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Approximate four-loop QCD corrections to the Higgs-boson production cross section

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Abstract

We study the soft and collinear (SV) contributions to inclusive Higgs-boson production in gluon-gluon fusion at four loops. Using recent progress for the quark and gluon form factors and Mellin moments of splitting functions, we are able to complete the soft-gluon enhanced contributions exactly in the limit of a large number of colours, and to a sufficiently accurate numerical accuracy for QCD. The four-loop SV contributions increase the QCD cross section at 14 TeV by 2.7% and 0.2% for the standard choices $\mu_R = m_H$ and $\mu_R = m_H/2$ of the renormalization scale, and reduce the scale uncertainty to below $\pm 3\%$. As by-products, we derive the complete $\delta(1-x)$ term for the gluon-gluon splitting function at four loops and its purely Abelian contributions at five loops, and provide a numerical result for the single pole of the four-loop gluon form factor in dimensional regularization. Finally we present the closely related fourth-order coefficients D_4 for the soft-gluon exponentiation of Higgs-boson and Drell-Yan lepton-pair production.

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The production of the Standard Model (SM) Higgs boson in proton-proton collisions and its subsequent decay are flagship measurements in run 2 of the Large Hadron Collider (LHC) [1, 2]. The main production mechanism for $pp \to H + X$ is the gluon-gluon fusion (ggF) process. The corresponding inclusive cross section serves as a benchmark for the achieved accuracy, both in the LHC experiments and for theoretical research. The radiative corrections in Quantum Chromodynamics (QCD) for the ggF process are large and have motivated significant efforts to improve the precision of the predictions. The QCD corrections are currently known to the next-to-next-to-next-to-leading order (N^{3}LO) in the effective theory for a large top-quark mass, $m_t \gg m_H$ [3, 4], and to next-to-next-to-leading order (NNLO) in the full theory for Higgs-boson masses $m_H \lesssim 2\, m_t$ [5–7].

Near the production threshold, for $z = m_H^2/\hat{s}$ close to unity, where m_H is the Higgs mass and $\sqrt{\hat{s}}$ the partonic center-of-mass energy, the QCD corrections to the ggF process are dominated by the well-known large logarithmic corrections. At n-th order they appear in the partonic cross section in the $\overline{\text{MS}}$ scheme as plus-distributions $\mathcal{D}_k = [(1-z)^{-1} \ln^k (1-z)]_+$ with $2n-1 \geq k \geq 0$, while the virtual contributions lead to $\delta(1-z)$ terms. In Mellin N-space, where N is the conjugate variable of z, the threshold logarithms read $\ln^k N$ with $2n \geq k \geq 1$, and the virtual contributions lead to a constant in N. The soft-virtual (SV) approximation to the partonic ggF cross section in N-space yields reliable predictions for the total Higgs production cross section, as has been demonstrated with comparisons to exact fixed-order results up to NNLO, see, e.g., Refs. [8, 9]. In addition, Mellin N-space lends itself to an all-order exponentiation of threshold contributions up to next-to-next-to-next-to-leading logarithmic (N^{3}LL) accuracy and beyond [10, 11].

These facts motivate the derivation of approximate QCD corrections to the ggF process at four loops in the effective theory, which can be achieved thanks to recent progress in the computation of QCD corrections for related quantities at the four-loop level. This comprises results for specific colour contributions, including quartic group invariants, and the planar limit of quark and gluon form factors [12–18], correlators of Wilson lines [19–21], splitting functions for the evolution of parton distributions (PDFs) [22–25], and, related, the knowledge of a low number of Mellin moments for the structure functions in deep-inelastic scattering (DIS), see Ref. [26].

Taken together, this knowledge enables us to determine precise numerical results for the complete SV approximation of the ggF process at four loops as well as partial information on terms suppressed by a power $1/N$ in Mellin N-space using physical evolution kernels at the same order [9]. The results are used to provide new predictions for the ggF cross section at the collision energy of 14 TeV, as planned for run 3 of the LHC. We also present the corresponding expression for the Drell-Yan (DY) process, $pp \to \gamma^* + X$, which is closely related to the ggF process in the threshold limit, and the N^{4}LL soft-gluon exponentiation coefficient D_4 for both processes. As by-products, we obtain a complete result for the so-called gluon virtual anomalous dimension, i.e. the $\delta(1-z)$ terms of the gluon-gluon splitting function at four loops in QCD, together with partial information at five loops, and we derive a numerical result for the single pole $1/\varepsilon$ of the dimensionally regulated gluon form factor at four loops.

The effective coupling of the Higgs boson to partons is described by the Lagrangian

$$\mathcal{L}_{\text{eff}} \;=\; -\frac{1}{4\upsilon}\, C(\mu_R^2)\, H\, G^a_{\mu\nu} G_a^{\mu\nu} \;, \tag{1}$$

where $\upsilon \simeq 246$ GeV is the Higgs vacuum expectation value in the SM and $G^a_{\mu\nu}$ denotes the gluon

field strength tensor. The inclusive hadronic cross section for Higgs-boson production at a center-of-mass energy $E_{cm} = \sqrt{S}$ is given in standard QCD factorization by

$$\sigma(S, m_H^2) \;=\; \tau \sum_{a,b} \int_0^1 \frac{dx_1}{x_1}\, \frac{dx_2}{x_2}\, f_{a/h_1}(x_1, \mu_F^2)\, f_{b/h_2}(x_2, \mu_F^2) \int_0^1 dz\, \delta\Big(z - \frac{\tau}{x_1 x_2}\Big) \times \\ \times\, \widetilde{\sigma}_0^H\, c_{ab}^H(z, \alpha_s(\mu_R^2), m_H^2/\mu_R^2, m_H^2/\mu_F^2)\;, \tag{2}$$

where $\tau = m_H^2/S$, and μ_F, μ_R are the mass-factorization and renormalization scales, and $f_{a/h}(x, \mu_F^2)$ the PDFs of the proton. The expansion in the strong coupling α_s of the large-m_t effective vertex for the Higgs coupling to gluons is included in $\widetilde{\sigma}_0^H$, viz

$$\widetilde{\sigma}_0^H \;=\; \frac{\pi C(\mu_R^2)^2}{8\, n_A\, \upsilon^2} \quad \text{with} \quad C(\mu_R^2) \;=\; -\frac{\alpha_s(\mu_R^2)}{3\pi} \Big\{ 1 + 11\, \frac{\alpha_s(\mu_R^2)}{4\pi} + \ldots \Big\}\;, \tag{3}$$

where $n_A = (n_c^2 - 1)$ denotes the dimension of the adjoint representation of the $SU(n_c)$ gauge group, and the matching coefficient $C(\mu_R^2)$ is fully known up to N^{4}LO in QCD ($n_c = 3$) [27][4]. The coefficient functions c_{ab}^H are expanded in powers of $a_s \equiv \alpha_s(\mu_R^2)/(4\pi)$,

$$c_{ab}^H(z, \alpha_s(\mu_R^2), m_H^2/\mu_R^2, m_H^2/\mu_F^2) \;=\; \sum_{n=0} a_s^n c_{ab}^{H,(n)}(z, m_H^2/\mu_R^2, m_H^2/\mu_F^2)\;, \tag{4}$$

where the leading order (LO) is $c_{ab}^{H,(0)} = \delta_{ag}\, \delta_{bg}\, \delta(1-z)$ and the corrections to N^{3}LO have been computed in Refs. [3, 4][5]. For $c_{gg}^{H,(4)}(z)$, at N^{4}LO, seven of the eight plus-distributions of the SV approximation are known. The coefficients of $\mathcal{D}_k$ for $7 \geq k \geq 2$ can be inferred from Ref. [8] and have been written down in Eq. (16) of Ref. [38] and that of $\mathcal{D}_1$ is fixed by the results of Ref. [39] and has been given in Eq. (13) of Ref. [40]. An approximate result for the $\mathcal{D}_0$ term has been provided before in Eqs. (2.13) and (2.14) of Ref. [9].

Here we present a new result for the latter coefficient for a general gauge group. The relevant Casimir invariants for $SU(n_c)$ are $C_A = n_c$, $C_F = (n_c^2 - 1)/(2n_c)$ and

$$\frac{d_A^{abcd} d_A^{abcd}}{n_A} \;=\; \frac{1}{24}\, n_c^2 (n_c^2 + 36)\;, \qquad \frac{d_F^{abcd} d_A^{abcd}}{n_A} \;=\; \frac{1}{48}\, n_c (n_c^2 + 6)\;. \tag{5}$$

With the recent progress at four loops on the pole structure of the QCD form factors, on splitting functions and on Mellin moments for DIS structure functions, and following the same procedure as employed for DIS structure functions in Ref. [11], the $\mathcal{D}_0$ term in $c_{gg}^{H,(4)}(z)$ can now be given as

$$c_{gg}^{H,(4)}\Big|_{\mathcal{D}_0} = \\ C_A^4 \Big(-\frac{40498399}{2187} + \frac{28613426}{729}\, \zeta_2 + \frac{10995352}{81}\, \zeta_3 + \frac{2598712}{81}\, \zeta_4 - \frac{7252952}{27}\, \zeta_2 \zeta_3 \\ + \frac{3411280}{9}\, \zeta_5 - \frac{656216}{3}\, \zeta_3^2 + \frac{1019381}{9}\, \zeta_6 - \frac{293488}{3}\, \zeta_3 \zeta_4 - 561344\, \zeta_2 \zeta_5 + 986440\, \zeta_7$$

[4] See Refs. [28–30] for previous work up to N^{3}LO in QCD.

[5] See Refs. [31–34] and [35–37] for previous work at lower orders in QCD.

$$
\begin{aligned}
&+\frac{1}{12}\, f^{\rm q}_{4,d_F^{abcd}d_A^{abcd}}\Big) + \frac{d_A^{abcd}d_A^{abcd}}{n_A}\Big(-2\, f^{\rm q}_{4,d_F^{abcd}d_A^{abcd}}\Big) + C_A^3 n_f \Big(\frac{17665315}{2916} - \frac{10870138}{729}\,\zeta_2 \\
&-\frac{3234580}{81}\,\zeta_3 - \frac{364960}{81}\,\zeta_4 + \frac{520864}{9}\,\zeta_2\zeta_3 - \frac{2216816}{27}\,\zeta_5 + \frac{263864}{9}\,\zeta_3^2 - \frac{578258}{27}\,\zeta_6 \\
&- b^{\rm q}_{4,n_f C_F^3} - 2\, b^{\rm q}_{4,n_f C_F^2 C_A} - \frac{1}{12}\, b^{\rm q}_{4,d_F^{abcd}d_F^{abcd}}\Big) + C_A^2 C_F n_f \Big(\frac{2798681}{486} - \frac{39658}{9}\,\zeta_2 \\
&-\frac{367508}{27}\,\zeta_3 - \frac{130640}{27}\,\zeta_4 + \frac{17120}{9}\,\zeta_2\zeta_3 + \frac{21904}{9}\,\zeta_5 + \frac{34064}{3}\,\zeta_3^2 - 988\,\zeta_6 + 4\, b^{\rm q}_{4,n_f C_F^2 C_A}\Big) \\
&+C_A C_F^2 n_f \Big(-\frac{27949}{54} - 632\,\zeta_2 + \frac{2240}{9}\,\zeta_3 + 668\,\zeta_4 + \frac{1024}{3}\,\zeta_2\zeta_3 - \frac{7744}{3}\,\zeta_5 \\
&-736\,\zeta_3^2 + \frac{29336}{9}\,\zeta_6 + 4\, b^{\rm q}_{4,n_f C_F^3}\Big) + n_f \frac{d_F^{abcd}d_A^{abcd}}{n_A}\Big(768 - \frac{9088}{3}\,\zeta_2 + \frac{10624}{9}\,\zeta_3 \\
&+\frac{1600}{3}\,\zeta_4 - 256\zeta_2\zeta_3 + \frac{43520}{9}\,\zeta_5 - \frac{2432}{3}\,\zeta_3^2 - \frac{2368}{9}\,\zeta_6 + 4\, b^{\rm q}_{4,d_F^{abcd}d_F^{abcd}}\Big) \\
&-C_A^2 n_f^2 \Big(\frac{1543153}{2916} - \frac{1171400}{729}\,\zeta_2 - \frac{176624}{81}\,\zeta_3 - \frac{1168}{9}\,\zeta_4 + \frac{71200}{27}\,\zeta_2\zeta_3 - \frac{34592}{9}\,\zeta_5\Big) \\
&-C_A C_F n_f^2 \Big(\frac{155083}{243} - \frac{5600}{9}\,\zeta_2 - \frac{4784}{9}\,\zeta_3 - \frac{160}{3}\,\zeta_4 + \frac{1280}{3}\,\zeta_2\zeta_3 - 32\,\zeta_5\Big) \\
&+C_A n_f^3 \Big(\frac{10432}{2187} - \frac{3200}{81}\,\zeta_2 - \frac{3680}{81}\,\zeta_3 + \frac{112}{9}\,\zeta_4\Big)\,. \qquad (6)
\end{aligned}
$$

Here the term $f^{\rm q}_{4,d_F^{abcd}d_A^{abcd}}$ is related to the eikonal anomalous dimension of the four-loop quark form factor, i.e., to the single pole in dimensional regularization, cf. Ref. [11]. The expressions $b^{\rm q}_{4,d_F^{abcd}d_F^{abcd}}$, $b^{\rm q}_{4,n_f C_F^2 C_A}$ and $b^{\rm q}_{4,n_f C_F^3}$ denote the four-loop coefficients of the respective colour factor in the quark virtual anomalous dimension $B^{\rm q}$, i.e.,the coefficient of $\delta(1-z)$ in the quark-quark splitting function P_{qq}. At n-th order, expanding in powers of $a_{\rm s} \equiv \alpha_{\rm s}(\mu_R^2)/(4\pi)$ analogous to Eq. (4), the flavor-diagonal splitting functions P_{ii} admit the large-z expansion [41,42] as

$$
P_{ii}^{(n-1)}(z) \;=\; A_n^{\rm i}\,\mathcal{D}_0 + B_n^{\rm i}\,\delta(1-z) + \ldots\,, \qquad i = {\rm q}, {\rm g}\,, \qquad (7)
$$

where the coefficients $A_n^{\rm i}$ are the well-known lightlike cusp anomalous dimensions. The quantities $f^{\rm q}_{4,d_F^{abcd}d_A^{abcd}}$, $b^{\rm q}_{4,d_F^{abcd}d_F^{abcd}}$, $b^{\rm q}_{4,n_f C_F^2 C_A}$ and $b^{\rm q}_{4,n_f C_F^3}$ are not known analytically. They drop out in the large-n_c limit of Eq. (6). Precise numerical estimates, i.e., $b^{\rm q}_{4,d_F^{abcd}d_F^{abcd}} = -143.6 \pm 0.2$, $b^{\rm q}_{4,n_f C_F^2 C_A} = -455.247 \pm 0.005$ and $b^{\rm q}_{4,n_f C_F^3} = 80.780 \pm 0.005$ have been given in Ref. [11] together with $f^{\rm q}_{4,d_F^{abcd}d_A^{abcd}} = -100 \pm 100$.

While the large-n_c limit of Eq. (6) is exact, the above general expression uses one assumption on the relation of quark and gluon form factors $\mathcal{F}^{\rm q}$ and $\mathcal{F}^{\rm g}$ in QCD in dimensional regularization, which, in the normalization of Eq. (4), admit the perturbative expansion as

$$
\mathcal{F}^p \;=\; 1 + a_{\rm s}\left(-\frac{1}{2\varepsilon^2}\,A_1^p - \frac{1}{2\varepsilon}\,G_1^p\right) + O(a_{\rm s}^2) \qquad i = {\rm q}, {\rm g}\,, \qquad (8)
$$

see e.g. Refs. [11,43] for the higher orders.

The logarithms of $\mathcal{F}^{\rm q}$ and $\mathcal{F}^{\rm g}$ contain double and single poles in ε, the former being controlled by their respective cusp anomalous dimensions, $A^{\rm q}$ and $A^{\rm g}$, cf. Eq. (7), which exhibit generalized Casimir scaling through four loops, see, e.g. [18,24]. The single poles on the other hand, which are proportional to functions $G_n^p(\varepsilon)$ at n-th order, are controlled by the collinear anomalous dimensions and can be converted to appropriate eikonal (Wilson line) quantities, after separation of the virtual anomalous dimensions $B^{\rm q}$ and $B^{\rm g}$, cf. Eq. (7), and terms proportional to the QCD β-function. In detail (see, e.g. [11,43]), the functions $G_n^p(\varepsilon)$ satisfy the following relations to five loops

$$\begin{aligned}
G_1^p &= 2\left(B_1^p - \delta_{pg}\beta_0\right) + f_1^p + \varepsilon f_{01}^p\,,\\
G_2^p &= 2\left(B_2^p - 2\delta_{pg}\beta_1\right) + (f_2^p + \beta_0 f_{01}^p) + \varepsilon f_{02}^p\,,\\
G_3^p &= 2\left(B_3^p - 3\delta_{pg}\beta_2\right) + (f_3^p + \beta_1 f_{01}^p + \beta_0 f_{02}^p) + \varepsilon f_{03}^p\,,\\
G_4^p &= 2\left(B_4^p - 4\delta_{pg}\beta_3\right) + (f_4^p + \beta_2 f_{01}^p + \beta_1 f_{02}^p + \beta_0 f_{03}^p) + \varepsilon f_{04}^p\,,\\
G_5^p &= 2\left(B_5^p - 5\delta_{pg}\beta_4\right) + (f_5^p + \beta_3 f_{01}^p + \beta_2 f_{02}^p + \beta_1 f_{03}^p + \beta_0 f_{04}^p) + \varepsilon f_{05}^p\,,
\end{aligned} \tag{9}$$

where the functions $f_{0n}^p(\varepsilon)$ at n loops are polynomials in ε and β_n are the coefficients of the QCD β-function normalized as in Eq. (4), i.e., $\beta(a_{\rm s}) = -\beta_0 a_{\rm s}^2 - \ldots$ with $\beta_0 = 11/3\, C_A - 2/3\, n_f$.

The eikonal anomalous dimensions $f^{\rm q}$ and $f^{\rm g}$ of these Wilson line quantities for quarks and gluons exhibit the same maximal non-Abelian colour structure as the cusp anomalous dimensions, a fact verified explicitly at lower fixed orders [43, 44] and generalized in Ref. [45]. Hence we assume here that also the expressions for $f^{\rm q}$ and $f^{\rm g}$ are related by generalized Casimir scaling at four loops (and beyond), in complete analogy to the cusp anomalous dimensions, $A^{\rm q}$ and $A^{\rm g}$, see also the recent work [46].

This leads immediately to the expression for the full colour dependence of the four-loop gluon virtual anomalous dimension $B_4^{\rm g}$ as

$$\begin{aligned}
B_4^{\rm g} \;=\;& \\
& C_A^4\Big(\, b^{\rm g}_{4,C_A^4}\Big) + \frac{d_A^{abcd}d_A^{abcd}}{n_A}\Big(\, b^{\rm g}_{4,d_A^{abcd}d_A^{abcd}}\Big) + n_f C_A^3\Big(-\frac{8075}{108} - \frac{6155}{54}\zeta_2 - \frac{22714}{27}\zeta_3 + \frac{7789}{18}\zeta_4\\
& + \frac{1874}{9}\zeta_2\zeta_3 + \frac{919}{9}\zeta_5 - \frac{1268}{3}\zeta_3^2 + \frac{1777}{54}\zeta_6 - \frac{1}{4}\, b^{\rm q}_{4,n_f C_F^3} - \frac{1}{2}\, b^{\rm q}_{4,n_f C_F^2 C_A} - \frac{1}{48}\, b^{\rm q}_{4,d_F^{abcd}d_F^{abcd}}\Big)\\
& + n_f C_A^2 C_F\Big(\frac{23566}{243} + \frac{4198}{27}\zeta_2 + \frac{8854}{27}\zeta_3 - \frac{27269}{27}\zeta_4 - \frac{2744}{9}\zeta_2\zeta_3 + \frac{6712}{9}\zeta_5 + \frac{1928}{3}\zeta_3^2\\
& - \frac{2879}{9}\zeta_6 + \, b^{\rm q}_{4,n_f C_F^2 C_A}\Big) + n_f C_A C_F^2\Big(-\frac{2723}{27} - 162\,\zeta_2 + \frac{2948}{9}\zeta_3 + 204\zeta_4 + \frac{256}{3}\zeta_2\zeta_3\\
& - 912\,\zeta_5 - 224\,\zeta_3^2 + \frac{6434}{9}\zeta_6 + \, b^{\rm q}_{4,n_f C_F^3}\Big) + n_f C_F^3\Big(23\Big) + n_f \frac{d_F^{abcd}d_A^{abcd}}{n_A}\Big(\frac{1952}{9} - \frac{2368}{3}\zeta_2\\
& + \frac{1312}{3}\zeta_3 + \frac{1016}{3}\zeta_4 + 544\zeta_2\zeta_3 - \frac{1520}{3}\zeta_5 - \frac{1496}{9}\zeta_6 + \, b^{\rm q}_{4,d_F^{abcd}d_F^{abcd}}\Big) + n_f^2 C_A^2\Big(\frac{1352}{81} + \frac{37}{27}\zeta_2\\
& + \frac{289}{27}\zeta_3 + \frac{200}{27}\zeta_4 - \frac{32}{9}\zeta_2\zeta_3 - \frac{8}{9}\zeta_5\Big) + n_f^2 C_A C_F\Big(\frac{3910}{243} + \frac{160}{9}\zeta_3\Big) + n_f^2 C_F^2\Big(\frac{338}{27} - \frac{176}{9}\zeta_3\Big)\\
& + n_f^2 \frac{d_F^{abcd}d_F^{abcd}}{n_A}\Big(-\frac{704}{9} + \frac{512}{3}\zeta_3\Big) + n_f^3 C_A\Big(\frac{5}{243}\Big) + n_f^3 C_F\Big(\frac{154}{243}\Big)\,,
\end{aligned} \tag{10}$$

where the n_f^3-dependent terms agree with Ref. [22]. In addition, we have checked that a numerical fit for the $d_F^{abcd}d_A^{abcd}$ term in the gluon-gluon splitting function [24] to the known Mellin moments nicely confirms the value quoted in Eq. (10). Altogether, we take this as strong indications on the correctness of the assumption made in the derivation of Eqs. (6) and (10). The remaining unknowns can be determined numerically as $b^{\rm g}_{4,C_A^4} = 1098 \pm 20$ and $b^{\rm g}_{4,d_A^{abcd}d_A^{abcd}} = -1125.6 \pm 1.0$ from the Mellin moments $N = 2$, 4, 6 and 8 obtained via DIS structure functions, see Ref. [25].

We also note that the purely Abelian (QED) contributions in $B_4^{\rm g}$ coincide with the respective terms in the four-loop QCD β-function [47, 48]. This concerns the colour factors $n_f C_F^3$, $n_f^2 C_F^2$, $n_f^3 C_F$ and $d_F^{abcd}d_F^{abcd}$ and is a direct consequence of the generalized Casimir scaling of $f^{\rm q}$ and $f^{\rm g}$, which implies that $f_4^{\rm g}$ must have only non-Abelian colour factors (terms proportional to C_A, $d_F^{abcd}d_A^{abcd}$ or $d_A^{abcd}d_A^{abcd}$). This reproduces a pattern already observed for $B_n^{\rm g}$ up to third order, $n \leq 3$, see [49], for all terms $n_f^k C_F^{n-k}$ with $1 \leq k \leq n-1$. At four loops, the colour factors $n_f C_F^3$ and $d_F^{abcd}d_F^{abcd}$ are unique in the single pole $1/\varepsilon$ of the gluon form factor, cf. in G_4^g in Eq. (9), and therefore, can be related directly to those in β_3 and, hence, $B_4^{\rm g}$. The other two colour factors in G_4^g, $n_f^2 C_F^2$ and $n_f^3 C_F$, do also receive contributions from lower orders. For instance $n_f^2 C_F^2$ terms are generated from $\beta_1 f_{02}^g$, but cancel in extraction of $B_4^{\rm g}$ from G_4^g.

With the help of Eqs. (9) and (10) we obtain the single pole in ε in the gluon form factor $\mathcal{F}_4^g$ at four loops as

$$
\begin{aligned}
\mathcal{F}_4^g\Big|_{1/\varepsilon} =& \\
& C_A^4\Big(-\frac{746918615}{104976}+\frac{595199}{216}\zeta_2+\frac{8305667}{1458}\zeta_3+\frac{975575}{972}\zeta_4+\frac{39811}{81}\zeta_2\zeta_3-\frac{781411}{405}\zeta_5 \\
& +\frac{41335}{54}\zeta_3\zeta_4-\frac{272338}{81}\zeta_3^2-\frac{739783}{144}\zeta_6-\frac{14629}{45}\zeta_2\zeta_5+\frac{563669}{126}\zeta_7+\frac{1}{192}\,f^{\rm q}_{4,d_F^{abcd}d_A^{abcd}} \\
& -\frac{1}{4}\,b^{\rm g}_{4,C_A^4}\Big)+\frac{d_A^{abcd}d_A^{abcd}}{n_A}\Big(-\frac{80}{9}+\frac{704}{3}\zeta_3-\frac{1}{8}\,f^{\rm q}_{4,d_F^{abcd}d_A^{abcd}}-\frac{1}{4}\,b^{\rm g}_{4,d_A^{abcd}d_A^{abcd}}\Big)+O(n_f) \\
=\;& C_A^4\Big(-1084.7\pm 5.5\Big)+\frac{d_A^{abcd}d_A^{abcd}}{n_A}\Big(567.3\pm 12.8\Big)+n_f\text{ terms}\,, \qquad (11)
\end{aligned}
$$

where all n_f-dependent contributions have been given analytically in Ref. [18].

The observed relation between the gluon virtual anomalous dimension $B_n^{\rm g}$ and the corresponding coefficient of the β-function for purely Abelian terms leads to new predictions at five loops. Using the expression for the β-function for a general gauge group at five loops [50–52] (the QCD result was obtained before in Ref. [53]), one deduces for the splitting function P_{gg} in Eq. (7)

$$
\begin{aligned}
B_5^{\rm g} =& \\
& n_f C_F^4\Big(-\frac{4157}{12}-64\,\zeta_3\Big)+n_f^2 C_F^3\Big(-\frac{2509}{18}-\frac{536}{3}\zeta_3+\frac{1160}{3}\zeta_5\Big) \\
& +n_f^2 C_F\frac{d_F^{abcd}d_F^{abcd}}{n_A}\Big(\frac{4160}{3}+\frac{5120}{3}\zeta_3-\frac{12800}{3}\zeta_5\Big)+n_f^3 C_F^2\Big(-\frac{4961}{324}+\frac{952}{27}\zeta_3-\frac{44}{3}\zeta_4\Big) \\
& +n_f^3\frac{d_F^{abcd}d_F^{abcd}}{n_A}\Big(\frac{1760}{9}-\frac{1312}{3}\zeta_3+128\,\zeta_4+\frac{640}{3}\zeta_5\Big)+n_f^4 C_F\Big(-\frac{107}{486}-\frac{8}{27}\zeta_3\Big) \\
& +\text{ terms with }\; C_A,\, d_A^{abcd}d_A^{abcd},\, d_F^{abcd}d_A^{abcd}\,.
\end{aligned}
$$

Predictions for the purely Abelian part of the gluon form factor at five loops are possible, e.g., for the $n_f C_F^4$ in $\mathcal{F}_5^g$, which can be read off from Eq. (12) using G_5^g in Eq. (9), while finite terms of $\mathcal{F}^g$ at lower orders are needed for other colour structures.

Beyond the SV approximation, predictions for the ggF cross section are possible using physical evolution kernels [9, 54, 55]. In z-space, this concerns terms enhanced as $\ln^k(1-z)$ with $2n-1 \geq k \geq 1$ at n-th order, or equivalently power suppressed contributions $(\ln^k N)/N$ in Mellin N-space. Such next-to-leading power threshold effects have also been studied in Refs. [56–63]. At N^{4}LO, these subleading terms in $c_{gg}^{H,(4)}(z)$ can be obtained from the physical evolution kernel K_{gg}, which one can define by re-expressing Eq. (2) as dimensionless 'structure functions' $\mathcal{F}_{ab}$, i.e.,

$$\sigma(S, m_H^2) \;=\; \textstyle\sum_{a,b} \widetilde{\sigma}_0^H \, \mathcal{F}_{ab} \;. \tag{12}$$

The kernel K_{gg} and its perturbative expansion for a scale choice $\mu_F = m_H$ are then given in terms of the splitting function P_{gg}, the β-function and the gluon coefficient function c_{gg}^H by

$$\begin{aligned} \frac{d}{d\ln m_H^2}\,\mathcal{F}_{gg} \;&=\; \left\{ 2P_{gg}(a_s) + \beta(a_s)\,\frac{dc_{gg}^H(a_s)}{da_s} \otimes \left(c_{gg}^H(a_s)\right)^{-1} \right\} \otimes \mathcal{F}_{gg} \\ &\equiv\; K_{gg} \otimes \mathcal{F}_{gg} \;\equiv\; \sum_{\ell=0} a_s^{\ell+1} K_{gg}^{(\ell)} \otimes \mathcal{F}_{gg} \end{aligned} \tag{13}$$

where $\otimes$ denotes the usual Mellin convolution and $\beta(a_s)$ the β-function as defined below Eq. (9).

The key feature of the kernel K_{gg} is its single-logarithmic enhancement. In z-space, this implies at n-th order that all terms $\ln^k(1-z)$ with $2n-1 \geq k \geq n+1$ have to cancel in Eq. (13) to all orders in $(1-z)$, which leads to predictions for coefficient function c_{gg}^H, cf. Ref. [9]. In Mellin N-space the leading large-N logarithms of the sub-dominant N^{-1} contributions in K_{gg} take the simple form,

$$\begin{aligned} K_{gg}^{(1)}\Big|_{N^{-1}} \;&=\; -\left(8\,\beta_0 C_A + 32\,C_A^2\right)\ln N + \;O(1)\;, \\ K_{gg}^{(2)}\Big|_{N^{-1}} \;&=\; -\left(16\,\beta_0^2 C_A + 112\,\beta_0 C_A^2\right)\ln^2 N + \;O(\ln N)\;, \\ K_{gg}^{(3)}\Big|_{N^{-1}} \;&=\; -\left(32\,\beta_0^3 C_A + \frac{896}{3}\,\beta_0^2 C_A^2\right)\ln^3 N + \;O(\ln^2 N)\;, \\ K_{gg}^{(4)}\Big|_{N^{-1}} \;&=\; -\left(64\,\beta_0^4 C_A + \xi_H^{(4)}\,\beta_0^3 C_A^2\right)\ln^4 N + \;O(\ln^3 N)\,, \end{aligned} \tag{14}$$

where the first three lines follow from the known coefficient functions $c_{gg}^{H,(n)}$ up to N^{3}LO. The expression for $K_{gg}^{(4)}$ contains an unknown coefficient $\xi_H^{(4)}$, to be determined at N^{4}LO by explicit computations (the corresponding coefficient for DIS has been obtained in Refs. [57,58])[6]. Eq. (14) predicts the following next-to-leading power threshold terms in the four-loop gluon coefficient function $c_{gg}^{H,(4)}$ for the ggF process at the scale $\mu_R = \mu_F = m_H$,

$$c_{gg}^{H,(4)}(z) \;=\; c_{gg}^{H,(4)}(z)\Big|_{\rm SV} - \frac{4096}{3}\,C_A^4 \ln^7(1-z) + \left\{ \frac{19712}{3}\,C_A^4 + \frac{3584}{3}\,C_A^3\,\beta_0 \right\} \ln^6(1-z)$$

[6] We note that the pattern of the ratios of the lower order coefficients is $112/32 = 7/2$ and $(896/3)/112 = 8/3$. A generalization of this pattern leads to an estimate of $\xi_H^{(4)} = 670 \pm 300$ with a conservative numerical uncertainty, which has a sub-percent effect on the $N^{-1}\ln^4 N$ coefficient in Eq. (16) below.

$$
\begin{aligned}
&+\left\{(-2240+23552\,\zeta_2)\,C_A^4-\frac{64576}{9}C_A^3\,\beta_0-\frac{1024}{3}\,C_A^2\,\beta_0^2\right\}\ln^5(1-z)\\
&+\left\{\left(\frac{80384}{9}-81520\zeta_2-\frac{313600}{3}\zeta_3\right)C_A^4+\left(\frac{104128}{9}-\frac{47408}{3}\zeta_2\right)C_A^3\,\beta_0\right.\\
&\qquad\left.+\left(1856+\frac{1}{4}\xi_H^{(4)}\right)C_A^2\,\beta_0^2+32\,C_A\,\beta_0^3+\frac{640}{3}\,C_A^2\,\beta_1\right\}\ln^4(1-z)\;+\;O\left(\ln^3(1-z)\right)
\end{aligned}
\tag{15}
$$

where $c_{gg}^{H,(4)}(z)|_{\rm SV}$ denotes the z-space SV approximation at N^{4}LO as discussion above.

We are now ready to assemble the results of Refs. [9, 38, 40] and Eqs. (6) and (14) for the inclusive ggF process at N^{4}LO. The resulting threshold expansion of $c_{gg}^{H,(4)}$ in N-space reads

$$
\begin{aligned}
\kappa_4\,c_{gg}^{H,(4)}(N)\;\simeq\;&\;0.55296\ln^8N+3.96654\ln^7N+21.2587\ln^6N+62.2985\ln^5N\\
&+150.141\ln^4N+212.443\ln^3N+255.911\ln^2N+(128.78\pm0.11)\ln N+\kappa_4\,g_{0,4}\\
&+N^{-1}\Big\{2.21184\ln^7N+19.6890\ln^6N+93.0439\ln^5N\\
&\quad+(256.454+132.25\,\kappa_4\,\xi_H^{(4)})\ln^4N+O(\ln^3N)\Big\}
\end{aligned}
\tag{16}
$$

with $\kappa_4=1/25000\simeq1/(4\pi)^4$, which approximately converts the coefficients to an expansion in α_s. We have inserted the QCD values of the group factors in Eq. (5) and above, used the physical value of $n_f=5$ light flavors at scales of order m_H^2, and truncated all exact numbers to six decimals. The quoted uncertainty in the coefficient of $\ln N$ stems entirely from $f^{\rm q}_{4,d_F^{abcd}d_A^{abcd}}$, as the uncertainties in the values of $b^{\rm q}_{4,d_F^{abcd}d_F^{abcd}}$, $b^{\rm q}_{4,n_fC_F^2C_A}$ and $b^{\rm q}_{4,n_fC_F^3}$ are completely negligible. The constant-N contribution $g_{0,4}$ has been estimated in Ref. [9] by three Padé approximants which yield a fairly wide spread of values suggesting $\kappa_4\,g_{0,4}=65\pm65$.

The N^{4}LO coefficient function in the SV approximation, together with the sub-dominant N^{-1} contributions in Mellin N-space, can be expected to provide a reliable approximation of the exact result. As pointed out earlier, the exact Mellin N-space result at lower orders resides inside the band spanned by the SV and SV+N^{-1} terms at moderately large N. This is shown in Fig. 1 (left) at N^{3}LO (for corresponding NLO and NNLO plots see Fig. 1 of Ref. [9]). The exact coefficient functions differ from the approximation based on the SV+N^{-1} terms by 0.44% at NLO, 0.83% at N^{2}LO and 1.15% at N^{3}LO at $N=12$. For smaller N values the difference between the exact results and the approximation based on the SV+N^{-1} terms is larger, however they always remain inside the SV and SV+N^{-1} band. At N^{4}LO, see Fig. 1 (right), the SV approximation of Eq. (16) is shown including the N-independent constant g_{04} and the known $1/N$ terms as specified in Eq. (16).

The predictions for the ggF cross sections at the collision energy of 14 TeV use a Higgs mass $m_H=125$ GeV, an on-shell top quark mass $m_t=172.5$ GeV, $n_f=5$ active quark flavors and the PDF sets ABMP16 [64] and MMHT2014 [65] using the `lhapdf` [66] interface. The PDF sets and as well as the value of the strong coupling constant α_s corresponding to the respective PDF set are taken order-independent at NNLO throughout. The prefactor $C(\mu_R^2)$ in Eq. (3) is improved with the full top-mass dependence of the Born cross section. The results up to N^{3}LO are computed with the program *iHixs* [67] which directly provides the cross sections in this rescaled effective field theory.

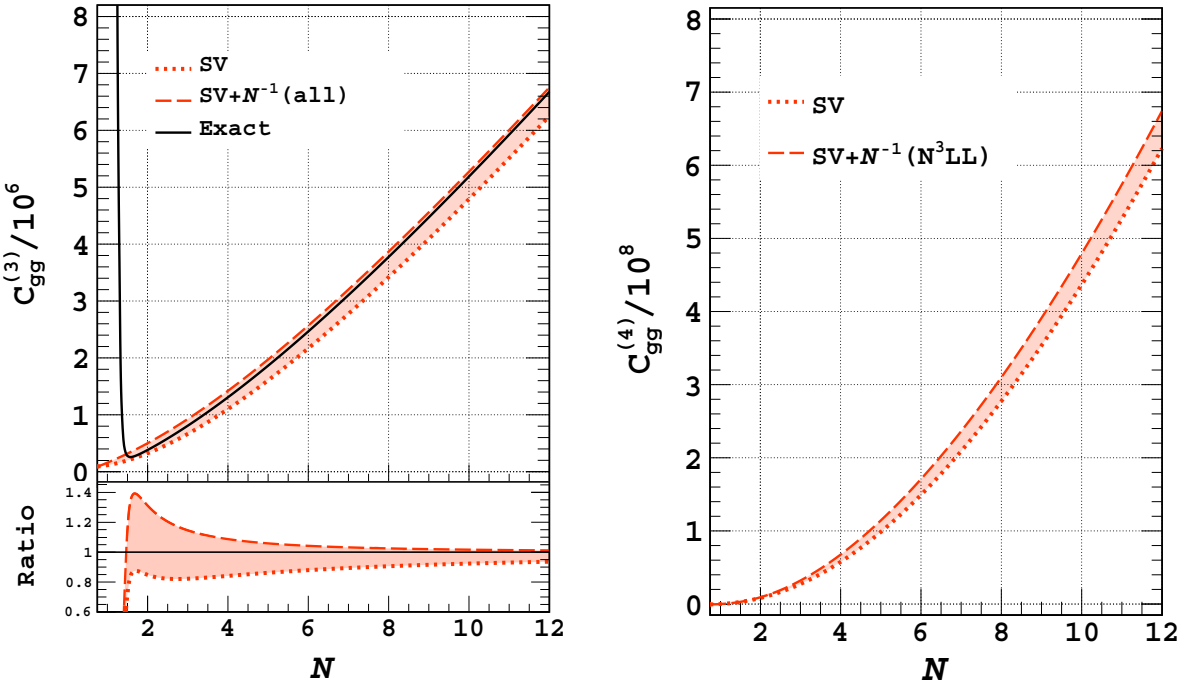

Figure 1: Soft-virtual (SV) and SV $+N^{-1}$ approximations to the N^{3}LO (left) and N^{4}LO (right) ggF coefficient functions. The N^{3}LO curves, where the N^{-1} contributions are complete, are compared to the exact result. At N^{4}LO the highest four N^{-1} logarithms are includes as given in Eq. (16).

The residual uncertainty in the SV approximation of the ggF cross section due to the coefficients $f^{\mathrm{q}}_{4,d_F^{abcd}d_A^{abcd}}$ in Eq. (6) and g_{04} in Eq. (16), which are currently least constrained, is practically negligible: the 100% error on g_{04} leads to an uncertainty of 0.3% for the cross section, while the 100% error on $f^{\mathrm{q}}_{4,d_F^{abcd}d_A^{abcd}}$ is completely negligible. The virtual anomalous dimension B_4^g only appears in the scale-dependent terms at N^{4}LO and its contribution vanishes for the central scale choice $\mu_R = \mu_F = m_H$. For the scale setting $\mu_R = \mu_F = m_H/2$ a change below 0.002% is observed for the cross-section at 14 TeV LHC due to the numerical uncertainty in B_4^g. Thus, precise predictions at N^{4}LO are now possible for all relevant kinematics and scale choices.

The impact of the SV corrections at N^{4}LO is shown in Fig. 2 (left) in a range of center-of-mass energies for two different choices of renormalization scale, $\mu_R = m_H$ and $\mu_R = m_H/2$, keeping always $\mu_F = m_H$. The corresponding K-factors, defined as the ratio of the SV corrections at N^{4}LO over the exact N^{3}LO result, are displayed in the lower panel. The SV correction at N^{4}LO increases the cross-section at the LHC with 14 TeV by 1.41 pb for the scale $\mu_R = m_H$ and by 0.08 pb for the scale $\mu_R = m_H/2$. The K-factor shows little dependence on the collision energy in the entire range of center-of-mass energies displayed in Fig. 2 (left). For the scale choice $\mu_R = m_H/2$, which is closer to the point of minimal sensitivity, the effect of higher order corrections is indeed small, leading to a K-factor close to unity, cf. also Ref. [9]. At the scale $\mu_R = m_H$ a K-factor of 1.027 is obtained for the LHC at 14 TeV.

In Fig. 2 (right) we show the dependence of the cross section for the $E_{\mathrm{CM}} = 14$ TeV on the renormalization scale μ_R. The μ_R dependence indeed decreases order by order in perturbation theory up to N^{4}LO. In the range $m_H/4 \leq \mu_R \leq 2m_H$ the effect of the μ_R variation decreases drastically

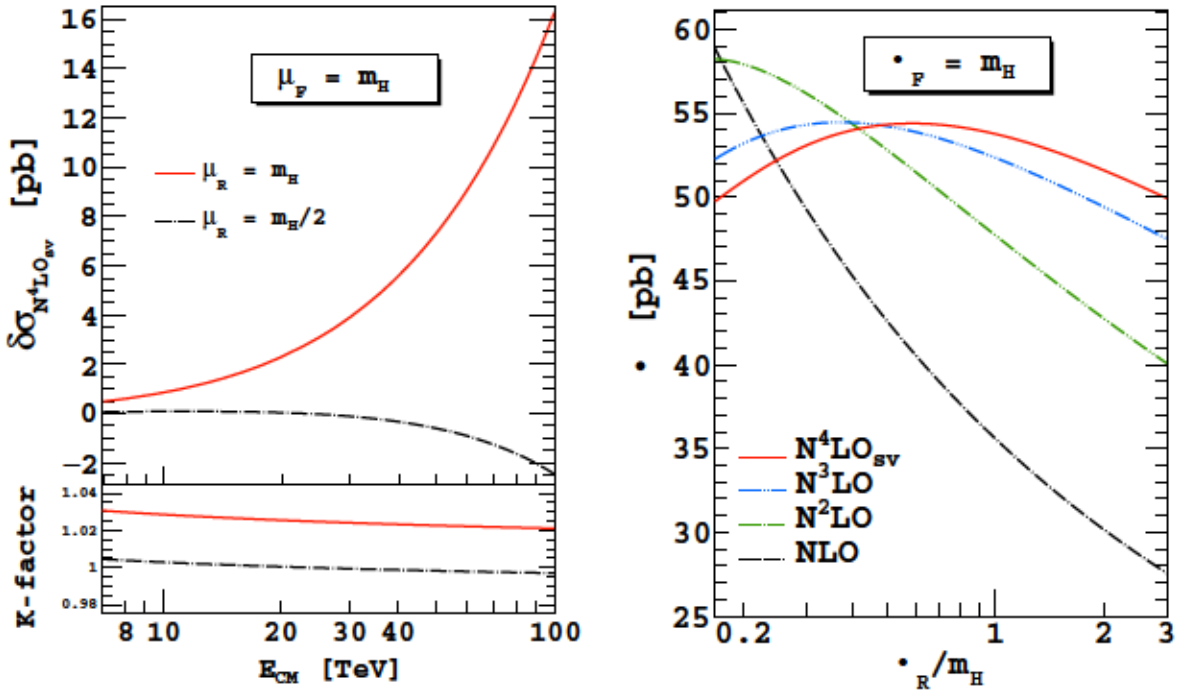

Figure 2: Left panel: The contribution of the SV approximation at N^{4}LO to the ggF cross section in proton-proton collisions as a function of the center-of-mass energy $E_{\rm CM}$ for $\mu_F = m_H$ at two renormalization scales, $\mu_R = m_H$ and $\mu_R = m_H/2$, using the MMHT2014 PDFs [65]. The corresponding K-factors with respect to N^{3}LO are shown in the lower panel. Right panel: The ggF cross section up to N^{4}LO with a variation of the renormalization scale μ_R for the LHC at 14 TeV using the same PDFs and $\mu_F = m_H$.

from $\pm 27\%$ at NLO and $\pm 14\%$ at NNLO to $\pm 5\%$ at N^{3}LO, while it amounts to less than $\pm 3\%$ at N^{4}LO. Also here the factorization scale is kept fixed, at $\mu_F = m_H$, since beyond NNLO only flavour non-singlet results have published for the QCD splitting functions $P_{ik}(z)$ [22, 23, 68], and PDFs fits have been limited to NNLO so far.

The uncertainty in the predicted ggF cross sections due to the truncation of the perturbation series is now, at N^{4}LO, smaller than that due to the use of different sets of PDFs and corresponding different values of the strong coupling constant α_s. For $\sqrt{S} = 14$ TeV, $m_H = 125$ GeV, the central scale $\mu_R = m_H$, and including the PDF uncertainties at N^{3}LO, one obtains

$$\begin{aligned} \sigma|_{\rm N^3LO} &= 49.6 \pm 0.5\ {\rm pb}, \qquad \sigma|_{\rm N^4LO} = 50.8\ {\rm pb}, \qquad {\rm ABMP16}, \\ \sigma|_{\rm N^3LO} &= 52.3 \pm 0.8\ {\rm pb}, \qquad \sigma|_{\rm N^4LO} = 53.8\ {\rm pb}, \qquad {\rm MMHT2014}, \end{aligned} \tag{17}$$

where the spread in predictions is due to different values of the strong coupling constant at NNLO corresponding to the different PDF sets used, i.e., $\alpha_s(M_Z) = 0.1147$ for ABMP16 and $\alpha_s(M_Z) = 0.1180$ for MMHT2014, and due to different gluon PDFs in the relevant kinematic range. These are consequences of different choices for the theoretical framework and assumptions on parameters used in the respective global fits, see Ref. [69], which lead to systematic shifts that are often significantly larger than the PDF and $\alpha_s(M_Z)$ uncertainties associated to individual PDF sets.

Due to the universality of threshold dynamics for colourless final states in hadronic collisions, relevant formulae for the Drell-Yan process, $pp \to \gamma^* + X$, can be easily obtained from the above

considerations, using Eq. (2) with the replacement $\widetilde{\sigma}_0^H\, c_{ab}^H \to \widetilde{\sigma}_0^{\gamma^*}\, c_{ab}^{\rm DY}$ with

$$\widetilde{\sigma}_0^{\gamma^*} \;=\; 4\pi\alpha^2/(3Q^2 n_c) \;\; . \tag{18}$$

Here α is the fine-structure constant of QED and Q^2 the virtuality of the produced photon γ^*. The coefficient functions $c_{ab}^{\rm DY}$ enjoy a perturbative expansive analogous to Eq. (4) with the leading order normalization $c_{ab}^{{\rm DY},(0)} = \delta_{aq}\,\delta_{b\bar{q}}\,\delta(1-z)$. The coefficients of $\mathcal{D}_k$ for $7 \geq k \geq 2$ of the four-loop term $c_{q\bar{q}}^{{\rm DY},(4)}$ can be found in Eq. (6) of Ref. [38] and that of $\mathcal{D}_1$ in Eq. (14) of Ref. [40].

We are now in the position to present the four-loop $\mathcal{D}_0$ term for the DY process. It is given by

$$\begin{aligned}
&c_{q\bar{q}}^{{\rm DY},(4)}\Big|_{\mathcal{D}_0} = \\
&\quad C_F^4\Big(32704\,\zeta_3 + 113152\,\zeta_2\zeta_3 - 196608\,\zeta_5 - 15360\,\zeta_3^2 - 491520\,\zeta_2\zeta_5 - 195584\,\zeta_3\zeta_4 \\
&\quad +983040\,\zeta_7\Big) + C_F^3 C_A\Big(-\frac{206444}{27} - \frac{32740}{9}\,\zeta_2 - \frac{746878}{9}\,\zeta_3 + \frac{146768}{9}\,\zeta_4 - \frac{1011088}{9}\,\zeta_2\zeta_3 \\
&\quad +274432\,\zeta_5 - \frac{484192}{3}\,\zeta_3^2 + \frac{356048}{3}\,\zeta_6 - 73728\,\zeta_2\zeta_5 + 76000\,\zeta_3\zeta_4\Big) \\
&\quad +C_F^2 C_A^2\Big(\frac{15086188}{729} - \frac{12535492}{729}\,\zeta_2 + \frac{3043898}{81}\,\zeta_3 + \frac{2522080}{81}\,\zeta_4 - \frac{2968640}{27}\,\zeta_2\zeta_3 \\
&\quad +\frac{1046528}{9}\,\zeta_5 - \frac{82592}{3}\,\zeta_3^2 - \frac{30184}{3}\,\zeta_6 + 3072\zeta_2\zeta_5 + \frac{60944}{3}\,\zeta_3\zeta_4\Big) + C_F C_A^3\Big(-\frac{28325071}{2187} \\
&\quad +\frac{5761670}{243}\,\zeta_2 + \frac{867584}{27}\,\zeta_3 - \frac{150632}{9}\,\zeta_4 - \frac{119624}{9}\,\zeta_2\zeta_3 - \frac{49840}{9}\,\zeta_5 - \frac{4664}{3}\,\zeta_3^2 + \frac{41789}{9}\,\zeta_6 \\
&\quad +832\,\zeta_2\zeta_5 + 1440\,\zeta_3\zeta_4 + 3400\,\zeta_7 + \frac{1}{12}\,f^{\rm q}_{4,d_F^{abcd}d_A^{abcd}}\Big) + \frac{d_F^{abcd}d_A^{abcd}}{n_F}\Big(-2\,f^{\rm q}_{4,d_F^{abcd}d_A^{abcd}}\Big) \\
&\quad +C_F^3 n_f\Big(-\frac{80221}{54} - \frac{25744}{27}\,\zeta_2 + \frac{95936}{9}\,\zeta_3 - \frac{11492}{9}\,\zeta_4 + \frac{189824}{9}\,\zeta_2\zeta_3 - \frac{130624}{3}\,\zeta_5 \\
&\quad +\frac{106336}{3}\,\zeta_3^2 - \frac{160840}{9}\,\zeta_6 + 4\,b^{\rm q}_{4,n_f C_F^3}\Big) + C_F^2 C_A n_f\Big(-\frac{955285}{1458} + \frac{3057110}{729}\,\zeta_2 - \frac{1222648}{81}\,\zeta_3 \\
&\quad -\frac{1261168}{81}\,\zeta_4 + \frac{306400}{9}\,\zeta_2\zeta_3 - 37616\zeta_5 + \frac{11728}{3}\,\zeta_3^2 - \frac{164}{3}\,\zeta_6 + 4\,b^{\rm q}_{4,n_f C_F^2 C_A}\Big) \\
&\quad +C_F C_A^2 n_f\Big(\frac{10761379}{2916} - \frac{2418814}{243}\,\zeta_2 - \frac{948884}{81}\,\zeta_3 + \frac{213280}{27}\,\zeta_4 + \frac{28064}{9}\,\zeta_2\zeta_3 - \frac{29552}{27}\,\zeta_5 \\
&\quad -\frac{9736}{9}\,\zeta_3^2 - \frac{32930}{27}\,\zeta_6 - \frac{1}{12}\,b^{\rm q}_{4,d_F^{abcd}d_F^{abcd}} - 2\,b^{\rm q}_{4,n_f C_F^2 C_A} - b^{\rm q}_{4,n_f C_F^3}\Big) \\
&\quad +n_f\frac{d_F^{abcd}d_F^{abcd}}{n_F}\Big(768 - \frac{9088}{3}\,\zeta_2 + \frac{10624}{9}\,\zeta_3 + \frac{1600}{3}\,\zeta_4 - 256\,\zeta_2\zeta_3 + \frac{43520}{9}\,\zeta_5 - \frac{2432}{3}\,\zeta_3^2 \\
&\quad -\frac{2368}{9}\,\zeta_6 + 4\,b^{\rm q}_{4,d_F^{abcd}d_F^{abcd}}\Big) + C_F^2 n_f^2\Big(-\frac{142769}{729} - \frac{99184}{729}\,\zeta_2 + \frac{113456}{81}\,\zeta_3 + \frac{23200}{27}\,\zeta_4 \\
&\quad -\frac{79360}{27}\,\zeta_2\zeta_3 + \frac{33056}{9}\,\zeta_5\Big) + C_F C_A n_f^2\Big(-\frac{898033}{2916} + \frac{293528}{243}\,\zeta_2 + \frac{87280}{81}\,\zeta_3 - \frac{1744}{3}\,\zeta_4 \\
&\quad -\frac{608}{9}\,\zeta_2\zeta_3 + \frac{608}{3}\,\zeta_5\Big) + C_F n_f^3\Big(\frac{10432}{2187} - \frac{3200}{81}\,\zeta_2 - \frac{3680}{81}\,\zeta_3 + \frac{112}{9}\,\zeta_4\Big)\;,
\end{aligned} \tag{19}$$

where the quartic Casimirs are normalized by the dimension of the fundamental representation of the $SU(n_c)$ gauge group, $n_F = n_c$. As Eq. (6), this result is exact in the large-n_c limit and has an amply sufficient numerical accuracy for all phenomenological applications in QCD.

The next-to-leading power threshold terms for the DY process can also be derived with the help of the corresponding physical evolution kernel K_{qq}, which exhibits the same simple form in Mellin N-space as Eq. (14) for the leading large-N logarithms of the N^{-1} contributions with the obvious replacement $C_A \to C_F$, see Ref. [9] for further details.

Finally, by combining our new result (6) with Eq. (2.13) in Ref. [9], and proceeding analogously with its Drell-Yan counterpart (19), we can derive the four-loop coefficient D_4 for the soft-gluon exponentiation of inclusive Higgs production via ggF and DY lepton-pair production. The two results are, as expected, related by generalized Casimir scaling [24] which reduces to standard 'numerical' C_A/C_F lower-order Casimir scaling in the their exact large-n_c limit:

$$
\begin{aligned}
D_4 \;=\; & \\
& C_I C_A^3 \Big(-\frac{28325071}{2187} + \frac{5761670}{243}\zeta_2 + \frac{867584}{27}\zeta_3 - \frac{150632}{9}\zeta_4 - \frac{119624}{9}\zeta_2\zeta_3 - \frac{49840}{9}\zeta_5 \\
& - \frac{4664}{3}\zeta_3^2 + \frac{41789}{9}\zeta_6 + 832\,\zeta_2\zeta_5 + 1440\,\zeta_3\zeta_4 + 3400\,\zeta_7 + \frac{1}{12} f^{\rm q}_{4,d_F^{abcd} d_A^{abcd}} \Big) \\
& - \frac{d_A^{abcd} d_I^{abcd}}{n_I} \Big(2\, f^{\rm q}_{4,d_F^{abcd} d_A^{abcd}} \Big) + n_f C_I C_A^2 \Big(\frac{10761379}{2916} - \frac{2418814}{243}\zeta_2 - \frac{948884}{81}\zeta_3 + \frac{213280}{27}\zeta_4 \\
& + \frac{28064}{9}\zeta_2\zeta_3 - \frac{29552}{27}\zeta_5 - \frac{9736}{9}\zeta_3^2 - \frac{32930}{27}\zeta_6 - \frac{1}{12} b^{\rm q}_{4,d_F^{abcd} d_F^{abcd}} - 2\, b^{\rm q}_{4,n_f C_F^2 C_A} - b^{\rm q}_{4,n_f C_F^3} \Big) \\
& + n_f C_I C_F C_A \Big(\frac{2149049}{486} - \frac{56222}{27}\zeta_2 - \frac{8932}{9}\zeta_3 - \frac{113360}{27}\zeta_4 + \frac{3808}{9}\zeta_2\zeta_3 + \frac{21904}{9}\zeta_5 \\
& + \frac{6800}{3}\zeta_3^2 - 1436\,\zeta_6 + 4\, b^{\rm q}_{4,n_f C_F^2 C_A} \Big) + n_f C_I C_F^2 \Big(-\frac{27949}{54} - 632\,\zeta_2 + \frac{2240}{9}\zeta_3 + 668\,\zeta_4 \\
& + \frac{1024}{3}\zeta_2\zeta_3 - \frac{7744}{3}\zeta_5 - 736\,\zeta_3^2 + \frac{29336}{9}\zeta_6 + 4\, b^{\rm q}_{4,n_f C_F^3} \Big) + n_f \frac{d_F^{abcd} d_I^{abcd}}{n_I} \Big(768 - \frac{9088}{3}\zeta_2 \\
& + \frac{10624}{9}\zeta_3 + \frac{1600}{3}\zeta_4 - 256\,\zeta_2\zeta_3 + \frac{43520}{9}\zeta_5 - \frac{2432}{3}\zeta_3^2 - \frac{2368}{9}\zeta_6 + 4\, b^{\rm q}_{4,d_F^{abcd} d_F^{abcd}} \Big) \\
& + n_f^2 C_I C_A \Big(-\frac{898033}{2916} + \frac{293528}{243}\zeta_2 + \frac{87280}{81}\zeta_3 - \frac{1744}{3}\zeta_4 - \frac{608}{9}\zeta_2\zeta_3 + \frac{608}{3}\zeta_5 \Big) \\
& + C_I n_f^2 C_F \Big(-\frac{110059}{243} + 384\,\zeta_2 + \frac{10768}{27}\zeta_3 + \frac{160}{3}\zeta_4 - 256\,\zeta_2\zeta_3 + 32\,\zeta_5 \Big) \\
& + C_I n_f^3 \Big(\frac{10432}{2187} - \frac{3200}{81}\zeta_2 - \frac{3680}{81}\zeta_3 + \frac{112}{9}\zeta_4 \Big)
\end{aligned}
\tag{20}
$$

with $C_I = C_F$, $d_I^{abcd} = d_F^{abcd}$ and $n_I = n_F$ for the DY case, and $C_I = C_A$, $d_I^{abcd} = d_A^{abcd}$ and $n_I = n_A$ for Higgs production. The lower-order coefficients can be found in Eqs. (33) - (35) of Ref. [8].

With this result, and the approximate values of Ref. [68] for the small effect of the five-loop cusp anomalous dimensions A_5, all ingredients are now available for extending the soft-gluon exponentiation to the next-to-next-to-next-to-next-to-leading logarithmic ($\mathrm{N^4LL}$) accuracy. The corresponding function g_5 can be inferred from the DIS result in Eq. (2.9) of Ref. [11] as described below Eq. (3.6) of Ref. [10].

Using recent progress on related fourth-order quantities, we have been able to determine the final soft-gluon enhanced contribution (6) to the N^{4}LO coefficient function for inclusive Higgs-boson production in gluon-gluon fusion in the heavy-top limit, and the corresponding result (19) for the Drell-Yan process $pp \to \gamma^* + X$. These results also fix the respective N^{4}LL coefficients D_4 for the soft-gluon exponentiation (20) which are related by the same fourth-order generalization of the well-known Casimir scaling observed before in the cusp anomalous dimensions, now completely known at this order [21]. Our results are exact in the limit of a large number of flavours n_c. Their colour-factor decomposition in full QCD involves a few quantities which are known only numerically at this point. The resulting uncertainties are practically negligible as can be seen from the $\ln N$ coefficient in Eq. (16) above.

We have employed the latter Mellin N-space results to add the N^{4}LO soft + virtual (SV) corrections to the known complete N^{3}LO results [3, 4] for the LHC at 14 TeV. With the effect of the only uncomputed quantity, the soft-gluon coefficient g_{04} for this process, being well below 1%, we find that the cross sections are enhanced by 2.7% for the scale choice $\mu_R = m_H$, while the results are almost unchanged for $\mu_R = 0.5\, m_H$. It should be noted that these values refer to the not entirely realistic case of an order-independent α_s-value and PDFs at $\mu = m_H$. The renormalization-scale variation, estimated using the interval $0.25 m_H \leq \mu_R \leq 2 m_H$, is reduced from about 5% at N^{3}LO to less than 3% at N^{4}LO. Based on similar calculations at lower orders, we definitely expect that difference between the present N-space SV approximation and the complete N^{4}LO coefficient function will amount to well below 1% of the total cross section.

As by-products of our analysis, we have derived the expression (10) for the four-loop gluon virtual anomalous dimension (and determined the corresponding purely Abelian contributions at five loops), and provided a sub-percent accurate value (11) for the hitherto unknown $1/\varepsilon$ coefficient of the matter-independent contribution to the four-loop gluon form factor for which the n_f-terms have been recently computed in Ref. [18].

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