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Spectrum of the reflection operators in different integrable structures

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Abstract

The reflection operators are the simplest examples of the non-local integrals of motion, which appear in many interesting problems in integrable CFT. For the so-called Fateev, quantum AKNS, paperclip and KdV integrable structures, they are built from the (chiral) reflection S-matrices for the Liouville and cigar CFTs. Here we give the full spectrum of the reflection operators associated with these integrable structures. We also obtained a relation between the reflection S-matrices of the cigar and Liouville CFTs. The results of this work are applicable for the description of the scaling behaviour of the Bethe states in exactly solvable lattice systems and may be of interest to the study of the Generalized Gibbs Ensemble associated with the above mentioned integrable structures.

1 Introduction

The problem of the simultaneous diagonalization of an infinite set of mutually commuting local Integrals of Motion (IM) naturally appears in the study of 1+1 dimensional integrable QFT. In the case of a scale invariant theory significant simplifications occur due to the presence of an infinite dimensional algebra of (extended) conformal symmetry [1]. For a finite-size 2D CFT, where the spatial coordinate is compactified on a circle, it is possible to give a mathematically satisfactory construction of an infinite set of local IM, whose simultaneous diagonalization becomes a well-defined problem within the representation theory of the associated conformal algebra. Different conformal algebras, as well as different sets of mutually commuting local IM, yield a variety of integrable structures in CFT.

Perhaps the simplest integrable structure involves the Virasoro algebra itself,

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\, n(n^2-1)\, \delta_{n+m,0}\,, \tag{1.1}$$

with the local IM being given by integrals over the local densities built out of the holomorphic field

$$T(u) = -\frac{c}{24} + \sum_{n=-\infty}^{\infty} L_n\, \mathrm{e}^{-nu} \qquad (u = x_2 + \mathrm{i}\, x_1)\ . \tag{1.2}$$

The explicit form for the first few IM reads as follows[1]

$$\mathbb{I}_1^{(\mathrm{KdV})} = \int_0^{2\pi} \frac{\mathrm{d}x_1}{2\pi}\, T\,, \quad \mathbb{I}_3^{(\mathrm{KdV})} = \int_0^{2\pi} \frac{\mathrm{d}x_1}{2\pi}\, T^2\,, \quad \mathbb{I}_5^{(\mathrm{KdV})} = \int_0^{2\pi} \frac{\mathrm{d}x_1}{2\pi}\, \Big(T^3 - \frac{c+2}{12}\,(\partial T)^2\Big)\,, \ldots\ . \tag{1.3}$$

They are referred to as the local IM for the quantum KdV integrable structure since, in the limit when the central charge $c \to \infty$ (which can be understood as a certain classical limit), the $\big\{\mathbb{I}_{2m-1}^{(\mathrm{KdV})}\big\}_{m=1}^{\infty}$ becomes the set of IM for the classical KdV equation [2–6]. The operators (1.3) act in the Verma module $\mathcal{V}_\Delta$ for the Virasoro algebra. An important property is that they leave the level subspaces $\mathcal{V}_\Delta^{(N)}$ of the Verma module invariant. Thus the problem of their simultaneous diagonalization can be restricted to $\mathcal{V}_\Delta^{(N)}$, where the IM are $\mathtt{par}_1(N) \times \mathtt{par}_1(N)$ dimensional mutually commuting matrices with $\mathtt{par}_1(N)$ being the number of integer partitions of N.

In ref. [7] the spectrum of the local IM (1.3) in $\mathcal{V}_\Delta^{(N)}$ was expressed in terms of the solutions of the algebraic system

$$\sum_{b\neq a}^{N} \frac{v_a\,(\,v_a^2 + (3+\alpha)(1+2\alpha) v_a v_b + \alpha(1+2\alpha)\, v_b^2\,)}{(v_a - v_b)^3} - \frac{v_a}{4} + \Delta = 0 \qquad (a = 1, \ldots, N)\ , \tag{1.4}$$

where α parameterizes the central charge

$$c = 1 - \frac{6\alpha^2}{\alpha+1}\ . \tag{1.5}$$

[1]In this work the subscript in the notation of the local IM $\mathbb{I}_s$ always indicates that the corresponding local density has the Lorentz spin $s+1$, e.g., in eq. (1.3) $\mathrm{spin}(T) = 2,\ \mathrm{spin}(T^2) = 4,\ \mathrm{spin}(T^3) = \mathrm{spin}\big((\partial T)^2\big) = 6$.

It was conjectured in that work and recently proved by D. Masoero [8] that for generic (complex) values of Δ and c the number of distinct, up to the action of the symmetric group S_N, solutions of the algebraic system (1.4) coincides with the number of integer partitions $\mathtt{par}_1(N)$. The eigenvalues of the local IM $\mathbb{I}^{(\mathrm{KdV})}_{2m-1}$ turn out to be certain symmetric polynomials of order $m-1$ w.r.t. $\{v_a\}_{a=1}^N$ that satisfy the set of equations (1.4). This implies that these distinct unordered sets can be used to label states, which form a basis in $\mathcal{V}_\Delta^{(N)}$ and simultaneously diagonalize the local IM (1.3):

$$\mathbb{I}^{(\mathrm{KdV})}_{2m-1}\,|\,\boldsymbol{v}\,\rangle = I^{(\mathrm{KdV})}_{2m-1}(\boldsymbol{v})\,|\,\boldsymbol{v}\,\rangle\ , \qquad \text{where} \qquad \boldsymbol{v}=(v_1,\ldots,v_N)\ . \tag{1.6}$$

Together with the local IM the integrable structures also involve the non-local IM. In the case of the quantum KdV integrable structure these were discussed in ref. [9]. Furthermore, in ref. [10], it was pointed out that the set of non-local IM contains an operator which is deeply related to the Liouville reflection S-matrix. Its construction is based on the observation that for $c\geq 25$ the field $T(u)$ (1.2) can be interpreted as the holomorphic component of the stress-energy tensor for the Liouville CFT. The configuration space of the Liouville theory contains a domain where the exponential interaction term is negligible and the Liouville field becomes a free massless field, whose chiral component is built out of the Heisenberg operators

$$[a_n,a_m]=\frac{n}{2}\ \delta_{n+m,0}\ . \tag{1.7}$$

In the asymptotic domain, the Virasoro generators are expressed in terms of the Heisenberg ones as

$$\begin{aligned} L_n &= \sum_{m\neq 0,n} a_m a_{n-m} + \big(2a_0+\mathrm{i}Q\,n\big)\,a_n \qquad\qquad (n\neq 0)\\ L_0 &= 2\sum_{m>0} a_{-m}a_m + a_0^2+\tfrac{1}{4}\,Q^2\ , \end{aligned} \tag{1.8}$$

where

$$Q=\sqrt{\frac{c-1}{6}}\ . \tag{1.9}$$

The Heisenberg algebra (1.7) can be represented in the Fock space $\mathcal{F}_P$ generated by the action of the creation operators on the Fock vacuum

$$a_n\,|P\rangle=0 \quad (\forall\ n>0)\ , \qquad a_0\,|P\rangle=P\,|P\rangle\ . \tag{1.10}$$

As a consequence of the relations (1.8), $\mathcal{F}_P$ is isomorphic to the Verma module $\mathcal{V}_\Delta$ with the highest weight Δ given in terms of the zero-mode momentum P as

$$\Delta=P^2+\tfrac{1}{4}\,Q^2. \tag{1.11}$$

Notice that Δ is independent of the sign of P. For $P>0$, $\mathcal{F}_P$ and $\mathcal{F}_{-P}$ can be interpreted as the space of "in" and "out" (chiral) asymptotic states of the Liouville CFT. It is natural to introduce the S-matrix intertwining the two spaces

$$\hat{s}_{\mathrm{L}}(P\,|\,Q)\ :\qquad \mathcal{F}_P\mapsto\mathcal{F}_{-P}\ , \tag{1.12}$$

where we explicitly indicate the dependence on the zero mode momentum P and the parameter Q. As was discussed in [10], this operator is fully determined by the conformal symmetry and the normalization condition

$$\hat{s}_{\rm L}(P\,|\,Q)\,|P\rangle = |-P\rangle\ . \tag{1.13}$$

One can introduce another intertwiner, the "C-conjugation", whose action is defined by the relations

$$\hat{C}\ :\qquad \hat{C}\,a_n\,\hat{C} = -a_n\ ,\qquad \hat{C}\,|P\rangle = |-P\rangle\ , \tag{1.14}$$

so that the operator $\hat{C}\hat{s}_{\rm L}$ acts invariantly in the Fock space. It turns out that the latter commutes with the local IM (1.3) provided that they are understood as operators in $\mathcal{F}_P$ via eq. (1.8) [10,11]. We define the reflection operator associated with the quantum KdV integrable structure as in ref. [11]

$$\mathbb{R}^{(\rm KdV)} = R_0\ \big[\hat{C}\hat{s}_{\rm L}(P|\,Q)\big]^{-1}\ . \tag{1.15}$$

The scalar factor R_0 coincides with the vacuum eigenvalue and its value is not essential for the purpose of this work. Thus we arrive at the problem of the calculation of the eigenvalues $R^{(\rm KdV)}(\boldsymbol{v})$:

$$\mathbb{R}^{(\rm KdV)}\,|\boldsymbol{v}\rangle_P = R^{(\rm KdV)}(\boldsymbol{v})\,|\boldsymbol{v}\rangle_P\ , \tag{1.16}$$

where we emphasize with the subscript "P" that $|\boldsymbol{v}\rangle_P$ should be considered as a state in the level subspace $\mathcal{F}_P^{(N)}\cong\mathcal{V}_\Delta^{(N)}$. Among the purposes of this work is to present an explicit description of $R^{(\rm KdV)}(\boldsymbol{v})$ in terms of the sets $\{v_a\}_{a=1}^N$ solving eqs.(1.4).

In the case of extended conformal symmetry the rôle of the symmetry algebra is usually played by a W-algebra generated by the holomorphic fields of (half-) integer Lorentz spin along with the spin-2 field similar to $T(u)$ (1.2) [12]. It should be kept in mind that for a given such algebra there are a number of integrable structures corresponding to the different sets of mutually commuting local IM that can be built out of the holomorphic fields. For example, in the case of the W_∞-algebra studied in [13], there are at least three different integrable structures: the quantum AKNS [14–17], paperclip [14, 18, 19] and Getmanov ones [20, 21]. For each of them there is an associated reflection operator(s) similar to (1.15). In this work we consider the eigenvalue problem for the reflection operators for the quantum AKNS and paperclip integrable structures. It turns out that the spectrum of these reflection operators as well as $\mathbb{R}^{(\rm KdV)}$ (1.15) can be obtained via certain limiting procedures and restrictions from that of the reflection operators introduced in ref. [22]. The latter appear within the integrable structure originally found by Fateev [14, 23, 24].

The paper is organized as follows. In sec. 2 we use the results of the recent work of Eremenko and Tarasov [25] to calculate the spectrum of the reflection operators in the Fateev integrable structure. As will be discussed in the subsequent section, this allows one to find the spectrum of the reflection operators in the quantum AKNS integrable structure through a certain limiting procedure. In sec. 3 we also list some basic facts about the quantum AKNS and Getmanov integrable structures. Sec. 4 is devoted to the reflection operator for the paperclip integrable structure. Its spectrum follows immediately from the results of sec. 2 by a specialization of the parameters. In turn, a proper restriction of the formulae in sec. 3 yields the spectrum of the reflection operator (1.15). This is the subject of consideration of sec. 5. Finally, the relation of the reflection operators to the different Hermitian forms associated with the quantum AKNS and paperclip integrable structures is briefly discussed in sec. 6.

2 Spectrum of the reflection operators in the Fateev integrable structure

The W-algebra associated with the exceptional Lie superalgebra $D(2,1;\alpha)$ that was studied in ref. [26] appeared in the description of the so-called corner-brane conformal boundary state in [24], where it was referred to as the "corner-brane W-algebra". This algebra of extended conformal symmetry admits the integrable structure that was found in [23] and further investigated in the papers [22, 24]. In particular, the construction of the corresponding reflection operators was given in sec. (6.1) from [22]. The reflection operators act invariantly in the level subspaces of the Fock space which is the highest weight representation of three copies of the Heisenberg algebra of the form (1.7), labeled by the three component zero-mode momentum. The problem of the calculation of the spectrum of the reflection operators was reduced to finding connection coefficients for a certain linear ODE. Let us first give an outline of the relevant results from ref. [22].

The starting point of that work is the so-called generalized hypergeometric equation

$$\big(-\partial_z^2+t(z)\big)\,\psi=0\ , \tag{2.1}$$

where

$$t(z)=t_0(z)+t_1(z) \tag{2.2}$$

with

$$\begin{aligned} t_0(z)\ &=\ -\sum_{i=1}^{3}\Big(\frac{\delta_i}{(z-z_i)^2}+\frac{c_i}{z-z_i}\Big) \\ t_1(z)\ &=\ \sum_{a=1}^{L}\Big(\frac{2}{(z-x_a)^2}-\frac{C_a}{z-x_a}\Big)\ . \end{aligned} \tag{2.3}$$

Treating the complex variable z as a coordinate on the Riemann sphere, the imposed conditions

$$\begin{aligned} &\sum_{i=1}^{3}c_i=-\sum_{a=1}^{L}C_a \\ &\sum_{i=1}^{3}(c_i\,z_i+\delta_i)=\sum_{a=1}^{L}\big(2-C_a x_a\big) \\ &\sum_{i=1}^{3}(c_i\,z_i^2+2\delta_i\,z_i)=\sum_{a=1}^{L}\big(4x_a-C_a\,x_a^2\big) \end{aligned} \tag{2.4}$$

imply that the north pole of the sphere corresponding to $z=\infty$ is a regular point for the ODE (2.1). Equations (2.4) can be used to express c_1,c_2,c_3 through the other parameters. Moreover it is assumed that $\{x_a\}_{a=1}^{L}$ and $\{C_a\}_{a=1}^{L}$ obey the system of algebraic equations

$$\begin{aligned} &C_a\,\Big[\,\frac{1}{4}\,C_a^2-t_0(x_a)-\sum_{b\neq a}^{L}\Big(\frac{2}{(x_a-x_b)^2}-\frac{C_b}{x_a-x_b}\Big)\Big] \\ &-t_0'(x_a)+\sum_{b\neq a}^{L}\left(\frac{4}{(x_a-x_b)^3}-\frac{C_b}{(x_a-x_b)^2}\right)=0 \qquad (a=1,\ldots,L)\ , \end{aligned} \tag{2.5}$$

where the prime stands for the derivative w.r.t. the argument. The above system guarantees that all the singularities at $z = x_a$ $(a = 1, \ldots, L)$ are apparent, i.e., any solution of (2.1) remains a single valued function in the vicinity of these singular points.

Let $\chi^{(i)}_\sigma(z)$ $(i = 1, 2, 3;\ \sigma = \pm)$ be solutions of (2.1) such that

$$\chi^{(i)}_\sigma \to \frac{1}{\sqrt{2p_i}}\ (z - z_i)^{\frac{1}{2}+\sigma p_i} \Big(1 + O(z - z_i)\Big) \qquad \text{as} \qquad z \to z_i \tag{2.6}$$

with the p_i defined by the relations

$$\delta_i = \tfrac{1}{4} - p_i^2\,. \tag{2.7}$$

The prefactor in eq. (2.6) is chosen to ensure the following normalization for $\chi^{(i)}_\sigma$:

$$\mathtt{W}[\chi^{(i)}_{\sigma'},\, \chi^{(i)}_\sigma] = \sigma\, \delta_{\sigma+\sigma',0}\ , \tag{2.8}$$

where $\mathtt{W}[f, g] = fg' - gf'$ stands for the Wronskian. With the restriction $0 < p_i < \frac{1}{2}$, the asymptotic conditions (2.6) define three different bases (for $i = 1, 2, 3$) in the two-dimensional linear space of solutions of (2.1). Let us combine the solutions for given i into the row vector

$$\boldsymbol{\chi}^{(i)} = (\chi^{(i)}_-, \chi^{(i)}_+)\,, \qquad i = 1, 2, 3\ . \tag{2.9}$$

Then the linear transformation relating any two sets of bases can be expressed in the form

$$\boldsymbol{\chi}^{(i)} = \boldsymbol{\chi}^{(j)}\ \boldsymbol{S}^{(j,i)}\ . \tag{2.10}$$

The matrices

$$\boldsymbol{S}^{(j,i)} = \begin{pmatrix} S^{(j,i)}_{--} & S^{(j,i)}_{-+} \\ S^{(j,i)}_{+-} & S^{(j,i)}_{++} \end{pmatrix} \tag{2.11}$$

obey the relations

$$\det\big(\boldsymbol{S}^{(j,i)}\big) = 1\,, \qquad \boldsymbol{S}^{(i,j)}\boldsymbol{S}^{(j,i)} = \boldsymbol{I}\,, \qquad \boldsymbol{S}^{(i,k)}\boldsymbol{S}^{(k,j)}\boldsymbol{S}^{(j,i)} = \boldsymbol{I}\,, \tag{2.12}$$

where in the last equality (i, j, k) is any cyclic permutation of $(1, 2, 3)$. In the case $L = 0$, the solutions of the ODE (2.1) are given in terms of the hypergeometric function and the matrices $\boldsymbol{S}^{(j,i)}|_{L=0}$ are well known (see, e.g., eqs.(2.35)-(2.37) and B.2 in [22]). For general $L = 1, 2, \ldots$ they can be represented in the form

$$S^{(2,1)}_{\sigma'\sigma} = G_L(\boldsymbol{X}\,|-\sigma p_1, \sigma' p_2, p_3)\ S^{(2,1)}_{\sigma'\sigma}\big|_{L=0} \qquad (\sigma, \sigma' = \pm)\,, \tag{2.13}$$

where the function G_L depends on the L cross-ratios $\boldsymbol{X} = (X_1, X_2, \ldots, X_L)$,

$$X_a = \frac{x_a - z_1}{x_a - z_3}\,\frac{z_2 - z_3}{z_2 - z_1}\ , \tag{2.14}$$

as well as the parameters (p_1, p_2, p_3).

The ODE (2.1)-(2.5) is covariant w.r.t. Möbius transformations of the Riemann sphere. This can be used to move the non-apparent singularities to the standard positions:

$$(z_1, z_2, z_3) = (0, 1, \infty)\ . \tag{2.15}$$

With this set up the cross-ratios X_a (2.14) coincide with x_a and

$$
\begin{aligned}
t_0(z) &= -\bigg[\,\frac{\delta_1}{z^2}+\frac{\delta_1+\delta_2-\delta_3-2L-\sum_{a=1}^{L}C_a\,(1-x_a)}{z}\\
&\quad +\frac{\delta_2}{(z-1)^2}-\frac{\delta_1+\delta_2-\delta_3-2L+\sum_{a=1}^{L}C_a\,x_a}{z-1}\,\bigg] \\
t_1(z) &= \sum_{a=1}^{L}\bigg(\frac{2}{(z-x_a)^2}-\frac{C_a}{z-x_a}\bigg)\ .
\end{aligned}
\tag{2.16}
$$

In what follows we will always assume the choice (2.15), so that

$$
G_L=G_L(\boldsymbol{x}\,|\,p_1,p_2,p_3)\ , \qquad \text{where} \qquad \boldsymbol{x}=\boldsymbol{X}=(x_1,x_2,\ldots,x_L)\ . \tag{2.17}
$$

The set of L equations (2.5) with $t_0(z)$, $t_1(z)$ given by (2.16) allows one to express $\{C_a\}_{a=1}^L$ in terms of the $\{x_a\}_{a=1}^L$. In ref. [22] it was conjectured that if this algebraic system is supplemented by the extra equations

$$
C_a=\frac{2-a_1}{x_a}+\frac{2-a_2}{x_a-1} \qquad\qquad (a=1,\ldots,L)\ , \tag{2.18}
$$

then for generic values of the parameters $a_1,\ a_2$ and (p_1,p_2,p_3), the combined system admits exactly $\texttt{par}_3(L)$ distinct (up to permutations) solutions for $\{x_a\}_{a=1}^L$. Here $\texttt{par}_3(L)$ denotes the number of partitions of L into integer parts of three kinds:

$$
\sum_{L=0}^{\infty}\texttt{par}_3(L)\ q^L=\prod_{m=1}^{\infty}\frac{1}{(1-q^m)^3}=1+3\,q+9\,q^2+22\,q^3+\ldots\ . \tag{2.19}
$$

The sets of solutions $\boldsymbol{x}=(x_1,\ldots,x_L)$ label the basis states in the level-L subspaces of the Fock space whose highest weight is parameterized by $\boldsymbol{p}=(p_1,p_2,p_3)$. This particular basis diagonalizes the local IM from the Fateev integrable structure:

$$
\mathbb{I}^{(\mathrm{F})}_{2m-1}\,|\boldsymbol{x}\rangle_{\boldsymbol{p}}=I^{(\mathrm{F})}_{2m-1}(\boldsymbol{x})\,|\boldsymbol{x}\rangle_{\boldsymbol{p}} \tag{2.20}
$$

and the eigenvalues $I^{(\mathrm{F})}_{2m-1}(\boldsymbol{x})$ turn out to be symmetric functions of x_a (for an illustration see eqs.(3.40)-(3.44), (4.17) from ref. [22]). Notice that these local IM depend additionally on the two parameters a_1 and a_2 which appear in eq.(2.18).

The reflection operators for the Fateev integrable structure were introduced in the work [22]. The paper considered twelve operators, denoted as $\mathbb{R}^{(k)}_{\sigma'\sigma}$ ($k=1,2,3$; $\sigma,\sigma'=\pm$), such that

$$
\mathbb{R}^{(k)}_{-\sigma'-\sigma}=\big[\mathbb{R}^{(k)}_{\sigma'\sigma}\big]^{-1}\ , \qquad\qquad \big[\,\mathbb{R}^{(k)}_{\sigma'\sigma},\mathbb{I}^{(\mathrm{F})}_{2m-1}\big]=0\ . \tag{2.21}
$$

For their explicit construction, we refer the reader to sec. (6.1) in [22]. The results of that work imply that the eigenvalues of $\mathbb{R}^{(3)}_{\sigma'\sigma}$ for the state $|\boldsymbol{x}\rangle_{\boldsymbol{p}}$ are given by

$$
\begin{aligned}
R^{(3)}_{++} &= \big[R^{(3)}_{--}\big]^{-1}=G_L(\boldsymbol{x}\,|\,p_1,p_2,p_3\,\|\,a_1,a_2)\\
R^{(3)}_{-+} &= \big[R^{(3)}_{+-}\big]^{-1}=G_L(\boldsymbol{x}\,|\,p_1,-p_2,p_3\,\|\,a_1,a_2)\ .
\end{aligned}
\tag{2.22}
$$

Note that, having made the specialization (2.18), the function

$$G_L = G_L(\boldsymbol{x}\,|p_1,p_2,p_3\,\|\,a_1,a_2) \tag{2.23}$$

depends additionally on the two parameters a_1 and a_2 as well as $\boldsymbol{x}=(x_1\,\ldots,x_L)$, which is now understood as one of the $\mathtt{par}_3(L)$ solutions of the system (2.5),(2.16) and (2.18).

It is possible to obtain the eigenvalues of the reflection operators $\mathbb{R}^{(k)}_{\sigma'\sigma}$ with $k=1,2$ through the modular transformation of the set $\boldsymbol{x}$. In particular, one has

$$\begin{aligned} R^{(2)}_{++} &= \big[R^{(2)}_{--}\big]^{-1} = G_L(\tilde{\boldsymbol{x}}\,|\,p_3,p_1,p_2\,\|\,a_3,a_1) \\ R^{(2)}_{-+} &= \big[R^{(2)}_{+-}\big]^{-1} = G_L(\tilde{\boldsymbol{x}}\,|\,p_3,-p_1,p_2\,\|\,a_3,a_1) \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} R^{(1)}_{++} &= \big[R^{(1)}_{--}\big]^{-1} = G_L(\tilde{\tilde{\boldsymbol{x}}}\,|\,p_2,p_3,p_1\,\|\,a_2,a_3) \\ R^{(1)}_{-+} &= \big[R^{(1)}_{+-}\big]^{-1} = G_L(\tilde{\tilde{\boldsymbol{x}}}\,|\,p_2,-p_3,p_1\,\|\,a_2,a_3)\ , \end{aligned} \tag{2.25}$$

where

$$\tilde{\boldsymbol{x}} = \big(\tfrac{1}{1-x_1},\,\ldots,\,\tfrac{1}{1-x_L}\big)\,, \qquad\qquad \tilde{\tilde{\boldsymbol{x}}} = \big(1-\tfrac{1}{x_1},\ldots,1-\tfrac{1}{x_L}\big)\ .$$

Also we use the parameter a_3 defined by

$$a_1+a_2+a_3=2\ . \tag{2.26}$$

Thus the problem of calculation of the spectrum of the reflection operators in the Fateev integrable structure is reduced to finding the functions G_L (2.13). For $L=1$ it was solved in Appendix A of ref. [22]. The results of the important work [25] allows one to derive an explicit analytical expression for G_L for any $L=1,2,\ldots$.

The ODE studied by Eremenko and Tarasov is more general than the one defined by eqs. (2.1), (2.16) and (2.5). Their result, specialized to the case considered here, implies that the basic solutions $\chi^{(i)}_\sigma$ are given in terms of the conventional hypergeometric function ${}_2F_1$. In particular, assuming that the non-apparent singularities are chosen as in (2.15), one has

$$\begin{aligned} \chi^{(1)}_- &= \frac{z^{\frac{1}{2}-p_1}\,(1-z)^{\frac{1}{2}-p_2}}{\sqrt{2p_1}\,P_{2L}(0)\prod_{a=1}^L(1-\frac{z}{x_a})} \\ &\times\ P_{2L}\big(z\,\tfrac{\mathrm{d}}{\mathrm{d}z}\big)\,{}_2F_1\big(\tfrac{1}{2}-L-p_1-p_2-p_3,\tfrac{1}{2}-L-p_1-p_2+p_3,1-2p_1;z\big) \\ \chi^{(1)}_+ &= \frac{z^{\frac{1}{2}-p_1}\,(1-z)^{\frac{1}{2}-p_2}}{\sqrt{2p_1}\,P_{2L}(2p_1)\prod_{a=1}^L(1-\frac{z}{x_a})} \\ &\times\ P_{2L}\big(z\,\tfrac{\mathrm{d}}{\mathrm{d}z}\big)\,z^{2p_1}\,{}_2F_1\big(\tfrac{1}{2}-L+p_1-p_2-p_3,\tfrac{1}{2}-L+p_1-p_2+p_3,1+2p_1;z\big)\ . \end{aligned} \tag{2.27}$$

Here $P_{2L}(D)$ is a certain polynomial of order $2L$ in the variable D such that $P_{2L}(D) = \prod_{b>a}(x_b - x_a)\ D^{2L} + \ldots$. It is given by the determinant of an $L \times L$ matrix

$$P_{2L}(D) = \det\big(x_a^{b-1}\, U_a(D+b)\big) \qquad (a, b = 1, \ldots, L)\ , \tag{2.28}$$

where $U_a(D)$ entering into the above formula are rather cumbersome functions:

$$\begin{aligned} U_a(D) &= (D-1)^2 - \bigg(\frac{x_a}{x_a-1}\ (2p_2+1) + 2p_1 + 2 - C_a x_a + \sum_{b\neq a} \frac{4x_a}{x_a-x_b}\bigg)\ (D-1) \\ &+ \frac{x_a^2}{2}\ C_a^2 + \Big((p_1+p_2)(p_1+p_2+1) - p_3^2 + \tfrac{1}{4} + 2L + \big(p_1 + \tfrac{3}{2} - (p_1+p_2+3)\, x_a\big)\, C_a\Big)\ \frac{x_a}{x_a-1} \\ &+ 2p_1 + 1 + \frac{x_a^2}{(x_a-1)^2}\ (2p_2+1) + \bigg(\sum_{b\neq a} \frac{2x_a}{x_a-x_b}\bigg)^2 \\ &+ \sum_{b\neq a}^{L} \bigg[\frac{x_a x_b (x_b-1)\, C_b}{(x_a-x_b)(x_a-1)} + \bigg(\frac{x_a(2p_2+1)}{x_a-1} + 2p_1 + 1 - C_a x_a\bigg)\ \frac{2x_a}{x_a-x_b}\bigg]\ . \end{aligned} \tag{2.29}$$

With this result at hand, it is straightforward to show that G_L (2.23) is given by

$$\begin{aligned} &G_L(\boldsymbol{x}\,|\,p_1, p_2, p_3\,\|\,a_1, a_2) = \\ &\frac{\prod_{b>a}(x_b-x_a)}{\det\big(x_a^{b-1}\, U_a(b)\big)} \prod_{b=1}^{L} \frac{x_b}{x_b-1}\big(p_1+p_2+p_3+b-\tfrac{1}{2}\big)\big(p_1+p_2-p_3+b-\tfrac{1}{2}\big)\bigg|_{C_a = \frac{2-a_1}{x_a} + \frac{2-a_2}{x_a-1}}\ . \end{aligned} \tag{2.30}$$

Thus formulae (2.22)-(2.26), (2.30), (2.29) provide a full description of the spectrum of the reflection operators $\mathbb{R}^{(k)}_{\sigma'\sigma}$. Recall that $\boldsymbol{x} = (x_1, \ldots, x_L)$ solves the algebraic system of equations (2.5), (2.16), (2.18). Needless to say that for the case $L = 1$ the general formula for the spectrum of the reflection operator turns out to be equivalent to the result obtained in ref. [22].

3 Quantum AKNS integrable structure

3.1 Spectrum of the reflection operators

As it was pointed out in ref. [17], the quantum AKNS integrable structure [15,16] possesses two reflection operators $\check{\mathbb{R}}$ and $\check{\mathbb{D}}$ that commute with the set of local IM $\big\{\mathbb{I}_m^{(\mathrm{AKNS})}\big\}_{m=1}^{\infty}$. In that work, the eigenvalues of the reflection operators were expressed in terms of the connection coefficients of the ODE, that can be obtained from (2.1) with $t(z)$ given by eqs. (2.2),(2.16),(2.5) through a certain limiting procedure. The latter is similar to that which brings the Gauss hypergeometric equation to the confluent one. For this reason we will refer to the limit as the confluent limit.

Consider the ODE (2.1) with $t(z)$ from (2.2),(2.16),(2.5) such that

$$z = \varepsilon\, w\ , \quad x_a = \varepsilon\, w_a\ , \quad C_a = -\varepsilon^{-1}\ \frac{n_a}{w_a}\ , \quad p_1 = p, \quad p_2 = \tfrac{1}{2}\,\mathrm{i}s + \varepsilon^{-1}\ , \quad p_3 = \tfrac{1}{2}\,\mathrm{i}s - \varepsilon^{-1} \tag{3.1}$$

and $\varepsilon \to 0$. A straightforward calculation leads to the equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}w^2} + \frac{p^2 - \frac{1}{4}}{w^2} + \frac{2\mathrm{i}s}{w} + 1 + \sum_{a=1}^{L} \left(\frac{2}{(w - w_a)^2} + \frac{n_a}{w(w - w_a)} \right) \right] \Psi = 0 \ . \tag{3.2}$$

Together with a regular singular point at $w = 0$, this ODE possesses an irregular singular point at $w = \infty$, as well as L additional apparent singularities characterized by the $2L$ complex parameters (w_a, n_a). The latter satisfy the following system of algebraic constraints

$$n_a \left(\tfrac{1}{4} n_a^2 - w_a^2 t_0^{(a)} \right) + w_a^3 t_1^{(a)} = 0 \qquad\qquad (a = 1, 2, \ldots, L) \, , \tag{3.3}$$

where

$$\begin{aligned} t_0^{(a)} &= \frac{p^2 - \frac{1}{4}}{w_a^2} + \frac{2\mathrm{i}s}{w_a} + 1 - \frac{n_a}{w_a^2} + \sum_{b \neq a} \left(\frac{2}{(w_a - w_b)^2} + \frac{n_b}{w_a(w_a - w_b)} \right) \\ t_1^{(a)} &= -2\, \frac{p^2 - \frac{1}{4}}{w_a^3} - \frac{2\mathrm{i}s}{w_a^2} + \frac{n_a}{w_a^3} - \sum_{b \neq a} \left(\frac{4}{(w_a - w_b)^3} + \frac{n_b\, (2w_a - w_b)}{w_a^2\, (w_a - w_b)^2} \right) \, , \end{aligned} \tag{3.4}$$

which is just the limiting form of eq. (2.5). This way we arrive at the ODE appearing in ref. [17]. Introduce two solutions of (3.2)-(3.4) by means of the asymptotic condition

$$\Psi_{\pm p} \to w^{\frac{1}{2} \pm p} \qquad \text{with} \qquad w \to 0 \ , \tag{3.5}$$

and define the connection coefficients $C_p^{(\pm,L)}$ as in eq.(38) from [17], i.e.,

$$\Psi_p(w) \to \begin{cases} C_p^{(+,L)}\ (+w)^{+\mathrm{i}s}\ \mathrm{e}^{+w} & \text{as} \quad \Re e(w) \to +\infty \\ C_p^{(-,L)}\ (-w)^{-\mathrm{i}s}\ \mathrm{e}^{-w} & \text{as} \quad \Re e(w) \to -\infty \end{cases} \ . \tag{3.6}$$

Taking the confluent limit of the solutions (2.27) one can show that the connection coefficients $C_p^{(\pm,L)}$ are given by the following formulae

$$C_p^{(\pm,L)} = C_p^{(\pm,0)}\ \ \frac{(\mp 1)^L\ \det\left(w_a^{b-1}\, \tilde{U}_a^{(\pm)}(b) \right)}{\prod_{a=1}^{L} w_a\ \prod_{b>a}(w_b - w_a)\ \prod_{a=1}^{L} \left(2p + 2a - 1 \pm 2\mathrm{i}s \right)} \ , \tag{3.7}$$

where

$$\begin{aligned} \tilde{U}_a^{(\pm)}(D) &= (D-1)^2 - \left(2p + 2 + n_a \mp 2w_a + \sum_{b \neq a}^{L} \frac{4w_a}{w_a - w_b} \right) (D - 1) \\ &+ \tfrac{1}{2} n_a^2 + \left(p + \tfrac{3}{2} \right) n_a \mp \left(n_a + 1 + 2p + 2\mathrm{i}s \right) w_a + 2p + 1 \\ &+ \left(\sum_{b \neq a}^{L} \frac{2w_a}{w_a - w_b} \right)^2 + \sum_{b \neq a}^{L} \left(2\, (2p + 1 + n_a \mp 2w_a) - n_b \right) \frac{w_a}{w_a - w_b} \end{aligned} \tag{3.8}$$

and

$$C_p^{(\pm,0)} = 2^{\pm\mathrm{i}s - p - \frac{1}{2}}\ \frac{\Gamma(1 + 2p)}{\Gamma(\frac{1}{2} + p \pm \mathrm{i}s)} \ . \tag{3.9}$$

The reflection operators $\check{\mathbb{R}}^{(\mathrm{AKNS})}$ and $\check{\mathbb{D}}^{(\mathrm{AKNS})}$ from the quantum AKNS integrable structure were normalized in ref. [17] in such a way that their vacuum (level zero) eigenvalues are equal to 1.[2] Then the eigenvalues at level L are expressed through the connection coefficients $C_p^{(\pm,L)}$ as follows

$$\begin{aligned} \check{R}^{(\mathrm{AKNS})} &= \frac{C_p^{(+,L)}}{C_p^{(+,0)}}\ \frac{C_p^{(-,L)}}{C_p^{(-,0)}}\bigg|_{n_a=n} \\ \check{D}^{(\mathrm{AKNS})} &= \frac{C_p^{(+,L)}}{C_p^{(+,0)}}\ \frac{C_p^{(-,0)}}{C_p^{(-,L)}}\bigg|_{n_a=n}\ . \end{aligned} \tag{3.10}$$

Combining these with (3.7), (3.8) and (3.3), (3.4) one obtains

$$\begin{aligned} \check{R}^{(\mathrm{AKNS})}(\boldsymbol{w}) &= \frac{(-1)^L}{\prod_{a=1}^L w_a^2}\ \frac{\det\big(w_a^{b-1}\,V_a^{(+)}(b)\big)\ \det\big(w_a^{b-1}\,V_a^{(-)}(b)\big)}{\prod_{b>a}(w_b-w_a)^2\prod_{a=1}^L(2p+2a-1+2\mathrm{i}s)\big(2p+2a-1-2\mathrm{i}s\big)} \\ \check{D}^{(\mathrm{AKNS})}(\boldsymbol{w}) &= (-1)^L\ \prod_{a=1}^L\frac{p+a-\frac{1}{2}-\mathrm{i}s}{p+a-\frac{1}{2}+\mathrm{i}s}\ \ \frac{\det\big(w_a^{b-1}\,V_a^{(+)}(b)\big)}{\det\big(w_a^{b-1}\,V_a^{(-)}(b)\big)}\ . \end{aligned} \tag{3.11}$$

Here

$$\begin{aligned} V_a^{(\pm)}(D) &= (D-1)^2-\bigg(2p+2+n\mp 2w_a+\sum_{b\neq a}^L\frac{4w_a}{w_a-w_b}\bigg)\ (D-1) \\ &+\ \tfrac{1}{2}\,n^2+\big(p+\tfrac{3}{2}\big)\ n\mp(n+1+2p\pm 2\mathrm{i}s)\ w_a+2p+1 \\ &+\ \bigg(\sum_{b\neq a}^L\frac{2w_a}{w_a-w_b}\bigg)^2+\ \big(\,4p+2\mp 4w_a+n\,\big)\ \sum_{b\neq a}^L\frac{w_a}{w_a-w_b} \end{aligned} \tag{3.12}$$

and the set $\boldsymbol{w}=\{w_a\}_{a=1}^L$ obeys the system of algebraic equations

$$\begin{aligned} &4n\,w_a^2\ +\ 8\mathrm{i}s\,(n+1)\,w_a-(n+2)\ \big((n+1)^2-4p^2\big) \qquad\qquad (3.13)\\ &\quad +\ 4\ \sum_{b\neq a}^L\frac{w_a\,(\,(n+2)^2\,w_a^2-n(2n+5)\,w_aw_b+n(n+1)\,w_b^2\,)}{(w_a-w_b)^3}=0 \qquad (a=1,\ldots,L)\ . \end{aligned}$$

Note that for $L=1$ eqs. (3.11), (3.12) are equivalent to the formulae (48) for $\check{R}^{(1)}(\boldsymbol{w})$ and $\check{D}^{(1)}(\boldsymbol{w})$ quoted in ref. [17].

To the best of our knowledge, the reflection operators for the quantum AKNS integrable structure have not been discussed in sufficient detail in the literature. For this reason, we'll elaborate some important points concerning them in the rest of this section.

[2]In what follows we will always use the "check" notation for the reflection operators associated with the different integrable structures in order to emphasize that they are normalized such that their vacuum eigenvalues are equal to one.

3.2 W_∞-algebra and the reflection S-matrix for the cigar CFT

In the case of the quantum AKNS integrable structure the rôle of the extended conformal symmetry is played by the W_∞-algebra from ref. [13]. The latter involves an infinite set of currents $W_j(u)$ with Lorentz spin $j = 2, 3, \ldots$ satisfying the infinite system of Operator Product Expansions (OPE) of the form

$$\begin{aligned}
W_2(u)\, W_2(0) &= \frac{c}{2u^4} + \frac{2}{u^2}\, W_2(0) + \frac{1}{u}\, \partial W_2(0) + O(1) \\
W_2(u)\, W_3(0) &= \frac{3}{u^2}\, W_3(0) + \frac{1}{u}\, \partial W_3(0) + O(1) \\
W_2(u)\, W_4(0) &= \frac{(c+10)(17c+2)}{15(c-2)\, u^4}\, W_2(0) + \frac{4}{u^2}\, W_4(0) + \frac{1}{u}\, \partial W_4(0) + O(1) \\
W_3(u)\, W_3(0) &= \frac{c(c+7)(2c-1)}{9(c-2)u^6} + \frac{(c+7)(2c-1)}{3(c-2)u^4}\, \big(W_2(u) + W_2(0)\big) + \frac{1}{u^2}\, \Big(W_4(u) + W_4(0) \\
&+ W_2^2(u) + W_2^2(0) - \frac{2c^2+22c-25}{30(c-2)}\, \big(\partial^2 W_2(u) + \partial^2 W_2(0)\big)\Big) + O(1) \\
&\ldots\ .
\end{aligned} \tag{3.14}$$

Note that the local field W_2^2, appearing in the last line of eq. (3.14) is a composite field built from the currents W_2 and is defined to be the first regular term in the OPE $W_2(u)W_2(0)$.

When the central charge $c > 2$, $W_j = W_j(x_2 + \mathrm{i}x_1)$ are holomorphic currents in the cigar non-linear sigma model [27, 28] defined on the space-time cylinder with $x_1 \sim x_1 + 2\pi$ and $-\infty < x_2 < +\infty$. This CFT admits a dual description based on the Euclidean action [29]

$$\tilde{\mathcal{A}}_{\rm cig} = \int_{-\infty}^{\infty} \mathrm{d}x_2 \int_0^{2\pi} \mathrm{d}x_1\, \Big(\frac{1}{4\pi}\, \big[\, (\partial_a\varphi)^2 + (\partial_a\vartheta)^2\, \big] + 2\mu\, \mathrm{e}^{-\sqrt{k}\varphi}\, \cos\big(\sqrt{k+2}\,\vartheta\big)\Big) \tag{3.15}$$

with the parameter k related to the central charge as

$$c = 2 + \frac{6}{k}\ . \tag{3.16}$$

Similar to the Liouville CFT, the configuration space of the model (3.15) contains an asymptotic domain in which the interaction term becomes negligible and φ, ϑ approach free massless fields. In particular

$$\begin{aligned}
\frac{1}{2}\Big(\frac{\partial}{\partial x_2} - \mathrm{i}\, \frac{\partial}{\partial x_1}\Big)\varphi \quad &\to \quad \partial\varphi_+ = -\mathrm{i} \sum_{m=-\infty}^{\infty} a_m\, \mathrm{e}^{-mu} \\
\frac{1}{2}\Big(\frac{\partial}{\partial x_2} - \mathrm{i}\, \frac{\partial}{\partial x_1}\Big)\vartheta \quad &\to \quad \partial\vartheta_+ = -\mathrm{i} \sum_{m=-\infty}^{\infty} b_m\, \mathrm{e}^{-mu}
\end{aligned} \tag{3.17}$$

and the $\{a_m\}_{m=-\infty}^{\infty}$ are a set of creation-annihilation operators satisfying the Heisenberg algebra commutation relations (1.7) and similarly for b_m. In terms of the asymptotic fields $\partial\varphi_+$ and

$\partial\vartheta_+$, the first two W-currents are given by the following expressions

$$W_2 = -(\partial\varphi_+)^2 - (\partial\vartheta_+)^2 - \frac{1}{\sqrt{k}}\,\partial^2\varphi_+ \tag{3.18}$$

$$W_3 = -\mathrm{i}\left[\frac{6k+4}{3k}\,(\partial\vartheta_+)^3 + 2\,(\partial\varphi_+)^2\partial\vartheta_+ - \sqrt{k}\,\partial^2\varphi_+\partial\vartheta_+ + \frac{k+2}{\sqrt{k}}\,\partial\varphi_+\partial^2\vartheta_+ + \frac{k+2}{6k}\,\partial^3\vartheta_+\right].$$

Let P_1 and P_2 be the eigenvalues of the zero-modes a_0 and b_0 respectively in the Fock representation of two copies of the Heisenberg algebra, $\mathcal{F}_{P_1,P_2} \equiv \mathcal{F}^{(a)}_{P_1} \otimes \mathcal{F}^{(b)}_{P_2}$. As in the Liouville CFT the Fock spaces $\mathcal{F}_{P_1,P_2}$ and $\mathcal{F}_{-P_1,P_2}$ can be interpreted as the space of "in" and "out" (chiral) asymptotic states. Note that, since the potential term in (3.15) becomes negligible as $\varphi \to +\infty$, the "in" asymptotic space corresponds to $P_1 < 0$. This is opposed to the convention used in the Introduction for the Liouville CFT, as in that case it was assumed that the exponential interaction term vanishes as the Liouville field turns to $-\infty$. In full analogy with the Liouville theory, one can introduce the reflection S-matrix for the cigar that intertwines the spaces of in and out asymptotic states. This operator admits the factorized structure

$$\hat{S}_{\rm cig} = S^{(0)}_{\rm cig}\ \ \hat{\bar{s}}_{\rm cig} \otimes \hat{s}_{\rm cig}\ , \tag{3.19}$$

where $S^{(0)}_{\rm cig}$ is a certain phase factor while $\hat{\bar{s}}_{\rm cig}$ and $\hat{s}_{\rm cig}$ are properly normalized operators acting in the chiral Fock spaces. In particular,

$$\hat{s}_{\rm cig}(P_1)\ :\quad \mathcal{F}_{P_1,P_2} \mapsto \mathcal{F}_{-P_1,P_2}\ ,\quad \hat{s}_{\rm cig}(P_1)\,|P_1,P_2\rangle = |-P_1,P_2\rangle \tag{3.20}$$

with $|P_1,P_2\rangle \equiv |P_1\rangle \otimes |P_2\rangle$ standing for the Fock vacuum. In the above formula we explicitly indicate the dependence of $\hat{s}_{\rm cig}$ on P_1, though it also depends on P_2 and the parameter k. The action of the operator $\hat{s}_{\rm cig}$ on the excited states is fully determined by the W_∞-symmetry and below we'll describe its construction, which is similar in spirit to that of the Liouville reflection S-matrix (1.12) discussed in ref. [10].

First of all we note that the higher spin W_j currents are generated through the OPE involving the W_j currents of lower spin, similar to how the current W_4 appears in the singular part of $W_3(u)W_3(0)$ in eq. (3.14). In fact, starting from W_2 and W_3, it is possible to generate all the W_j by recursively computing OPEs. Since the currents W_2 and W_3 can be represented via the Heisenberg generators using eq. (3.18), the Fock space $\mathcal{F}_{P_1,P_2}$ possesses the structure of the Verma module for the W_∞-algebra. The latter is defined by means of the Fourier coefficients of $W_j(u)$:

$$W_j(u) = \sum_{n=-\infty}^{\infty} \widetilde{W}_j(m)\ \mathrm{e}^{-mu}\ . \tag{3.21}$$

Namely starting from the highest weight vector $|\,\boldsymbol{\varpi}\rangle$ satisfying the conditions

$$\widetilde{W}_j(m)\,|\,\boldsymbol{\varpi}\rangle = 0\ ,\qquad \widetilde{W}_j(0)\,|\,\boldsymbol{\varpi}\rangle = \varpi_j\,|\,\boldsymbol{\varpi}\rangle \qquad (j=2,3;\ m=1,2,\ldots) \tag{3.22}$$

the Verma module $\mathcal{V}_{\boldsymbol{\varpi}}$ is constructed by taking all linear combinations of the basis vectors of the form

$$\boldsymbol{v}_I = \widetilde{W}_2(-i_1)\ldots\widetilde{W}_2(-i_m)\,\widetilde{W}_3(-i'_1)\ldots\widetilde{W}_3(-i'_{m'})|\,\boldsymbol{\varpi}\rangle\ , \tag{3.23}$$

where I stands for the multi-index $I = (i_1, \ldots, i_m, i'_1, \ldots, i'_{m'})$ such that $1 \leq i_1 \leq i_2 \leq \ldots \leq i_m$ and $1 \leq i'_1 \leq i'_2 \leq \ldots \leq i'_{m'}$.

In view of eq. (3.18), it is easy to see that $\mathcal{F}_{\pm P_1, P_2}$ is isomorphic to the Verma module of the W_∞-algebra $\mathcal{V}_{\boldsymbol{\varpi}}$ with the highest weights $\boldsymbol{\varpi} = (\varpi_2, \varpi_3)$ related to the zero mode momenta (P_1, P_2) as

$$\begin{aligned} \varpi_2 &= P_1^2 + P_2^2 - \frac{1}{12} \\ \varpi_3 &= 2P_2 \left(P_1^2 + \frac{3k+2}{3k}\, P_2^2 - \frac{2k+1}{12k} \right) . \end{aligned} \tag{3.24}$$

Hence the vectors $\boldsymbol{v}_I$ defined in eq. (3.23) form a basis in $\mathcal{F}_{\pm P_1, P_2}$. On the other hand, the Fock space contains a natural basis that is obtained by acting with the Heisenberg creation operators on the vacuum

$$\boldsymbol{e}_I(P_1) = a_{-i_1} \ldots a_{-i_m}\, b_{-i'_1} \ldots b_{-i'_{m'}} |P_1, P_2\rangle \,, \tag{3.25}$$

where again $I = (i_1, \ldots, i_m, i'_1, \ldots, i'_{m'})$ with $1 \leq i_1 \leq i_2 \leq \ldots \leq i_m$ and $1 \leq i'_1 \leq i'_2 \leq \ldots \leq i'_{m'}$. The two bases (3.23) and (3.25) are, of course, linearly related:

$$\boldsymbol{v}_J = \boldsymbol{e}_I(P_1)\, \Theta^I{}_J(P_1) \ . \tag{3.26}$$

The matrix elements of the chiral part of the cigar reflection S-matrix are expressed in terms of $\Theta^I{}_J(P_1)$ as

$$\big[\hat{s}_{\rm cig}\big]^J{}_I(P_1, P_2|\, Q_{\rm cig}) = \Theta^J{}_A(-P_1)\ \big[\Theta^{-1}\big]^A{}_I(P_1) \ . \tag{3.27}$$

In the l.h.s. of this equation we have indicated the dependence of the matrix elements of $\hat{s}_{\rm cig}$ on the zero-mode momenta (P_1, P_2) and the parameter

$$Q_{\rm cig} \equiv -\frac{1}{\sqrt{k}} \ . \tag{3.28}$$

Explicit formulae for the matrix $\Theta^J{}_A$ at the levels $L = 1, 2$ are presented in the appendix.

3.3 Local IM and reflection operators $\check{\mathbb{R}}$ in the quantum AKNS and Getmanov integrable structures

The CFT (3.15) admits an integrable deformation which is described by the action [14]

$$\tilde{\mathcal{A}}_{\rm LR} = \int_{-\infty}^{\infty} {\rm d}x_2 \int_0^R {\rm d}x_1 \, \left(\frac{1}{4\pi}\ \big[\, (\partial_a \varphi)^2 + (\partial_a \vartheta)^2 \,\big] + 2\mu\ {\rm e}^{-\sqrt{k}\varphi}\ \cos\big(\sqrt{k+2}\vartheta\big) + \mu'\, {\rm e}^{\frac{2\varphi}{\sqrt{k}}} \right). \tag{3.29}$$

With the additional term the theory is not scale invariant as the parameter $\big(\mu^{\frac{2}{k}} \mu'\big)^{\frac{k}{2(k+2)}}$ has dimensions of $[mass]$. Thus the compactification length for the space coordinate $x_1 \sim x_1 + R$ can no longer be painlessly rescaled to 2π as it was done in (3.15). It is believed [14] that (3.29) provides a dual description for the model, whose classical limit is governed by the (Euclidean) action

$$\mathcal{A}_{\rm LR} = \frac{k}{4\pi}\ \int_{-\infty}^{\infty} {\rm d}x_2 \int_0^R {\rm d}x_1 \, \left(\frac{|\partial_a Z|^2}{1 + |Z|^2} + \mathfrak{m}^2\, |Z|^2 \right) \qquad (k \to +\infty) \ . \tag{3.30}$$

Here $Z = X + \mathrm{i}Y$ is a complex field and the mass parameter $\mathfrak{m} \propto \left(\mu^{\frac{2}{k}}\mu'\right)^{\frac{k}{2(k+2)}}$. The model (3.30) is well known in the theory of classically integrable systems, where it goes under the name of the Lund-Regge (complex sinh-Gordon I) model [30–32]. Furthermore, there are strong arguments in support of the integrability of the quantum theory (3.29) as well. Among other things that this implies, it is expected that the quantum Lund-Regge model admits two sets of local IM $\{\mathbb{I}_m^{(\mathrm{LR})}\}_{m=1}^{+\infty}$, $\{\mathbb{I}_m^{(\mathrm{LR})}\}_{m=-\infty}^{-1}$. In the short distance limit each of these sets becomes the set of local IM from the AKNS integrable structure associated with the W_∞-algebra. In particular,

$$\lim_{R\to 0}\left(\frac{R}{2\pi}\right)^m \mathbb{I}_m^{(\mathrm{LR})} = \mathbb{I}_m^{(\mathrm{AKNS})} \qquad (m = 1, 2, \ldots) \tag{3.31}$$

and the first few representatives are expressed in terms of the W-currents as follows [14, 15]

$$\begin{aligned}
\mathbb{I}_1^{(\mathrm{AKNS})} &= \int_0^{2\pi}\frac{\mathrm{d}x_1}{2\pi}\, W_2 \\
\mathbb{I}_2^{(\mathrm{AKNS})} &= \frac{3k}{2(3k+2)}\int_0^{2\pi}\frac{\mathrm{d}x_1}{2\pi}\, W_3 \\
\mathbb{I}_3^{(\mathrm{AKNS})} &= \frac{k}{(2k+1)(5k+4)}\int_0^{2\pi}\frac{\mathrm{d}x_1}{2\pi}\,\Big(k\, W_4 + (2k+1)\, W_2^2\Big)\ .
\end{aligned} \tag{3.32}$$

The construction of the reflection operator for the AKNS integrable structure is based on the observation that the theory (3.29) can be considered as a deformation of the Liouville plus free massless CFT by the term $\propto \mu$. This implies that the AKNS local IM (3.32) should admit an alternative description in terms of the two fields

$$\begin{aligned}
T(u) &= -(\partial\varphi_+)^2 + Q_{\mathrm{L}}\,\partial^2\varphi_+\ , \qquad \text{where}\ \ Q_{\mathrm{L}} = \frac{1+k}{\sqrt{k}} \\
J(u) &= \mathrm{i}\,\partial\vartheta_+\ .
\end{aligned} \tag{3.33}$$

Indeed, it is not difficult to check that

$$\begin{aligned}
\mathbb{I}_1^{(\mathrm{AKNS})} &= \int_0^{2\pi}\frac{\mathrm{d}x_1}{2\pi}\,\big(J^2 + T\big) \\
\mathbb{I}_2^{(\mathrm{AKNS})} &= \int_0^{2\pi}\frac{\mathrm{d}x_1}{2\pi}\,\Big(J^3 + \frac{3k}{3k+2}\,J\,T\Big) \\
\mathbb{I}_3^{(\mathrm{AKNS})} &= \int_0^{2\pi}\frac{\mathrm{d}x_1}{2\pi}\,\Big(J^4 - \frac{k^2+4k+2}{5k+4}\,(\partial J)^2 + \frac{6k}{5k+4}\,J^2\,T + \frac{k}{5k+4}\,T^2\Big)\ .
\end{aligned} \tag{3.34}$$

Notice that in the above formulae the overall multiplicative normalization of the local IM is fixed such that

$$\mathbb{I}_m^{(\mathrm{AKNS})} = \int_0^{2\pi}\frac{\mathrm{d}x_1}{2\pi}\,\big(J^{m+1} + \ldots\big)\ , \tag{3.35}$$

where the dots stand for terms containing lower powers of the current J and its derivatives.

Consider the first line in eq. (3.33). The r.h.s. is identical to the expression for the stress energy momentum tensor in the Liouville CFT with central charge $c_{\mathrm{L}} = 1 + 6\,Q_{\mathrm{L}}^2$, written in terms of the free Liouville asymptotic field. This suggests that the reflection operator commuting with the local IM (3.34) is built form, together with $\hat{s}_{\rm cig}(P_1,P_2|\,Q_{\rm cig})$ given by eq. (3.27), the operator $\big(\hat{s}_{\rm L}^{(a)}\otimes \mathbf{1}_b\big)$. The latter is to be understood as the operator which commutes with all the modes $\{b_m\}_{m=-\infty}^{\infty}$, and acts as the Liouville reflection S-matrix $\hat{s}_{\rm L}$ (1.12) on vectors of the form $a_{-i_1}\ldots a_{-i_m}|P_1,P_2\rangle$ $(1\le i_1\le i_2\le\ldots\le i_m)$. It turns out that

$$\check{\mathbb{R}}^{(\rm AKNS)}=\hat{s}_{\rm cig}(-P_1,P_2|\,Q_{\rm cig})\,\big(\hat{s}_{\rm L}^{(a)}(P_1|\,Q_{\rm L})\otimes\mathbf{1}_b\big)\quad\text{with}\quad Q_{\rm L}=\frac{1+k}{\sqrt{k}}\,,\qquad Q_{\rm cig}=-\frac{1}{\sqrt{k}}\,,\tag{3.36}$$

acts invariantly in the Fock space $\mathcal{F}_{P_1,P_2}\cong\mathcal{V}_{\boldsymbol{\varpi}}$ and commutes with the local IM (3.34):

$$\big[\,\check{\mathbb{R}}^{(\rm AKNS)},\mathbb{I}_m^{(\rm AKNS)}\,\big]=0\ .\tag{3.37}$$

For completeness let us recall the construction of the Liouville reflection S-matrix $\hat{s}_{\rm L}(P|Q)$ [10]. As was mentioned in the Introduction the Fock space $\mathcal{F}_P$, defined as the highest weight representation of the Heisenberg algebra (1.7) with highest weight vector $|P\rangle$ (1.10), is isomorphic to the Verma module $\mathcal{V}_\Delta$ for the Virasoro algebra with Δ as in eq. (1.11). Hence the two different bases

$$\boldsymbol{t}_I=L_{-i_1}\ldots L_{-i_m}|P\rangle\tag{3.38}$$

and

$$\boldsymbol{e}_I(P)=a_{-i_1}\ldots a_{-i_m}|P\rangle\,,\tag{3.39}$$

one associated with the Virasoro algebra and the other with the Heisenberg algebra, are linearly related. Using the explicit expressions for the Virasoro generators in terms of the Heisenberg ones (1.8), it is not difficult to read off the components of the matrix $\Omega^I{}_J(P)$ connecting the two bases,

$$\boldsymbol{t}_J=\boldsymbol{e}_I(P)\ \Omega^I{}_J(P)\ .\tag{3.40}$$

Then the chiral part of the Liouville reflection S-matrix is given by

$$\big[\hat{s}_{\rm L}\big]^J{}_I(P|\,Q)=\Omega^J{}_A(-P)\ \big[\Omega^{-1}\big]^A{}_I(P)\ .\tag{3.41}$$

Explicit formulae for $\Omega^J{}_A$ at levels $L=1,2,3$ can be found in the appendix.

The AKNS local IM (3.31) and the reflection operator (3.36) admit a heuristic interpretation based on the integrable QFT (3.29), where k is a positive coupling constant. At the same time it is not difficult to see that for a given level L the finite matrices of the local IM, $\mathbb{I}_m^{(\rm AKNS)}$, as well as the matrix (3.36) are rational functions of $Q_{\rm cig}=-1/\sqrt{k}$ and (P_1,P_2) (see, e.g., Appendix). Hence the commutativity conditions (3.37) and $\big[\mathbb{I}_m^{(\rm AKNS)},\mathbb{I}_{m'}^{(\rm AKNS)}\big]=0$ must remain unchanged for any complex values of these variables. The first line in eq. (3.11) gives the eigenvalues of the operator (3.36) provided that

$$P_1=\frac{p}{\sqrt{n+2}}\ ,\qquad P_2=\frac{s}{\sqrt{n}}\ ,\qquad \sqrt{k}=-{\rm i}\,\sqrt{n+2}\ .\tag{3.42}$$

Finishing this subsection let us note that the model (3.15) admits another integrable deformation of the form

$$\tilde{\mathcal{A}}_{\rm G}=\int_{-\infty}^{\infty}{\rm d}x_2\int_0^R{\rm d}x_1\,\Big(\frac{1}{4\pi}\,\big[\,(\partial_a\varphi)^2+(\partial_a\vartheta)^2\,\big]+2\mu\,{\rm e}^{-\sqrt{k}\varphi}\,\cos\big(\sqrt{k+2}\vartheta\big)+\mu'\,{\rm e}^{\frac{4\varphi}{\sqrt{k}}}\Big),\tag{3.43}$$

which gives the dual description for the Getmanov (complex sinh-Gordon II) model [32]

$$\mathcal{A}_{\rm G}=\frac{k}{4\pi}\int_{-\infty}^{\infty}{\rm d}x_2\int_0^R{\rm d}x_1\,\bigg(\frac{|\partial_a Z|^2}{1+|Z|^2}+\mathfrak{m}^2\,|Z|^2\,\big(1+|Z|^2\big)\bigg)\qquad\qquad (k\to+\infty)\ . \quad (3.44)$$

Thus, as was mentioned in the Introduction, there is another integrable structure associated with the W_∞-algebra. The Getmanov integrable structure contains the set of local IM $\{\mathbb{I}_{2m-1}^{(\rm G)}\}_{m=1}^{\infty}$ such that

$$\begin{aligned}\mathbb{I}_1^{(\rm G)}&=\int_0^{2\pi}\frac{{\rm d}x_1}{2\pi}\,W_2 &&(3.45)\\ \mathbb{I}_3^{(\rm G)}&=\frac{k^2}{4(k+2)(2k+1)(2k+3)}\int_0^{2\pi}\frac{{\rm d}x_1}{2\pi}\,\Big((k+6)\,W_4+4\,(2k+1)\,W_2^2\Big)\ .\end{aligned}$$

The latter are related to the local IM from the massive theory (3.44) similarly to the way the AKNS local IM appear in the context of the Lund-Regge (complex sinh-Gordon I) model (see eq. (3.31)). The reflection operator for the Getmanov integrable structure is given by the formula analogous to (3.36)

$$\check{\mathbb{R}}^{(\rm G)}=\hat{s}_{\rm cig}(-P_1,P_2|\,Q_{\rm cig})\,\big(\hat{s}_{\rm L}^{(a)}(P_1|\,Q'_{\rm L})\otimes\mathbf{1}_b\big)\quad\text{with}\ \ Q'_{\rm L}=\frac{k+4}{2\sqrt{k}}\,,\quad Q_{\rm cig}=-\frac{1}{\sqrt{k}}\ . \quad (3.46)$$

3.4 Reflection operators $\check{\mathbb{C}}^{(\pm)}$ and $\check{\mathbb{D}}^{(\rm AKNS)}$

Introduce a new set of holomorphic fields $(\partial\chi_+,\partial\eta_+)$ related to the basic fields $(\partial\varphi_+,\partial\vartheta_+)$ (3.17) through the (complex) orthogonal transformation

$$\partial\chi_+=\sqrt{\frac{k+2}{2}}\,\partial\varphi_++{\rm i}\sqrt{\frac{k}{2}}\,\partial\vartheta_+\ ,\qquad \partial\eta_+=-{\rm i}\,\sqrt{\frac{k}{2}}\,\partial\varphi_++\sqrt{\frac{k+2}{2}}\,\partial\vartheta_+ \qquad (3.47)$$

and define the currents

$$J_1={\rm i}\,\partial\chi_+\ ,\qquad J_2=-(\partial\eta_+)^2+\frac{{\rm i}}{\sqrt{2}}\,\partial^2\eta_+\ . \qquad (3.48)$$

Notice that the spin-2 field J_2 generates the Virasoro algebra with central charge $c=-2$. For this particular value the commuting system of local IM $\{\mathbb{I}_{2m-1}^{(c=-2)}\}_{m=1}^{\infty}$ given by (1.3) with T substituted by J_2 and c set to -2 can be extended to the system $\{\mathbb{I}_m^{(c=-2)}\}_{m=1}^{\infty}$ with $\mathbb{I}_{2m}^{(c=-2)}$ being the integrals over the local densities of odd Lorentz spin built out of the field $\partial\eta_+$. For example

$$\mathbb{I}_2^{(c=-2)}=-{\rm i}\int_0^{2\pi}\frac{{\rm d}x_1}{2\pi}\,(\partial\eta_+)^3\ . \qquad (3.49)$$

As was discussed in Appendix C in ref. [15], it is possible to rewrite the local IM from the quantum AKNS integrable structure (3.32) in the following form

$$\begin{aligned}
\mathbb{I}_1^{(\mathrm{AKNS})} &= \mathbb{I}_1^{(c=-2)}[\partial\eta_+] + \int_0^{2\pi} \frac{\mathrm{d}x_1}{2\pi}\, J_1^2 \\
\mathbb{I}_2^{(\mathrm{AKNS})} &= \frac{4k+2}{3k+2}\sqrt{\frac{k+2}{2}}\, \mathbb{I}_2^{(c=-2)}[\partial\eta_+] + \frac{\mathrm{i}\sqrt{2k}}{3k+2}\int_0^{2\pi} \frac{\mathrm{d}x_1}{2\pi}\, \big(k\, J_1^3 + 3\,(k+1)\, J_2 J_1\big) \\
&\ldots \\
\mathbb{I}_m^{(\mathrm{AKNS})} &= c_m\, \mathbb{I}_m^{(c=-2)}[\partial\eta_+] + \int_0^{2\pi} \frac{\mathrm{d}x}{2\pi}\, T_{m+1}\big(J_1, J_2\big)\ ,
\end{aligned} \tag{3.50}$$

where c_m are some k-dependent constants and T_{m+1} is a certain local differential polynomial built out of the currents J_1, J_2 (3.48).

Let $\mathcal{F}_P^{(\eta)}$ be the space of representation for the Heisenberg operators

$$\eta_m = -\mathrm{i}\sqrt{\frac{k}{2}}\, a_m + \sqrt{\frac{k+2}{2}}\, b_m \tag{3.51}$$

and $\hat{s}_{\mathrm{L}}^{(\eta)}(P|\frac{\mathrm{i}}{\sqrt{2}})$ be the Liouville reflection S-matrix intertwining $\mathcal{F}_P^{(\eta)}$ and $\mathcal{F}_{-P}^{(\eta)}$. Also, similar to (1.14), introduce the operator $\hat{C}^{(\chi)}$ of the C-conjugation for the Heisenberg generators

$$\chi_m = \sqrt{\frac{k+2}{2}}\, a_m + \mathrm{i}\sqrt{\frac{k}{2}}\, b_m\ . \tag{3.52}$$

Then we define

$$\check{\mathbb{C}} = \Big[\hat{C}^{(\chi)} \otimes \hat{s}_{\mathrm{L}}^{(\eta)}\big(-P_\eta\,\big|\,\tfrac{\mathrm{i}}{\sqrt{2}}\big)\Big]\, \Big[\hat{s}_{\mathrm{L}}^{(a)}(P_1\,|\,Q_{\mathrm{L}}) \otimes \hat{C}^{(b)}\Big]\ , \tag{3.53}$$

where

$$P_\eta = -\mathrm{i}\sqrt{\frac{k}{2}}\, P_1 + \sqrt{\frac{k+2}{2}}\, P_2\ , \qquad Q_{\mathrm{L}} = \frac{1+k}{\sqrt{k}}\ . \tag{3.54}$$

Notice that the second factor in the square brackets $[\ldots]$ in the r.h.s. of (3.53) acts from $\mathcal{F}_{P_1}^{(a)} \otimes \mathcal{F}_{P_2}^{(b)}$ to the space $\mathcal{F}_{-P_1}^{(a)} \otimes \mathcal{F}_{-P_2}^{(b)}$. The latter is equivalent to $\mathcal{F}_{-P_\chi}^{(\chi)} \otimes \mathcal{F}_{-P_\eta}^{(\eta)}$ with P_η given by (3.54) and $P_\chi = \sqrt{\frac{k+2}{2}}\, P_1 + \mathrm{i}\sqrt{\frac{k}{2}}\, P_2$. Since the first factor intertwines $\mathcal{F}_{-P_\chi}^{(\chi)} \otimes \mathcal{F}_{-P_\eta}^{(\eta)}$ back to $\mathcal{F}_{P_\chi}^{(\chi)} \otimes \mathcal{F}_{P_\eta}^{(\eta)} \equiv \mathcal{F}_{P_1}^{(a)} \otimes \mathcal{F}_{P_2}^{(b)}$, the operator (3.53) acts invariantly in the Fock space. Formula (3.50) suggests that $\check{\mathbb{C}}$ commutes with the local IM from the quantum AKNS integrable structure.

Up until now k was assumed to be a positive real number and $\sqrt{k}$, $\sqrt{k+2}$ were understood to be the arithmetic square roots. The analytic continuation of the operator (3.53) to the domain $k < -2$ using eq. (3.42) with $n > 0$ requires one to specify the branch of $\sqrt{k+2}$. Let us set

$$\sqrt{k+2} = -\mathrm{i}\sqrt{n} \qquad\qquad (\sqrt{n} > 0)\ . \tag{3.55}$$

This results in the operator

$$\check{\mathbb{C}}^{(+)} = \Big[\hat{C}^{(\chi)} \otimes \hat{s}_{\rm L}^{(\eta)}\big(\tfrac{p+{\rm i}\,s}{\sqrt{2}}\,\big|\,\tfrac{\rm i}{\sqrt{2}}\big)\Big]\ \Big[\hat{s}_{\rm L}^{(a)}\big(\tfrac{p}{\sqrt{n+2}}\,\big|\,\tfrac{n+1}{{\rm i}\sqrt{n+2}}\big)\otimes \hat{C}^{(b)}\Big]\ . \tag{3.56}$$

If one were to choose the other branch of the square root in eq. (3.55) one would obtain the operator $\check{\mathbb{C}}^{(-)}$. It is easy to see that

$$\check{\mathbb{C}}^{(-)} = \big(\mathbf{1}_a \otimes \hat{C}^{(b)}\big)\,\mathbb{C}^{(+)}\big(\mathbf{1}_a\otimes \hat{C}^{(b)}\big)\ . \tag{3.57}$$

The eigenvalues of $\check{\mathbb{C}}^{(\pm)}$ are given by $C_p^{(\pm,L)}/C_p^{(\pm,0)}\big|_{n_a=n}$ with $C_p^{(\pm,L)}$ defined by eqs. (3.7), (3.8). The operators $\check{\mathbb{R}}^{(\rm AKNS)}$ and $\check{\mathbb{D}}^{(\rm AKNS)}$ with eigenvalues (3.10), are simply expressed in terms of $\check{\mathbb{C}}^{(\pm)}$:

$$\check{\mathbb{R}}^{(\rm AKNS)} = \check{\mathbb{C}}^{(+)}\,\check{\mathbb{C}}^{(-)}\,,\qquad \check{\mathbb{D}}^{(\rm AKNS)} = \check{\mathbb{C}}^{(+)}\,\big[\check{\mathbb{C}}^{(-)}\big]^{-1}\ . \tag{3.58}$$

Combining the above with eqs. (3.56), (3.57) yields

$$\check{\mathbb{D}}^{(\rm AKNS)} = \Big[\hat{C}^{(\chi)}\otimes \hat{s}_{\rm L}^{(\eta)}\big(\tfrac{p+{\rm i}\,s}{\sqrt{2}}\,\big|\,\tfrac{\rm i}{\sqrt{2}}\big)\Big]\big(\mathbf{1}_a\otimes\hat{C}^{(b)}\big)\Big[\hat{C}^{(\chi)}\otimes \hat{s}_{\rm L}^{(\eta)}\big(-\tfrac{p-{\rm i}\,s}{\sqrt{2}}\,\big|\,\tfrac{\rm i}{\sqrt{2}}\big)\Big]\big(\mathbf{1}_a\otimes\hat{C}^{(b)}\big)\ . \tag{3.59}$$

At the same time, due to the relation (3.36), it is possible to use (3.56)-(3.58) to express the cigar reflection S-matrix in terms of the Liouville one. A straightforward calculation leads to the remarkable formula

$$\begin{aligned}\hat{s}_{\rm cig}\big(P_1,P_2\,|\,Q\big) &= \Big[\hat{C}^{(\chi)}\otimes\hat{s}_{\rm L}^{(\eta)}\Big(\tfrac{\rm i}{\sqrt{2}\,Q}\,P_1-\tfrac{\sqrt{1+2Q^2}}{\sqrt{2}\,Q}\,P_2\,\Big|\,\tfrac{\rm i}{\sqrt{2}}\Big)\Big]\ \Big[\hat{s}_{\rm L}^{(a)}\big(-P_1\,\big|\,-Q-Q^{-1}\big)\otimes\mathbf{1}_b\Big]\\ &\times\ \Big[\hat{C}^{(\chi)}\otimes\hat{s}_{\rm L}^{(\eta)}\Big(\tfrac{\rm i}{\sqrt{2}\,Q}\,P_1+\tfrac{\sqrt{1+2Q^2}}{\sqrt{2}\,Q}\,P_2\,\Big|\,\tfrac{\rm i}{\sqrt{2}}\Big)\Big]\ .\end{aligned} \tag{3.60}$$

Here P_1, P_2 and Q can be taken to be arbitrary complex numbers, while the set of Heisenberg generators $\{\eta_m,\chi_m\}_{m\neq0}$ are related to the set $\{a_m,b_m\}_{m\neq0}$ as

$$\begin{aligned}\eta_m &= \tfrac{\rm i}{\sqrt{2}\,Q}\,a_m+\tfrac{\sqrt{1+2Q^2}}{\sqrt{2}\,Q}\,b_m\\ \chi_m &= \tfrac{\sqrt{1+2Q^2}}{\sqrt{2}\,Q}\,a_m-\tfrac{\rm i}{\sqrt{2}\,Q}\,b_m\ .\end{aligned} \tag{3.61}$$

4 Spectrum of the reflection operator in the paperclip integrable structure

As was mentioned in the Introduction there are at least three different integrable structures associated with the W_∞-algebra. Two of them are the AKNS and Getmanov integrable structures that were discussed in the previous section. The third one is related to the so-called sausage model [33], whose classical action reads as follows

$$\mathcal{A}_{\rm saus} = \frac{k}{4\pi}\int_{-\infty}^{\infty}{\rm d}x_2\int_0^R{\rm d}x_1\ \frac{(1-\lambda)\ |\partial_a Z|^2}{(1+\lambda|Z|^2)(1+|Z|^2)}\qquad\qquad (k\to+\infty)\ , \tag{4.1}$$

where $0<\lambda<1$ is some constant.[3] The dual description of the sausage model was originally proposed by Al. Zamolodchikov [29] and can also be considered as a massive integrable

[3]The mass scale in the quantum theory (4.1) appears through the mechanism of dimensional transmutation of the bare coupling λ.

deformation of (3.15):

$$\tilde{\mathcal{A}}_{\rm saus} = \int_{-\infty}^{\infty} \mathrm{d}x_2 \int_0^R \mathrm{d}x_1 \, \Big(\frac{1}{4\pi} \left[(\partial_a\varphi)^2 + (\partial_a\theta)^2 \right] + 4\mu \, \cosh\big(\sqrt{k}\,\varphi\big) \cos\big(\sqrt{k+2}\,\vartheta\big) \Big) \,. \tag{4.2}$$

In the short distance limit the set of local IM for the massive theory become $\big\{\mathbb{I}^{(\rm pc)}_{2m-1}\big\}_{m=1}^{\infty}$, whose first few members are given in terms of the W_∞-currents as [14, 18]

$$\begin{aligned} \mathbb{I}_1^{(\rm pc)} &= \int_0^{2\pi} \frac{\mathrm{d}x_1}{2\pi} \, W_2 \\ \mathbb{I}_3^{(\rm pc)} &= \frac{k}{(k+2)(2k+1)(3k+2)} \int_0^{2\pi} \frac{\mathrm{d}x_1}{2\pi} \, \Big(k\, W_4 + (2k+1)(3k+4)\, W_2^2 \Big) \,. \end{aligned} \tag{4.3}$$

These local IM play an important rôle in the description of the so-called paperclip boundary state [18] and we use the superscript "(pc)" for their notation. The associated reflection operator, which commutes with the paperclip IM, can be expressed in the form

$$\big(\check{\mathbb{R}}^{(\rm pc)}\big)^2 = \big[\hat{s}_{\rm cig}(-P_1,P_2|-Q_{\rm cig})\, \hat{s}_{\rm cig}(P_1,P_2|\,Q_{\rm cig})\big]^{-1} \qquad \text{with} \qquad Q_{\rm cig} = -\frac{1}{\sqrt{k}} \,. \tag{4.4}$$

The reason why we define the r.h.s. to be the square of the reflection operator is the following. It is not difficult to check that the simultaneous change of sign of P and $Q_{\rm cig}$ corresponds to the C-conjugation of $\hat{s}_{\rm cig}$:

$$\hat{s}_{\rm cig}(-P_1,P_2|-Q_{\rm cig}) = (\hat{C}^{(a)} \otimes \mathbf{1}_b)\, \hat{s}_{\rm cig}(P_1,P_2\,|\,Q_{\rm cig})\, (\hat{C}^{(a)} \otimes \mathbf{1}_b) \tag{4.5}$$

(recall that $(\hat{C}^{(a)} \otimes \mathbf{1}_b)$ acts as in (1.14) on the modes a_m and as the identity operator on the b-modes). From this it immediately follows that $\check{\mathbb{R}}^{(\rm pc)}$ can be written in the form similar to (1.15)

$$\check{\mathbb{R}}^{(\rm pc)} = \big[(\hat{C}^{(a)} \otimes \mathbf{1}_b)\, \hat{s}_{\rm cig}(P_1,P_2|\,Q_{\rm cig})\big]^{-1} \,. \tag{4.6}$$

The spectrum of the reflection operator (4.6) is obtained from the general result quoted in sec. 2 by the following specialization of the parameters (for details, see ref. [19])

$$p_1 = -\mathrm{i}\sqrt{k}\,P_1, \qquad p_2 = \sqrt{k+2}\,P_2 \ , \quad p_3 = 0 \ , \qquad a_1 = -k \ , \qquad a_2 = k+2 \ . \tag{4.7}$$

Notice that in this case the system of algebraic equations (2.5), (2.16), (2.18) for $\boldsymbol{x} = (x_1,\ldots,x_L)$ is expected to possess $\mathtt{par}_2(L)$ different (up to permutations) solutions, where $\mathtt{par}_2(L)$ is the number of bipartitions of L. Each solution $\boldsymbol{x}$ corresponds to a certain eigenvector $|\boldsymbol{x}\rangle_{\boldsymbol{p}} \in \mathcal{F}_{P_1,P_2}$ of the reflection operator (4.6), and the corresponding eigenvalue $\check{R}^{(\rm pc)}(\boldsymbol{x})$ is such that

$$\begin{aligned} \big(\check{R}^{(\rm pc)}(\boldsymbol{x})\big)^2 &= G_L\big(\boldsymbol{x}\,|-\mathrm{i}\sqrt{k}\ P_1, +\sqrt{k+2}\,P_2, 0\,\|\,{-k}, k+2\big) \\ &\times\ G_L\big(\boldsymbol{x}\,|-\mathrm{i}\sqrt{k}\ P_1, -\sqrt{k+2}\,P_2, 0\,\|\,{-k}, k+2\big) \ . \end{aligned} \tag{4.8}$$

Here the function $G_L(\boldsymbol{x}\,|\,p_1,p_2,p_3\,\|\,a_1,a_2)$ is defined in eqs. (2.30), (2.29).

5 Spectrum of the reflection operator in the quantum KdV integrable structure

The local IM in the quantum KdV integrable structure (1.3) appear in the short distance limit of the quantum sinh-Gordon model

$$\mathcal{A}_{\rm shG} = \int_{-\infty}^{\infty} \mathrm{d}x_2 \int_0^R \mathrm{d}x_1 \, \Big(\frac{1}{4\pi} \, (\partial_a \varphi)^2 + 2\mu \, \cosh\big(2b\varphi\big) \Big) \ . \tag{5.1}$$

Similar to eq. (3.31) one has

$$\lim_{R\to 0} \left(\frac{R}{2\pi}\right)^{2m-1} \mathbb{I}^{(\rm shG)}_{2m-1} = \mathbb{I}^{(\rm KdV)}_{2m-1} \qquad\qquad (m=1,2,\ldots) \ . \tag{5.2}$$

Here the relation between the coupling constant b entering (5.1) and Q, which parameterizes c in eq. (1.3) as $c = 1 + 6\,Q^2$, is given by

$$Q = b^{-1} + b \ . \tag{5.3}$$

The local IM $\big\{\mathbb{I}^{(\rm KdV)}_{2m-1}\big\}_{m=1}^{\infty}$ act invariantly in the level subspace of the Fock space $\mathcal{F}_P$ and commute with the operator $\hat{s}_{\rm L}(-P|-Q)\,\hat{s}_{\rm L}(P|\,Q)$. Taking into account the relation

$$\hat{s}_{\rm L}(-P|-Q) = \hat{C}\,\hat{s}_{\rm L}(P|\,Q)\,\hat{C}\,, \tag{5.4}$$

where $\hat{C}$ denotes the C-conjugation (1.14), one concludes that the reflection operator in eq.(1.15) commutes with all members of the set $\big\{\mathbb{I}^{(\rm KdV)}_{2m-1}\big\}_{m=1}^{\infty}$. Its eigenvalues $R^{(\rm KdV)}(\boldsymbol{v})$ (1.16) follow from the results of sec. 3.1. They can be obtained through a certain reduction of eqs. (3.11)-(3.13) that will be described below.

Consider the algebraic system (3.13) in the case with even $L = 2N$ and $s = 0$. It admits solutions such that

$$w_{2N+1-a} = -w_a \qquad \text{with} \qquad a = 1, \ldots, N \ . \tag{5.5}$$

Using the set $\{v_a\}_{a=1}^N$ defined by the formula

$$w_a^2 = -\frac{(n+2)^2}{2n}\, v_a \qquad\qquad (a=1,\ldots,N)\ , \tag{5.6}$$

equations (3.13) can be rewritten in the form (1.4), provided that the parameters are identified as follows

$$\alpha = -\frac{n}{n+2}\ , \qquad \Delta = \frac{4\,p^2 - n^2}{8(n+2)}\ . \tag{5.7}$$

In connection with this, let us note that the above reduction brings the ODE

$$\Big[-\frac{\mathrm{d}^2}{\mathrm{d}w^2} + \frac{p^2-\frac{1}{4}}{w^2} + \frac{2\mathrm{i}s}{w} + 1 + \sum_{a=1}^{L} \Big(\frac{2}{(w-w_a)^2} + \frac{n}{w(w-w_a)} \Big) + \lambda^{-2-n}\, w^n \Big]\, \Psi = 0 \ , \tag{5.8}$$

which plays the central rôle in the quantum AKNS integrable structure (see [17] for details), to the form

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}w^2} + \frac{p^2 - \frac{1}{4}}{w^2} + 1 + \sum_{a=1}^{N} \left(\frac{4(w^2 + w_a^2)}{(w^2 - w_a^2)^2} + \frac{2n}{w^2 - w_a^2} \right) + \lambda^{-2-n}\, w^n \right] \Psi = 0\ . \tag{5.9}$$

The latter is equivalent to the Schrödinger equation with Monster potentials associated with the quantum KdV integrable structure [7]. Indeed, the change of variables

$$\Psi(w) = y^{\frac{\alpha}{2}}\ \tilde{\Psi}(y)\ , \qquad w = \frac{y^{\alpha+1}}{\alpha+1} \tag{5.10}$$

transforms the ODE (5.9) to

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V_{\mathrm{Monst}}(y) + E \right) \tilde{\Psi}(y) = 0\ , \tag{5.11}$$

where

$$V_{\mathrm{Monst}}(y) = \frac{\ell(\ell+1)}{y^2} + y^{2\alpha} - 2\, \frac{\mathrm{d}^2}{\mathrm{d}y^2} \sum_{a=1}^{N} \log\left(y^{2\alpha+2} - \tfrac{\alpha+1}{\alpha}\, v_a \right)\ . \tag{5.12}$$

Here the parameters of the Schrödinger operator are related to those of the original ODE (5.9) as

$$E = \frac{4}{(n+2)^2} \left(\frac{n+2}{2\lambda} \right)^{n+2}\ , \qquad \ell = \frac{2p}{n+2} - \frac{1}{2}\ , \qquad \alpha = -\frac{n}{n+2}\ , \tag{5.13}$$

while the set $\{v_a\}_{a=1}^{N}$ satisfies (1.4) with $\Delta = \frac{(2\ell+1)^2 - 4\alpha^2}{16(\alpha+1)}$.

The specialization of (3.12) to the case (5.5),(5.6) gives

$$\begin{aligned} V_a^{(\pm)}(D) \ &=\ (D-1)^2 - \left(2p + 2 + n \mp 2\mathrm{i}\,(n+2) \sqrt{\frac{v_a}{2n}} + 2 + \sum_{b\neq a}^{N} \frac{8v_a}{v_a - v_b} \right) (D-1) \\ &+\ \tfrac{1}{2}\, n^2 + \left(p + \tfrac{3}{2} \right) n \mp \mathrm{i}\,(n+1+2p)\,(n+2) \sqrt{\frac{v_a}{2n}} + 2p + 1 \\ &+\ \left(1 + \sum_{b\neq a}^{N} \frac{4v_a}{v_a - v_b} \right)^2 + \left(4p + 2 \mp 4\,\mathrm{i}\,(n+2) \sqrt{\frac{v_a}{2n}} + n \right) \left(\frac{1}{2} + \sum_{b\neq a}^{N} \frac{2v_a}{v_a - v_b} \right)\ . \end{aligned} \tag{5.14}$$

Now we define

$$V_a(D) = \begin{cases} V_a^{(+)}(D) & \text{for} \quad a = 1, \ldots, N \\ V_{2N+1-a}^{(-)}(D) & \text{for} \quad a = N+1, \ldots, 2N \end{cases} \tag{5.15}$$

and

$$w_a = \begin{cases} +\mathrm{i}\,(n+2)\ \sqrt{\frac{v_a}{2n}} & \text{for} \quad a = 1, \ldots, N \\ -\mathrm{i}\,(n+2)\ \sqrt{\frac{v_{2N+1-a}}{2n}} & \text{for} \quad a = N+1, \ldots, 2N \end{cases}\ . \tag{5.16}$$

Then the eigenvalues $\check{R}^{(\mathrm{KdV})}(\boldsymbol{v})$ of the normalized reflection operator in the quantum KdV integrable structure,

$$\check{\mathbb{R}}^{(\mathrm{KdV})} = \big[\hat{C}\hat{s}_{\mathrm{L}}(P|\,Q)\big]^{-1} \ , \tag{5.17}$$

in the N^{th}-level subspace $\mathcal{F}_P^{(N)}$ of the Fock space with

$$P = -\frac{p}{\sqrt{2(n+2)}} \ , \qquad Q = -\frac{\mathrm{i}\,n}{\sqrt{2(n+2)}} \tag{5.18}$$

are given by

$$\check{R}^{(\mathrm{KdV})}(\boldsymbol{v}) = \frac{\det\big(w_a^{b-1}\,V_a(b)\big)}{\prod_{a=1}^{2N} w_a\ \prod_{b>a}(w_b-w_a)\ \prod_{a=1}^{2N}\big(2p+2a-1\big)} \ . \tag{5.19}$$

In ref. [11] the eigenvalues $\check{R}^{(\mathrm{KdV})}$ were quoted for the level $N=1$ and $N=2$ in eqs.(7.9) and (7.10) respectively. The expressions are equivalent to formula (5.19) provided that the parameters in that work are identified with those in the current paper as $\beta=\sqrt{\frac{n+2}{2}}$, $\rho=-\frac{n}{\sqrt{2(n+2)}}$ and the notation p from [11] coincides with P. Also in [11] a simple formula was presented for the product of the eigenvalues for a given level N, i.e., $\det_N\big(\check{\mathbb{R}}^{(\mathrm{KdV})}\big)$. It reads as

$$\det{}_N\big(\check{\mathbb{R}}^{(\mathrm{KdV})}\big) = \prod_{\substack{1\le j,m\le N\\ jm\le N}}\left[\frac{2P+m\beta^{-1}-j\beta}{2P-m\beta^{-1}+j\beta}\right]^{\mathtt{par}_1(N-mj)} \qquad \left(\beta=\sqrt{\frac{n+2}{2}}\,\right) \ , \tag{5.20}$$

where $\mathtt{par}_1(N)$ is the number of integer partitions of N.

Significant simplifications occur for the quantum KdV integrable structure when the central charge $c=-2$. In this case all the eigenvalues (5.19) take the form

$$\check{R}^{(c=-2)}(\boldsymbol{v}) = \prod_{j=1}^{J}\frac{\sqrt{2}\,P+n_j^{(-)}-\frac{1}{2}}{\sqrt{2}\,P-n_j^{(+)}+\frac{1}{2}} \ . \tag{5.21}$$

Here $\big\{n_j^{(\pm)}\big\}$ are two sets of integers satisfying the conditions $1\le n_1^{(\pm)}<n_2^{(\pm)}<\ldots<n_J^{(\pm)}$ and

$$N=\sum_{j=1}^{J}\big(n_j^{(+)}+n_j^{(-)}-1\big) \ . \tag{5.22}$$

In fact, the sets $\big\{n_j^{(\pm)}\big\}$ can be used to classify the states $|\boldsymbol{v}\rangle_P$ for any $c\le 1$. The integers which appear in the exact Bohr-Sommerfeld quantization condition for the Schrödinger equation with the Monster potentials (5.11) are expressed through these numbers (for details, see Appendix A in ref. [7]).

Equation (5.20) or/and (5.21) combined with the formula (3.59) explains the simple form of the determinant of the operator $\check{\mathbb{D}}^{(\mathrm{AKNS})}$ in the level subspaces $L=1,2$ which appear in eqs.(80),(81) from ref. [17]. It is possible to show that for general L

$$\det{}_L\big(\check{\mathbb{D}}^{(\mathrm{AKNS})}\big) = \prod_{N=1}^{L}\prod_{\substack{1\le j,m\\ jm\le N}}\left[\frac{(2p-2\mathrm{i}s+2m-j)\,(2p+2\mathrm{i}s-2m+j)}{(2p+2\mathrm{i}s+2m-j)\,(2p-2\mathrm{i}s-2m+j)}\right]^{\mathtt{par}_1(N-mj)\,\mathtt{par}_1(L-N)} \tag{5.23}$$

It would be remiss not to mention the other integrable structure associated with the Virasoro algebra. The first representatives from the corresponding set of local IM, $\big\{\mathbb{I}^{(\mathrm{BD})}_{6m-5}, \mathbb{I}^{(\mathrm{BD})}_{6m-1}\big\}_{m=1}^{\infty}$, are given by

$$\mathbb{I}^{(\mathrm{BD})}_1 \;=\; \int_0^{2\pi} \frac{\mathrm{d}x_1}{2\pi}\, T = \int_0^{2\pi}\frac{\mathrm{d}x}{2\pi}\,\tilde{T} \tag{5.24}$$

$$\mathbb{I}^{(\mathrm{BD})}_5 \;=\; \int_0^{2\pi} \frac{\mathrm{d}x_1}{2\pi}\, \Big(T^3+\frac{8-\tilde{c}}{8}\,(\partial T)^2\Big) = \int_0^{2\pi} \frac{\mathrm{d}x_1}{2\pi}\, \Big(\tilde{T}^3+\frac{8-c}{8}\,(\partial \tilde{T})^2\Big)\ ,$$

where

$$T=-(\partial\varphi_+)^2+Q\,\partial^2\varphi_+\ , \qquad \tilde{T}=-(\partial\varphi_+)^2+\tilde{Q}\,\partial^2\varphi_+\ . \tag{5.25}$$

Each of the holomorphic fields T and $\tilde{T}$ generates the Virasoro algebra with different central charges $c=1+6Q^2$ and $\tilde{c}=1+6\tilde{Q}^2$ respectively. The latter are not independent, but satisfy the quadratic relation

$$4\,(\tilde{c}^2+c^2)-17\,\tilde{c}c+117\,(\tilde{c}+c)+504=0 \tag{5.26}$$

so that parameterizing Q as $Q=b+b^{-1}$, the tilde counterpart is given by $\tilde{Q}=-2b+(-2b)^{-1}$. The above integrable structure is related to the Bullough-Dodd model described by the action

$$\mathcal{A}_{\mathrm{BD}}=\int_{-\infty}^{\infty}\mathrm{d}x_2\int_0^R\mathrm{d}x_1\,\Big(\frac{1}{4\pi}\,(\partial_a\varphi)^2+\mu\,\mathrm{e}^{2b\varphi}+\tilde{\mu}\,\mathrm{e}^{-4b\varphi}\Big)\ . \tag{5.27}$$

One can introduce the reflection operator for the Bullough-Dodd integrable structure, commuting with the local IM, as

$$\check{\mathbb{R}}^{(\mathrm{BD})}=\hat{s}_{\mathrm{L}}(-P\,|\,\tilde{Q})\,\hat{s}_{\mathrm{L}}(P\,|\,Q)\ . \tag{5.28}$$

6 Reflection operators and Hermitian structures

Up till now we have been focused on the spectral problem for the different commuting families of operators and, as such, there has not been any mention of the Hermitian structures consistent with these integrable structures. By consistent, among other things, we take to mean that with respect to the formal Hermitian conjugation in the algebra of extended conformal symmetry, the (properly normalized) local IM are Hermitian operators. For the integrable structures considered in this paper there are several natural Hermitian structures, which are related to one another via the reflection operators. Let us first illustrate this for the simplest case of the KdV integrable structure, following ref. [11].

The Virasoro algebra commutation relations (1.1) admit the natural Hermitian conjugation given by

$$L_m^\star=L_{-m}\ . \tag{6.1}$$

It is not difficult to see that $\big[T(u)\big]^\star=T(-u^*)$ and hence the KdV IM (1.3) are Hermitian operators with respect to the conjugation (6.1). On the other hand, in view of eq. (1.8) that expresses the Virasoro generators in terms of the Heisenberg ones, we can consider another conjugation

$$a_m^\dagger=a_{-m} \tag{6.2}$$

that is consistent with the commutation relations (1.7). It is easy to see from eq. (1.8) that for real Q, i.e., $c\geq 1$, the dagger conjugation of L_m is identical to (6.1). However as $c<1$ this conjugation acts highly non-trivially on the Virasoro generators. Nevertheless, it is possible to show that the KdV local IM are Hermitian w.r.t. the dagger conjugation as well (see, e.g., [11])

$$\Big[\mathbb{I}^{(\mathrm{KdV})}_{2m-1}\Big]^{\dagger} = \Big[\mathbb{I}^{(\mathrm{KdV})}_{2m-1}\Big]^{\star} = \mathbb{I}^{(\mathrm{KdV})}_{2m-1} \ . \tag{6.3}$$

Each of the conjugations (6.1) and (6.2) can be used to introduce a Hermitian form in the space $\mathcal{F}_P$ for real P. Namely, for any vectors $\boldsymbol{\psi}_{1,2} \in \mathcal{F}_P$, the forms $\mathbf{V}(\boldsymbol{\psi}_2, \boldsymbol{\psi}_1)$ and $\mathbf{H}(\boldsymbol{\psi}_2, \boldsymbol{\psi}_1)$ are defined uniquely by

$$\mathbf{V}(\boldsymbol{\psi}_2, L_n\boldsymbol{\psi}_1) = \mathbf{V}(L_{-n}\boldsymbol{\psi}_2, \boldsymbol{\psi}_1) \, , \qquad \mathbf{H}(\boldsymbol{\psi}_2, a_n\boldsymbol{\psi}_1) = \mathbf{H}(a_{-n}\boldsymbol{\psi}_2, \boldsymbol{\psi}_1) \tag{6.4}$$

along with the normalization condition

$$\mathbf{V}(\boldsymbol{\psi}, \boldsymbol{\psi}) = \mathbf{H}(\boldsymbol{\psi}, \boldsymbol{\psi}) = 1 \qquad \text{for} \qquad \boldsymbol{\psi} = |P\rangle \ . \tag{6.5}$$

It turns out that $\mathbf{V}$ and $\mathbf{H}$ coincide when the central charge $c \geq 1$. However for $c < 1$, as was pointed out in the work [11], they are related through the reflection operator as

$$\mathbf{H}\big(\boldsymbol{\psi}_2, \boldsymbol{\psi}_1\big) = \mathbf{V}\big(\boldsymbol{\psi}_2, \check{\mathbb{R}}^{(\mathrm{KdV})}\boldsymbol{\psi}_1\big) \ . \tag{6.6}$$

For the W_∞-algebra, whose commutation relations are encoded by the infinite set of OPEs (3.14), the natural Hermitian conjugation, similar to (6.1), reads as

$$\big[\widetilde{W}_j(m)\big]^{\ddagger} = \widetilde{W}_j(-m) \ . \tag{6.7}$$

Since the W_∞-algebra is bosonized by means of two copies of the Heisenberg algebra, the analogue of the dagger conjugation (6.2) is now

$$a_m^{\dagger} = a_{-m} \, , \qquad b_m^{\dagger} = b_{-m} \ . \tag{6.8}$$

For the case with the central charge $c \geq 2$, i.e., the parameter k in (3.16) is real positive, the two conjugations coincide and the AKNS and paperclip local IMs are Hermitian w.r.t. both of them. The situation is more complicated when $c < 2$. It is easy to see from eqs. (3.18), (3.32) that when k is negative or more generally a complex number, $\mathbb{I}_2^{(\mathrm{AKNS})}$ cannot possibly be Hermitian under the conjugation (6.8). At the same time, as it follows from the discussion in ref. [18], it turns out that the system of paperclip local IM (4.3) are still Hermitian under the dagger conjugation when k is a negative real number. For $c < 2$, i.e., for real negative k we found that:

$$\mathbf{H}\big(\boldsymbol{\psi}_2, \boldsymbol{\psi}_1\big) = \mathbf{W}\big(\boldsymbol{\psi}_2, \check{\mathbb{R}}^{(\mathrm{pc})}\boldsymbol{\psi}_1\big) \ . \tag{6.9}$$

Here the Hermitian form $\mathbf{H}$ is defined through the relation

$$\mathbf{H}(\boldsymbol{\psi}_2, a_m\boldsymbol{\psi}_1) = \mathbf{H}(a_{-m}\boldsymbol{\psi}_2, \boldsymbol{\psi}_1) \, , \qquad \mathbf{H}(\boldsymbol{\psi}_2, b_m\boldsymbol{\psi}_1) = \mathbf{H}(b_{-m}\boldsymbol{\psi}_2, \boldsymbol{\psi}_1) \, , \tag{6.10}$$

while for $\mathbf{W}$ the corresponding equation is

$$\mathbf{W}\big(\boldsymbol{\psi}_2, \widetilde{W}_j(m)\boldsymbol{\psi}_1\big) = \mathbf{W}\big(\widetilde{W}_j(-m)\boldsymbol{\psi}_2, \boldsymbol{\psi}_1\big) \qquad (j = 2, 3, \ldots) \ . \tag{6.11}$$

In the above formulae the vectors $\boldsymbol{\psi}_{1,2} \in \mathcal{F}_{P_1,P_2}$ with real P_1, P_2 and the Hermitian forms are assumed to be normalized as

$$\mathbf{W}(\boldsymbol{\psi},\boldsymbol{\psi}) = \mathbf{H}(\boldsymbol{\psi},\boldsymbol{\psi}) = 1 \qquad \text{for} \qquad \boldsymbol{\psi} = |P_1,P_2\rangle\ . \tag{6.12}$$

Finally let us take a closer look at (3.34), where J and T are defined by (3.33). It is straightforward to see that if we introduce the conjugation

$$\big[J(u)\big]^\star = J(-u^*)\,, \qquad \big[T(u)\big]^\star = T(-u^*)\,, \tag{6.13}$$

then the AKNS local IM are Hermitian for any real k. The Hermitian form $\mathbf{HV}$ that is consistent with the conjugation (6.13), i.e.,

$$\mathbf{HV}\big(\boldsymbol{\psi}_2, J(u)\boldsymbol{\psi}_1\big) = \mathbf{HV}\big(J(-u^*)\boldsymbol{\psi}_2,\boldsymbol{\psi}_1\big)\,, \quad \mathbf{HV}\big(\boldsymbol{\psi}_2, T(u)\boldsymbol{\psi}_1\big) = \mathbf{HV}\big(T(-u^*)\boldsymbol{\psi}_2,\boldsymbol{\psi}_1\big) \tag{6.14}$$

and normalized similar to (6.12) coincides with $\mathbf{W}$ and $\mathbf{H}$ for $c \geq 2$. However in the domain $c < 2$, in full analogy with (6.9), the following relation holds true

$$\mathbf{HV}\big(\boldsymbol{\psi}_2,\boldsymbol{\psi}_1\big) = \mathbf{W}\big(\boldsymbol{\psi}_2, \check{\mathbb{R}}^{(\mathrm{AKNS})}\boldsymbol{\psi}_1\big)\ . \tag{6.15}$$

7 Conclusion

Together with the commuting family of local IM, the reflection operator(s) are one of the key ingredients in a variety of integrable structures of Conformal Field Theory. In this work we used the results of ref. [25] to compute the spectrum of the reflection operators in the Fateev integrable structure. Since the quantum AKNS, KdV and paperclip integrable structures can be obtained through certain reductions of the Fateev one, we were able to find the spectrum of their associated reflection operators as well. Another result that deserves to be mentioned is the remarkable formula (3.60) connecting the reflection S-matrices of the cigar and Liouville CFTs.

As was demonstrated in the papers [11, 17], the reflection operators are a powerful tool for the study of the scaling limit of the Bethe states in integrable spin chains. We believe that this is one of the major applications for the results of our work. In particular, the quantum AKNS integrable structure occurs in the context of the alternating spin chain [17], while the KdV one is related to the spin-$\frac{1}{2}$ XXZ model [11]. Our result on the spectrum of the paperclip integrable structure is directly applicable to the study of the critical behaviour of the Fateev-Zamolodchikov $\mathbb{Z}_n$-invariant spin chain (for details see, e.g., [19]). The most general case of the Fateev integrable structure is expected to occur in the scaling limit of the higher spin integrable $\mathfrak{sl}(2)$ chains.

We mentioned the construction of the reflection operator for the quantum Bullough-Dodd integrable structure. Though it has yet to be carried out, the calculation of its spectrum would be valuable for the study of the scaling behaviour of the Izergin-Korepin spin chain [34]. Similar to how the quantum KdV integrable structure is obtained from a reduction of the AKNS one, the Bullough-Dodd can be derived through a reduction of the quantum Boussinesq integrable structure [35, 36]. The system of algebraic equations whose solution sets label the eigenstates for the Boussinesq integrable structure was obtained recently in the works [37, 38]. In this case the eigenvalues of the reflection operator are expressed in terms of the connection coefficients of a certain class of third order ODEs. For the quantum Getmanov integrable structure the spectrum of the reflection operator is likewise related to another class of linear third order differential equations [21]. Hence the extension of the results of Eremenko and Tarasov to third and higher order ODEs is of special interest in this regard. Note that the higher order ODEs occur in the computation of the spectrum of the reflection operator in the integrable hierarchy for the Toda theories associated with the affine Lie (super)algebras [39–41].

Another application is related to the recent interest in constructing the Generalized Gibbs ensemble for an integrable QFT [42,43]. Going beyond the $c=\infty$ limit, the formulation of the Generalized Gibbs ensemble in all likelihood requires one to properly account for the reflection operator and perhaps other non-local IM.

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Explicit formulae for $\Omega^I{}_J$, $\Theta^I{}_J$ at the first few levels

Here we present explicit formulae for the matrices $\Omega^I{}_J$ (3.40) for $L=1,2,3$ and $\Theta^I{}_J$ (3.26) for $L=1,2$, which are the building blocks for the Liouville (3.41) and cigar (3.27) reflection S-matrices.

The oscillator basis for the Fock space $\mathcal{F}_P$ reads as

$$\boldsymbol{e}_{(1)}=a_{-1}|P\rangle$$

for level one;

$$\boldsymbol{e}_{(1,1)}=a_{-1}^2|P\rangle\,,\qquad \boldsymbol{e}_{(2)}=a_{-2}|P\rangle$$

for the case $L=2$; and

$$\boldsymbol{e}_{(1,1,1)}=a_{-1}^3|P\rangle\,,\qquad \boldsymbol{e}_{(1,2)}=a_{-1}a_{-2}|P\rangle\,,\qquad \boldsymbol{e}_{(3)}=a_{-3}|P\rangle$$

for the third level. As was mentioned in the main body of the text, the Fock space admits the structure of the Verma module for the Virasoro algebra, with $|P\rangle$ being the highest weight vector with the conformal dimension $\Delta=P^2+\frac{1}{4}Q^2$ and the central charge $c=1+6Q^2$. The Virasoro basis is obtained by acting with the L_{-n} with $n>0$ on $|P\rangle$. Using eqs. (1.8) that express the Virasoro generators in terms of the Heisenberg ones, this basis can be re-written in terms of the $\boldsymbol{e}_I$ presented above. A straightforward calculation gives

$$\boldsymbol{t}_{(1)}\equiv L_{-1}|P\rangle=(2P-\mathrm{i}Q)\boldsymbol{e}_{(1)}\,,$$

while

$$\boldsymbol{t}_{(1,1)} \equiv L_{-1}^2|P\rangle = (2P-\mathrm{i}Q)^2\boldsymbol{e}_{(1,1)} + (2P-\mathrm{i}Q)\boldsymbol{e}_{(2)}\,, \quad \boldsymbol{t}_{(2)} \equiv L_{-2}|P\rangle = \boldsymbol{e}_{(1,1)} + 2(P-\mathrm{i}Q)\boldsymbol{e}_{(2)}\ .$$

At the third level one has

$$\begin{aligned}
\boldsymbol{t}_{(1,1,1)} \equiv L_{-1}^3|P\rangle &= (2P-\mathrm{i}Q)^3\,\boldsymbol{e}_{(1,1,1)} + 3(2P-\mathrm{i}Q)^2\,\boldsymbol{e}_{(1,2)} + 2(2P-\mathrm{i}Q)\,\boldsymbol{e}_{(3)}\\
\boldsymbol{t}_{(1,2)} \equiv L_{-1}L_{-2}|P\rangle &= (2P-\mathrm{i}Q)\,\boldsymbol{e}_{(1,1,1)} + 2\big(1+(P-\mathrm{i}Q)(2P-\mathrm{i}Q)\big)\,\boldsymbol{e}_{(1,2)} + 4(P-\mathrm{i}Q)\,\boldsymbol{e}_{(3)}\\
\boldsymbol{t}_{(3)} \equiv L_{-3}|P\rangle &= 2\,\boldsymbol{e}_{(1,2)} + (2P-3\mathrm{i}Q)\,\boldsymbol{e}_{(3)}\ .
\end{aligned}$$

The matrix elements ${\Omega^I}_J$ (3.40) are easily read off from the above formulae. For level $L=1$:

$${\Omega^{(1)}}_{(1)} = 2P-\mathrm{i}Q\ .$$

At the second level

$$\boldsymbol{\Omega} = \begin{pmatrix} {\Omega^{(1,1)}}_{(1,1)} & {\Omega^{(1,1)}}_{(2)} \\ {\Omega^{(2)}}_{(1,1)} & {\Omega^{(2)}}_{(2)} \end{pmatrix} = \begin{pmatrix} (2P-\mathrm{i}Q)^2 & 1 \\ 2P-\mathrm{i}Q & 2(P-\mathrm{i}Q) \end{pmatrix}\ ,$$

while at the third level, using similar matrix notation,

$$\boldsymbol{\Omega} = \begin{pmatrix} (2P-\mathrm{i}Q)^3 & 2P-\mathrm{i}Q & 0 \\ 3(2P-\mathrm{i}Q)^2 & 2+2(P-\mathrm{i}Q)(2P-\mathrm{i}Q) & 2 \\ 2(2P-\mathrm{i}Q) & 4(P-\mathrm{i}Q) & 2P-3\mathrm{i}Q \end{pmatrix}\ .$$

The computation for ${\Theta^I}_J$ (3.26) is completely analogous to that of ${\Omega^I}_J$ described above. In this case one needs to relate the basis (3.23) built from the Fourier modes of the currents $W_2(u)$ and $W_3(u)$ with the one obtained by acting with the Heisenberg modes a_{-m}, b_{-m} $(m>0)$ on the Fock vacuum $|P_1,P_2\rangle$, see eq. (3.25). A straightforward calculation using eqs. (3.18), (3.21) and (3.17) allows one to express the W-modes in terms of the Heisenberg generators. The result reads as

$$\begin{aligned}
\widetilde{W}_2(m) &= \sum_{j=-\infty}^{\infty}\big(a_j\,a_{m-j} + b_j\,b_{m-j}\big) + \mathrm{i}m\,Q\,a_m \qquad\qquad (m\neq 0)\\
\widetilde{W}_3(m) &= \frac{4Q^2+6}{3}\sum_{i+j+l=m} :b_ib_jb_l: + 2\sum_{i+j+l=m} :a_ia_j:b_l - \frac{\mathrm{i}}{Q}\sum_{l=-\infty}^{\infty} l\,\big(a_l\,b_{m-l} - (1+2Q^2)\,b_l\,a_{m-l}\big)\\
&- \frac{2+Q^2+m^2(1+2Q^2)}{6}\,b_m \qquad\qquad (m\neq 0)\ .
\end{aligned}$$

Here $Q=-\frac{1}{\sqrt{k}}$ and the monomials occurring in the sums are normal ordered so that the creation operators are always placed to the left of the annihilation operators. At the first level, the basis vectors

$$\boldsymbol{v}_{(1)} = \widetilde{W}_2(-1)|P_1,P_2\rangle, \qquad \boldsymbol{v}_{(1')} = \widetilde{W}_3(-1)|P_1,P_2\rangle$$

are expressed in terms of the Heisenberg basis:

$$\boldsymbol{e}_{(1)} = a_{-1}|P_1,P_2\rangle, \qquad \boldsymbol{e}_{(1')} = b_{-1}|P_1,P_2\rangle$$

as

$$
\begin{aligned}
\boldsymbol{v}_{(1)} &= (2P_1 - \mathrm{i}\,Q)\,\boldsymbol{e}_{(1)} + 2P_2\,\boldsymbol{e}_{(1')} \\
\boldsymbol{v}_{(1')} &= P_2\left(4P_1 + \frac{\mathrm{i}}{Q}\right)\boldsymbol{e}_{(1)} + B\,\boldsymbol{e}_{(1')}\,,
\end{aligned}
$$

where

$$
B = 2(2Q^2+3)P_2^2 + P_1\big(2P_1 - \mathrm{i}(2Q+Q^{-1})\big) - \frac{1}{2}\,(Q^2+1)\ .
$$

Hence the matrix elements $\Theta^I{}_J$ (3.26) at level $L=1$ are given by

$$
\boldsymbol{\Theta} = \left(\begin{array}{cc} \Theta^{(1)}{}_{(1)} & \Theta^{(1)}{}_{(1')} \\ \Theta^{(1')}{}_{(1)} & \Theta^{(1')}{}_{(1')} \end{array}\right) = \left(\begin{array}{cc} 2P_1 - \mathrm{i}Q & P_2\left(4P_1 + \frac{\mathrm{i}}{Q}\right) \\ 2P_2 & B \end{array}\right)\ .
$$

The level $L=2$ subspace of the Fock space $\mathcal{F}_{P_1,P_2}$ is five dimensional. It is spanned by the states

$$
\boldsymbol{e}_{(1,1)} = a_{-1}^2|P_1,P_2\rangle, \qquad \boldsymbol{e}_{(2)} = a_{-2}|P_1,P_2\rangle\,, \qquad \boldsymbol{e}_{(1',1')} = b_{-1}^2|P_1,P_2\rangle\,,
$$

$$
\boldsymbol{e}_{(2')} = b_{-2}|P_1,P_2\rangle\,, \qquad \boldsymbol{e}_{(1,1')} = a_{-1}b_{-1}|P_1,P_2\rangle\ .
$$

There are five vectors of the form (3.23) belonging to this subspace:

$$
\boldsymbol{v}_{(1,1)} = \widetilde{W}_2^2(-1)|P_1,P_2\rangle, \qquad \boldsymbol{v}_{(2)} = \widetilde{W}_2(-2)|P_1,P_2\rangle\,, \qquad \boldsymbol{v}_{(1',1')} = \widetilde{W}_3^2(-1)|P_1,P_2\rangle\,,
$$

$$
\boldsymbol{v}_{(2')} = \widetilde{W}_3(-2)|P_1,P_2\rangle\,, \qquad \boldsymbol{v}_{(1,1')} = \widetilde{W}_2(-1)\widetilde{W}_3(-1)|P_1,P_2\rangle\ .
$$

In terms of the Heisenberg basis, they are given by

$$
\begin{aligned}
\boldsymbol{v}_{(1,1)} &= (2P_1-\mathrm{i}Q)^2\,\boldsymbol{e}_{(1,1)} + (2P_1-\mathrm{i}Q)\,\boldsymbol{e}_{(2)} + 4P_2^2\,\boldsymbol{e}_{(1',1')} + 2P_2\,\boldsymbol{e}_{(2')} + 4P_2(2P_1-\mathrm{i}Q)\,\boldsymbol{e}_{(1,1')} \\
\boldsymbol{v}_{(2)} &= \boldsymbol{e}_{(1,1)} + 2(P_1-\mathrm{i}Q)\,\boldsymbol{e}_{(2)} + \boldsymbol{e}_{(1',1')} + 2P_2\,\boldsymbol{e}_{(2')} \\
\boldsymbol{v}_{(2')} &= 2P_2\,\boldsymbol{e}_{(1,1)} - \frac{2\mathrm{i}P_2}{Q}\,(2\mathrm{i}P_1Q-1)\boldsymbol{e}_{(2)} + 2P_2(2Q^2+3)\,\boldsymbol{e}_{(1',1')} + \frac{1}{2Q}\big[4Q\,P_1^2 \\
&+ 4Q(2Q^2+3)\,P_2^2 - 4\mathrm{i}(2Q^2+1)\,P_1 - Q(3Q^2+2)\,\big]\,\boldsymbol{e}_{(2')} + 2(2P_1-\mathrm{i}Q)\,\boldsymbol{e}_{(1,1')} \\
\boldsymbol{v}_{(1,1')} &= -\frac{\mathrm{i}P_2}{Q}\,(2P_1-\mathrm{i}Q)(4\mathrm{i}Q\,P_1-1)\,\boldsymbol{e}_{(1,1)} - \frac{\mathrm{i}P_2}{Q}\,(4\mathrm{i}QP_1-1)\,\boldsymbol{e}_{(2)} \\
&+ \frac{P_2}{Q}\,\big[4Q\,P_1^2 + 4Q(2Q^2+3)\,P_2^2 - 2\mathrm{i}(2Q^2+1)\,P_1 - Q(Q^2+1)\big]\boldsymbol{e}_{(1',1')} \\
&+ \frac{1}{2Q}\,\big[4Q\,P_1^2 + 4Q\,(2Q^2+3)\,P_2^2 - 2\mathrm{i}(2Q^2+1)\,P_1 - Q(Q^2+1)\big]\,\boldsymbol{e}_{(2')} \\
&+ \frac{1}{2Q}\,\big[8Q\,P_1^3 + 8Q\,(2Q^2+5)\,P_1P_2^2 - 4\mathrm{i}(3Q^2+1)\,P_1^2 - 4\mathrm{i}(2Q^4+3Q^2-1)\,P_2^2 \\
&- 2Q(3Q^2+2)\,P_1 + \mathrm{i}Q^2(Q^2+1)\big]\,\boldsymbol{e}_{(1,1')}
\end{aligned}
$$

The expression for $\boldsymbol{v}_{(1',1')}$ is rather cumbersome and reads as

$$
\begin{aligned}
\boldsymbol{v}_{(1',1')} &= \frac{1}{2Q^2}\big[32Q^2\,P_1^2P_2^2 + 16\mathrm{i}Q\,P_1P_2^2 + 4Q^2\,P_1^2 + 2(4Q^4+6Q^2-1)\,P_2^2 - 2\mathrm{i}Q(1+2Q^2)P_1 \\
&- Q^2(1+Q^2)\big]\,\boldsymbol{e}_{(1,1)} + \frac{1}{4Q^2}\big[16Q^2\,P_1^3 + 16Q^2(2Q^2+5)P_1\,P_2^2 - 4\mathrm{i}Q(2Q^2-1)\,P_1^2 \\
&+ 4\mathrm{i}Q\,(4Q^4+12Q^2+11)\,P_2^2 + 2\,(2Q^4+6Q^2+3)\,P_1 - \mathrm{i}Q(Q^2+1)(2Q^2+3)\big]\,\boldsymbol{e}_{(2)} \\
&+ \frac{1}{4Q^2}\big[16Q^2\,P_1^4 + 16Q^2(2Q^2+3)^2\,P_2^4 + 32Q^2(2Q^2+3)\,P_1^2P_2^2 - 16\mathrm{i}Q(2Q^2+1)\,P_1^3 \\
&- 16\mathrm{i}Q(2Q^2+1)(2Q^2+3)\,P_1P_2^2 - 4(2Q^4+1)P_1^2 + 8\,Q^2\,(Q^2+2)(2Q^2+3)\,P_2^2 \\
&- 4\mathrm{i}Q(Q^2+2)(2Q^2+1)\,P_1 - Q^2\,(5+8Q^2+3Q^4)\big]\,\boldsymbol{e}_{(1',1')} + \frac{P_2}{2Q^2}\,\big[8Q^2(2Q^2+5)\,P_1^2 \\
&+ 8Q^2(2Q^2+3)^2\,P_2^2 - 4\mathrm{i}Q(4Q^4+12Q^2+5)\,P_1 - (4Q^6+10Q^4+2Q^2-3)\big]\,\boldsymbol{e}_{(2')} \\
&+ \frac{P_2}{Q^2}\big[16Q^2\,P_1^3 + 16Q^2(2Q^2+3)\,P_1P_2^2 - 4\mathrm{i}Q(4Q^2+1)\,P_1^2 + 4\mathrm{i}Q(2Q^2+3)\,P_2^2 \\
&- 2(2Q^4-4Q^2-1)\,P_1 - \mathrm{i}Q(Q^2-1)\big]\,\boldsymbol{e}_{(1,1')}\ .
\end{aligned}
$$

The matrix elements $\Theta^I{}_J$ (3.26) at the second level follow immediately from the above formulae.

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