}
$$

$$
\begin{align*}
& +263040 r_{Z}^{2}+30000 \pi^{2} r_{Z}^{2}-28800 \ln (2) r_{Z}^{2}-915648 r_{Z}^{3}-94200 \pi^{2} r_{Z}^{3}+22080 \ln (2) r_{Z}^{3} \\
& +1546624 r_{Z}^{4}+184560 \pi^{2} r_{Z}^{4}+424320 \ln (2) r_{Z}^{4}-1103872 r_{Z}^{5}-218880 \pi^{2} r_{Z}^{5} \\
& \left.-1320960 \ln (2) r_{Z}^{5}+546816 r_{Z}^{6}+111360 \pi^{2} r_{Z}^{6}+768000 \ln (2) r_{Z}^{6}-163840 r_{Z}^{7}\right] \\
& +C_{0}\left(1-2 r_{Z}\right)\left(1-4 r_{Z}\right)\left[46080\left(1-2 r_{Z}\right)^{3} r_{p_{T}}\left(2-9 r_{Z}+20 r_{Z}^{2}-12 r_{Z}^{3}-8 r_{Z}^{4}+16 r_{Z}^{5}\right)\right. \\
& \left.-2880\left(1-2 r_{Z}\right)\left(1-4 r_{Z}\right)\left(31-99 r_{Z}-1242 r_{Z}^{2}+8912 r_{Z}^{3}-23696 r_{Z}^{4}+29840 r_{Z}^{5}-16608 r_{Z}^{6}+2176 r_{Z}^{7}\right)\right] \\
& +r_{Z}\left(1-4 r_{Z}\right) \sqrt{\frac{1-r_{Z}}{r_{Z}}} \arctan \left(\frac{2 \sqrt{\left(1-r_{Z}\right) r_{Z}}}{1-2 r_{Z}}\right)\left[9 6 0 ( 1 - 4 r _ { Z } ) \left(-687+8843 r_{Z}\right.\right. \\
& \left.-46162 r_{Z}^{2}+126356 r_{Z}^{3}-195416 r_{Z}^{4}+174304 r_{Z}^{5}-95360 r_{Z}^{6}+30720 r_{Z}^{7}\right) \\
& \left.-46080\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(-6+45 r_{Z}-152 r_{Z}^{2}+264 r_{Z}^{3}-176 r_{Z}^{4}+16 r_{Z}^{5}\right)\right] \\
& +\left(1-2 r_{Z}\right) \ln \left(2-4 r_{Z}\right)\left[1 9 2 0 \left(-170+3251 r_{Z}-26282 r_{Z}^{2}+117196 r_{Z}^{3}-314896 r_{Z}^{4}\right.\right. \\
& \left.+524464 r_{Z}^{5}-549440 r_{Z}^{6}+394688 r_{Z}^{7}-240512 r_{Z}^{8}+96768 r_{Z}^{9}\right) \\
& \left.\left.-92160\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(-1+13 r_{Z}-70 r_{Z}^{2}+200 r_{Z}^{3}-280 r_{Z}^{4}+128 r_{Z}^{5}\right)\right]\right] \text {, }  \tag{A13}\\
& b_{A A, \ln }^{(5,0)}=\frac{16 \pi}{5\left(1-2 r_{Z}\right)^{2} r_{Z}^{2}}\left(-1+4 r_{Z}+6 r_{p_{T}}\right)\left(1-2 r_{Z}+4 r_{Z}^{2}\right) \text {, }  \tag{A14}\\
& b_{A A}^{(5,1)}=\frac{32 \pi}{135\left(1-2 r_{Z}\right)^{2} r_{Z}^{2}}\left[53-318 r_{Z}+846 r_{Z}^{2}-848 r_{Z}^{3}-108 r_{p_{T}}\left(1-2 r_{Z}+4 r_{Z}^{2}\right)\right],  \tag{A15}\\
& b_{A A}^{(6,1)}=\frac{32 \pi^{2}}{315\left(1-2 r_{Z}\right)^{6} r_{Z}^{2}}\left[19-266 r_{Z}+1730 r_{Z}^{2}-6504 r_{Z}^{3}+14648 r_{Z}^{4}-19072 r_{Z}^{5}+12128 r_{Z}^{6}-2816 r_{Z}^{7}\right. \\
& \left.+4\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(9-54 r_{Z}+168 r_{Z}^{2}-208 r_{Z}^{3}+128 r_{Z}^{4}\right)\right],  \tag{A16}\\
& b_{A A}^{(7,0)}=\frac{\pi}{793800\left(1-r_{Z}\right)\left(1-2 r_{Z}\right)^{7}\left(1-4 r_{Z}\right)^{4} r_{Z}^{2}}\left[( 1 - r _ { Z } ) ( 1 - 2 r _ { Z } ) ( 1 - 4 r _ { Z } ) \left[-48296976+1306935 \pi^{2}\right.\right. \\
& +68546520 \ln (2)+1381163616 r_{Z}-33980310 \pi^{2} r_{Z}-1782209520 \ln (2) r_{Z}-19953987184 r_{Z}^{2} \\
& +396287640 \pi^{2} r_{Z}^{2}+20934631200 \ln (2) r_{Z}^{2}+178890721728 r_{Z}^{3}-2743009920 \pi^{2} r_{Z}^{3} \\
& -146825159040 \ln (2) r_{Z}^{3}-1038970811008 r_{Z}^{4}+12546379440 \pi^{2} r_{Z}^{4}+682278602880 \ln (2) r_{Z}^{4} \\
& +3951328802304 r_{Z}^{5}-39832823520 \pi^{2} r_{Z}^{5}-2194951852800 \ln (2) r_{Z}^{5}-9798614287104 r_{Z}^{6} \\
& +89300171520 \pi^{2} r_{Z}^{6}+4941762577920 \ln (2) r_{Z}^{6}+15459576751104 r_{Z}^{7}-140100468480 \pi^{2} r_{Z}^{7} \\
& -7665966120960 \ln (2) r_{Z}^{7}-14595082354688 r_{Z}^{8}+147641840640 \pi^{2} r_{Z}^{8}+7817008496640 \ln (2) r_{Z}^{8} \\
& +7124726562816 r_{Z}^{9}-94379765760 \pi^{2} r_{Z}^{9}-4715940741120 \ln (2) r_{Z}^{9}-1110704586752 r_{Z}^{10} \\
& \left.+27358248960 \pi^{2} r_{Z}^{10}+1270490726400 \ln (2) r_{Z}^{10}-100576788480 r_{Z}^{11}\right] \\
& +2\left(1-r_{Z}\right)\left(1-2 r_{Z}\right)\left(1-4 r_{Z}\right) r_{p_{T}}\left[12875128-2482515 \pi^{2}-32397960 \ln (2)-322712656 r_{Z}\right. \\
& +54615330 \pi^{2} r_{Z}+712755120 \ln (2) r_{Z}+3481217888 r_{Z}^{2}-530889660 \pi^{2} r_{Z}^{2}-7045775520 \ln (2) r_{Z}^{2} \\
& -21890762240 r_{Z}^{3}+3016742400 \pi^{2} r_{Z}^{3}+41557608960 \ln (2) r_{Z}^{3}+89967740544 r_{Z}^{4}-11128456080 \pi^{2} r_{Z}^{4} \\
& -162222883200 \ln (2) r_{Z}^{4}-251512872192 r_{Z}^{5}+27788412960 \pi^{2} r_{Z}^{5}+434414211840 \ln (2) r_{Z}^{5} \\
& +477015472640 r_{Z}^{6}-47130431040 \pi^{2} r_{Z}^{6}-795771594240 \ln (2) r_{Z}^{6}-592554727424 r_{Z}^{7} \\
& +52573812480 \pi^{2} r_{Z}^{7}+963913574400 \ln (2) r_{Z}^{7}+437616467968 r_{Z}^{8}-35230325760 \pi^{2} r_{Z}^{8} \\
& -705638277120 \ln (2) r_{Z}^{8}-137703424000 r_{Z}^{9}+10818662400 \pi^{2} r_{Z}^{9}+235343216640 \ln (2) r_{Z}^{9} \\
& \left.-8257536000 r_{Z}^{10}\right] \\
& +201600\left(1-2 r_{Z}\right)^{3}\left(1-4 r_{Z}\right)^{4}\left[-356+3 \pi^{2}+520 \ln (2)\right] r_{p_{T}}^{2}\left(-1+r_{Z}\right)\left(1-2 r_{Z}+4 r_{Z}^{2}\right) \\
& +C_{0}\left(1-r_{Z}\right)\left(1-2 r_{Z}\right)^{2}\left(1-4 r_{Z}\right)^{2}\left[-20160\left(1-4 r_{Z}\right)\left(1773-13705 r_{Z}-15494 r_{Z}^{2}+542860 r_{Z}^{3}\right.\right. \\
& \left.-2672760 r_{Z}^{4}+6652528 r_{Z}^{5}-9490464 r_{Z}^{6}+7595840 r_{Z}^{7}-2952064 r_{Z}^{8}+311808 r_{Z}^{9}\right) \\
& \left.+645120\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(96-601 r_{Z}+1800 r_{Z}^{2}-4344 r_{Z}^{3}+7592 r_{Z}^{4}-5664 r_{Z}^{5}-32 r_{Z}^{6}+1600 r_{Z}^{7}\right)\right]
\end{align*}
$$

$$
\begin{align*}
& +r_{Z}\left(1-4 r_{Z}\right)^{2} \sqrt{\frac{1-r_{Z}}{r_{Z}}} \arctan \left(\frac{2 \sqrt{\left(1-r_{Z}\right) r_{Z}}}{1-2 r_{Z}}\right)\left[-6720\left(1-4 r_{Z}\right)\left(35337-616496 r_{Z}\right.\right. \\
& +4722401 r_{Z}^{2}-20896358 r_{Z}^{3}+59069448 r_{Z}^{4}-111353552 r_{Z}^{5}+142045200 r_{Z}^{6} \\
& \left.-122427488 r_{Z}^{7}+70734336 r_{Z}^{8}-26438656 r_{Z}^{9}+5122048 r_{Z}^{10}\right) \\
& -215040\left(1-r_{Z}\right)\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(-1044+10253 r_{Z}-43216 r_{Z}^{2}\right. \\
& \left.\left.+104204 r_{Z}^{3}-158912 r_{Z}^{4}+150832 r_{Z}^{5}-70976 r_{Z}^{6}+9408 r_{Z}^{7}\right)\right] \\
& -\ln \left(2-4 r_{Z}\right)\left(1-r_{Z}\right)\left(1-2 r_{Z}\right)\left[1 3 4 4 0 \left(9662-255657 r_{Z}+3011806 r_{Z}^{2}-20841376 r_{Z}^{3}\right.\right. \\
& +94139672 r_{Z}^{4}-291779424 r_{Z}^{5}+635459136 r_{Z}^{6}-982191360 r_{Z}^{7}+1084817792 r_{Z}^{8} \\
& \left.-874003712 r_{Z}^{9}+538395136 r_{Z}^{10}-253452288 r_{Z}^{11}+66985984 r_{Z}^{12}\right) \\
& +430080\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(-251+4211 r_{Z}-30950 r_{Z}^{2}+132676 r_{Z}^{3}-370528 r_{Z}^{4}\right. \\
& \left.\left.\left.+699712 r_{Z}^{5}-871904 r_{Z}^{6}+647168 r_{Z}^{7}-220160 r_{Z}^{8}+18432 r_{Z}^{9}\right)\right]\right],  \tag{A17}\\
& b_{A A, \ln }^{(7,0)}=\frac{8 \pi}{35\left(1-2 r_{Z}\right)^{4} r_{Z}^{2}}\left(-1+4 r_{Z}+6 r_{p_{T}}\right)\left(39-234 r_{Z}+672 r_{Z}^{2}-920 r_{Z}^{3}+592 r_{Z}^{4}\right) \text {, }  \tag{A18}\\
& b_{A A}^{(7,1)}=\frac{16 \pi}{945\left(1-2 r_{Z}\right)^{4} r_{Z}^{2}}\left[2079-20790 r_{Z}+91822 r_{Z}^{2}-218096 r_{Z}^{3}+269368 r_{Z}^{4}-137024 r_{Z}^{5}\right. \\
& \left.-4 r_{p_{T}}\left(1169-7014 r_{Z}+20000 r_{Z}^{2}-27624 r_{Z}^{3}+17840 r_{Z}^{4}\right)\right] \text {, }  \tag{A19}\\
& b_{A A}^{(8,1)}=\frac{32 \pi^{2}}{945\left(1-2 r_{Z}\right)^{8} r_{Z}^{2}}\left[233-4194 r_{Z}+34914 r_{Z}^{2}-174856 r_{Z}^{3}+575264 r_{Z}^{4}\right. \\
& -1278272 r_{Z}^{5}+1903168 r_{Z}^{6}-1816064 r_{Z}^{7}+992768 r_{Z}^{8}-235520 r_{Z}^{9} \\
& +4\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(11-110 r_{Z}+572 r_{Z}^{2}-1544 r_{Z}^{3}+2288 r_{Z}^{4}-1184 r_{Z}^{5}+192 r_{Z}^{6}\right) \\
& \left.-64\left(1-2 r_{Z}\right)^{4} r_{p_{T}}^{2}\left(1-2 r_{Z}+4 r_{Z}^{2}\right)\right] \text {, }  \tag{A20}\\
& b_{A A, \ln }^{(9,0)}=\frac{-2 \pi}{315\left(1-2 r_{Z}\right)^{6} r_{Z}^{2}}\left[2621-36694 r_{Z}+225124 r_{Z}^{2}-793216 r_{Z}^{3}+1727344 r_{Z}^{4}-2317408 r_{Z}^{5}+1757888 r_{Z}^{6}\right. \\
& -610560 r_{Z}^{7}-2 r_{p_{T}}\left(8183-81830 r_{Z}+373652 r_{Z}^{2}-964400 r_{Z}^{3}+1472912 r_{Z}^{4}-1224416 r_{Z}^{5}+457920 r_{Z}^{6}\right) \\
& \left.+3200 r_{p_{T}}^{2}\left(1-2 r_{Z}\right)^{2}\left(1-2 r_{Z}+4 r_{Z}^{2}\right)\right] . \tag{A21}
\end{align*}
$$

for the expansion of the axial-vector component. The corresponding coefficients in the expansion of the vector part read

$$
\begin{align*}
a_{V V}^{(3,0)}= & \frac{16 \pi}{3\left(1-2 r_{Z}\right)^{2}}\left(2-5 r_{Z}\right),  \tag{A22}\\
a_{V V}^{(5,0)}= & \frac{8 \pi}{15\left(1-2 r_{Z}\right)^{4}}\left[34-217 r_{Z}+492 r_{Z}^{2}-412 r_{Z}^{3}+8\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\right]  \tag{A23}\\
a_{V V}^{(7,0)}= & \frac{2 \pi}{105\left(1-2 r_{Z}\right)^{6}}\left[1314-13549 r_{Z}+57240 r_{Z}^{2}-124296 r_{Z}^{3}+141056 r_{Z}^{4}-70032 r_{Z}^{5}\right. \\
& \left.+16\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(11-72 r_{Z}+148 r_{Z}^{2}\right)\right],  \tag{A24}\\
b_{V V}^{(2,1)}= & \frac{128 \pi^{2}}{9\left(1-2 r_{Z}\right)^{2}}\left(2-5 r_{Z}\right),  \tag{A25}\\
b_{V V}^{(3,0)}= & -\frac{\pi}{9\left(1-2 r_{Z}\right)^{3}\left(1-4 r_{Z}\right)^{2}}\left[4 ( 1 - 2 r _ { Z } ) ( 1 - 4 r _ { Z } ) \left(136+21 \pi^{2}+56 \ln (2)-680 r_{Z}\right.\right. \\
& -8\left(1-2 r_{Z}\right)\left(1-4 r_{Z}\right)^{2}\left(-136+3 \pi^{2}+168 \ln (2)\right) r_{p_{T}} \\
& -128 C_{0}\left(1-2 r_{Z}\right)^{2}\left(1-4 r_{Z}\right)^{2}\left(-3+4 r_{Z}^{2}\right) \\
& +64\left(1-4 r_{Z}\right)^{2} \sqrt{\frac{1-r_{Z}}{r_{Z}}} \arctan \left(\frac{2 \sqrt{\left(1-r_{Z}\right) r_{Z}}}{1-2 r_{Z}}\right)\left(-5+46 r_{Z}-96 r_{Z}^{2}+32 r_{Z}^{3}\right) \\
& \left.+512\left(1-2 r_{Z}\right) \ln \left(2-4 r_{Z}\right)\left(2-23 r_{Z}+86 r_{Z}^{2}-112 r_{Z}^{3}+24 r_{Z}^{4}\right)\right]
\end{align*}
$$

$$
\begin{aligned}
& b_{V V}^{(7,0)}=-\frac{\pi}{198450\left(1-2 r_{Z}\right)^{7}\left(1-4 r_{Z}\right)^{4}\left(1-r_{Z}\right)^{2}}\left[-\left(1-2 r_{Z}\right)\left(1-4 r_{Z}\right)\left(1-r_{Z}\right)\left[122096632-127575 \pi^{2}\right.\right. \\
&-92998920 \ln (2)-2848934592 r_{Z}+1630125 \pi^{2} r_{Z}+2155938120 \ln (2) r_{Z}+29029779592 r_{Z}^{2} \\
&-2152710 \pi^{2} r_{Z}^{2}-21749266560 \ln (2) r_{Z}^{2}-169738450784 r_{Z}^{3}-74541600 \pi^{2} r_{Z}^{3} \\
&+125261747520 \ln (2) r_{Z}^{3}+628426159680 r_{Z}^{4}+590919840 \pi^{2} r_{Z}^{4}-453823735680 \ln (2) r_{Z}^{4} \\
&-1530688965120 r_{Z}^{5}-2089568880 \pi^{2} r_{Z}^{5}+1072845164160 \ln (2) r_{Z}^{5}+2460756033152 r_{Z}^{6} \\
&+3889861920 \pi^{2} r_{Z}^{6}-1656277002240 \ln (2) r_{Z}^{6}-2533430536704 r_{Z}^{7}-3445787520 \pi^{2} r_{Z}^{7} \\
&+1613260615680 \ln (2) r_{Z}^{7}+1538421684224 r_{Z}^{8}+694310400 \pi^{2} r_{Z}^{8}-902101401600 \ln (2) r_{Z}^{8} \\
&\left.-445393100800 r_{Z}^{9}+435456000 \pi^{2} r_{Z}^{9}+220520939520 \ln (2) r_{Z}^{9}+25144197120 r_{Z}^{10}\right]
\end{aligned}
$$

$$
-2\left(1-2 r_{Z}\right)\left(1-4 r_{Z}\right)\left(1-r_{Z}\right)^{2} r_{p_{T}}\left[8486296-785295 \pi^{2}-11199720 \ln (2)-222273632 r_{Z}\right.
$$

$$
+17777340 \pi^{2} r_{Z}+267660960 \ln (2) r_{Z}+2469324096 r_{Z}^{2}-173313000 \pi^{2} r_{Z}^{2}-2810969280 \ln (2) r_{Z}^{2}
$$

$$
-14926447104 r_{Z}^{3}+941371200 \pi^{2} r_{Z}^{3}+16619420160 \ln (2) r_{Z}^{3}+52801418112 r_{Z}^{4}-3062631600 \pi^{2} r_{Z}^{4}
$$

$$
-58795390080 \ln (2) r_{Z}^{4}-108303542784 r_{Z}^{5}+5922262080 \pi^{2} r_{Z}^{5}+122121377280 \ln (2) r_{Z}^{5}
$$

$$
+116805871616 r_{Z}^{6}-6230165760 \pi^{2} r_{Z}^{6}-134713743360 \ln (2) r_{Z}^{6}-48174653440 r_{Z}^{7}
$$

$$
\left.+2704665600 \pi^{2} r_{Z}^{7}+58835804160 \ln (2) r_{Z}^{7}-2064384000 r_{Z}^{8}\right]
$$

$$
\begin{align*}
& b_{V V, \ln }^{(3,0)}=0, \\
& b_{V V}^{(3,1)}=0, \\
& b_{V V}^{(4,1)}=\frac{128 \pi^{2}}{45\left(1-2 r_{Z}\right)^{4}}\left(2-11 r_{Z}+36 r_{Z}^{2}-56 r_{Z}^{3}+4\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\right), \\
& b_{V V}^{(5,0)}=-\frac{\pi}{2025\left(1-r_{Z}\right)\left(1-2 r_{Z}\right)^{5}\left(1-4 r_{Z}\right)^{3}}\left[-2\left(1-2 r_{Z}\right)\left(1-4 r_{Z}\right)\left(1-r_{Z}\right)\left(186008-6345 \pi^{2}-177000 \ln (2)\right.\right. \\
& -2712968 r_{Z}+88830 \pi^{2} r_{Z}+2535360 \ln (2) r_{Z}+15732800 r_{Z}^{2}-498420 \pi^{2} r_{Z}^{2}-14241120 \ln (2) r_{Z}^{2} \\
& -45725792 r_{Z}^{3}+1443960 \pi^{2} r_{Z}^{3}+39511680 \ln (2) r_{Z}^{3}+68432384 r_{Z}^{4}-2246400 \pi^{2} r_{Z}^{4} \\
& \left.-55011840 \ln (2) r_{Z}^{4}-45982208 r_{Z}^{5}+1537920 \pi^{2} r_{Z}^{5}+31488000 \ln (2) r_{Z}^{5}+6266880 r_{Z}^{6}\right) \\
& +36\left(1-2 r_{Z}\right)\left(1-4 r_{Z}\right)\left(1-r_{Z}\right) r_{p_{T}}\left(-8024+585 \pi^{2}+11400 \ln (2)+97248 r_{Z}-6900 \pi^{2} r_{Z}\right. \\
& -130080 \ln (2) r_{Z}-384928 r_{Z}^{2}+28860 \pi^{2} r_{Z}^{2}+505440 \ln (2) r_{Z}^{2}+532224 r_{Z}^{3}-49440 \pi^{2} r_{Z}^{3} \\
& \left.-718080 \ln (2) r_{Z}^{3}-88576 r_{Z}^{4}+27840 \pi^{2} r_{Z}^{4}+192000 \ln (2) r_{Z}^{4}-40960 r_{Z}^{5}\right) \\
& +C_{0}\left(1-r_{Z}\right)\left(1-2 r_{Z}\right)^{2}\left(1-4 r_{Z}\right)\left[2880\left(1-4 r_{Z}\right)\left(53-564 r_{Z}+2000 r_{Z}^{2}-2368 r_{Z}^{3}-336 r_{Z}^{4}+1088 r_{Z}^{5}\right)\right. \\
& \left.-46080\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(3-24 r_{Z}+40 r_{Z}^{2}+8 r_{Z}^{3}\right)\right] \\
& +\left(1-r_{Z}\right)\left(1-2 r_{Z}\right) \ln \left(2-4 r_{Z}\right)\left[-3840\left(-80+1549 r_{Z}-12538 r_{Z}^{2}\right.\right. \\
& \left.+54340 r_{Z}^{3}-134752 r_{Z}^{4}+187264 r_{Z}^{5}-126752 r_{Z}^{6}+24192 r_{Z}^{7}\right) \\
& \left.-92160\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(-1+16 r_{Z}-64 r_{Z}^{2}+80 r_{Z}^{3}\right)\right] \\
& +\sqrt{\frac{1-r_{Z}}{r_{Z}}} \arctan \left(\frac{2 \sqrt{\left(1-r_{Z}\right) r_{Z}}}{1-2 r_{Z}}\right)\left[4 8 0 ( 1 - 4 r _ { Z } ) ^ { 2 } \left(-318+4749 r_{Z}-29882 r_{Z}^{2}\right.\right. \\
& \left.+103460 r_{Z}^{3}-212040 r_{Z}^{4}+249152 r_{Z}^{5}-144896 r_{Z}^{6}+30720 r_{Z}^{7}\right) \\
& \left.\left.+23040\left(1-2 r_{Z}\right)^{2}\left(1-4 r_{Z}\right)\left(1-r_{Z}\right) r_{p_{T}}\left(-5+54 r_{Z}-184 r_{Z}^{2}+200 r_{Z}^{3}+16 r_{Z}^{4}\right)\right]\right], \\
& b_{V V, \ln }^{(5,0)}=\frac{64 \pi}{5\left(1-2 r_{Z}\right)^{2}}\left(-1+4 r_{Z}+6 r_{p_{T}}\right) \text {, }  \tag{A31}\\
& b_{V V}^{(5,1)}=\frac{128 \pi}{135\left(1-2 r_{Z}\right)^{2}}\left(-52+103 r_{Z}-108 r_{p_{T}}\right) \text {, }  \tag{A32}\\
& b_{V V}^{(6,1)}=\frac{128 \pi^{2}}{315\left(1-2 r_{Z}\right)^{6}}\left[-58+619 r_{Z}-2624 r_{Z}^{2}+5540 r_{Z}^{3}-5552 r_{Z}^{4}+1648 r_{Z}^{5}+4\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(-5+6 r_{Z}+32 r_{Z}^{2}\right)\right], \tag{A33}
\end{align*}
$$

$$
\begin{align*}
& +201600\left(1-2 r_{Z}\right)^{3}\left(1-4 r_{Z}\right)^{4}\left(1-r_{Z}\right)^{2}\left[-356+3 \pi^{2}+520 \ln (2)\right] r_{p_{T}}^{2} \\
& +C_{0}\left(1-r_{Z}\right)^{2}\left(1-2 r_{Z}\right)^{2}\left(1-4 r_{Z}\right)^{2}\left[1 0 0 8 0 ( 1 - 4 r _ { Z } ) \left(1879-29708 r_{Z}+186948 r_{Z}^{2}\right.\right. \\
& \left.-592848 r_{Z}^{3}+968208 r_{Z}^{4}-663104 r_{Z}^{5}-57920 r_{Z}^{6}+155904 r_{Z}^{7}\right) \\
& \left.-322560\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(85-896 r_{Z}+3708 r_{Z}^{2}-7176 r_{Z}^{3}+5312 r_{Z}^{4}+800 r_{Z}^{5}\right)\right] \\
& +\left(1-4 r_{Z}\right)^{2} \sqrt{\frac{1-r_{Z}}{r_{Z}}} \arctan \left(\frac{2 \sqrt{\left(1-r_{Z}\right) r_{Z}}}{1-2 r_{Z}}\right)\left[-1680\left(1-4 r_{Z}\right)\left(14088-262050 r_{Z}\right.\right. \\
& +2162807 r_{Z}^{2}-10516934 r_{Z}^{3}+33617664 r_{Z}^{4}-74234384 r_{Z}^{5}+114844848 r_{Z}^{6} \\
& \left.-121868128 r_{Z}^{7}+82948992 r_{Z}^{8}-31822336 r_{Z}^{9}+5122048 r_{Z}^{10}\right) \\
& -53760\left(1-r_{Z}\right)\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(504-7341 r_{Z}+44374 r_{Z}^{2}-142180 r_{Z}^{3}\right. \\
& \left.\left.+253056 r_{Z}^{4}-229360 r_{Z}^{5}+68704 r_{Z}^{6}+9408 r_{Z}^{7}\right)\right] \\
& +\ln \left(2-4 r_{Z}\right)\left(1-r_{Z}\right)^{2}\left(1-2 r_{Z}\right)\left[1 3 4 4 0 \left(2522-68893 r_{Z}+832786 r_{Z}^{2}-5869624 r_{Z}^{3}+26678552 r_{Z}^{4}\right.\right. \\
& \left.-81564240 r_{Z}^{5}+169456096 r_{Z}^{6}-235064576 r_{Z}^{7}+205653632 r_{Z}^{8}-98444288 r_{Z}^{9}+16746496 r_{Z}^{10}\right) \\
& +215040\left(1-2 r_{Z}\right)^{2} r_{p_{T}}\left(17-772 r_{Z}+9460 r_{Z}^{2}-51664 r_{Z}^{3}\right. \\
& \left.\left.\left.+146272 r_{Z}^{4}-212992 r_{Z}^{5}+124160 r_{Z}^{6}+9216 r_{Z}^{7}\right)\right]\right],  \tag{A34}\\
& b_{V V, \ln }^{(7,0)}=\frac{32 \pi}{35\left(1-2 r_{Z}\right)^{4}}\left(-1+4 r_{Z}+6 r_{p_{T}}\right)\left(11-72 r_{Z}+148 r_{Z}^{2}\right) \text {, }  \tag{A35}\\
& b_{V V}^{(7,1)}=\frac{64 \pi}{945\left(1-2 r_{Z}\right)^{4}}\left[-2128+12817 r_{Z}-25428 r_{Z}^{2}+16060 r_{Z}^{3}+4 r_{p_{T}}\left(-413+2408 r_{Z}-4460 r_{Z}^{2}\right)\right],  \tag{A36}\\
& b_{V V}^{(8,1)}=\frac{128 \pi^{2}}{945\left(1-2 r_{Z}\right)^{8}}\left[-430+6217 r_{Z}-38796 r_{Z}^{2}+135616 r_{Z}^{3}-286976 r_{Z}^{4}+367712 r_{Z}^{5}-261632 r_{Z}^{6}+74752 r_{Z}^{7}\right. \\
& \left.+4 r_{p_{T}}\left(1-2 r_{Z}\right)^{2}\left(-43+290 r_{Z}-764 r_{Z}^{2}+808 r_{Z}^{3}+48 r_{Z}^{4}\right)-64 r_{p_{T}}^{2}\left(1-2 r_{Z}\right)^{4}\right],  \tag{A37}\\
& b_{V V, \ln }^{(9,0)}=\frac{8 \pi}{315\left(1-2 r_{Z}\right)^{6}}\left[187+564 r_{Z}-19112 r_{Z}^{2}+92640 r_{Z}^{3}-186896 r_{Z}^{4}+152640 r_{Z}^{5}\right. \\
& \left.+r_{p_{T}}\left(-482-12992 r_{Z}+95984 r_{Z}^{2}-233344 r_{Z}^{3}+228960 r_{Z}^{4}\right)-3200\left(1-2 r_{Z}\right)^{2} r_{p_{T}}^{2}\right] . \tag{A38}
\end{align*}
$$

## Appendix B: Subtractions

In this appendix, we give the functions $s_{i}$ with $i \in\{V V, A A\}$ used to subtract the threshold logarithms. We write them in terms of auxiliary subtraction functions $s_{n}, n \in \mathbb{N}$, i.e.

$$
\begin{equation*}
s_{i}^{(2)}(z)=\sum_{n=2}^{\infty} C_{i, n} s_{n}(z), \tag{B1}
\end{equation*}
$$

where the coefficients $C_{i, n}$ are constants and $s_{n}{ }^{z \rightarrow 1} \bar{z}^{\frac{n}{2}} \ln (\bar{z})+\mathcal{O}\left(\bar{z}^{\frac{n+1}{2}}\right)$ in the threshold region. We construct these auxiliary functions based on the known analytical results for the vacuum polarization function. The subtraction functions and their threshold expansions are

$$
\begin{align*}
s_{2}(z) & =-\frac{16(1-z) \Pi^{(1), v}(z)}{3 z} \\
& { }_{z \rightarrow 1}(1-z) \ln (1-z)-\frac{8}{\pi}(1-z)^{3 / 2}+\frac{1}{3}(1-z)^{2} \ln (1-z)-\frac{8}{9 \pi}(-5+18 \ln (2))(1-z)^{5 / 2} \\
& -\frac{16}{3 \pi}(1-z)^{5 / 2} \ln (1-z)-\frac{2}{3}(1-z)^{3} \ln (1-z)+\frac{1}{675 \pi}(14653-26280 \ln (2))(1-z)^{7 / 2} \\
& -\frac{548}{45 \pi}(1-z)^{7 / 2} \ln (1-z)-2(1-z)^{4} \ln (1-z)+\mathcal{O}\left((1-z)^{9 / 2}\right),  \tag{B2}\\
s_{4}(z) & =-\frac{8}{81 \pi^{2}} \frac{54 \pi^{2}(1-z)^{2} \Pi^{(1), v}(z)-41 z}{z^{2}} \\
& { }^{z \rightarrow 1}(1-z)^{2} \ln (1-z)-\frac{8}{\pi}(1-z)^{5 / 2}+\frac{4}{3}(1-z)^{3} \ln (1-z)-\frac{16}{9 \pi}(2+\ln (512))(1-z)^{7 / 2}
\end{align*}
$$

$$
\begin{align*}
& -\frac{16}{3 \pi}(1-z)^{7 / 2} \ln (1-z)+\frac{2}{3}(1-z)^{4} \ln (1-z)+\mathcal{O}\left((1-z)^{9 / 2}\right)  \tag{B3}\\
s_{5}(z) & =-\frac{32(1-z)^{3} G(z) \Pi^{(1), v}(z)}{3 \pi z^{2}}+\frac{656}{81 \pi^{3} z} \\
& \stackrel{z \rightarrow 1}{\asymp}\left[-\frac{11}{8}+\ln (8)+\frac{3}{2 \pi^{2}}\left(-2+7 \zeta_{3}\right)\right](1-z)^{5 / 2}+(1-z)^{5 / 2} \ln (1-z) \\
& -\frac{2}{\pi}(1-z)^{3} \ln (1-z)+\frac{1}{48}\left(-145+264 \ln (2) \cdot+\frac{376+924 \zeta_{3}}{\pi^{2}}\right)(1-z)^{7 / 2}+\frac{11}{6}(1-z)^{7 / 2} \ln (1-z) \\
& -\frac{28}{3 \pi}(1-z)^{4} \ln (1-z)+\mathcal{O}\left((1-z)^{9 / 2}\right)  \tag{B4}\\
s_{6}(z) & =-\frac{16(1-z)^{3} \Pi^{(1), v}(z)}{3 z^{3}}+\frac{328}{81 \pi^{2} z^{2}}-\frac{6404}{675 \pi^{2} z} \\
& \stackrel{z \rightarrow 1}{\asymp}(1-z)^{3} \ln (1-z)-\frac{8}{\pi}(1-z)^{7 / 2}+\frac{7}{3}(1-z)^{4} \ln (1-z)+\mathcal{O}\left((1-z)^{9 / 2}\right)  \tag{B5}\\
s_{7}(z) & =-\frac{32(1-z)^{4} G(z) \Pi^{(1), v}(z)}{3 \pi z^{3}}+\frac{656}{81 \pi^{3} z^{2}}-\frac{131672}{6075 \pi^{3} z} \\
& \stackrel{z \rightarrow 1}{\asymp}\left[-\frac{11}{8}+\ln (8)+\frac{3}{2 \pi^{2}}\left(-2+7 \zeta_{3}\right)\right](1-z)^{7 / 2} \\
& +(1-z)^{7 / 2} \ln (1-z)-\frac{2}{\pi}(1-z)^{4} \ln (1-z)+\mathcal{O}\left((1-z)^{9 / 2}\right)  \tag{B6}\\
s_{8}(z) & =-\frac{16(1-z)^{4} \Pi^{(1), v}(z)}{3 z^{4}}+\frac{328}{81 \pi^{2} z^{3}}-\frac{27412}{2025 \pi^{2} z^{2}}+\frac{7773424}{496125 \pi^{2} z} \\
& { }^{\prime} \rightarrow 1  \tag{B7}\\
& (1-z)^{4} \ln (1-z)+\mathcal{O}\left((1-z)^{9 / 2}\right)
\end{align*}
$$

where we have used the symbol $\asymp$ to denote that terms analytical in $(1-z)$ have been dropped on the right-hand side, and we only use subtractions for the logarithmic terms, hence no subtraction functions $s_{1}(z)$ and $s_{3}(z)$ are necessary. We have used

$$
\begin{equation*}
G(z)=\frac{1}{2 z \sqrt{1-1 / z}} \ln \left(\frac{\sqrt{1-1 / z}-1}{\sqrt{1-1 / z}+1}\right) \tag{B8}
\end{equation*}
$$

and $\Pi^{(1), v}$ is the well-known two-loop correction to the vacuum polarization [53] in the convention of [54]. The functions $s_{i}$ in eqs. $\overline{\mathrm{B} 2}$ - B 7 are constant as $z \rightarrow 0$ and only diverge logarithmically as $z \rightarrow \infty$.
[1] N. Kauer and G. Passarino, JHEP 08, 116 (2012), 1206.4803.
[2] F. Caola and K. Melnikov, Phys. Rev. D88, 054024 (2013), 1307.4935.
[3] J. M. Campbell, R. K. Ellis, and C. Williams, JHEP 04, 060 (2014), 1311.3589.
[4] C. Englert and M. Spannowsky, Phys. Rev. D90, 053003 (2014), 1405.0285.
[5] C. Englert, Y. Soreq, and M. Spannowsky, JHEP 05, 145 (2015), 1410.5440.
[6] J. M. Campbell, R. K. Ellis, and C. Williams, Phys. Rev. D89, 053011 (2014), 1312.1628.
[7] M. Aaboud et al. (ATLAS), Phys. Lett. B786, 223 (2018), 1808.01191.
[8] A. M. Sirunyan et al. (CMS), Phys. Rev. D99, 112003 (2019), 1901.00174.
[9] D. de Florian et al. (LHC Higgs Cross Section Working

Group) (2016), 1610.07922.
[10] J. S. Gainer, J. Lykken, K. T. Matchev, S. Mrenna, and M. Park, Phys. Rev. D91, 035011 (2015), 1403.4951.
[11] A. Azatov, C. Grojean, A. Paul, and E. Salvioni, Zh. Eksp. Teor. Fiz. 147, 410 (2015), [J. Exp. Theor. Phys.120,354(2015)], 1406.6338.
[12] M. Buschmann, D. Goncalves, S. Kuttimalai, M. Schonherr, F. Krauss, and T. Plehn, JHEP 02, 038 (2015), 1410.5806.
[13] A. Azatov, C. Grojean, A. Paul, and E. Salvioni, JHEP 09, 123 (2016), 1608.00977.
[14] M. Spira, A. Djouadi, D. Graudenz, and P. M. Zerwas, Nucl. Phys. B453, 17 (1995), hep-ph/9504378.
[15] R. Harlander and P. Kant, JHEP 12, 015 (2005), hepph/0509189.
[16] C. Anastasiou, S. Beerli, S. Bucherer, A. Daleo, and Z. Kunszt, JHEP 01, 082 (2007), hep-ph/0611236.
[17] U. Aglietti, R. Bonciani, G. Degrassi, and A. Vicini, JHEP 01, 021 (2007), hep-ph/0611266.
[18] J. Davies, R. Gröber, A. Maier, T. Rauh, and M. Steinhauser (2019), 1906.00982.
[19] R. V. Harlander, M. Prausa, and J. Usovitsch (2019), 1907.06957.
[20] E. W. N. Glover and J. J. van der Bij, Nucl. Phys. B321, 561 (1989).
[21] F. Caola, J. M. Henn, K. Melnikov, A. V. Smirnov, and V. A. Smirnov, JHEP 06, 129 (2015), 1503.08759.
[22] A. von Manteuffel and L. Tancredi, JHEP 06, 197 (2015), 1503.08835.
[23] F. Caola, K. Melnikov, R. Röntsch, and L. Tancredi, Phys. Rev. D92, 094028 (2015), 1509.06734.
[24] F. Caola, M. Dowling, K. Melnikov, R. Röntsch, and L. Tancredi, JHEP 07, 087 (2016), 1605.04610.
[25] M. Grazzini, S. Kallweit, M. Wiesemann, and J. Y. Yook, JHEP 03, 070 (2019), 1811.09593.
[26] K. Melnikov and M. Dowling, Phys. Lett. B744, 43 (2015), 1503.01274.
[27] J. M. Campbell, R. K. Ellis, M. Czakon, and S. Kirchner, JHEP 08, 011 (2016), 1605.01380.
[28] R. Gröber, A. Maier, and T. Rauh, JHEP 03, 020 (2018), 1709.07799.
[29] S. Borowka, N. Greiner, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, U. Schubert, and T. Zirke, Phys. Rev. Lett. 117, 012001 (2016), [Erratum: Phys. Rev. Lett.117,no.7,079901(2016)], 1604.06447.
[30] S. Borowka, N. Greiner, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, and T. Zirke, JHEP 10, 107 (2016), 1608.04798.
[31] G. Heinrich, S. P. Jones, M. Kerner, G. Luisoni, and E. Vryonidou, JHEP 08, 088 (2017), 1703.09252.
[32] J. Baglio, F. Campanario, S. Glaus, M. Mühlleitner, M. Spira, and J. Streicher, Eur. Phys. J. C79, 459 (2019), 1811.05692.
[33] R. Bonciani, G. Degrassi, P. P. Giardino, and R. Gröber, Phys. Rev. Lett. 121, 162003 (2018), 1806.11564.
[34] J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, JHEP 03, 048 (2018), 1801.09696.
[35] J. Davies, G. Mishima, M. Steinhauser, and D. Wellmann, JHEP 01, 176 (2019), 1811.05489.
[36] X. Xu and L. L. Yang, JHEP 01, 211 (2019), 1810.12002.
[37] J. Davies, G. Heinrich, S. P. Jones, M. Kerner, G. Mishima, M. Steinhauser, and D. Wellmann (2019), 1907.06408.
[38] J. M. Campbell, R. K. Ellis, E. Furlan, and R. Röntsch, Phys. Rev. D90, 093008 (2014), 1409.1897.
[39] J. M. Campbell, R. K. Ellis, and G. Zanderighi, JHEP 12, 056 (2007), 0710.1832.
[40] V. Costantini, B. De Tollis, and G. Pistoni, Nuovo Cim. A2, 733 (1971).
[41] S. Catani, Phys. Lett. B427, 161 (1998), hepph/9802439.
[42] M. Beneke and V. A. Smirnov, Nucl. Phys. B522, 321 (1998), hep-ph/9711391.
[43] B. Jantzen, JHEP 12, 076 (2011), 1111.2589.
[44] P. Nogueira, J. Comput. Phys. 105, 279 (1993).
[45] J. A. M. Vermaseren (2000), math-ph/0010025.
[46] K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B192, 159 (1981).
[47] A. V. Smirnov, Comput. Phys. Commun. 189, 182 (2015), 1408.2372.
[48] S. Laporta, Int. J. Mod. Phys. A15, 5087 (2000), hepph/0102033.
[49] M. Beneke, J. Piclum, and T. Rauh, Nucl. Phys. B880, 414 (2014), 1312.4792.
[50] J. Fleischer and O. V. Tarasov, Z. Phys. C64, 413 (1994), hep-ph/9403230.
[51] B. Agarwal, Two-loop amplitudes for $g g \rightarrow Z Z$ with full top mass dependence, HXSWG Offshell and Interference Meeting (27th May 2019).
[52] B. Agarwal and A. von Manteuffel, The two-loop amplitudes for $g g \rightarrow Z Z$ with full top-quark mass dependence, to appear.
[53] A. O. G. Källén and A. Sabry, Kong. Dan. Vid. Sel. Mat. Fys. Med. 29, 1 (1955).
[54] Y. Kiyo, A. Maier, P. Maierhöfer, and P. Marquard, Nucl. Phys. B823, 269 (2009), 0907.2120.


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    ${ }^{1}$ Note that the indirect way of constraining the Higgs width is not entirely model-independent (4) 5.

[^1]:    ${ }^{2}$ Recently, an independent numerical calculation 32, several approximations 33 which are consistent with the earlier results

[^2]:    and a combined result 37 have appeared.

[^3]:    ${ }^{3}$ Note that this subtraction differs at order $\epsilon^{0}$ from the one given in eq. (2.14) of 27.

[^4]:    ${ }^{4}$ To make the double-triangle contribution shown in Fig. 1 anomaly free we have to consider doublets of quarks and not just a single (top) quark, but we omit this technicality here since the double-triangle contribution is known and not considered below.

