From Momentum Expansions to Post-Minkowskian Hamiltonians by Computer Algebra Algorithms

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Abstract

The post-Newtonian and post-Minkowskian solutions for the motion of binary mass systems in gravity can be derived in terms of momentum expansions within effective field theory approaches. In the post-Minkowskian approach the expansion is performed in the ratio G_N/r, retaining all velocity terms completely, while in the post-Newtonian approach only those velocity terms are accounted for which are of the same order as the potential terms due to the virial theorem. We show that it is possible to obtain the complete post-Minkowskian expressions completely algorithmically, under most general purely mathematical conditions from a finite number of velocity terms and illustrate this up to the third post-Minkowskian order given in [1].

1   Introduction

The use of a non-relativistic effective field theory [2–9], provides one way to derive the equations of motion of a binary mass system within the post-Newtonian (PN) approach. Currently all corrections are known up to the 4th post-Newtonian order [3–7] and first corrections due to the static potential at 5PN [8,9]. Previously, the results up to the 4th post-Newtonian order have already been derived using different methods, see Refs. [10,11] and references therein, and first results up to the 5th post-Newtonian order have been obtained in [12] recently.

The general principle is to expand in the ratio G_N/r, with G_N Newton's constant and r denoting the distance between the two point masses m_1 and m_2, and to retain all velocity corrections up to the order implied by the virial theorem [13] $G_N m_1 m_2/r \sim \mathbf{v}_1^2 m_1 + \mathbf{v}_2^2 m_2$. In the post-Minkowskian (PM) approach [1,14–16] the expansion is performed in G_N/r retaining all velocity corrections as closed form expressions. Recently calculations have been performed up to the third post-Minkowskian order, cf. [1,16], in the center of momentum frame using isotropic coordinates $\mathbf{p}.\mathbf{r} = 0$. Here the Hamiltonian reads[1]

$$H(\mathbf{p}, \mathbf{r}) = \sqrt{m_1^2 + \mathbf{p}^2} + \sqrt{m_2^2 + \mathbf{p}^2} + V(\mathbf{p}, \mathbf{r}), \tag{1}$$

$$V(\mathbf{p}, \mathbf{r}) = \sum_{k=1}^{\infty} V_k(\mathbf{p}) \frac{G_N^k}{|\mathbf{r}|^k}, \quad V_k(\mathbf{p}) = \sum_{l=0}^{\infty} a_k(l) x^l, \tag{2}$$

where x denotes an appropriate expansion variable, which will be defined below.

The question arises, whether these corrections can also be obtained using the effective field theory approach, in which one usually can only expand up to finite terms in the velocity. In this note we show that this is indeed possible algorithmically under the following three sufficient conditions.

1. There exist recurrences for the coefficients $a_k(l)$ of Eq. (2) in l, up to a finite number of polynomial terms in x.

2. The recurrence or its associated differential equation factorizes at first order.

3. The dependence of $V_k(x)$ on $\rho = m_1/m_2$ is rational.

Here we will not use any special additional physical conditions, e.g. on expected structures of the solution, but follow a purely mathematical approach instead. We will use the method of guessing, see e.g. [17], to obtain the corresponding difference equation, which is then solved by applying difference field theory as implemented in the package `Sigma` [18,19]. The final expressions are then obtained by performing one infinite sum and adjusting one polynomial by initial conditions. In perturbative Quantum Chromodynamics (QCD) the method of guessing has been successfully applied to problems which are much more voluminous then the present ones, see Refs. [20,21], and are based on up to $O(5000 - 8000)$ input values.

Concerning the integration, the master integrals in the zero-dimensional case are simpler to perform than for the momentum resummed expressions. Moreover, all the master integrals in the case one expands in the momentum are already known up to five-loop order from the post-Newtonian approach. The challenge for the present method lies in the expansion up to moderately large powers in the momentum.

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[1]It has been shown in [15] that the results of [1] are in accordance with the energy- and angular momentum-orbital frequency relation, $E(\omega)$ and $L(\omega)$, obtained in the post-Newtonian approach up to $O((G_N M \omega)^2)$, with $M = m_1 + m_2$. We have verified that the results in [1], when compared to the post-Newtonian results in ADM coordinates in [10], agree up to terms of $O(G_N^3 (\mathbf{p}^2)^2)$.

In the effective field theory approach one usually starts out to work in harmonic coordinates. Here higher order time derivatives of the velocities occur, which may be eliminated by using the equation of motion to obtain a first order Lagrangian resp. Hamiltonian. This operation also induces a coordinate transformation [22]. One may then transform to coordinates which are connected to ADM coordinates [10].The structure of the transformation matrices between the different systems can be fixed by confronting a corresponding ansatz with the results for observables being calculated in both frames up to the desired perturbative order. Here one well suited observable is the scattering angle [23].

In the following we will demonstrate how the potentials of Eq. (2) can be completely recovered algorithmically from a finite number of expansion coefficients $a_k(l)$. We will first consider the equal mass case $m_1 = m_2$, and then turn to the general case. In an appendix we discuss how some special sums occurring can be carried out.

2   The equal mass case

We first study the case $m_1 = m_2 \equiv m$, introduce the variable

$$x = \frac{\mathbf{p}^2}{m^2} \tag{3}$$

and normalize V_k by a factor $1/m_1^{k+1}$ to obtain dimensionless quantities. We will keep this normalization in the unequal mass case as well.

Within the effective field theory approach we sort the contributions keeping the terms of $O((G/r)^k)$ in the kth post-Minkowskian order and retain a finite number of terms in $x^l\big|_{l=0}^{M}$, with M a sufficiently large integer, which will be specified below. Next we seek for a recursion relation for the coefficients $a_k(l) \in \mathbb{Q}$ in l for $k = 1, 2, 3$. We apply the method of guessing to a finite set of theses coefficients, which allows one to obtain this recurrence of order o and degree d,

$$\sum_{n=0}^{o} Q_n(l) f[l + n] = 0. \tag{4}$$

and to check its validity. Here Q_n are polynomials in l maximally of degree d. Whenever the corresponding recurrences are first order factorizable in difference fields, they can be solved in terms of iterated sums defined over hypergeometric products by the package `Sigma` [18, 19]. For this o initial values are needed, which are given by a subset of the expansion coefficients $a_k(l)$. In this way we obtain a closed form for the expansion coefficients $a_k(l)$, possibly up to a finite number of expansion terms in x. The latter function is a polynomial. By performing a Taylor series expansion of the result, one can fix these terms. Finally, we perform the infinite sum analytically and obtain the closed form expressions for the potentials at the kth post-Minkowskian order. Here the important point is, that the reconstruction is possible using a finite number of terms.

The simplest example is V_1. The momentum expansion yields the series

$$
\begin{aligned}
V_1(x) \;\simeq\; & -1 - 7x - x^2 + x^3 - x^4 + x^5 - x^6 + x^7 - x^8 + x^9 - x^{10} + x^{11} - x^{12} + x^{13} - x^{14} \\
& + x^{15} - x^{16} + x^{17} - x^{18} + x^{19} - x^{20} + O(x^{21}), \tag{5} \\
S_1 \;=\; & \{-1, -7, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1\}. \tag{6}
\end{aligned}
$$

In the effective field theory approach one would obtain the sequence S_1 by expanding in x into a formal Taylor series.

By guessing we obtain the recurrence

$$f_1[n] + f_1[n+1] = 0, \tag{7}$$

with the solution

$$\bar{a}_1(l) = (-1)^{l+1}. \tag{8}$$

Here the potential (2) has the representation

$$V_k(\mathbf{p}) = p_k(x) + \sum_{l=0}^{\infty} \bar{a}_k(l) x^l, \tag{9}$$

where $p_k(x)$ is a polynomial. The coefficients of $p_k(x)$ are determined comparing the Taylor expansions of (9) with those of (2). For this step we need 8 initial values. The corresponding solution reads then

$$V_1 = p_1(x) - \frac{1}{1+x}, \tag{10}$$

and $p_1(x)$ is given by

$$p_1(x) = -8x. \tag{11}$$

The same algorithm can now be applied at 2PM and 3PM. In Table 1 we list the respective numbers of minimally necessary input values to establish the corresponding difference equations without any further assumption together with their orders and degrees.

	order	degree	# input values
1PM	1	0	8
2PM	2	5	24
3PM	3	15	54

Table 1: Characteristics of the recurrences and number of input values for the different post-Minkowskian orders in the equal mass case.

The exact value of the minimal number of input parameters needed at low numbers, as is the case here, is determined experimentally. Usually one works with a larger number of expansion coefficients. Here we wanted to display the minimal value needed, since the expansions in x in the effective field theory approach is not yet possible fully algorithmically.

The recurrences for V_2 and V_3 read

$$(10713 - 278n - 3688n^2 - 400n^3 + 16n^4 - 96n^5) f_2[n]$$
$$+ (4461 - 10188n - 8648n^2 - 672n^3 + 80n^4 - 192n^5) f_2[n+1]$$
$$+ (-6252 - 9910n - 4960n^2 - 272n^3 + 64n^4 - 96n^5) f_2[n+2] = 0 \tag{12}$$
$$Q_1\, f_3[n] + Q_2\, f_3[n+1] + Q_3\, f_3[n+2] + Q_4\, f_3[n+3] = 0, \tag{13}$$

with

$$
\begin{aligned}
Q_1 &= 126903309120 + 327090111984n + 199501827192n^2 - 15839063268n^3 \\
&\quad + 125598633964n^4 + 319201064194n^5 + 244500413870n^6 + 74947793534n^7 \\
&\quad - 2304037362n^8 - 7916007828n^9 - 1912314952n^{10} - 69778816n^{11} \\
&\quad + 36357088n^{12} + 6925120n^{13} + 938880n^{14} + 84480n^{15},
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
Q_2 &= 120213548280 + 370215834660n + 215909250030n^2 - 222044191596n^3 \\
&\quad - 29242273581n^4 + 508977450525n^5 + 530659013385n^6 + 196490815287n^7 \\
&\quad + 1030457202n^8 - 20244382200n^9 - 5290755840n^{10} - 189159456n^{11} \\
&\quad + 115021824n^{12} + 22664640n^{13} + 2985600n^{14} + 253440n^{15},
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
Q_3 &= -101452470840 - 39208126464n + 158218785864n^2 - 262227440529n^3 \\
&\quad - 511020293886n^4 + 6335298708n^5 + 336175588032n^6 + 179465093733n^7 \\
&\quad + 10546876182n^8 - 17246382984n^9 - 4999683456n^{10} - 174901344 * n^{11} \\
&\quad + 123193344n^{12} + 24786240n^{13} + 3154560n^{14} + 253440n^{15},
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
Q_4 &= 185299500960 + 326857962960n + 377719564176n^2 + 3846793164n^3 \\
&\quad - 343897751366n^4 - 172310662748n^5 + 55100830352n^6 + 58117627580n^7 \\
&\quad + 6784432878n^8 - 5022804012n^9 - 1624176328n^{10} - 53736544n^{11} \\
&\quad + 44718688n^{12} + 9046720n^{13} + 1107840n^{14} + 84480n^{15}.
\end{aligned} \tag{17}
$$

We find for $\bar{a}_2(l)$ and $\bar{a}_3(l)$

$$
\bar{a}_2(l) = 6(-1)^{l+1} - \frac{\left(87 + 96l + 168l^2 + 128l^3 - 16l^4\right)}{(2l-3)(2l-1)} \left(\frac{-1}{4}\right)^{l+1} \frac{(2l)!}{(l!)^2}, \tag{18}
$$

$$
\begin{aligned}
\bar{a}_3(l) &= \frac{1}{6}\left(-114 - 94l - 15l^2 + l^3\right)(-1)^l + \frac{\left(12 + 22l + l^2 + 14l^3 + 11l^4\right)}{(l-1)l(1+2l)(3+2l)} \frac{(-1)^l 2^{3+2l}(l!)^2}{(2l)!} \\
&\quad + \frac{3(5+2l)\left(-283 - 470l - 312l^2 - 40l^3 + 16l^4\right)}{(1+l)(2+l)(-1+2l)} \frac{(-1)^l 2^{-2-2l}(2l)!}{(l!)^2}.
\end{aligned} \tag{19}
$$

Note that the expression for $\bar{a}_3$ is valid for $l > 1$ only. The remaining sum for $V_i(x)|_{i=1}^3$ can be simply performed using `Mathematica` and more specialized software, which may be necessary for higher post-Minkowskian orders, is not yet needed.

The lower coefficients can be determined using the corresponding input values. The polynomials $p_{2(3)}(x)$ are

$$
p_2(x) = -30x, \tag{20}
$$

$$
p_3(x) = -\frac{5}{2} - \frac{1027x}{8} - \frac{43917x^2}{80}. \tag{21}
$$

By using (2) we finally reconstruct the functions given in [1] Eqs. (10.10) in the equal mass case:[2]

$$
V_1 = -8x - \frac{1}{1+x}, \tag{22}
$$

$$
V_2 = -\frac{6\left(1 + 5x + 5x^2\right)}{1+x} + \frac{1}{4}\frac{\left(1 + 8x + 8x^2\right)\left(29 + 72x + 40x^2\right)}{(1+x)^{5/2}}, \tag{23}
$$

---

[2]The logarithm in (24) can be transformed into a arcsinh-function.

$$
\begin{aligned}
V_3 \;=\;& -\frac{3424x^2}{3} + \frac{3\left(-1+\sqrt{1+x}\right)}{x} + \frac{80}{3}x\left(-25+27\sqrt{1+x}\right)\\
&+\frac{-1+8(1+x)-23(1+x)^3-68(1+x)^4+3\left(34+22x-55x^2-40x^3\right)(1+x)^{3/2}}{(1+x)^4}\\
&+\frac{2\log\left(\sqrt{x}+\sqrt{1+x}\right)\left[-11+16x(1+x)(-1+4x(1+x))\right]}{\sqrt{x}(1+x)^{3/2}}.
\end{aligned}
\tag{24}
$$

3 The general case

While the determination of multi-variate recurrence relations is possible in certain cases, their solution is much more difficult than the one in the uni-variate case and the theory behind is less known. Therefore, we propose a different way to solve the general case. In performing the momentum series in isotropic coordinates we can keep the masses different without any effort. This allows to study the reconstruction similar to the one in Section 2 after fixing the ratio $\rho = m_1/m_2$ to a rational number or an integer. Except particular degenerate cases one will always find the same functional structure in a chosen kinematical variable ξ. The choice of primes for ρ serves this purpose.

Starting with the asymmetric kinematic variable $\xi = \mathbf{p}^2/m_1^2$ the kinematic square roots $\sqrt{m_i^2 + \mathbf{p}^2}$, $i = 1, 2$ remain. They cause a problem in the reconstruction, since the expansion coefficients for the function

$$
\sqrt{1+x}\sqrt{1+\rho^2 x} = \sum_{k=0}^{\infty} b(k)x^k
\tag{25}
$$

are

$$
b(k) = \sum_{l=0}^{k} \binom{\frac{1}{2}}{l}\binom{\frac{1}{2}}{k-l}\rho^{2l} = \binom{\frac{1}{2}}{k}{}_2F_1\left[\begin{array}{c}-\frac{1}{2},\,-k\\ \frac{3}{2}-k\end{array};\rho^2\right].
\tag{26}
$$

$b(k)$ is not a hypergeometric term since $b(k+1)/b(k)$ is not a rational function (of fixed numerator and denominator degree). One may obtain a recurrence for $b(n)$ but it is not solvable within our available difference field algorithms.

We rather choose a more physical variable appropriate for the two–mass case[3],

$$
\begin{aligned}
z(x) \;=\;& \frac{1+\rho^2}{4\rho} + \frac{\rho x}{2} + \frac{1}{2}\sqrt{1+x}\sqrt{1+\rho^2 x}, \quad x = \frac{\mathbf{p}^2}{m_1^2}
\tag{27}\\
\;=\;& \frac{(E_1+E_2)^2}{4m_1m_2},
\tag{28}
\end{aligned}
$$

with $E_i = \sqrt{\mathbf{p}^2 + m_i^2}$, cf. also [26], which has been also used in [27] recently. The inversion of (27) reads

$$
x \equiv \frac{[(1-\rho)^2-4\rho z][(1+\rho)^2-4\rho z]}{16\rho^3 z} = \frac{(E^2-M_+^2)(E^2-M_-^2)}{4E^2 m_1^2},
\tag{29}
$$

where $E = E_1 + E_2$ and $M_\pm = m_1 \pm m_2$.

---

[3]One often does this in massive calculations. Landau variables are early examples for this [24]. Of course, after this change of variable also related integrals can then be performed in simpler function spaces [25].

It is also useful to change to the variable $\bar{z}$, given by

$$\bar{z} = z - \frac{(m_1 + m_2)^2}{4 m_1 m_2}, \tag{30}$$

which vanishes for $\mathbf{p}^2 \to 0$ like the expansion variable in the equal mass case. Note also that in these variables one obtains the contributions to the potential in a very compact form.

Now, the further dependence on ρ is at most a rational function given by integer coefficients.

	order	degree	# input values
1PM	2	0	8
2PM	4	12	45
3PM	9	26	120

Table 2: Characteristics of the recurrences and number of input values for the different post-Minkowskian orders in the general case.

Let us consider the cases $m_1 = 3m_2$ and $m_1 = 5m_2$ for V_1 as examples. We obtain the recurrences

$$9 f[n] - 4 f[2 + n] = 0, \quad 25 f[n] - 36 f[2 + n] = 0, \tag{31}$$

from 8 initial values, which are one order higher than in the equal mass case. This is potentially expected, since certain structural simplifications w.r.t. the variable x can occur in the equal mass case. Still we always seek the lowest order recurrences. By similar steps as performed in Section 2 we obtain

$$V_1\left(z; \rho = 3\right) = -\frac{128 - 480z + 369z^2}{27z(4 - 9z^2)}, \quad V_1\left(x; \rho = 5\right) = -\frac{10368 - 18720z + 7825z^2}{125z(36 - 25z^2)}, \tag{32}$$

with the polynomial contributions

$$p_1^{(2)}(z; \rho = 3) = \frac{41}{27}\frac{1}{z}, \quad p_1^{(2)}(z; \rho = 5) = \frac{313}{125}\frac{1}{z}. \tag{33}$$

For sufficiently asymmetric choices of rational ratios of the masses this structure remains and is given by

$$V_1\left(z; \frac{m_1}{m_2} = \rho\right) = \frac{c_1(\rho) + c_2(\rho)z + c_3(\rho)z^2}{\rho^3 z\left[\left(1 - \rho^2\right)^2 - 16\rho^2 z^2\right]}. \tag{34}$$

Here the dependence on ρ in the denominator is easily visible. The functions $c_i(\rho)$, $i = 1, 2, 3$ are polynomials up to degree $d = 8$. One obtains

$$V_1(z; \rho) = \frac{-(1 - \rho^2)^4 + 8(1 - \rho^2)^2\rho(1 + \rho^2)z - 16\rho^2(1 + \rho^4)z^2}{2\rho^3 z[(1 - \rho^2)^2 - 16\rho^2 z^2]}. \tag{35}$$

To determine the polynomial coefficients for ρ in the numerators, one removes the denominator and solves the corresponding system of linear equations, which one extends as long as the coefficient

matrix is not degenerate. Using the variable $\bar{z}$ instead of z will give a similar result in the case of V_1. Finally, one transforms back from z to x, cf. (27).

In a similar way one proceeds for V_2 and V_3. Here we display the results for $\rho = 3$ only. The same structures with different values of the coefficients are obtained choosing other odd prime ratios ρ. Here and in the following we use the variable $\bar{z}$. One obtains

$$V_2(\bar{z}; \rho = 3) \;\; = \;\; -\frac{4P_5}{(2+\bar{z})^2(4+3\bar{z})} + \frac{P_6 P_7}{12(2+\bar{z})^6(4+3\bar{z})^{5/2}}, \tag{36}$$

with

$$
\begin{aligned}
P_5 &= 5\bar{z}^4 + 25\bar{z}^3 + 49\bar{z}^2 + 44\bar{z} + 16, & (37) \\
P_6 &= 8\bar{z}^4 + 40\bar{z}^3 + 73\bar{z}^2 + 56\bar{z} + 16, & (38) \\
P_7 &= 360\bar{z}^6 + 3720\bar{z}^5 + 15629\bar{z}^4 + 34588\bar{z}^3 + 42804\bar{z}^2 + 28192\bar{z} + 7744. & (39)
\end{aligned}
$$

The polynomial not obtained by guessing reads

$$p_2^{(2)}(\bar{z}; \rho = 3) = \frac{20}{9} - \frac{20}{3}\bar{z}. \tag{40}$$

The numerator polynomials of type P_5 have degree 5 and those of $P_6 \cdot P_7$ have 4×8 in ρ.

Likewise, one proceeds in the case of V_3.

$$
\begin{aligned}
V_3(\bar{z}; \rho = 3) \;\; = \;\; & -\frac{(4+3\bar{z})P_{10}}{27\bar{z}(1+\bar{z})(2+\bar{z})^5(2+3\bar{z})^5} + \frac{2P_9\,(4+3\bar{z})^{5/2}}{9\bar{z}(1+\bar{z})(2+\bar{z})^3(2+3\bar{z})^3} \\
& + \frac{2(4+3\bar{z})P_8}{9(2+\bar{z})(2+3\bar{z})}\frac{\operatorname{arcsinh}(\sqrt{\bar{z}})}{\sqrt{\bar{z}(1+\bar{z})}},
\end{aligned} \tag{41}
$$

with the polynomials

$$
\begin{aligned}
P_8 &= 64\bar{z}^4 + 128\bar{z}^3 + 48\bar{z}^2 - 16\bar{z} - 11 & (42) \\
P_9 &= 720\bar{z}^7 + 4120\bar{z}^6 + 8718\bar{z}^5 + 9055\bar{z}^4 + 4947\bar{z}^3 + 1363\bar{z}^2 + 156\bar{z} + 4 & (43) \\
P_{10} &= 277344\bar{z}^{13} + 3690576\bar{z}^{12} + 21772044\bar{z}^{11} + 75259086\bar{z}^{10} \\
& \quad + 169789579\bar{z}^9 + 263423972\bar{z}^8 + 288178779\bar{z}^7 + 223910983\bar{z}^6 + 122572992\bar{z}^5 \\
& \quad + 46092412\bar{z}^4 + 11326448\bar{z}^3 + 1657920\bar{z}^2 + 120832\bar{z} + 3072. & (44)
\end{aligned}
$$

The associated polynomial reads

$$p_3^{(2)}(\bar{z}; \rho = 3) = \frac{995}{216} - \frac{280333\bar{z}}{15552} - \frac{1526641\bar{z}^2}{23040}. \tag{45}$$

Some infinite sums are most economically calculated using relations of the type given in the appendix. Others are recognized as special $_pF_q$-functions, which finally reduce to elementary functions, cf. [28]. The numerator polynomials in ρ have a degree up to $d = 2$ for the arcsinh-term, $d = 3$ for the term proportional to $(\frac{1}{4}(\rho+1)^2 + \rho\bar{z})^{5/2}$, and $d = 12$ for the rational term. Finally, one substitutes back from $\bar{z}$ to x.

We mention that in [12, 15, 29], new special constants contribute which did not yet emerge in the above results. Terms resulting from the local-in-time dynamics introduce the new special constant π^2 from 3PN on which will show up in the post-Minkowskian series from 4PM on. The non-local-in-time dynamics, starting at 4PN, will introduce the new special constants

$\ln(2), \ln(3), \ln(5)$ etc., together with the Euler-Mascheroni constant[4], γ_E, the former of which are well-known to belong to the cyclotomic extension [30] of the multiple zeta values [31]. Especially they are related to the (linear combinations) of the digamma function $\psi(k/l)$, $k, l \in \mathbb{N} \backslash \{0\}$ or its derivatives. Constants of this type and their associated one–dimensional functions emerge in many massive higher order calculations in QCD, cf. [32, 33]. One should note that observables in the post-Newtonian approach, as e.g. $E(\omega)$, the energy-rotation frequency relation at the last stable orbit, contain also π^2 terms from 3PN onward, cf. [10]. At 3PN they cannot come from the tail terms. Additionally, from the non-local-in-time dynamics, the power expansion $(G_N/r)^n$ breaks down and also receives contributions of $O[(G_N/r)^n \ln(G_N/r)]$.

4   Conclusions

We have shown that one may determine the expansion coefficients of the potential in the post-Minkowskian approach from the velocity expansion in the effective field theory approach by finite terms, using very general mathematical algorithms. Here the approach works without any special further assumptions and even allows automation. In the two–mass case the choice of appropriate symmetric variables is important. The recurrences determining the solutions for the potential up to the 3PM are relatively compact compared to the ones needed in characteristic examples known from massless and massive calculations at three-loop order in QCD [20, 21]. Correspondingly, the solutions can be found by comparatively low numbers of initial values and the reconstruction of the solutions proceeds very fast and does not require large computational resources. The method of guessing [17], implemented in `Sage` [34], delivers the recurrences for the expansion coefficients for the potentials V_k, which can be solved using the package `Sigma` [18, 19]. The final reconstruction requires to perform one more infinite sum and the adjustment of one polynomial. This method can also be applied to the calculation of individual amplitudes, if their momentum expansion can be performed. Here one can refer to the respective differential equations in the expansion parameter and apply the method of large moments introduced in Ref. [21].

A   Appendix

Let us consider the calculation of the following typical example sums which appear in the reconstruction of V_3 in the unequal mass case

$$\sigma_1(x) = \sum_{n=1}^{\infty} n^2 \frac{(-x)^n}{2^{3+n}} \sum_{k=1}^{n} \frac{3^k \binom{2k}{k}}{(2k-1)2^{3k}}, \tag{46}$$

$$\sigma_2(x) = \sum_{n=1}^{\infty} \left(\frac{-x}{2} \right)^n \sum_{k=1}^{n} \frac{2^{3k}}{\binom{2k}{k}k}. \tag{47}$$

One uses the identity

$$\sum_{n=1}^{\infty} x^n \sum_{l=1}^{n} f(l) = \frac{1}{1-x} \sum_{n=1}^{\infty} f(n) x^n. \tag{48}$$

---

[4]Calculating loop-integrals in D dimensions in momentum space, γ_E, only appears in the spherical factor S_ε, which is set to unity at the end of the calculation working in the $\overline{\text{MS}}$ scheme. This seems not to be the case in gravity. Here the tail terms result in genuine γ_E contributions.

Relations of this kind were considered in [32, 35] before. Furthermore,

$$\sum_{n=1}^{\infty} n x^n f(n) = x \frac{d}{dx} \sum_{n=1}^{\infty} x^n f(n) \tag{49}$$

holds, which is used to absorb the powers of n.

One finally obtains

$$\sigma_1(x) = \frac{x\left\{32 + x\left[92 + 3x\left(12 - 3x + 8\sqrt{4+3x}\right) - 16\sqrt{4+3x}\right] - 64\sqrt{4+3x}\right\}}{32(2+x)^3(4+3x)^{3/2}}, \tag{50}$$

$$\sigma_2(x) = -\frac{4x}{\sqrt{1+x}(2+x)} \frac{\operatorname{arcsinh}[\sqrt{x}]}{\sqrt{x}}. \tag{51}$$

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