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Yang-Mills Theory**

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# Nonplanar Cusp and Transcendental Anomalous Dimension at Four Loops in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

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We compute the nonplanar contribution to the universal anomalous dimension of the  $SU(4)$ -singlet twist-two operators in  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory at four loops through Lorentz spin 18. From this, we numerically evaluate the nonplanar contribution to the four-loop lightlike cusp anomalous dimension and derive the transcendental  $\zeta_3$  and  $\zeta_5$  parts of the universal anomalous dimension for arbitrary Lorentz spin in analytic form. As for the lightlike cusp anomalous dimension and the  $\zeta_5$  part of the universal anomalous dimension, we confirm previous results.

The AdS/CFT correspondence [1–3], also known as holographic duality, has been one of the most active and tantalizing research topics in high-energy theory over the past two decades. This implies that quantum gravity in anti-de Sitter space, with constant negative curvature, is equivalent to a lower-dimensional nongravitational quantum field theory of conformal type,  $\mathcal{N} = 4$  supersymmetric Yang-Mills (SYM) theory, living on the boundary of that gravitational space. The AdS/CFT correspondence has led to a plethora of intriguing physical insights and powerful novel methods of calculation [4–16]. The latter allow us to solve longstanding problems not only in supersymmetric toy models, but also in real theories of nature, such as quantum chromodynamics (QCD) [17–19].

So far, investigations of the AdS/CFT correspondence have largely been confined to the planar limit, in which Feynman diagrams of planar topologies contribute, while nonplanar topologies are far more difficult to tackle. It is obviously of paramount interest to go beyond the planar limit, as this will allow us to significantly deepen and consolidate our understanding of the AdS/CFT correspondence and to access as-yet unexplored regions of it.

Quantities of key interest include the anomalous dimensions of the operators, composed of the quantum fields of  $\mathcal{N} = 4$  SYM theory, that are of leading twist, twist two, and are singlets under the internal symmetry group  $SU(4)$ . These operators are sorted by their Lorentz spin  $j$ , which counts the covariant derivatives, and are multiplicatively renormalized, sharing the same universal anomalous dimension  $\gamma_{\text{uni}}(j)$ , which just depends on  $j$ . Nonplanar contributions to the latter can be obtained by directly computing, by means of advanced computerized methods, the relevant Feynman diagrams in perturbation theory in powers of the gauge coupling  $g$ .

The study of the renormalization of composite operators in  $\mathcal{N} = 4$  SYM theory has led to the discovery of the relation of this problem with exactly solvable models [20].

The integrability in the planar limit was intensively studied and established from both sides of the AdS/CFT correspondence (see Ref. [15] for a review and Refs. [16, 21] for the recently developed quantum spectral curve approach). In the nonplanar case, integrability-based methods have been considered, in general, in Refs. [22, 23]. Nonplanar contributions to anomalous dimensions serve as a welcome laboratory for stringent tests of the ideas and models thus proposed. This provides a strong motivation for our present work.

Once a general result for the universal anomalous dimension is established, it is interesting to study its analytical properties and particular limits. The most interesting one,  $j \rightarrow \infty$ , yields the lightlike cusp anomalous dimension  $\gamma_{\text{cusp}}$  [24], which can be computed by alternative methods, too. The planar part of  $\gamma_{\text{cusp}}$  has been found to all orders a long time ago, through the asymptotic Bethe *Ansatz* equation [11]. Recently, its nonplanar part has been established through four loops, at  $\mathcal{O}(g^8)$ , via the Sudakov form factor, numerically in Refs. [25, 26] and analytically in Ref. [27], and via the lightlike polygonal Wilson loops, again analytically, in Ref. [28]. At four loops in QCD, at  $\mathcal{O}(\alpha_s^4)$  in the strong-coupling constant  $\alpha_s$ , the quark cusp anomalous dimension in the planar limit has been found via the quark form factor in Ref. [29], its contribution with quartic fundamental color factor has been obtained, again via the quark form factor, in Ref. [30], and the complete quark and gluon cusp anomalous dimensions have been established via their counterpart in  $\mathcal{N} = 4$  SYM theory in Ref. [28] and via the massless quark and gluon form factors in Ref. [31].

Explicit knowledge of  $\gamma_{\text{uni}}(j)$  for general value of  $j$  would unfold the nonplanar anatomy of the anomalous dimensions in  $\mathcal{N} = 4$  SYM theory. A possible avenue to this goal is to evaluate  $\gamma_{\text{uni}}(j)$  for as many values of  $j$  as possible and to try and extract from this the general result. In  $\mathcal{N} = 4$  SYM theory, nonplanarity appears for the first time at  $\mathcal{O}(g^8)$ . In Refs. [32–34], the nonplanar

contributions to  $\gamma_{\text{uni}}(j)$  at  $\mathcal{O}(g^8)$  were analytically calculated for the first three nontrivial values  $j = 4, 6, 8$ . Recently, these results have been confirmed and extended to  $j = 10$  applying the method of asymptotic expansions to the four-point functions of length-two half-Bogomol'nyi-Prasad-Sommerfield operators [35]. The purpose of this Letter is to push this endeavor as far as possible, which turns out to be through  $j = 18$ , thanks to cutting-edge technology and computing power. We will thus be able to reconstruct the general coefficients of  $\zeta_3$  and  $\zeta_5$  and to obtain an independent numerical result for  $\gamma_{\text{cusp}}$  at  $\mathcal{O}(g^8)$ . The former is new, and the latter confirm previous findings in Ref. [32] and Refs. [25–28], respectively.

Specifically, the set of local, gauge-invariant, SU(4)-singlet, twist-two operators of definite Lorentz spin  $j$  in  $\mathcal{N} = 4$  SYM theory reads:

$$\mathcal{O}_{\mu_1, \dots, \mu_j}^\lambda = \hat{S} \bar{\lambda}_i^a \gamma_{\mu_1} \mathcal{D}_{\mu_2} \cdots \mathcal{D}_{\mu_j} \lambda^{a, i}, \quad (1)$$

$$\mathcal{O}_{\mu_1, \dots, \mu_j}^g = \hat{S} G_{\rho\mu_1}^a \mathcal{D}_{\mu_2} \mathcal{D}_{\mu_3} \cdots \mathcal{D}_{\mu_{j-1}} G_{\mu_j}^{a, \rho}, \quad (2)$$

$$\mathcal{O}_{\mu_1, \dots, \mu_j}^\phi = \hat{S} \bar{\phi}_r^a \mathcal{D}_{\mu_1} \mathcal{D}_{\mu_2} \cdots \mathcal{D}_{\mu_j} \phi^{a, r}, \quad (3)$$

where the spinors  $\lambda_i$  refer to the gauginos, the field strength tensor  $G_{\mu\nu}$  to the gauge fields,  $\phi_r$  are the complex scalar fields of extended supersymmetry, and  $\mathcal{D}_\mu$  are covariant derivatives. The indices  $i = 1, \dots, 4$  and  $r = 1, \dots, 3$  refer to the SU(4) and SO(6)  $\simeq$  SU(4) groups of internal symmetry, respectively. The symbol  $\hat{S}$  implies a symmetrization of the respective tensor in the Lorentz indices  $\mu_1, \dots, \mu_j$  and a subtraction of all its possible traces. As mentioned above, these operators form the multiplicatively renormalized operators, whose anomalous dimensions are expressed through the so-called universal anomalous dimension up to integer argument shifts [36],

$$\gamma_{\text{uni}}(j) = \sum_{n=1}^{\infty} \gamma_{\text{uni}}^{(n-1)}(j) g^{2n}, \quad g^2 = \frac{\lambda}{16\pi^2}, \quad (4)$$

where  $\lambda = g_{\text{YM}}^2 N_c$  is the 't Hooft coupling constant.

In the planar limit,  $\gamma_{\text{uni}}(j)$  is analytically known for arbitrary  $j$  through seven loops [37–42] and for special values of  $j$  even through ten loops [43–50], e.g. for  $j = 4$  corresponding to the Konishi operator. In the latter case, we quote the result through four loops [43–45] here

$$\gamma_{\text{Konishi, planar}} = \gamma_{\text{uni, planar}}(4) = 12g^2 - 48g^4 + 336g^6 + 96g^8(-26 + 6\zeta_3 - 15\zeta_5) + \mathcal{O}(g^{10}). \quad (5)$$

As for the nonplanar contributions to  $\gamma_{\text{uni}}(j)$  at  $\mathcal{O}(g^8)$ , the state of the art is given by [32–34]

$$\gamma_{\text{uni, np}}^{(3)}(4) = -360\zeta_5 \frac{48}{N_c^2}, \quad (6)$$

$$\gamma_{\text{uni, np}}^{(3)}(6) = \frac{25}{9}(21 + 70\zeta_3 - 250\zeta_5) \frac{48}{N_c^2}, \quad (7)$$

$$\gamma_{\text{uni, np}}^{(3)}(8) = \frac{49}{600}(1357 + 4340\zeta_3 - 11760\zeta_5) \frac{48}{N_c^2}, \quad (8)$$

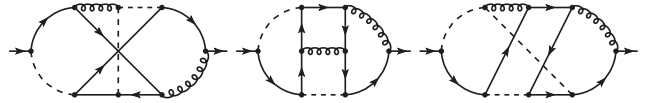


FIG. 1: Typical Feynman diagrams contributing to  $\gamma_{\text{uni, np}}^{(3)}(j)$ . The operators are inserted in the lines or gauge vertices.

where we have pulled out common factors. If such a factorization were preserved for the higher  $j$  values, this could considerably simplify the procedure of finding the general form of  $\gamma_{\text{uni, np}}^{(3)}(j)$ . In fact, the prefactors in Eqs. (6)–(8) resemble the harmonic sums  $\sum_{i=1}^{j-2} \frac{1}{i}$  for  $j = 4, 6, 8$ , with values  $3/2, 25/12, 49/20$ , and harmonic sums are also expected to appear as building blocks of  $\gamma_{\text{uni, np}}^{(3)}(j)$ , as explained below.

In this Letter, we extend Eqs. (6)–(8) by the next five terms. Our computational procedure is similar to Refs. [32–34]. We work in the programming language FORM [51]. Specifically, we generate all the contributing Feynman diagrams with DIANA [52] based on QGRAF [53], evaluate the color traces with COLOR [54], reduce the occurring scalar integrals to the master integrals of Ref. [55] with the custom-made program package BAMBA based on the Laporta algorithm [56], and reduce the propagator-type diagrams to fully massive tadpole diagrams using infrared rearrangement [57] (see also Refs. [55, 58, 59] for details). Typical Feynman diagrams are depicted in Fig. 1. This setup allows us to proceed to  $j = 10$  only. Further progress is enabled by the recently developed package FORCER [60] based on FORM [51], which was also used to compute anomalous dimensions of twist-two operators in QCD [61, 62]. We may thus proceed to  $j = 18$ , where about one year of running on the modern high-performance computing facilities available to us is required. Altogether, we have

$$\gamma_{\text{uni, np}}^{(3)}(10) = \left( \frac{220854227}{1411200} + \frac{27357}{56}\zeta_3 - \frac{579121}{490}\zeta_5 \right) \frac{48}{N_c^2}, \quad (9)$$

$$\gamma_{\text{uni, np}}^{(3)}(12) = \left( \frac{28337309747461}{144027072000} + \frac{345385183}{571536}\zeta_3 - \frac{54479161}{39690}\zeta_5 \right) \frac{48}{N_c^2}, \quad (10)$$

$$\gamma_{\text{uni, np}}^{(3)}(14) = \left( \frac{9657407179406311}{4149351360000} + \frac{158654990663}{224532000}\zeta_3 - \frac{7399612441}{4802490}\zeta_5 \right) \frac{48}{N_c^2}, \quad (11)$$

$$\gamma_{\text{uni, np}}^{(3)}(16) = \left( \frac{74429504651244877}{280496151936000} + \frac{205108095887}{256864608}\zeta_3 - \frac{1372958223289}{811620810}\zeta_5 \right) \frac{48}{N_c^2}, \quad (12)$$

$$\gamma_{\text{uni,np}}^{(3)}(18) = \left( \frac{8122582838282649980649377}{27516111512617728000000} \right. \quad (13)$$

$$\left. + \frac{72169501556777041}{81811377648000} \zeta_3 - \frac{5936819760481}{3246483240} \zeta_5 \right) \frac{48}{N_c^2}.$$

Our result for  $j = 10$  agrees with Ref. [35]. In contrast to Eqs. (6)–(8), it is not possible to extract common factors in Eqs. (9)–(13), so that our expectations regarding factorization have to be dropped.

Equipped with the information contained in Eqs. (6)–(13), we now try to reconstruct the general form of  $\gamma_{\text{uni,np}}^{(3)}(j)$ , i.e. to determine the  $j$  dependence of the coefficients of  $\zeta_5$  and  $\zeta_3$  and the rational reminder in the *Ansatz*

$$\gamma_{\text{uni,np}}^{(3)}(j) = \left[ \gamma_{\text{uni,np},\zeta_5}^{(3)}(j) \zeta_5 + \gamma_{\text{uni,np},\zeta_3}^{(3)}(j) \zeta_3 \right. \quad (14)$$

$$\left. + \gamma_{\text{uni,np,rational}}^{(3)}(j) \right] \frac{48}{N_c^2}.$$

For this purpose, we adopt a powerful method based on number theory which has been proposed in Ref. [63] and successfully applied to the reconstruction of anomalous dimensions in  $\mathcal{N} = 4$  SYM theory [41, 42, 64] and QCD in Ref. [65] and after in Refs. [61, 66]. This method is based on the assumption that  $\gamma_{\text{uni,np},\zeta_5}^{(3)}(j)$ ,  $\gamma_{\text{uni,np},\zeta_3}^{(3)}(j)$ , and  $\gamma_{\text{uni,np,rational}}^{(3)}(j)$  in Eq. (14) are linear combinations of certain basis functions with certain coefficients. As for the basis functions and coefficients, we are guided by several heuristic observations.

As for the basis functions, in the case of anomalous dimensions of twist-two operators in  $\mathcal{N} = 4$  SYM theory, these are known to be generalized harmonic sums, defined as [67, 68]

$$S_{a_1, \dots, a_n}(M) = \sum_{j=1}^M \frac{[\text{sign}(a_1)]^j}{j^{|a_1|}} S_{a_2, \dots, a_n}(j), \quad (15)$$

where the indices  $a_1, \dots, a_n$  may take all (positive and negative) integer values, except for  $-1$ . The weight or *transcendentality*  $\ell$  of the sum  $S_{a_1, \dots, a_n}$  is defined as the sum of the absolute values of its indices,  $\ell = |a_1| + \dots + |a_n|$ , and the weight of a product of generalized harmonic sums is equal to the sum of their weights.

For twist-two operators, there is an additional simplification, thanks to the so-called generalized Gribov-Lipatov reciprocity [69–72], which reflects the symmetry of the underlying processes under the crossing of scattering channels. As a consequence, the harmonic sums can enter the anomalous dimensions only in the form of special combinations satisfying the above-mentioned property by themselves. In practice, this allows us to impose restrictions on the choice of basis functions leaving us with a smaller number of so-called binomial harmonic

sums, defined as [67]

$$S_{a_1, \dots, a_n}(N) = \sum_{j=1}^N (-1)^{j+N} \binom{N}{j} \binom{N+j}{j} S_{a_1, \dots, a_n}(j). \quad (16)$$

They only have positive-integer indices, while their transcendentality is the same as for usual harmonic sums. There are  $2^{\ell-1}$  binomial harmonic sums at transcendentality  $\ell$ . While reciprocity has not yet been rigorously proven, counterexamples have not been encountered either. In particular, the anomalous dimensions of the twist-two operators in the planar limit of  $\mathcal{N} = 4$  SYM theory, which are known through seven loops, may all be represented in terms of binomial harmonic sums and their derivatives. In the planar limit of QCD, the non-singlet quark anomalous dimensions at four loops can also be written in terms of binomial harmonic sums and their derivatives [61]. Furthermore, the Feynman diagrams contributing to  $\gamma_{\text{uni,np}}^{(3)}(j)$  at subleading order in  $N_c$  also contribute to its planar counterpart, as may be seen from Fig. 1. These observations suggest that reciprocity should work for the case at hand, which is also confirmed by a nontrivial self-consistency test within our calculation, as explained below.

According to the maximal-transcendentality principle [36], the anomalous dimensions of twist-two operators at  $\ell$ th order in  $\mathcal{N} = 4$  SYM theory are of transcendentality  $2\ell - 1$ , which is seven for our case of  $\ell = 4$ . Thus,  $\gamma_{\text{uni,np},\zeta_5}^{(3)}(j)$ ,  $\gamma_{\text{uni,np},\zeta_3}^{(3)}(j)$ , and  $\gamma_{\text{uni,np,rational}}^{(3)}(j)$  in Eq. (14) are of transcendentality 2, 4, and 7; i.e., they are composed of 2, 8, and 64 binomial harmonic sums of the respective transcendentality.

As for the coefficients in front of the basis functions, inspection of the expressions of the  $j$ -dependent anomalous dimensions that are already known reveals that they are usually small integer numbers. So, in general, we obtain a system of Diophantine equations. If the number of equations is equal to the number of variables, then we can solve such a system exactly. However, this requires the knowledge of the anomalous dimensions for a large number of fixed  $j$  values. Fortunately, the system of Diophantine equations can be solved with the help of special methods from number theory even if the number of equations is less than the number of variables. In fact, we may then apply the Lenstra-Lenstra-Lovasz algorithm [73], which allows us to reduce the matrix obtained from the system of Diophantine equations to a form in which the rows are the solutions of the system with the minimal Euclidean norm.

Equation (6) is sufficient to fix the two coefficients in the *Ansatz* for  $\gamma_{\text{uni,np},\zeta_5}^{(3)}(j)$ . The result,

$$\gamma_{\text{uni,np},\zeta_5}^{(3)}(j) = -40 S_1^2(j-2), \quad (17)$$

thus obtained a long time ago [32] has been confirmed by all subsequent results in Eqs. (7)–(13). To determine the

eight coefficients in the *Ansatz* for  $\gamma_{\text{uni,np},\zeta_3}^{(3)}(j)$ , we need five input relations. Using Eqs. (6)–(10), we find

$$\gamma_{\text{uni,np},\zeta_3}^{(3)}(j) = 8(8\mathbb{S}_4 - 9\mathbb{S}_{1,3} - 3\mathbb{S}_{2,2} - 4\mathbb{S}_{3,1} + 4\mathbb{S}_{1,1,2} + 5\mathbb{S}_{1,2,1} - \mathbb{S}_{2,1,1}), \quad (18)$$

where  $\mathbb{S}_{\mathbf{a}} = \mathbb{S}_{\mathbf{a}}(j-2)$ , which is in agreement with Eqs. (11)–(13). By the way, this nicely supports our suggestion that reciprocity should work for  $\gamma_{\text{uni,np}}^{(3)}(j)$  as well. At any rate, the eight inputs from Eqs. (6)–(13) uniquely fix Eqs. (17) and (18). Unfortunately, these inputs do not yet suffice to determine the coefficients of the 64 binomial harmonic sums of transcendentality 7 in  $\gamma_{\text{uni,np,rational}}^{(3)}(j)$  beyond all doubt via the number theoretical procedure outlined above.

Nevertheless, we may exploit the information encoded in Eqs. (6)–(13) to numerically recover the nonplanar contribution to the cusp anomalous dimension with useful precision. To this end, we proceed along the lines of Refs. [61, 62] and approximately reconstruct the four-loop splitting function. We recall that the  $n$ -loop splitting function  $P^{(n)}(x)$  is related to the anomalous dimension of the respective twist-two spin- $j$  operator, with  $j = 2, 4, 6, \dots$ , by a Mellin transformation,

$$\gamma^{(n)}(j) = - \int_0^1 dx x^{j-1} P^{(n)}(x), \quad (19)$$

where the negative sign is a standard convention.

In QCD, the diagonal splitting functions at  $n$  loops, in general, assume the following form in the limit  $x \rightarrow 1$  [71]:

$$P_{kk}^{(n-1)}(x) = \frac{A_k^{(n)}}{(1-x)_+} + B_k^{(n)} \delta(1-x) + C_k^{(n)} \ln(1-x) + D_k^{(n)} + \mathcal{O}[(1-x) \ln^\ell(1-x)], \quad (20)$$

where  $k = q, g$  and the  $+$  distribution is defined as usual,  $\int_0^1 dx f(x)/(1-x)_+ = \int_0^1 dx [f(x) - f(1)]/(1-x)$ .  $A_q^{(n)}$  and  $A_g^{(n)}$  are the  $n$ -loop quark and gluon cusp anomalous dimensions, respectively [24]. In  $\mathcal{N} = 4$  SYM theory, the splitting functions, being related to the anomalous dimensions through the Mellin transformation in Eq. (19), satisfy the maximal-transcendentality principle [36], and we may use Eq. (20), with  $kk$  replaced by np. Since  $C_k^{(n)}$  and  $D_k^{(n)}$  can be predicted from lower-order information [71] and nonplanarity appears for the first time at  $n = 4$ , we have  $C_{\text{np}}^{(4)} = D_{\text{np}}^{(4)} = 0$ . Following Refs. [61, 62], we make *Ansätze* for approximations of the splitting function  $P_{\text{uni,np}}^{(3)}(x)$ , which consist of (i) the two large- $x$  parameters  $A_{\text{np}}^{(4)}$  and  $B_{\text{np}}^{(4)}$ , (ii) two out of the three large- $x$  logarithms  $(1-x) \ln^k(1-x)$  with  $k = 1, 2, 3$ , (iii) two out of the three small- $x$  logarithms  $\ln^k x$  with  $k = 1, 2, 3$ , and (iv) two out of the five polynomials  $(1-x)x^k$  with  $k = 0, \dots, 4$ . These are  $\binom{3}{2} \binom{3}{2} \binom{5}{2} = 90$  trial functions with eight coefficients

each, which we pin down using the eight available inputs in Eqs. (6)–(13). For each coefficient, we determine, from the values thus resulting, the central value to be half of the sum of the largest and smallest ones and the error to be half of their difference. Using all 90 solutions, we find  $A_{\text{np}}^{(4)} = -48 \times (98.1 \pm 5.8)$  and  $B_{\text{np}}^{(4)} = 48 \times (203.6 \pm 32.4)$ . We may considerably improve these results by rejecting 20 unlikely solutions, involving particularly large coefficients, to obtain  $A_{\text{np}}^{(4)} = -48 \times (97.5 \pm 0.6)$  and  $B_{\text{np}}^{(4)} = 48 \times (207.0 \pm 3.0)$ . The former result nicely agrees with the one from Refs. [27, 28, 31],  $A_{\text{np}}^{(4)} = -48(992\pi^6/315 + 1152\zeta_3^2) = -48 \times 97.75$ , while the latter is new. We emphasize that our method of computation is completely independent from Refs. [25–28, 31]. Our final result for the cusp anomalous dimension through four loops is

$$\gamma_{\text{cusp}} = 8g^2 - 26.32g^4 + 190.49g^6 - \left(1874.86 + (97.5 \pm 0.6) \frac{48}{N_c^2}\right)g^8 + \mathcal{O}(g^{10}). \quad (21)$$

To summarize, using modern computational techniques, we have considerably advanced our knowledge of the nonplanar sector of  $\mathcal{N} = 4$  SYM theory by studying the universal anomalous dimension of the local, gauge-invariant, SU(4)-singlet, twist-two operators of definite Lorentz spin  $j$  at four loops. Specifically, we have pushed the state of the art from  $j = 10$  [32–35] to  $j = 18$  upon providing a first independent confirmation of the recent result for  $j = 10$  [35]. The four new terms for  $j = 12, \dots, 18$  are all in agreement with the generic coefficient of  $\zeta_5$  already derived in Ref. [32]. The new information allowed us to uniquely determine also the generic coefficient of  $\zeta_3$ , but it does not yet suffice to pin down the generic expression of the rational term. However, following Refs. [61, 62], we managed to find a rather precise numerical result for the  $j \rightarrow \infty$  limit of the universal anomalous dimension by considering the  $x \rightarrow 1$  limit of the corresponding splitting function. The result for the cusp anomalous dimension thus obtained agrees with previous determinations based on very different approaches [25–28, 31].

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