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The fifth-order post-Newtonian Hamiltonian dynamics of two-body systems from an effective field theory approach: potential contributions

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Abstract

We calculate the potential contributions of the motion of binary mass systems in gravity to the fifth post–Newtonian order ab initio using coupling and velocity expansions within an effective field theory approach based on Feynman amplitudes starting with harmonic coordinates and using dimensional regularization. Furthermore, the singular and logarithmic tail contributions are calculated. We also consider the non–local tail contributions. Further steps towards the complete calculation are discussed and first comparisons are given to results in the literature.

1 Introduction

The measurement of gravitational wave signals from merging black holes and neutron stars [1] has been a recent milestone in astrophysics. The different gravitational wave detectors are reaching higher and higher sensitivity [2], which requests to provide more detailed predictions at the theoretical side. Currently in binary Hamiltonian dynamics the level of the 4th post–Newtonian (PN) order has been fully understood and agreeing results have been obtained using a variety of different computation techniques in quite a series of gauges which lead to identical predictions in all key observables [3–9]. Moreover, it has been shown by applying canonical transformations [9], that all descriptions are dynamically equivalent. The different approaches can only be compared either by using canonical transformations, which requires local representations, or by calculating observables.

At the level of the 5th post–Newtonian order, first two agreeing results on the static potential in the harmonic gauge were calculated [10, 11]. Later partial results were derived using different matching techniques for the Hamiltonian in the effective one body (EOB) approach in [12, 13].[1] Here two parameters, $\bar{d}_5$ and a_6, which are of $O(\nu^2)$, with $\nu = m_1 m_2/(m_1+m_2)^2$, remained yet undetermined.

The conserved Hamiltonian of the motion of binary mass systems in gravity has the following expansion

$$H = \sum_{k=0}^{\infty} H_{\mathrm{kPN}}, \tag{1}$$

where k labels the post–Newtonian order,[2] with $H_{\mathrm{0PN}} \equiv H_{\mathrm{N}}$. From $k = 4$ onward H_{kPN} consists out of the term due to potential interactions, H^{pot}, and the tail terms, H^{tail},

$$H_{\mathrm{kPN}} = H^{\mathrm{pot}}_{\mathrm{kPN}} + H^{\mathrm{tail}}_{\mathrm{kPN}}. \tag{2}$$

In effective field theory approaches[3] based on Feynman diagrams this is the most natural decomposition. In [12, 13] another decomposition has been chosen into the so–called non–local terms $H^{\mathrm{nl}}_{\mathrm{kPN}}$ and the local terms $H^{\mathrm{loc}}_{\mathrm{kPN}}$,

$$H_{\mathrm{kPN}} = H^{\mathrm{loc}}_{\mathrm{kPN}} + H^{\mathrm{nl}}_{\mathrm{kPN}}. \tag{3}$$

The non–local terms are fully contained in the tail terms and the local contributions are given by the local parts of the tail terms and the potential contributions.

In the present paper we calculate the 5PN potential corrections and some first parts of the 5PN tail terms using an effective field theory (EFT) approach; for related reviews see [24–28]. Here we follow Ref. [29]. [4] A series of technical details for the calculation of the potential terms have already been given in Refs. [9, 11] before. In the case of the tail terms one first applies the multi–pole expansion valid for the far zone [3,13,15,25,27,31–39] to the respective post–Newtonian order and then applies EFT methods to calculate their contribution, cf. [40]. Expansions of this type generally belong to the operator product expansions [41]. In the calculation one also applies the method of expansion by regions [42, 43].

In the present paper observables at 5PN such as the energy and periastron advance at circular orbits could not yet be calculated in complete form, since a series of differences with

[1] First results at 6PN order have been given in [14–16] recently. There is also a lot of activity in calculating post–Minkowskian corrections, cf. [14] Ref. [12], and [17–23].

[2] Here we do not deal with conserved half PN contributions occurring from 5.5 PN onward.

[3] For the 4PN calculations see [3, 4, 11].

[4] Following the ideas in [30].

the literature have still to be fully clarified. This concerns rational terms contributing to the tail term. However, we obtain all other contributions, including the π^2 contributions to the yet undetermined constants $\bar{d}_5$ and a_6, in [13]. Furthermore, quite a series of comparisons could be performed with the literature.

The paper is organized as follows. In Section 2 we describe the calculation of the 5PN potential terms and present the associated Hamiltonian $H_{5\text{PN}}^{\text{pot}}$ in the harmonic gauge. We use dimensional regularization in $D = 4 - 2\varepsilon$ dimensions. It is this method which allows a particular elegant merging of the potential and tail contributions in the conservative Hamiltonian, as we will show below. Already at 3PN the contributions to H^{pot} have poles in $1/\varepsilon$, cf. [44]. From 4PN corresponding poles also appear in the tail terms. We will discuss the main aspects of the 5PN tail term in Section 3 and construct a pole–free Hamiltonian in Section 4. Here we show that the poles in the combined Hamiltonian can be transformed away by a canonical transformation. In Section 5 we compare to results given in the literature and discuss open questions. A canonical transformation from harmonic to EOB coordinates is performed. We derive the non–local tail contributions within our approach and calculate their contribution to the binding energy and to periastron advance in the circular case. We then turn to the contributions to periastron advance from the potential terms and derive the π^2 contributions to the previously unknown constants $\bar{d}_5$ and a_6 and summarize our present results for the circular binding energy and periastron advance. Furthermore we briefly discuss the remaining contributions to the tail term. Section 6 contains the conclusions. In the appendices some technical aspects are presented on the merging of the potential and tail terms using the method of expansion by regions. As well we present longer formulae, which are used in the present calculation.

2 The potential contributions to the Hamiltonian

The calculation of the 5PN corrections is performed in the same way as has been described in Refs. [9, 11]. Starting from the Einstein–Hilbert Lagrangian, we parameterize the metric $g_{\mu\nu}$ according to Ref. [29] in terms of scalar, vector and tensor fields, and work in the harmonic gauge.[5] The Feynman diagrams are generated using `QGRAF` [45]. The Lorentz algebra is carried out using `Form` [46] and we perform the integration by parts (IBP) reduction to master integrals using the code `Crusher` [47]. Table 1 gives an overview on the present calculation.

#loops	QGRAF	source irred.	no source loops	no tadpoles	masters
0	3	3	3	3	0
1	72	72	72	72	1
2	3286	3286	3286	2702	1
3	81526	62246	60998	41676	1
4	545812	264354	234934	116498	7
5	332020	128080	101570	27582	4

Table 1: Numbers of contributing diagrams at the different loop levels and master integrals.

From the graphs generated by `QGRAF` one has to remove the source reducible graphs, graphs with source loops and tadpoles. In this way the 962719 initial diagrams reduce to 188533 diagrams.

[5] At a later stage technical steps will require to move away from the harmonic gauge, which we will explain in detail.

The computation time amounts to about one week, including the time for the IBP reduction, on an `Intel(R) Xeon(R) CPU E5-2643 v4` and it grows exponentially with the loop order. Most of the CPU time is needed to perform the time derivatives. Only one non–trivial master integral contributes, see [8, 11].

One first obtains a Lagrange function of mth order still containing the accelerations a_i and time derivatives thereof. They are removed by using first double zero insertions [48, 49] together with partial integration and the remaining linear accelerations by a shift [25, 49–51], cf. [9]. By this operation we leave harmonic coordinates. A Legendre transformation leads then to the potential contributions of the Hamiltonian, which still contains pole terms in the dimensional parameter ε.[6] The reduced Hamiltonian in the cms is given by

$$\hat{H} = \frac{H - Mc^2}{\mu c^2}, \tag{4}$$

with c the velocity of light, $M = m_1 + m_2$ the rest mass of the binary system and $\mu = m_1 m_2/M$,

$$\begin{aligned}
\hat{H}^{\rm pot}_{\rm 5PN} = & -\frac{21p^{12}}{1024} + \frac{5}{16r^6} - \frac{125p^2}{16r^5} - \frac{499p^4}{64r^4} - \frac{161p^6}{32r^3} - \frac{445p^8}{256r^2} - \frac{77p^{10}}{256r} + \frac{17(p.n)^2}{4r^5} + \frac{29p^2(p.n)^2}{8r^4} \\
& + \frac{21p^4(p.n)^2}{16r^3} + \frac{5p^6(p.n)^2}{32r^2} - \frac{(p.n)^4}{8r^4} + \frac{1}{\varepsilon}\Biggl\{\nu^2\Biggl[-\frac{520909}{37800r^6} - \frac{698242p^2}{4725r^5} + \frac{592957p^4}{2520r^4} \\
& - \frac{13583p^6}{336r^3} + \frac{23569(p.n)^2}{540r^5} - \frac{1895597p^2(p.n)^2}{2520r^4} + \frac{23047p^4(p.n)^2}{112r^3} + \frac{16223(p.n)^4}{28r^4} \\
& - \frac{130p^2(p.n)^4}{r^3} - \frac{91(p.n)^6}{6r^3}\Biggr] + \nu\Biggl[-\frac{272309}{12600r^6} + \frac{22439p^2}{12600r^5} - \frac{49023p^4}{560r^4} + \frac{1173p^6}{80r^3} \\
& - \frac{210947(p.n)^2}{2520r^5} + \frac{25169p^2(p.n)^2}{105r^4} - \frac{2271p^4(p.n)^2}{80r^3} - \frac{13059(p.n)^4}{70r^4} - \frac{81p^2(p.n)^4}{r^3} \\
& + \frac{77(p.n)^6}{r^3}\Biggr] + \nu^3\Biggl[\frac{28811p^2}{210r^5} - \frac{297509p^4}{2520r^4} - \frac{6889p^6}{360r^3} - \frac{3068(p.n)^2}{7r^5} + \frac{352834p^2(p.n)^2}{315r^4} \\
& + \frac{16538p^4(p.n)^2}{105r^3} - \frac{304669(p.n)^4}{240r^4} - \frac{18979p^2(p.n)^4}{56r^3} + \frac{2891(p.n)^6}{12r^3}\Biggr]\Biggr\} \\
& + \nu\Biggl[\frac{231p^{12}}{1024} - \frac{253555919}{529200r^6} - \frac{1457872519p^2}{2116800r^5} + \frac{2128837091p^4}{1411200r^4} + \frac{11206267p^6}{141120r^3} + \frac{937p^8}{32r^2} \\
& + \frac{805p^{10}}{256r} + \pi^2\Biggl(\frac{70399}{1152r^6} + \frac{65291p^2}{1152r^5} - \frac{1328147p^4}{12288r^4} - \frac{7719p^6}{4096r^3} + \frac{6649(p.n)^2}{576r^5} \\
& + \frac{5042575p^2(p.n)^2}{6144r^4} + \frac{58887p^4(p.n)^2}{4096r^3} - \frac{3293913(p.n)^4}{4096r^4} - \frac{89625p^2(p.n)^4}{4096r^3} \\
& + \frac{42105(p.n)^6}{4096r^3}\Biggr) + \ln\left(\frac{r}{r_0}\right)\Biggl(-\frac{272309}{1050r^6} + \frac{22439p^2}{1260r^5} - \frac{49023p^4}{70r^4} + \frac{3519p^6}{40r^3} - \frac{210947(p.n)^2}{252r^5} \\
& + \frac{201352p^2(p.n)^2}{105r^4} - \frac{6813p^4(p.n)^2}{40r^3} - \frac{52236(p.n)^4}{35r^4} - \frac{486p^2(p.n)^4}{r^3} + \frac{462(p.n)^6}{r^3}\Biggr) \\
& + \frac{467022407(p.n)^2}{2116800r^5} - \frac{2385014243p^2(p.n)^2}{282240r^4} - \frac{162949463p^4(p.n)^2}{235200r^3} - \frac{589p^6(p.n)^2}{16r^2}
\end{aligned}$$

[6] As also the case in renormalizable quantum field theories, Langrangians and Hamiltonians are in general no observables and are generally singular.

$$
-\frac{35p^8(p.n)^2}{256r}+\frac{1895797259(p.n)^4}{235200r^4}+\frac{31715507p^2(p.n)^4}{23520r^3}+\frac{8951p^4(p.n)^4}{384r^2}
-\frac{627281(p.n)^6}{960r^3}-\frac{5117p^2(p.n)^6}{320r^2}+\frac{159(p.n)^8}{28r^2}\Bigg]+\nu^2\Bigg[-\frac{231p^{12}}{256}+\frac{295859}{1050r^6}
+\frac{1652383903p^2}{529200r^5}-\frac{420686323p^4}{132300r^4}+\frac{3605263p^6}{29400r^3}-\frac{11535p^8}{128r^2}-\frac{2865p^{10}}{256r}
+\ln\left(\frac{r}{r_0}\right)\Bigg(-\frac{520909}{3150r^6}-\frac{1396484p^2}{945r^5}+\frac{592957p^4}{315r^4}-\frac{13583p^6}{56r^3}+\frac{23569(p.n)^2}{54r^5}
-\frac{1895597p^2(p.n)^2}{315r^4}+\frac{69141p^4(p.n)^2}{56r^3}+\frac{32446(p.n)^4}{7r^4}-\frac{780p^2(p.n)^4}{r^3}-\frac{91(p.n)^6}{r^3}\Bigg)
+\pi^2\Bigg(\frac{11573}{768r^6}-\frac{121315p^2}{768r^5}+\frac{2076041p^4}{12288r^4}+\frac{29987p^6}{4096r^3}+\frac{200359(p.n)^2}{768r^5}
-\frac{5962205p^2(p.n)^2}{6144r^4}-\frac{172311p^4(p.n)^2}{4096r^3}+\frac{2617363(p.n)^4}{4096r^4}+\frac{127125p^2(p.n)^4}{4096r^3}
+\frac{14175(p.n)^6}{4096r^3}\Bigg)-\frac{944072707(p.n)^2}{264600r^5}+\frac{35606467999p^2(p.n)^2}{2116800r^4}-\frac{1945067p^4(p.n)^2}{2450r^3}
+\frac{4969p^6(p.n)^2}{64r^2}+\frac{275p^8(p.n)^2}{256r}-\frac{1742633989(p.n)^4}{117600r^4}+\frac{848889p^2(p.n)^4}{1568r^3}+\frac{925p^4(p.n)^4}{24r^2}
+\frac{15p^6(p.n)^4}{128r}+\frac{18031(p.n)^6}{3360r^3}-\frac{8331p^2(p.n)^6}{160r^2}+\frac{751(p.n)^8}{28r^2}\Bigg]+\nu^3\Bigg[\frac{1617p^{12}}{1024}
-\frac{298537367p^2}{151200r^5}+\frac{617770201p^4}{423360r^4}+\frac{108551131p^6}{4233600r^3}+\frac{16283p^8}{256r^2}+\frac{3995p^{10}}{256r}+\pi^2
\times\Bigg(-\frac{2339p^2}{192r^5}+\frac{98447p^4}{3072r^4}-\frac{20259p^6}{1024r^3}-\frac{16111(p.n)^2}{192r^5}+\frac{131231p^2(p.n)^2}{1536r^4}
+\frac{106947p^4(p.n)^2}{1024r^3}-\frac{361499(p.n)^4}{1024r^4}-\frac{30075p^2(p.n)^4}{1024r^3}-\frac{65625(p.n)^6}{1024r^3}\Bigg)
+\ln\left(\frac{r}{r_0}\right)\Bigg(\frac{28811p^2}{21r^5}-\frac{297509p^4}{315r^4}-\frac{6889p^6}{60r^3}-\frac{30680(p.n)^2}{7r^5}+\frac{2822672p^2(p.n)^2}{315r^4}
+\frac{33076p^4(p.n)^2}{35r^3}-\frac{304669(p.n)^4}{30r^4}-\frac{56937p^2(p.n)^4}{28r^3}+\frac{2891(p.n)^6}{2r^3}\Bigg)+\frac{966353501(p.n)^2}{151200r^5}
-\frac{3656476457p^2(p.n)^2}{235200r^4}-\frac{2369976949p^4(p.n)^2}{1411200r^3}+\frac{177p^6(p.n)^2}{256r^2}-\frac{221p^8(p.n)^2}{64r}
+\frac{14035555739(p.n)^4}{705600r^4}+\frac{373945981p^2(p.n)^4}{94080r^3}-\frac{125225p^4(p.n)^4}{768r^2}-\frac{3p^6(p.n)^4}{128r}
-\frac{14830647(p.n)^6}{4480r^3}+\frac{136977p^2(p.n)^6}{1280r^2}-\frac{15p^4(p.n)^6}{128r}-\frac{289839(p.n)^8}{4480r^2}-\frac{35p^2(p.n)^8}{256r}\Bigg]
+\nu^4\Bigg[-\frac{1155p^{12}}{1024}-\frac{593p^6}{32r^3}+\frac{6649p^8}{256r^2}-\frac{1615p^{10}}{256r}+\frac{549p^4(p.n)^2}{32r^3}-\frac{62143p^6(p.n)^2}{256r^2}
$$

$$
+\frac{867p^8(p.n)^2}{256r}-\frac{5749p^2(p.n)^4}{96r^3}-\frac{3p^6(p.n)^4}{64r}+\frac{652381p^4(p.n)^4}{768}-\frac{17623(p.n)^6}{240r^3}
-\frac{1178329p^2(p.n)^6}{1280r^2}-\frac{45p^4(p.n)^6}{128r}+\frac{1443091(p.n)^8}{4480r^2}+\frac{105p^2(p.n)^8}{128r}\Bigg]+\nu^5\Bigg[\frac{231p^{12}}{1024}-\frac{63p^{10}}{256r}
-\frac{35p^8(p.n)^2}{256r}-\frac{15p^6(p.n)^4}{128r}-\frac{15p^4(p.n)^6}{128r}-\frac{35p^2(p.n)^8}{256r}-\frac{63(p.n)^{10}}{256r}\Bigg], \qquad (5)
$$

with

$$
r_0=\frac{e^{-\gamma_E/2}}{2\sqrt{\pi}\mu_1}, \qquad (6)
$$

where γ_E is the Euler–Mascheroni constant and μ_1 the mass scale accounting for Newton's constant $G_N \to G_N\mu_1^{-2\varepsilon}$ in D dimensions. The corresponding contributions up to 4PN have been presented in [9] before. We rescale

$$
p = p_{\rm phys}/(\mu c), \quad r=\left(G_N M/c^2\right) r_{\rm phys}, \qquad (7)
$$

where p and r are now the rescaled (dimensionless) cms momentum and the distance of the two masses, with $\vec{n}=\vec{r}/r$. In the following we will as widely as possible work with dimensionless quantities.

Pole and logarithmic contributions appear at $O(\nu), O(\nu^2)$ and $O(\nu^3)$, in accordance with the lower PN orders, where also always one more order in ν contributes from 3PN onward. We will see in Section 4 that the tail term is only singular for $O(\nu)$ and $O(\nu^2)$ at 5PN.

In the Schwarzschild limit, $\nu \to 0$, one obtains the following contributions,

$$
\hat{H}_{\rm 5PN}^{\rm Schw} = -\frac{21p^{12}}{1024}-\frac{77p^{10}}{256r}-\frac{445p^8}{256r^2}-\frac{161p^6}{32r^3}-\frac{499p^4}{64r^4}-\frac{125p^2}{16r^5}+\frac{5p^6(p.n)^2}{32r^2}+\frac{21p^4(p.n)^2}{16r^3}
+\frac{29p^2(p.n)^2}{8r^4}-\frac{(p.n)^4}{8r^4}+\frac{17(p.n)^2}{4r^5}+\frac{5}{16r^6}, \qquad (8)
$$

in agreement with the expansion of Eq. (30), [9], to 5PN, cf. [52, 53].

3 Remarks on the tail term

We will derive a pole–free Hamiltonian at 5PN in Section 4. For this we will add the singular and logarithmic terms of the tail term, $\hat{H}_{\rm 5PN}^{\rm tail,sing,log}$, to the potential term $\hat{H}_{\rm 5PN}^{\rm pot}$. Since these contributions are calculated, by different methods, either in the far zone (FZ) or the near zone (NZ), the question arises whether potential overlap contributions have to be considered. We remind that the calculation is performed in D dimensions, not using any other regularization.

One may apply the method of expansion by regions, which has been introduced for the asymptotic expansion of Feynman integrals for bound states in the non–relativistic limit in [42, 43]. Here each loop integral is split into four *distinct* momentum regions, which are denoted as hard, soft, potential, and ultrasoft. Integrals over the hard and soft region correspond to quantum corrections and are not considered in the context of classical gravity.

The potential region, characterized by the momentum scaling

$$
|k_0| \sim \frac{v}{R}, \qquad |k_i| \sim \frac{1}{R}, \qquad (9)
$$

with $v \in [v_1, v_2]$ and $v = v_{\rm phys}/c$ the typical velocities, is also referred to as orbital region. Here k_i and R are not rescaled. However, we set the associated action variable to 1.[7] It can be identified with the near zone of the literature (i.e. the potential terms). In the ultrasoft (or radiation region), corresponding to the far zone (i.e. the tail terms), momenta exhibit the uniform four–momentum scaling

$$|k_\mu| \sim \frac{v}{R} \,. \tag{10}$$

The kinematic region of the potential term is

$$k_i \in D_{\rm pot} = \left[-\infty, -\tfrac{1}{R}\right] \cup \left[\tfrac{1}{R}, \infty\right], \tag{11}$$

with R of the order of separation the binary system. Likewise, the one of the tail term is

$$k_i \in D_{\rm us} = \left[-\tfrac{1}{R}, \tfrac{1}{R}\right]. \tag{12}$$

In the former region the exchanged fields are potential gravitons and in the latter region ultrasoft gravitons. One performs a Taylor expansion of the integrands according to the respective momentum scaling in v up to the respective post–Newtonian order by observing that

$$v^2 \sim \frac{1}{r}. \tag{13}$$

For the tail terms this expansion includes the multi–pole expansion, which we will discuss below.[8] Let us introduce the operators $T_{\rm pot}$ and $T_{\rm us}$, which describe the Taylor expansions (with a few Laurent–terms in some cases) in the potential region and the ultrasoft region. In the post–Newtonian expansion they are given, more precisely, by

$$T_i^{\rm N} I_0(v) := \theta(N) \sum_{k=0}^{\infty} T_{i,k} v^k, \tag{14}$$

with the quantifier $\theta(N)$ truncating the series at a maximal term $v^N, N \in \mathbb{N}$, which is idempotent $\theta^l(N) \equiv \theta(N)$. Here the coefficients $T_{i,k}$ denote the expansion coefficients of the function $I_0(v)$. The corresponding integrals have the following form

$$I_1 = \int_{D_{\rm pot}} dk_i T^N_{\rm pot} I + \int_{D_{\rm us}} dk_i T^N_{\rm us} I \tag{15}$$

for each of the D components k_i, where I denotes the original integrand. One further obtains

$$I_1 = \int_{-\infty}^{+\infty} dk_i T^N_{\rm pot} I - \int_{D_{\rm us}} dk_i T^N_{\rm pot} I + \int_{-\infty}^{+\infty} dk_i T^N_{\rm us} I - \int_{D_{\rm pot}} dk_i T^N_{\rm us} I. \tag{16}$$

In the respective domains $D_{\rm pot}$ and $D_{\rm us}$ one may further apply the operators $T_{\rm pot}$ and $T_{\rm us}$ given the post–Newtonian accuracy one is working in. One then obtains

$$I_1 = \int_{-\infty}^{+\infty} dk_i T^N_{\rm pot} I - \int_{D_{\rm us}} dk_i T_{\rm us} T^N_{\rm pot} I + \int_{-\infty}^{+\infty} dk_i T^N_{\rm us} I - \int_{D_{\rm pot}} dk_i T_{\rm pot} T^N_{\rm us} I. \tag{17}$$

[7] In the quantum field theoretic case this would correspond to $\hbar = 1$.

[8] As is often the case in EFT representations, the corresponding expansions are not just kinematic. An important example in this respect is the light-cone expansion [54]. In the most simple case of the twist–2 contributions its results are also obtained by the QCD improved parton model, resulting from a kinematic expansion. This is much more subtle at higher twist, where partonic pictures require further conditions to give the same result, cf. [55] for a survey.

Here the 2nd and 4th term are the overlap integrals. Eq. (17) can be further arranged to

$$I_1 = \int_{-\infty}^{+\infty} dk_i T^N_{\rm pot} I + \int_{-\infty}^{+\infty} dk_i T^N_{\rm us} I - \int_{-\infty}^{+\infty} dk_i T^N_{\rm pot} T^N_{\rm us} I, \tag{18}$$

provided that

$$T^N_{\rm us} T^N_{\rm pot} - T^N_{\rm pot} T^N_{\rm us} = 0 \tag{19}$$

holds, which we prove in Appendix A. Furthermore, the operation $T_{\rm us} T_{\rm pot}$ leads to scaleless integrands, implying that the last term in Eq. (18) vanishes in D dimensions, see Appendix A.

We finally would would like to make some remarks on the relation on the multi–pole expansion [30] in the far zone to the ultrasoft region. One is starting from the full theory of general relativity in harmonic coordinates, i.e. the bulk action

$$S_{\rm GR,bulk} = 2\Lambda^2 \int d^D x \, \sqrt{-g} \left(R - \frac{1}{2} \Gamma^\mu \Gamma_\mu \right). \tag{20}$$

Here $\Lambda = c^2 \mu_1^\varepsilon / \sqrt{32\pi G_N}$, R is the Ricci scalar, $\Gamma^\mu = \Gamma^\mu_{\alpha\beta} g^{\alpha\beta}$ with $\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma} (g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma})$ is the Christoffel symbol. (20) is coupled to compact objects via the action

$$S_{\rm pp} = -\sum_{a=1}^{2} m_a \int d\tau_a, \tag{21}$$

with proper times τ_1, τ_2, one decomposes the metric into

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{\Lambda} (H_{\mu\nu} + h_{\mu\nu}), \tag{22}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. The momenta associated with $H_{\mu\nu}$ are of the potential type and the momenta of the $h_{\mu\nu}$ fields are ultrasoft, cf. Eqs. (11) and (12). The resulting loop integrals therefore have the same form as Eq. (15), as obtained from the asymptotic expansion.

The action of the full theory is matched to the non–relativistic general relativity (NRGR) action by

$$S_{\rm NRGR} = S_{\rm NRGR,bulk} + S_{{\rm NRGR},h^0} + S_{{\rm NRGR},h^1} + \mathcal{O}\left(h^2\right), \tag{23}$$

with

$$S_{{\rm NRGR},h^0} = \int dt \, (T - V_{\rm NZ}), \tag{24}$$

$$S_{{\rm NRGR},h^1} = \frac{1}{2\Lambda} \int d^D x \, T^{\mu\nu} h_{\mu\nu}, \tag{25}$$

where there are no potential modes anymore. Here T denotes the kinetic term and $V_{\rm NZ}$ the near–zone potential. $S_{\rm NRGR,bulk}$ is the same as the general relativity bulk action $S_{\rm GR,bulk}$ from Eq. (20), but without the potential contributions to the metric. Both $V_{\rm NZ}$ and the effective stress–energy tensor $T^{\mu\nu}$ are fixed by requiring that the NRGR action produces the same predictions as the asymptotic expansion of the full theory. In other words, the integrals over the potential region are absorbed into $V_{\rm NZ}$ and $T^{\mu\nu}$.

We now elaborate on the relation to the multi–pole expansion. Consider

$$h_{\mu\nu}(x) = \int \frac{d^D p}{(2\pi)^D} e^{ipx} h_{\mu\nu}(p), \tag{26}$$

where the momentum is ultrasoft by definition, i.e.

$$\vec{p}\vec{x} \sim v. \tag{27}$$

We can Taylor expand the exponential in Eq. (26) to obtain

$$h_{\mu\nu}(x) = \int \frac{d^D p}{(2\pi)^D} e^{-ip_0 x_0} \left(\sum_{n=0}^{N} \frac{(i\vec{p}\vec{x})^n}{n!} + \mathcal{O}\left(v^{N+1}\right) \right) h_{\mu\nu}(p)\,. \tag{28}$$

Rewriting

$$i^n p_{i_1} \dots p_{i_n} = \left[\partial x_{i_1} \dots \partial x_{i_n} e^{i\vec{p}\vec{x}} \right]_{\vec{x}=0} \tag{29}$$

now yields

$$\begin{aligned} h_{\mu\nu}(x) &= \sum_{n=0}^{N} \frac{1}{n!} x_{i_1} \dots x_{i_n} \left[\partial x_{i_1} \dots \partial x_{i_n} \int_{-\infty}^{\infty} \frac{d^D p}{(2\pi)^D} e^{ipx} h_{\mu\nu}(p) \right]_{\vec{x}=0} + \mathcal{O}\left(v^{N+1}\right) \\ &= \sum_{n=0}^{N} \frac{1}{n!} x_{i_1} \dots x_{i_n} \left[\partial x_{i_1} \dots \partial x_{i_n} h_{\mu\nu}(x) \right]_{\vec{x}=0} + \mathcal{O}\left(v^{N+1}\right). \end{aligned} \tag{30}$$

Inserting this expression back into the linear ultrasoft action Eq. (25) we retrieve the familiar starting point of the multi–pole expansion

$$S_{\mathrm{NRGR},h^1} = \frac{1}{2\Lambda} \int d^D x\, T^{\mu\nu}(x) \sum_{n=0}^{N} \frac{1}{n!} x_{i_1} \dots x_{i_n} \left[\partial x_{i_1} \dots \partial x_{i_n} h_{\mu\nu}(x) \right]_{\vec{x}=0} + \mathcal{O}\left(v^{N+1}\right)\,, \tag{31}$$

with an explicit velocity power counting. The remaining steps in the multi–pole expansion are standard. In short, one defines moments

$$M_n^{\mu\nu} = \int d^{D-1}\vec{x}\, T^{\mu\nu}(x) x_{i_1} \dots x_{i_n}\,, \tag{32}$$

and decomposes them into irreducible $SO(3)$ spherical tensors, choosing a symmetric trace–free (STF) basis, e.g. [3, 36]. The tail terms are represented by the multi–pole expansion valid in the far zone. In our treatment we will follow Refs. [38, 40].

4 The pole-free Hamiltonian at 5PN

It is convenient to work with pole–free Hamiltonians and we add the singular and logarithmic pieces of the Hamiltonian of the tail term in 5PN, $\hat{H}_{\mathrm{5PN}}^{\mathrm{tail,sing,log}}$,

$$\begin{aligned} \hat{H}_{\mathrm{5PN}}^{\mathrm{tail,sing,log}} = & \frac{1}{\varepsilon} \Bigg\{ \left(\frac{16\nu}{105} - \frac{332\nu^2}{105} \right) \frac{1}{r^6} + \left[\left(\frac{236\nu}{35} - \frac{212\nu^2}{35} \right) p^2 - \left(\frac{684\nu}{35} + \frac{1264\nu^2}{105} \right) (p.n)^2 \right] \frac{1}{r^5} \\ & + \left[\left(\frac{533\nu}{21} + \frac{706\nu^2}{21} \right) p^4 - \left(\frac{7732\nu}{35} + \frac{10936\nu^2}{105} \right) p^2 (p.n)^2 + \left(\frac{6197\nu}{35} + \frac{2656\nu^2}{35} \right) \times \right. \\ & \left. (p.n)^4 \right] \frac{1}{r^4} + \left[\left(\frac{94\nu}{15} - \frac{94\nu^2}{5} \right) p^6 + \left(-\frac{172\nu}{5} + \frac{516\nu^2}{5} \right) p^4 (p.n)^2 + \left(26\nu - 78\nu^2 \right) \right. \end{aligned}$$

$$\times p^2(p.n)^4\Bigg]\frac{1}{r^3}\Bigg\}$$

$$+\Bigg\{\left[\frac{128\nu}{105}-\frac{2656\nu^2}{105}\right]\frac{1}{r^6}+\Bigg[\left(\frac{1416\nu}{35}-\frac{1272\nu^2}{35}\right)p^2-\left(\frac{4104\nu}{35}+\frac{2528\nu^2}{35}\right)$$

$$\times(p.n)^2\Bigg]\frac{1}{r^5}+\Bigg[\left(\frac{2132\nu}{21}+\frac{2824\nu^2}{21}\right)p^4-\left(\frac{30928\nu}{35}+\frac{43744\nu^2}{105}\right)p^2(p.n)^2$$

$$+\left(\frac{24788\nu}{35}+\frac{10624\nu^2}{35}\right)(p.n)^4\Bigg]\frac{1}{r^4}+\Bigg[\left(\frac{188\nu}{15}-\frac{188\nu^2}{5}\right)p^6+\Bigg(-\frac{344\nu}{5}$$

$$+\frac{1032\nu^2}{5}\Bigg)p^4(p.n)^2+\Bigg(52\nu-156\nu^2\Bigg)p^2(p.n)^4\Bigg]\frac{1}{r^3}\Bigg\}\ln\left(\frac{r}{r_0}\right) \tag{33}$$

to $\hat{H}^{\rm pot}_{\rm 5PN}$. For all contributions resulting into Eq. (33) we agree with the integrals in the multi–pole expansion in [40].

The sum of the potential term and this contribution is not pole–free yet, as is the case from 3PN onward, cf. [9]. However, after performing the following canonical transformation a pole–free Hamiltonian is obtained, which is not the case for $H^{\rm pot}_{\rm 5PN}$ and $H^{\rm tail,sing}_{\rm 5PN}$ individually. By this transformation one further moves away from the harmonic coordinates, which were used at the starting point of the calculation. Still a prediction of all observables is possible. Moreover, the comparison with EOB results becomes simpler, since they are given in pole–free form [13].

Following the formalism described in Ref. [9], Eqs. (38–41), one obtains the corresponding generating function

$$\begin{aligned}G(p^2,p.n,r;\varepsilon) \;=\; & p.n\Bigg\{\frac{1}{\varepsilon}\Bigg\{-t_3\frac{17\nu}{6r^2}+t_4\Bigg[\frac{1}{r^2}\left(\frac{1}{90}\nu(585+4\nu)p^2-\frac{1}{3}\nu(12+37\nu)(p.n)^2\right)\\ & -\frac{\nu(65+264\nu)}{30r^3}\Bigg]+t_5\Bigg[\frac{1}{r^3}\Bigg(-\frac{1}{420}\nu\big(-43-69088\nu+52078\nu^2\big)p^2\\ & +\frac{\nu\big(1785-204868\nu+264212\nu^2\big)(p.n)^2}{1260}\Bigg)+\frac{1}{r^2}\Bigg(\\ & -\frac{\nu\big(-127218+374300\nu+70007\nu^2\big)p^4}{5040}-\frac{1}{12}\nu\big(132-26\nu+413\nu^2\big)(p.n)^4\\ & +\frac{\nu\big(-1680+36624\nu+43567\nu^2\big)p^2(p.n)^2}{840}\Bigg)+\frac{\nu(889917+388114\nu)}{37800r^4}\Bigg]\Bigg\}\Bigg\}.\end{aligned} \tag{34}$$

Furthermore, we transform the logarithmic part to explicitly match the structure of the non–local contribution from the tail term in harmonic coordinates,

$$\delta H^{\rm 4+5PN}_{\rm log}=2\frac{G_N^3E}{c^{10}}\left(\frac{1}{5}I^{(3)}(t)^2+\frac{1}{189c^2}O^{(4)}(t)^2+\frac{16}{45c^2}J^{(3)}(t)^2\right)\ln\left(\frac{r}{r_0}\right), \tag{35}$$

see also Section 5.2. Here the multi–pole moments I_{ab}, O_{abc} and J_{ab} are those of Eq. (2.4) in [13], with indices contracted, and E is the total energy. The corresponding transformation reads

$$G(p^2,p.n,r;\ln(r/r_0)) \;=\; p.n\Bigg\{\ln\left(\frac{r}{r_0}\right)\Bigg\{-t_3\frac{17\nu}{r^2}+t_4\Bigg[\frac{1}{r^2}\Bigg(\frac{\nu(585+4\nu)p^2}{15}-2\nu(12+37\nu)$$

$$
\times(p.n)^2\Big) - \frac{4\nu(73+264\nu)}{15r^3}\Bigg] + t_5\Bigg[\frac{1}{r^3}\Bigg(-\frac{2}{105}\nu\big(7557 - 65496\nu
+52078\nu^2\big)p^2 + \frac{2}{315}\nu\big(21345 - 195484\nu + 264212\nu^2\big)(p.n)^2\Bigg)
+\frac{1}{r^2}\Bigg(-\frac{1}{840}\nu\big(-106162 + 311132\nu + 70007\nu^2\big)p^4 + \frac{1}{140}\nu\big(1232
+27888\nu + 43567\nu^2\big)p^2(p.n)^2 - \frac{1}{2}\nu\big(132 - 26\nu + 413\nu^2\big)(p.n)^4\Bigg)
+\frac{\nu(886461+331090\nu)}{3780r^4}\Bigg]\Bigg\}\Bigg\}. \qquad (36)
$$

Here $t_i, i = 1...5$ labels the ith post–Newtonian order.

The pole–free Hamiltonian based on the above contributions is then given by[9]

$$
\begin{aligned}
\hat{H}_{5PN}^{\rm polefree} = & -\frac{21p^{12}}{1024} + \frac{5}{16r^6} - \frac{125p^2}{16r^5} - \frac{499p^4}{64r^4} - \frac{161p^6}{32r^3} - \frac{445p^8}{256r^2} - \frac{77p^{10}}{256r} \\
& +\nu\Bigg[\frac{231p^{12}}{1024} - \frac{279775133}{529200r^6} - \frac{1450584679p^2}{2116800r^5} + \frac{2010713771p^4}{1411200r^4} + \frac{11206267p^6}{141120r^3} + \frac{937p^8}{32r^2} \\
& +\frac{805p^{10}}{256r} + \ln\left(\frac{r}{r_0}\right)\Bigg(\frac{64}{105r^6} - \frac{18944p^2}{105r^5} + \frac{1796p^4}{105r^4} + \frac{19136(p.n)^2}{105r^5} - \frac{10664p^2(p.n)^2}{105r^4} \\
& +\frac{2748(p.n)^4}{35r^4}\Bigg) + \pi^2\Bigg(\frac{70399}{1152r^6} + \frac{65291p^2}{1152r^5} - \frac{1328147p^4}{12288r^4} - \frac{7719p^6}{4096r^3} + \frac{6649(p.n)^2}{576r^5} \\
& +\frac{5042575p^2(p.n)^2}{6144r^4} + \frac{58887p^4(p.n)^2}{4096r^3} - \frac{3293913(p.n)^4}{4096r^4} - \frac{89625p^2(p.n)^4}{4096r^3} \\
& +\frac{42105(p.n)^6}{4096r^3}\Bigg) - \frac{34541593(p.n)^2}{2116800r^5} - \frac{2395722563p^2(p.n)^2}{282240r^4} - \frac{62196341p^4(p.n)^2}{78400r^3} \\
& -\frac{589p^6(p.n)^2}{16r^2} - \frac{35p^8(p.n)^2}{256r} + \frac{631107353(p.n)^4}{78400r^4} + \frac{31226291p^2(p.n)^4}{23520r^3} \\
& +\frac{8951p^4(p.n)^4}{384r^2} - \frac{563921(p.n)^6}{960r^3} - \frac{5117p^2(p.n)^6}{320r^2} + \frac{159(p.n)^8}{28r^2}\Bigg] + \nu^2\Bigg[-\frac{231p^{12}}{256} \\
& +\frac{72454}{225r^6} + \frac{1353196483p^2}{529200r^5} - \frac{787300061p^4}{264600r^4} + \frac{3605263p^6}{29400r^3} - \frac{11535p^8}{128r^2} - \frac{2865p^{10}}{256r} \\
& -\ln\left(\frac{r}{r_0}\right)\Bigg(\frac{256}{105r^6} + \frac{3392p^2}{105r^5} - \frac{432p^4}{35r^4} - \frac{2992(p.n)^2}{105r^5} - \frac{6824p^2(p.n)^2}{105r^4} + \frac{496(p.n)^4}{7r^4}\Bigg) \\
& +\pi^2\Bigg(\frac{5453}{768r^6} - \frac{121315p^2}{768r^5} + \frac{2076041p^4}{12288r^4} + \frac{29987p^6}{4096r^3} + \frac{200359(p.n)^2}{768r^5} - \frac{172311p^4(p.n)^2}{4096r^3} \\
& -\frac{5962205p^2(p.n)^2}{6144r^4} + \frac{2617363(p.n)^4}{4096r^4} + \frac{127125p^2(p.n)^4}{4096r^3} + \frac{14175(p.n)^6}{4096r^3}\Bigg)
\end{aligned}
$$

[9]Note that $\hat{H}_{5PN}^{\rm polefree}$ does not yet contain the complete local Hamiltonian.

$$
\begin{aligned}
&-\frac{857318207(p.n)^2}{264600r^5}+\frac{34200172759p^2(p.n)^2}{2116800r^4}-\frac{5034763p^4(p.n)^2}{9800r^3}+\frac{4969p^6(p.n)^2}{64r^2}\\
&+\frac{275p^8(p.n)^2}{256r}-\frac{4989943687(p.n)^4}{352800r^4}+\frac{2674877p^2(p.n)^4}{7840r^3}+\frac{925p^4(p.n)^4}{24r^2}+\frac{15p^6(p.n)^4}{128r}\\
&-\frac{25649(p.n)^6}{3360r^3}-\frac{8331p^2(p.n)^6}{160r^2}+\frac{751(p.n)^8}{28r^2}\Bigg]+\nu^3\Bigg[\frac{1617p^{12}}{1024}-\frac{238966727p^2}{151200r^5}\\
&+\frac{127702733p^4}{84672r^4}+\frac{108551131p^6}{4233600r^3}+\frac{16283p^8}{256r^2}+\frac{3995p^{10}}{256r}+\pi^2\Bigg(-\frac{2339p^2}{192r^5}+\frac{98447p^4}{3072r^4}\\
&-\frac{20259p^6}{1024r^3}-\frac{16111(p.n)^2}{192r^5}+\frac{131231p^2(p.n)^2}{1536r^4}+\frac{106947p^4(p.n)^2}{1024r^3}-\frac{361499(p.n)^4}{1024r^4}\\
&-\frac{30075p^2(p.n)^4}{1024r^3}-\frac{65625(p.n)^6}{1024r^3}\Bigg)+\frac{758233181(p.n)^2}{151200r^5}-\frac{10374288811p^2(p.n)^2}{705600r^4}\\
&-\frac{2207947669p^4(p.n)^2}{1411200r^3}+\frac{177p^6(p.n)^2}{256r^2}-\frac{221p^8(p.n)^2}{64r}+\frac{12810612439(p.n)^4}{705600r^4}\\
&+\frac{355111837p^2(p.n)^4}{94080r^3}-\frac{125225p^4(p.n)^4}{768r^2}-\frac{3p^6(p.n)^4}{128r}-\frac{13905527(p.n)^6}{4480r^3}\\
&+\frac{136977p^2(p.n)^6}{1280r^2}-\frac{15p^4(p.n)^6}{128r}-\frac{289839(p.n)^8}{4480r^2}-\frac{35p^2(p.n)^8}{256r}\Bigg]+\nu^4\Bigg[-\frac{1155p^{12}}{1024}\\
&-\frac{593p^6}{32r^3}+\frac{6649p^8}{256r^2}-\frac{1615p^{10}}{256r}+\frac{549p^4(p.n)^2}{32r^3}-\frac{62143p^6(p.n)^2}{256r^2}+\frac{867p^8(p.n)^2}{256r}\\
&-\frac{5749p^2(p.n)^4}{96r^3}+\frac{652381p^4(p.n)^4}{768r^2}-\frac{3p^6(p.n)^4}{64r}-\frac{17623(p.n)^6}{240r^3}-\frac{1178329p^2(p.n)^6}{1280r^2}\\
&-\frac{45p^4(p.n)^6}{128r}+\frac{1443091(p.n)^8}{4480r^2}+\frac{105p^2(p.n)^8}{128r}\Bigg]+\nu^5\Bigg[\frac{231p^{12}}{1024}-\frac{63p^{10}}{256r}-\frac{35p^8(p.n)^2}{256r}\\
&-\frac{15p^6(p.n)^4}{128r}-\frac{15p^4(p.n)^6}{128r}-\frac{35p^2(p.n)^8}{256r}-\frac{63(p.n)^{10}}{256r}\Bigg]+\frac{17(p.n)^2}{4r^5}+\frac{29p^2(p.n)^2}{8r^4}\\
&+\frac{21p^4(p.n)^2}{16r^3}+\frac{5p^6(p.n)^2}{32r^2}-\frac{(p.n)^4}{8r^4}\ . \qquad (37)
\end{aligned}
$$

By this we have shown in explicit form the cancellation of the singularities originally occurring in harmonic coordinates, for reasons of *regularization* only. In the case of the binary point–mass problem up to 5PN order no singularities survive requiring another method to be removed. At 4PN this has also been shown in Ref. [9], see also [56]. Furthermore, logarithmic terms do now only occur at $O(\nu)$ and $O(\nu^2)$.

5 Comparison to the literature

In the following we perform a series of comparisons with results in the literature.

5.1 Canonical transformation to EOB

Let us first compare to the EOB results of Ref. [13], Eq. (11.8), for the contributions at $O(\nu^0)$ and $O(\nu^3)$ and higher given in EOB coordinates in complete form.[10] These terms do not receive contributions due to tail terms and one can therefore just refer to the pole–free Hamiltonian of Section 4 to construct the canonical transformation.

It is given by

$$
\begin{aligned}
G(p^2, p.n, r) \;=\; & p.n \Bigg\{ t_1 \Bigg\{ \nu \left(-\frac{1}{2} + \frac{1}{2} p^2 r \right) - 1 \Bigg\} + t_2 \Bigg\{ \nu \Bigg(-\frac{5}{4} p^2 + \frac{5}{4r} - \frac{1}{8} p^4 r \\
& -\frac{1}{2}(p.n)^2 \Bigg) + \nu^2 \left(\frac{1}{4} p^2 - \frac{1}{4r} - \frac{1}{8}(p.n)^2 \right) \Bigg\} + t_3 \Bigg\{ \nu \Bigg(\frac{\frac{29}{6} p^2 + \frac{4}{3}(p.n)^2}{r} \\
& + \frac{7}{16} p^4 + \frac{1795 - 63\pi^2}{72 r^2} + \frac{1}{16} p^6 r + \frac{1}{4} p^2 (p.n)^2 \Bigg) \\
& + \nu^2 \Bigg(\frac{-\frac{19}{24} p^2 - \frac{83}{24}(p.n)^2}{r} - \frac{37}{96} p^4 - \frac{3}{16 r^2} - \frac{1}{16} p^6 r + \frac{7}{24} p^2 (p.n)^2 \\
& - \frac{1}{4}(p.n)^4 \Bigg) + \nu^3 \left(\frac{\frac{5}{24} p^2 - \frac{1}{8}(p.n)^2}{r} - \frac{1}{48} p^4 - \frac{3}{16 r^2} + \frac{1}{96} p^2 (p.n)^2 \right) \Bigg\} \\
& + t_4 \Bigg\{ \nu^3 \Bigg(\frac{1}{r} \left[-\frac{143}{16} p^4 + \frac{493}{32} p^2 (p.n)^2 + \frac{31}{80}(p.n)^4 \right] + \frac{-\frac{15}{64} p^2 - \frac{15}{64}(p.n)^2}{r^2} \\
& + \frac{229}{384} p^6 + \frac{7}{96 r^3} - \frac{1}{48} p^8 r + \frac{61}{384} p^4 (p.n)^2 - \frac{1}{16} p^2 (p.n)^4 \Bigg] \\
& + \nu^4 \Bigg[\frac{1}{r} \left(-\frac{1}{24} p^4 + \frac{1}{48} p^2 (p.n)^2 + \frac{1}{48}(p.n)^4 \right) + \frac{\frac{5}{24} p^2 - \frac{1}{8}(p.n)^2}{r^2} - \frac{1}{6 r^3} \\
& + \frac{1}{96} p^4 (p.n)^2 - \frac{7}{128} p^2 (p.n)^4 + \frac{5}{128}(p.n)^6 \Bigg] \Bigg\} \\
& + t_5 \Bigg\{ \nu^3 \Bigg[\frac{1}{r^3} \Bigg(-\frac{p^2 \big(-202645909 + 1786050\pi^2 \big)}{100800} \\
& + \frac{\big(-84723358 + 562625\pi^2 \big)(p.n)^2}{33600} \Bigg) + \frac{1}{r^2} \Bigg(-\frac{p^4 \big(502480088 + 75854205\pi^2 \big)}{6773760} \\
& + \frac{p^2 \big(-2251644296 + 120889125\pi^2 \big)(p.n)^2}{5644800} + \frac{\big(17505304 + 16879275\pi^2 \big)(p.n)^4}{2257920} \Bigg) \\
& + \frac{1}{r} \left(\frac{7447}{128} p^6 + \frac{235}{12} p^4 (p.n)^2 - \frac{21323}{384} p^2 (p.n)^4 + \frac{4937}{420}(p.n)^6 \right) + \frac{595 p^8}{1536} \\
& + \frac{-186973 + 12400\pi^2}{640 r^4} - \frac{5}{48} p^{10} r - \frac{347}{384} p^6 (p.n)^2 + \frac{109}{256} p^4 (p.n)^4
\end{aligned}
$$

[10]For definiteness we use the minimal choice (8.24) of the flexibility parameters.

$$
-\frac{155}{256}p^2(p.n)^6\Bigg] + \nu^4\Bigg[\frac{1}{r^2}\Bigg(-\frac{7121}{640}p^4 + \frac{105479p^2(p.n)^2}{5760} - \frac{7073(p.n)^4}{1440}\Bigg]
+\frac{1}{r}\Bigg(\frac{14675}{384}p^6 - \frac{100025p^4(p.n)^2}{1152} + \frac{484729p^2(p.n)^4}{5760} - \frac{63677(p.n)^6}{2688}\Bigg)
+\frac{-\frac{7}{40}p^2 + \frac{3829(p.n)^2}{2880}}{r^3} - \frac{1927}{768}p^8 + \frac{13}{96r^4} + \frac{145}{768}p^6(p.n)^2 + \frac{29}{128}p^4(p.n)^4
-\frac{203}{768}p^2(p.n)^6 + \frac{7}{48}(p.n)^8\Bigg] + \nu^5\Bigg[\frac{1}{r^2}\Bigg(-\frac{77p^4}{1280} + \frac{23}{720}p^2(p.n)^2 + \frac{697(p.n)^4}{11520}\Bigg)
+\frac{1}{r}\Bigg(-\frac{1}{384}p^6 + \frac{13}{576}p^4(p.n)^2 - \frac{199p^2(p.n)^4}{1440} + \frac{13}{128}(p.n)^6\Bigg) + \frac{1}{r^3}\Bigg(\frac{107}{480}p^2
-\frac{377(p.n)^2}{2880}\Bigg) + \frac{1}{768}p^8 - \frac{31}{192r^4} + \frac{p^6(p.n)^2}{4608} + \frac{37p^4(p.n)^4}{7680} - \frac{7p^2(p.n)^6}{1536}\Bigg]\Bigg\}\Bigg\}. \quad (38)
$$

In this way we confirm all the contributions of $O(\nu^0)$ and $O(\nu^3)$ or higher given in [13] by an explicit Feynman diagram calculation ab initio.

5.2 The non-local terms

Next we turn to the non–local terms defined in [13], cf. Eq. (3). We perform the eccentricity expansion of the non–local contributions for $\langle \delta H^{\rm nl}_{4+5\rm PN}\rangle$ with

$$
\frac{\langle \delta H^{\rm nl}_{4+5\rm PN}\rangle}{Mc^2} = \frac{n}{2\pi Mc^2}\int_0^{\frac{2\pi}{n}} dt\ \delta H(t) \equiv F_{4+5\rm PN}(a_r, e_t) \quad (39)
$$

$$
= \frac{\nu^2}{a_r^5}\left[\mathcal{A}_{4\rm PN} + \mathcal{B}_{4\rm PN}\ln(a_r)\right] + \frac{\nu^2}{a_r^6}\left[\mathcal{A}_{5\rm PN} + \mathcal{B}_{5\rm PN}\ln(a_r)\right] \quad (40)
$$

starting from harmonic coordinates. Here a_r is the semimajor axis of the orbit, which we rescaled by $a_r = a_{r,\rm phys}c^2/(G_N M)$. It appears in the parameterization of the radial coordinate distance r in the form $r = a_r[1 - e_r\cos(u)]$, where e_r denotes the "radial eccentricity" of the orbit and u the "eccentric anomaly". The Kepler equation reads $n \cdot t = 1 - e_t \sin(u)$, with $n = 2\pi/P$. Here P is the orbital period and t the coordinate time defines the eccentricity e_t and one uses standard relations otherwise, cf. [57].

In the limit of vanishing eccentricity e_t we obtain the following contribution for $\langle \delta H^{\rm nl}_{5\rm PN}\rangle$, Eq. (2.12), [13],

$$
\frac{\langle \delta H^{\rm nl}_{4+5\rm PN}\rangle}{Mc^2} = \frac{\nu^2}{a_r^5}\left\{-\frac{32}{5}(\ln(a_r) - 2\gamma_E) + \frac{128}{5}\ln(2)\right\} + \frac{\nu^2}{a_r^6}\left\{\left(\frac{5854}{105} + \frac{56}{5}\nu\right)(\ln(a_r) - 2\gamma_E)
-\left(\frac{25276}{105} - \frac{912}{35}\nu\right)\ln(2) + \left(\frac{243}{14} - \frac{486}{7}\nu\right)\ln(3) + \frac{32}{5}\nu - \frac{96}{5}\right\}, \quad (41)
$$

which agrees with [13]. The terms up to $O(e_t^{20})$ are given in Appendix B. They agree with the expansion coefficients of Table I of [13] up to $O(e_t^{10})$ (in the harmonic gauge).[11] The non–local

[11] In Ref. [13] also the corresponding expressions in EOB coordinates are discussed.

contribution to the energy for circular orbits is then obtained by[12]

$$\frac{E_{\rm nl}^{\rm circ}}{\mu c^2} = \nu\Biggl\{\left[-\frac{64}{5}(\ln(j)-\gamma_E)+\frac{128}{5}\ln(2)\right]\frac{\eta^8}{j^{10}}+\Biggl[\frac{32}{5}+\frac{28484}{105}\ln(2)+\nu\Biggl(\frac{32}{5}+\frac{112}{5}(\ln(j) -\gamma_E)+\frac{912}{35}\ln(2)-\frac{486}{7}\ln(3)\Biggr)+\frac{243}{14}\ln(3)-\frac{15172}{105}(\ln(j)-\gamma_E)\Biggr]\frac{\eta^{10}}{j^{12}}\Biggr\}, \tag{42}$$

with

$$a_r = j^2 - 4\eta^2 + O\left(\eta^4\right), \tag{43}$$

with $j = J_{\rm phys}c/(G_N M)$. Here we have also introduced the dimensionless quantity η^2, accounting for $1/c^2$.

The contributions up to $O(e_t^2)$ are needed below to derive periastron advance for circular motion. One may now further express the variables a_r and e_t in terms of the normalized Delaunay variables [58] i_r, i_ϕ and $i_{r\phi}$, cf. [16], Eq. (A11), with $i_{r\phi} = i_r + i_\phi$, $i_\phi = j$. By this one obtains

$$\hat{F}(i_r, j) = F(a_r, e_t)\ . \tag{44}$$

The variables $\hat{H}, i_r$ and i_ϕ are related by Euler's chain rule

$$\left.\frac{\partial \hat{H}}{\partial i_\phi}\right|_{i_r}\left.\frac{\partial i_\phi}{\partial i_r}\right|_{\hat{H}}\left.\frac{\partial i_r}{\partial \hat{H}}\right|_{i_\phi} = -1 \tag{45}$$

since $\hat{H}$ depends only on i_r and i_ϕ and therefore a function f exists with $f(\hat{H}, i_r, i_\phi) = 0$. By applying the chain rule one obtains the periastron advance, K, defined in (46)[13] One obtains

$$K = \frac{1}{\Omega_R}\left.\frac{\partial \hat{H}(i_r, j)}{\partial i_\phi}\right|_{i_r} \tag{46}$$

$$\Omega_R = \left.\frac{\partial \hat{H}(i_r, j)}{\partial i_r}\right|_{j}. \tag{47}$$

Here $\hat{H}$ denotes the complete Hamiltonian. One may express

$$K = K_{\rm loc} + K_{\rm nl}, \tag{48}$$

$$\hat{H} = \hat{H}_{\rm loc} + \hat{H}_{\rm nl}, \tag{49}$$

where $K_{\rm nl}$ starts at 4PN and Ω_R receives non–local (nl) contributions from 4PN on, cf. (47). The contributions to $K_{\rm loc}$ are calculated in Section 5.3. For $K_{4+5\rm PN}^{\rm nl}$ the 4PN non–local contributions to Ω_R are necessary beyond the 1PN (local) correction

$$\Omega_R^{\rm loc,1PN} = i_{r\phi}^{-3}\left[1+\frac{(3+\nu)i_\phi+18 i_r}{2 i_\phi i_{r\phi}^2}\eta^2\right]+O\left(\eta^4\right), \tag{50}$$

[12]Note a difference to Eq. (8.27), [16] in the $\ln(2)\nu^2$ term at 5PN.
[13]Note that also a related quantity, $k = K - 1$, is sometimes denoted by periastron advance.

[16] with $\Omega_R = (G_N M/c^3)\Omega_{R,\rm phys}$. We first calculate $\Omega_R^{\rm nl,4PN}$, setting $i_\phi = j$, for circular orbits

$$\begin{aligned}
\Omega_R^{\rm nl,4PN} &= \left.\frac{\partial \hat{H}^{\rm nl,4PN}(i_r, j)}{\partial i_r}\right|_{j;\ i_r=0} \\
&= -\frac{64}{10}\nu\frac{\eta^8}{j^{11}}\left[13 + \frac{37}{6}(\ln(j) - \gamma_E) + \frac{203}{6}\ln(2) - \frac{729}{16}\ln(3)\right]. \qquad (51)
\end{aligned}$$

The Newtonian term of Ω_R for circular orbits is $1/j^3$. Since we are only considering the 4 and 5PN contributions, the post–Newtonian expansion of $1/\Omega_R$ can be done separately for (50) and (51), keeping the Newtonian contribution. The second term hits the $O(\eta^2)$ term of $K_{\rm loc}$ (61). In this way Eq. (8.21) in [16] needs a slight extension.

We obtain

$$\begin{aligned}
K_{\rm 4PN}^{\rm nl}(j) &= -\frac{64}{10}\nu\frac{\eta^8}{j^8}\left[-11 - \frac{157}{6}(\ln(j) - \gamma_E) + \frac{37}{6}\ln(2) + \frac{729}{16}\ln(3)\right] \qquad (52) \\
K_{\rm 5PN}^{\rm nl}(j) &= -\frac{64}{10}\nu\frac{\eta^{10}}{j^{10}}\left[-\frac{59723}{336} - \frac{9421}{28}[\ln(j) - \gamma_E] + \frac{7605}{28}\ln(2) + \frac{112995}{224}\ln(3)\right. \\
&\left. + \left(\frac{2227}{42} + \frac{617}{6}[\ln(j) - \gamma_E] - \frac{7105}{6}\ln(2) + \frac{54675}{112}\ln(3)\right)\nu\right], \qquad (53)
\end{aligned}$$

which is calculated in a different way than the local contributions. The representations are, however, equivalent. Here one has

$$\begin{aligned}
a_r &= i_{r\phi}^2 - 2\frac{3i_r + 2i_\phi}{i_\phi}\eta^2 + O\left(\eta^4\right), \qquad (54) \\
e_t^2 &= \frac{i_r}{i_{r\phi}^2}\left[i_r + 2i_\phi + 2\frac{i_r(\nu - 1) + i_\phi(2\nu - 5)}{i_{r\phi}^2}\eta^2\right] + O\left(\eta^4\right), \qquad (55)
\end{aligned}$$

cf. [16]. Eq. (54) turns into (43) for circular orbits ($i_r \to 0$). Eq. (52) agrees with Eq. (5.7) in [6], see also the expression of the related function $\rho(x)$ in [5] and Eqs. (52,53) agree with Eq. (8.29) of [16].

5.3 Periastron advance: local terms

The local contribution to periastron advance is obtained by

$$K_{\rm loc} = -\frac{1}{\pi}\frac{\partial}{\partial j}\int_{r_{\rm min}}^{r_{\rm max}} dr\sqrt{R(r, \hat{E}, j)}. \qquad (56)$$

Here $\hat{E}$ results from (4) by $H \to E$ and

$$R(r, \hat{E}, j)\Big|_{\rm 5PN} = A + \frac{2B}{r} + \frac{C}{r^2} + \eta^2\frac{D_1}{r^3} + \sum_{k=1}^{4}\eta^{2(k+1)}\left[\frac{D_{2k}}{r^{2k+2}} + \frac{D_{2k+1}}{r^{2k+3}}\right]. \qquad (57)$$

It is convenient to refer to the local terms rather than to a separation of potential and tail terms. The former ones have no logarithmic terms and the corresponding integrals are therefore somewhat simpler. The logarithmic terms have already been dealt with in Section 5.2.

The integrand of (56) has the form

$$\sqrt{R(r,\hat{E},j)} = \frac{1}{r}\sqrt{Ar^2+2Br+C} + \eta^2 \frac{D_1}{2r^2\sqrt{Ar^2+2Br+C}} + O\left(\eta^4\right). \tag{58}$$

The relation

$$\hat{E} = \hat{H}(p^2,(p.n)^2,r) \tag{59}$$

is solved iteratively for $R(r,\hat{E},j) = (p.n)^2$ by applying

$$p^2 = (p.n)^2 + \frac{j^2}{r^2}, \tag{60}$$

through which the functions A, B, C and D_k become polynomials in $\hat{E}$ and j. The integral (56) is usually solved by a mapping to a contour integral [59] applying the residue theorem, expanding in η^2 up to 5PN. Except of the integral for the Newtonian term, involving only A, B and C, all other integrals have only one residue at $r = 0$, see Appendix C.

We calculate the local contribution to periastron advance starting from harmonic coordinates and compare to Eq. (F5) of [16] resulting from the local EOB Hamiltonian Eq. (11.8) [13]. This is necessary to fix the notion of the parameters $\bar{d}_5$ and a_6 in $K(E,j)_{\rm loc,f}$ to 5PN. We rather use $K(E,j)_{\rm loc,f}$ than $K^{\rm circ}_{\rm loc,5PN}$ to test three relations between the parameters $\bar{d}_5$ and a_6, which is advantageous.

To 4PN one obtains

$$\begin{aligned} K(\hat{E},j)^{\leq 4\rm PN}_{\rm loc,f} \;=\; & 1 + \frac{3}{j^2}\eta^2 + \left[\left(\frac{15}{2} - 3\nu\right)\frac{\hat{E}}{j^2} + \left(\frac{105}{4} - \frac{15\nu}{2}\right)\frac{1}{j^4}\right]\eta^4 + \left[\left(\frac{15}{4}(1-\nu) + 3\nu^2\right)\frac{\hat{E}^2}{j^2}\right.\\ & + \left(\frac{315}{2} + \left(\frac{123\pi^2}{64} - 218\right)\nu + \frac{45}{2}\nu^2\right)\frac{\hat{E}}{j^4} + \left(\frac{1155}{4} + \left(\frac{615\pi^2}{128} - \frac{625}{2}\right)\nu\right.\\ & \left.\left. + \frac{105}{8}\nu^2\right)\frac{1}{j^6}\right]\eta^6 + \left[\left(\frac{15}{4} - 3\nu\right)\nu^2\frac{\hat{E}^3}{j^2} + \left(\frac{4725}{16} + \left(\frac{35569\pi^2}{2048} - \frac{20323}{24}\right)\nu\right.\right.\\ & \left. + \left(\frac{4045}{8} - \frac{615\pi^2}{128}\right)\nu^2 - 45\nu^3\right)\frac{\hat{E}^2}{j^4} + \left(-\frac{525\nu^3}{8} + \left(\frac{35065}{16} - \frac{615\pi^2}{16}\right)\nu^2\right.\\ & \left. + \left(\frac{257195\pi^2}{2048} - \frac{293413}{48}\right)\nu + \frac{45045}{16}\right)\frac{\hat{E}}{j^6} + \left(\frac{225225}{64} + \left(\frac{2975735\pi^2}{24576}\right.\right.\\ & \left.\left.\left. - \frac{1736399}{288}\right)\nu + \left(\frac{132475}{96} - \frac{7175\pi^2}{256}\right)\nu^2 - \frac{315}{16}\nu^3\right)\frac{1}{j^8}\right]\eta^8. \end{aligned} \tag{61}$$

We also rederived the periastron advance starting with the ADM Hamiltonian [3] and confirm the result given in [6, 16]. For the 5PN terms we obtain the partial result

$$\begin{aligned} K(\hat{E},j)^{5\rm PN}_{\rm loc,f} \;\propto\; & \left\{\left[\frac{15\nu^2}{16} - \frac{15}{4}\nu^3 + 3\nu^4\right]\frac{\hat{E}^4}{j^2} + \left[\frac{3465}{16} + \frac{15829\pi^2}{256}\nu - \frac{35569\pi^2}{1024}\nu^2 + \left(\frac{1107\pi^2}{128}\right.\right.\right.\\ & \left.\left. - \frac{7113}{8}\right)\nu^3 + 75\nu^4\right]\frac{\hat{E}^3}{j^4} + \left[\frac{315315}{32} + \frac{4899565\pi^2}{4096}\nu - \frac{3289285\pi^2}{1024}\nu^2 + \left(\frac{35055\pi^2}{256}\right.\right. \end{aligned}$$

$$-\frac{240585}{32}\Bigg)\nu^3+\frac{1575}{8}\nu^4\Bigg)\frac{\hat{E}^2}{j^6}+\Bigg[\frac{765765}{16}+\frac{16173395\pi^2}{8192}\nu-\frac{77646205\pi^2}{8192}\nu^2$$
$$+\left(\frac{121975\pi^2}{512}-\frac{271705}{24}\right)\nu^3+\frac{2205}{16}\nu^4\Bigg]\frac{\hat{E}}{j^8}+\Bigg[\frac{2909907}{64}+\frac{1096263\pi^2}{1024}\nu$$
$$-\frac{87068961\pi^2}{16384}\nu^2+\left(\frac{90405\pi^2}{1024}-\frac{127995}{32}\right)\nu^3+\frac{3465}{128}\nu^4\Bigg]\frac{1}{j^{10}}\Bigg\}\eta^{10}, \tag{62}$$

leaving out the few rational terms at $O(\nu)$ and $O(\nu^2)$ still to be calculated in complete form. The terms given in (62) agree with those of Ref. [16] considering our results in (63) and (64).

Eq. (62) allows to derive the π^2 contributions to $\bar{d}_5$ and a_6 to which we turn now.

5.4 The π^2 contributions

The π^2–contributions stem from the potential terms. They also contribute to the yet open constants $\bar{d}_5$ and a_6 in [13] at $O(\nu^2)$. They can be extracted from the corresponding contribution to the binding energy for circular motion (for a_6) and the circular periastron advance [6, 60, 61], respectively. One obtains

$$\bar{d}_5 = r_{\bar{d}_5}+\frac{306545}{512}\pi^2 \tag{63}$$
$$a_6 = r_{a_6}+\frac{25911}{256}\pi^2 \tag{64}$$

from Eqs. (62,65,66). We also agree with the π^2 contributions of $O(\nu)$. Let us finally summarize the terms for the binding energy $E^{\rm circ}(j)$ and the periastron advance $K^{\rm circ}(j)$ at circular orbits obtained in the present calculation,

$$\begin{aligned}\frac{E^{\rm circ}(j)}{\mu c^2} =& -\frac{1}{2j^2}+\left(-\frac{\nu}{8}-\frac{9}{8}\right)\frac{1}{j^4}\eta^2+\left(-\frac{\nu^2}{16}+\frac{7\nu}{16}-\frac{81}{16}\right)\frac{1}{j^6}\eta^4+\Bigg[-\frac{5\nu^3}{128}+\frac{5\nu^2}{64}+\Bigg(\frac{8833}{384}\\ &-\frac{41\pi^2}{64}\Bigg)\nu-\frac{3861}{128}\Bigg]\frac{1}{j^8}\eta^6+\Bigg[-\frac{7\nu^4}{256}+\frac{3\nu^3}{128}+\left(\frac{41\pi^2}{128}-\frac{8875}{768}\right)\nu^2+\Bigg(\frac{989911}{3840}\\ &-\frac{6581\pi^2}{1024}\Bigg)\nu-\frac{53703}{256}\Bigg]\frac{1}{j^{10}}\eta^8+\Bigg[\left(r^{\rm E}_{\nu^2}+\frac{132979\pi^2}{2048}\right)\nu^2-\frac{21\nu^5}{1024}+\frac{5\nu^4}{1024}\\ &+\left(\frac{41\pi^2}{512}-\frac{3769}{3072}\right)\nu^3+\left(r^{\rm E}_{\nu}-\frac{31547\pi^2}{1536}\right)\nu-\frac{1648269}{1024}\Bigg]\frac{1}{j^{12}}\eta^{10}+\frac{E^{\rm circ}_{\rm nl}}{\mu c^2}+O\left(\eta^{12}\right),\end{aligned} \tag{65}$$

$$\begin{aligned}K^{\rm circ}(j) =& 1+3\frac{1}{j^2}\eta^2+\left(\frac{45}{2}-6\nu\right)\frac{1}{j^4}\eta^4+\left[\frac{405}{2}+\left(-202+\frac{123}{32}\pi^2\right)\nu+3\nu^2\right]\frac{1}{j^6}\eta^6\\ &+\left[\frac{15795}{8}+\left(\frac{185767}{3072}\pi^2-\frac{105991}{36}\right)\nu+\left(-\frac{41}{4}\pi^2+\frac{2479}{6}\right)\nu^2\right]\frac{1}{j^8}\eta^8+\Bigg[\frac{161109}{8}\\ &+\left(r^{\rm K}_{\nu}+\frac{488373}{2048}\pi^2\right)\nu+\left(r^{\rm K}_{\nu^2}-\frac{1379075}{1024}\pi^2\right)\nu^2\\ &+\left(-\frac{1627}{6}+\frac{205}{32}\pi^2\right)\nu^3\Bigg]\frac{1}{j^{10}}\eta^{10}+K^{\rm nl}_{4+5{\rm PN}}(j)+O\left(\eta^{12}\right),\end{aligned} \tag{66}$$

cf. also [60], Eq. (5.25), and [6], Eq. (5.9). Note that there are correlations between the rational quantities $r^{\rm E,K}_{\nu,\nu^2}$ and $r_{\bar{d}_5,a_6}$.

To obtain the corresponding relations in terms of the variable $x = (G_N M\Omega_\phi/c^3)^{2/3}$, with Ω_ϕ the angular frequency, one may apply $j = j(x)$, Eq. (8.31) of [16]. In particular one has in the Schwarzschild limit of (65, 66)

$$\frac{E^{\rm Schw.}(x)}{\mu c^2} = \frac{1-2x}{\sqrt{1-3x}} - 1, \tag{67}$$

$$K^{\rm Schw.}(x) = \frac{1}{\sqrt{1-6x}}, \tag{68}$$

where (67) has been given in [27]. The relation for $K^{\rm Schw.}(j)$ has been given in [61], Eq. (A8).

5.5 Comparison to the other contributions to the tail terms

Let us mention that we have recalculated the contributions to the tail term given in [40] in D dimensions, but do not agree with all terms in the rational (local) contributions. A further detailed comparison has to be performed, which will be given elsewhere.

There are multi–pole moment contributions containing products of Levi–Civita tensors ε_{ijk}, which have to be dealt with in $D-1$ dimensions in the present approach, see also [40,62]. Despite the fact that products of two (or more) Levi–Civita symbols, here in Euclidean space, are turned into determinants of $D-1$ Kronecker symbols [63] it is known from almost all applications in Quantum Field Theory, that a so–called finite renormaliztion has to be performed to re-establish the Ward–identities.[14] The Larin method [64] is one consistent way (i.e. a non–degenerative way) to perform this analytic continuation to D dimensions.[15] Also this aspect still needs further study.

We finally mention that for the circular binding energy and periastron advance the yet differing results in the tail terms yield a numerical effect on the difference of $O(1\%)$ or less. This remaining difference still has to be settled analytically.

6 Conclusions

We have presented the 5PN potential contributions to the Hamiltonian of binary motion in gravity starting form the harmonic gauge and a part of the 5PN tail term. The calculation has thoroughly been performed in D dimensions, based on 188533 Feynman diagrams using effective field theory methods, as a calculation ab initio. The singular and logarithmic contributions to the 5PN tail terms have been calculated. We have shown the explicit cancellation of the singularities between both contributions, performing an additional canonical transformation to a pole–free Hamiltonian. We have shown in an explicit calculation how to match the potential and the tail terms, using dimensional regularization. Here the overlap–terms are canceling.

Comparisons to the literature have been performed. Firstly, we have shown that all terms of $O(\nu^0)$ and $O(\nu^3)$ and higher agree with the results presented in the literature. At $O(\nu^2)$ we determined the π^2 contributions to $\bar{d}_5$ and a_6. Furthermore, we also agree with the logarithmic tail and potential terms and the π^2–terms at $O(\nu)$ and the effect of the non–local terms on $E_{\rm circ,5PN}$

[14]In the present application to gravity one should not call this 'finite renormalization', since no renormalization is performed. However, the contribution to the observables will be different comparing the 3–dimensional properly regulated result with that obtained in $D-1$ dimensions.

[15]Other prescriptions were given in [65].

and $K_{\rm circ,5PN}$. We still observe a few differences in the purely rational (local) contributions to the tail term comparing to the present literature, which have to be clarified to obtain the complete 5PN result.

A Joining the potential and the tail term and the method of expansion by regions

We now prove that the Taylor expansions in the potential and ultrasoft region commute and that the occurring overlap integrals are indeed scaleless and vanish in D dimensions. Here we use arguments given in [43].

The general form of the contributing integrands I is

$$I = \frac{\exp\big(ik[x(t_1) - x(t_2)]\big)}{k^2 \prod_{i=1}^{P_{\rm us}} (k+p_i)^2 \prod_{i=1}^{P_{\rm pot}} (k+q_i)^2} \mathcal{J}\big(\{q_i\},\{p_i\}\big)\mathcal{P}\big(k,\{q_i\},\{p_i\},v_1,v_2\big)\,. \tag{69}$$

Apart from the loop momentum k, I depends on ultrasoft momenta $p_i, 1 \leq i \leq P_{\rm us}$ and potential momenta $q_i, 1 \leq i \leq P_{\rm pot}$. The loop is associated with one of the two worldlines, whose four–position at the time t is given by $x(t)$. $\mathcal{J}$ is a function with the same structure as I itself, but independent of k. Finally, $\mathcal{P}$ denotes a polynomial in the momenta and worldline velocities v_1, v_2. Our aim is to show that

$$T^N_{\rm pot} T^N_{\rm us} I = T^N_{\rm us} T^N_{\rm pot} I = \sum_{i=0}^{i_{\max}(N)} \frac{\mathcal{J}_i \mathcal{P}_i}{\vec{k}^{2i}}\,, \tag{70}$$

according to the power counting in the respective region. The $\mathcal{P}_i$ are polynomials in the components of the four–vectors appearing in Eq. (69) and the $\mathcal{J}_i$ are independent of k. This structure implies

$$\int_{\mathbb{R}^d} d^d\vec{k}\; T^N_{\rm pot} T^N_{\rm us} I = 0\,. \tag{71}$$

We first note that

$$T^N_{\rm pot} f\, g = (T^N_{\rm pot}\, f)(T^N_{\rm pot}\, g)\,, \tag{72}$$

and similar for $T^N_{\rm us}$. This allows us to expand each factor in Eq. (69) separately. $\mathcal{J}$ is independent of k, and $\mathcal{P}$ is a polynomial, so trivially

$$T^N_{\rm pot} T^N_{\rm us} \mathcal{J} = T^N_{\rm us} T^N_{\rm pot} \mathcal{J} = \mathcal{J}\,, \tag{73}$$

$$T^N_{\rm pot} T^N_{\rm us} \mathcal{P} = T^N_{\rm us} T^N_{\rm pot} \mathcal{P} = \mathcal{P}\,. \tag{74}$$

For the propagators without additional momenta, we obtain

$$T^N_{\rm pot} T^N_{\rm us} \frac{1}{k^2} = T^N_{\rm pot} \frac{1}{k^2} = \sum_{n=0}^{\frac{N}{2}} \frac{k_0^{2n}}{(\vec{k}^2)^{n+1}} = T^N_{\rm us} \sum_{n=0}^{\frac{N}{2}} \frac{k_0^{2n}}{(\vec{k}^2)^{n+1}} = T^N_{\rm us} T^N_{\rm pot} \frac{1}{k^2}\,, \tag{75}$$

and similar

$$T^N_{\rm pot} T^N_{\rm us} \frac{1}{(k+p_i)^2} = T^N_{\rm pot} \frac{1}{(k+p_i)^2} = \sum_{n=0}^{N} \frac{T^N_{\rm pot}\left[(k_0+p_{i0})^2 - 2\vec{k}\vec{p}_i - \vec{p}_i^{\,2}\right]^n}{(\vec{k}^2)^{n+1}}\,, \tag{76}$$

$$T^N_{\rm us} T^N_{\rm pot} \frac{1}{(k+p_i)^2} = T^N_{\rm us} \sum_{n=0}^{N} \frac{T^N_{\rm pot}\left[(k_0+p_{i0})^2 - 2\vec{k}\vec{p}_i - \vec{p}_i^{\,2}\right]^n}{(\vec{k}^2)^{n+1}}$$

$$= \sum_{n=0}^{N} \frac{T^N_{\rm pot}\big[(k_0+p_{i0})^2 - 2\vec{k}\vec{p}_i - \vec{p}_i^{\,2}\big]^n}{(k^2)^{n+1}}\,. \tag{77}$$

With respect to the remaining propagators, we first observe the absence of poles in v, i.e.

$$T^Z_{\rm pot}\frac{1}{(k+q_i)^2} = T^Z_{\rm pot}T^N_{\rm us}\frac{1}{(k+q_i)^2} = T^Z_{\rm us}\frac{1}{(k+q_i)^2} = T^Z_{\rm us}T^N_{\rm pot}\frac{1}{(k+q_i)^2} = 0 \qquad \text{for all } Z<0\,, \tag{78}$$

and note the following algebraic properties of the Taylor expansion operators.

$$T^N_{\rm pot}P = P\,T^{N-1}_{\rm pot}\,, \qquad P\in\{k_0,p_{i0},q_{i0},\vec{p}_i\} \tag{79}$$

$$T^N_{\rm pot}P = P\,T^{N}_{\rm pot}\,, \qquad P\in\{\vec{k},\vec{q}_i\} \tag{80}$$

$$T^N_{\rm us}P = P\,T^{N-1}_{\rm us}\,, \qquad P\in\{k_0,p_{i0},q_{i0},\vec{k},\vec{p}_i\} \tag{81}$$

$$T^N_{\rm us}P = P\,T^{N}_{\rm us}\,, \qquad P\in\{\vec{q}_i\}. \tag{82}$$

We now show that

$$T^N_{\rm pot}T^M_{\rm us}\frac{1}{(k+q_i)^2} = T^M_{\rm us}T^N_{\rm pot}\frac{1}{(k+q_i)^2} \tag{83}$$

by induction over $N+M$. The case $N+M=0$ is straightforward. For $N+M\geq 1$ we observe

$$\begin{aligned} T^N_{\rm pot}T^M_{\rm us}\frac{1}{(k+q_i)^2} &= \frac{T^N_{\rm pot}}{\vec{q}_i^{\,2}}\left(1+\big[(k_0+q_0)^2-\vec{k}^2\big]T^{M-2}_{\rm us}\frac{1}{(k+q_i)^2} - 2\,\vec{k}\vec{q}_i\,T^{M-1}_{\rm us}\frac{1}{(k+q_i)^2}\right),\\ &= \frac{1}{\vec{q}_i^{\,2}}\left(1+(k_0+q_0)^2T^{N-2}_{\rm pot}T^{M-2}_{\rm us}\frac{1}{(k+q_i)^2}\right.\\ &\qquad \left. -\vec{k}^2T^N_{\rm pot}T^{M-2}_{\rm us}\frac{1}{(k+q_i)^2} - 2\,\vec{k}\vec{q}\,T^N_{\rm pot}\,T^{M-1}_{\rm us}\frac{1}{(k+q_i)^2}\right), \end{aligned} \tag{84}$$

and similar

$$\begin{aligned} T^M_{\rm us}T^N_{\rm pot}\frac{1}{(k+q_i)^2} &= \frac{T^M_{\rm us}}{\vec{q}_i^{\,2}}\left(1+(k_0+q_0)^2T^{N-2}_{\rm pot}\frac{1}{(k+q_i)^2} - (\vec{k}^2+2\,\vec{k}\vec{q}_i)\,T^{N}_{\rm pot}\frac{1}{(k+q_i)^2}\right),\\ &= \frac{1}{\vec{q}_i^{\,2}}\left(1+(k_0+q_0)^2T^{M-2}_{\rm us}T^{N-2}_{\rm pot}\frac{1}{(k+q_i)^2}\right.\\ &\qquad \left. -\vec{k}^2T^{M-2}_{\rm us}T^{N}_{\rm pot}\frac{1}{(k+q_i)^2} - 2\,\vec{k}\vec{q}_i\,T^{M-1}_{\rm us}T^N_{\rm pot}\frac{1}{(k+q_i)^2}\right)\\ &= T^N_{\rm pot}T^M_{\rm us}\frac{1}{(k+q_i)^2}\,, \end{aligned} \tag{85}$$

where we have used the induction hypothesis

$$T^n_{\rm pot}T^m_{\rm us}\frac{1}{(k+q_i)^2} = T^m_{\rm us}T^n_{\rm pot}\frac{1}{(k+q_i)^2} \quad \text{for } n+m<N+M \tag{86}$$

in the last step.

Furthermore, we find

$$T^N_{\rm pot}T^N_{\rm us}\frac{1}{(k+q_i)^2} = \sum_{n=0}^{N}\frac{T^N_{\rm pot}T^N_{\rm us}\big[(k_0+q_{i0})^2-2\vec{q}_i\vec{k}-\vec{k}^2\big]^n}{(q_i^2)^{n+1}}\,, \tag{87}$$

where the numerators are simply polynomials in the components of k and q_i.

Finally, the exponential (69) can be expanded by observing

$$\vec{x}(t) \sim R, \tag{88}$$

i.e.

$$\vec{k}\vec{x}(t) \sim 1 \tag{89}$$

in the potential region and

$$\vec{k}\vec{x}(t) \sim v \tag{90}$$

in the ultrasoft region. This yields

$$\begin{aligned} T^N_{\rm pot} T^N_{\rm us} \exp\big(ik[x(t_1) - x(t_2)]\big) &= T^N_{\rm pot} \exp\big(-ik_0 c(t_1 - t_2)\big) \sum_{n=0}^{N} \frac{\big(i\vec{k}[\vec{x}(t_1) - \vec{x}(t_2)]\big)^n}{n!} \\ &= \exp\big(-ik_0 c(t_1 - t_2)\big) \sum_{n=0}^{N} \frac{\big(i\vec{k}[\vec{x}(t_1) - \vec{x}(t_2)]\big)^n}{n!}, \end{aligned} \tag{91}$$

$$\begin{aligned} T^N_{\rm us} T^N_{\rm pot} \exp\big(ik[x(t_1) - x(t_2)]\big) &= T^N_{\rm us} \exp\big(ik[x(t_1) - x(t_2)]\big) \\ &= \exp\big(-ik_0 c(t_1 - t_2)\big) \sum_{n=0}^{N} \frac{\big(i\vec{k}[\vec{x}(t_1) - \vec{x}(t_2)]\big)^n}{n!}. \end{aligned} \tag{92}$$

We have now shown that the Taylor expansions commute for each of the factors in Eq. (69) and that the product of all expanded factors Eqs. (73–76), (87), (91) has the required form Eq. (70).

B The eccentricity expansion of the non–local terms

We have recalculated the eccentricity expansion of the non–local terms using standard representations given in [13, 16, 66]. To $O(e_t^{20})$ we obtain

$$\begin{aligned} F_{\rm 4PN}(a_r, e_t) \;=\; & \frac{\nu^2}{a_r^5}\Bigg\{ -\frac{32}{5}(\ln(a_r) - 2\gamma_E) + \frac{128}{5}\ln(2) + e_t^2\Bigg[-\frac{176}{5} - \frac{628}{15}(\ln(a_r) - 2\gamma_E) \\ & + \frac{296}{15}\ln(2) + \frac{729}{5}\ln(3)\Bigg] + e_t^4\Bigg[-\frac{2681}{15} - 121(\ln(a_r) - 2\gamma_E) + \frac{29966}{15}\ln(2) \\ & - \frac{13851}{20}\ln(3)\Bigg] + e_t^6\Bigg[-\frac{90017}{180} - \frac{763}{3}(\ln(a_r) - 2\gamma_E) - \frac{116722}{15}\ln(2) \\ & + \frac{419661}{320}\ln(3) + \frac{1953125}{576}\ln(5)\Bigg] + e_t^8\Bigg[-\frac{306433}{288} - \frac{3605}{8}(\ln(a_r) - 2\gamma_E) \\ & + \frac{5381201}{180}\ln(2) + \frac{26915409}{2560}\ln(3) - \frac{83984375}{4608}\ln(5)\Bigg] + e_t^{10}\Bigg[-\frac{18541327}{9600} \\ & - \frac{114807}{160}(\ln(a_r) - 2\gamma_E) - \frac{4697998651}{54000}\ln(2) - \frac{138733913079}{2048000}\ln(3) \\ & + \frac{18736328125}{442368}\ln(5) + \frac{678223072849}{18432000}\ln(7)\Bigg] + e_t^{12}\Bigg[-\frac{364045577}{115200} - \frac{679679}{640} \\ & \times\big(-2\gamma_E + \ln(a_r)\big) + \frac{110301092701}{216000}\ln(2) + \frac{1437894581679}{8192000}\ln(3) \end{aligned}$$

$$
\begin{aligned}
&-\frac{100439453125}{1769472}\ln(5)-\frac{678223072849}{2949120}\ln(7)\Bigg]+e_t^{14}\Bigg[-\frac{775035553}{161280}-\frac{95381}{64}\\
&\times\big(-2\gamma_E+\ln\left(a_r\right)\big)-\frac{38217199661503}{15876000}\ln(2)+\frac{996383367472131}{3211264000}\ln(3)\\
&+\frac{155008544921875}{3121348608}\ln(5)+\frac{4122918059849071}{6370099200}\ln(7)\Bigg]+e_t^{16}\Bigg[-\frac{992166951}{143360}\\
&-\frac{1028313}{512}\big(-2\gamma_E+\ln\left(a_r\right)\big)+\frac{2885944821108703}{381024000}\ln(2)\\
&-\frac{199179784215931689}{51380224000}\ln(3)+\frac{35973968505859375}{49941577728}\ln(5)\\
&-\frac{558878193388980017}{509607936000}\ln(7)\Bigg]+e_t^{18}\Bigg[-\frac{13153280515}{1376256}-\frac{21477885}{8192}\big(-2\gamma_E\\
&+\ln\left(a_r\right)\big)-\frac{2994261720973969703}{164602368000}\ln(2)+\frac{1114170738835916825 1}{822083584000}\ln(3)\\
&-\frac{660797547149658203125}{115065395085312}\ln(5)+\frac{1113553185418308608323}{880602513408000}\ln(7)\\
&+\frac{814027493868397611133 21}{43149523156992000}\ln(11)\Bigg]+e_t^{20}\Bigg[-\frac{30115692673}{2359296}-\frac{1639570933}{491520}\\
&\times\big(-2\gamma_E+\ln\left(a_r\right)\big)+\frac{321972869963109745379}{7054387200000}\ln(2)\\
&-\frac{1923804568946611360809}{82208358400000}\ln(3)+\frac{8504215528488159179 6875}{4142354223071232}\ln(5)\\
&-\frac{312452950136178001875 83}{29353417113600000}\ln(7)-\frac{132686481500548810614713 23}{862990463139840000}\ln(11)\Bigg]\Bigg\}\\
&+O(e_t^{22}),
\end{aligned}
\tag{93}
$$

$$
\begin{aligned}
&F_{5\mathrm{PN}}(a_r,e_t)=\\
&\quad\frac{\nu^2}{a_r^6}\Bigg\{\frac{2}{105}(2927+588\nu)(\ln(a_r)-2\gamma_E)+\frac{4}{105}(-6319+684\nu)\ln(2)\\
&\quad-\frac{243}{14}(-1+4\nu)\ln(3)-\frac{96}{5}+\frac{32\nu}{5}+e_t^2\Bigg[\frac{2}{105}(8004+11935\nu)(\ln(a_r)-2\gamma_E)\\
&\quad-\frac{4}{105}(-14268+100247\nu)\ln(2)+\frac{729}{70}(-76+129\nu)\ln(3)-\frac{5441}{35}+\frac{4672\nu}{21}\Bigg]\\
&\quad+e_t^4\Bigg[\frac{1}{70}(-14003+78435\nu)(\ln(a_r)-2\gamma_E)+\frac{1}{105}(-599911+3476231\nu)\ln(2)\\
&\quad-\frac{729(2085+34396\nu)}{4480}\ln(3)-\frac{9765625(-1+4\nu)}{2688}\ln(5)-\frac{1160639}{840}+\frac{59756\nu}{35}\Bigg]\\
&\quad+e_t^6\Bigg[\frac{1}{30}(-53391+100135\nu)(\ln(a_r)-2\gamma_E)-\frac{73}{945}(-226057+3011889\nu)\ln(2)\\
&\quad-\frac{243}{896}(-95283+292001\nu)\ln(3)+\frac{78125(-5557+43575\nu)}{24192}\ln(5)-\frac{15761437}{2520}
\end{aligned}
$$

$$+\frac{474653\nu}{72}\Bigg] + e_t^8\Bigg[\frac{1}{64}(-356481 + 490280\nu)(\ln(a_r) - 2\gamma_E)$$
$$+\frac{(-1814239887 + 33331273432\nu)}{30240}\ln(2) + \frac{729(-181288681 + 1304180292\nu)}{1146880}$$
$$\times\ln(3) - \frac{78125(325441 + 43174300\nu)}{6193152}\ln(5) - \frac{96889010407(-1 + 4\nu)}{884736}\ln(7)$$
$$-\frac{168508293}{8960} + \frac{2591779\nu}{144}\Bigg] + e_t^{10}\Bigg[+\frac{561}{320}(-7258 + 8547\nu)(\ln(a_r) - 2\gamma_E)$$
$$+\frac{(1135478771202 - 6934343243023\nu)}{756000}\ln(2)$$
$$-\frac{2187(-1924018874 + 43692700941\nu)}{28672000}\ln(3) + \frac{1953125(69962 + 1244683\nu)}{2064384}$$
$$\times\ln(5) + \frac{282475249(-244698 + 1611757\nu)}{110592000}\ln(7) - \frac{1492974817}{33600} + \frac{255777929\nu}{6400}\Bigg]$$
$$+e_t^{12}\Bigg[+\left(-\frac{3926923}{160} + \frac{6733727\nu}{256}\right)(\ln(a_r) - 2\gamma_E) + \left(-\frac{30140932254133}{3402000}\right.$$
$$\left.+\frac{109595282746879\nu}{1814400}\right)\ln(2) + \left(\frac{431854060307859}{131072000} - \frac{37854312670341\nu}{6553600}\right)\ln(3)$$
$$-\left(\frac{47376871796875}{891813888} + \frac{60736572265625\nu}{37158912}\right)\ln(5) + \left(\frac{30496744521746713}{21233664000}\right.$$
$$\left.-\frac{90299298142188709\nu}{5308416000}\right)\ln(7) - \frac{20691354791}{230400} + \frac{3566474429\nu}{46080}\Bigg]$$
$$+e_t^{14}\Bigg[\left(-\frac{190225893}{4480} + \frac{5466461\nu}{128}\right)(\ln(a_r) - 2\gamma_E) + \left(\frac{4290237684292667}{133358400}\right.$$
$$\left.-\frac{19381110496948853\nu}{74088000}\right)\ln(2) + \left(-\frac{2171545303773930003}{8991539200}\right.$$
$$\left.+\frac{5487909996792714903\nu}{44957696000}\right)\ln(3) + \left(\frac{248411703945390625}{43698880512}\right.$$
$$\left.-\frac{34494852294921875\nu}{1618477056}\right)\ln(5) + \left(-\frac{5548114000135259}{2359296000}\right.$$
$$\left.+\frac{2952253684936181 29\nu}{7077888000}\right)\ln(7) - \frac{122529921403}{752640} + \frac{6267828367\nu}{46080}\Bigg]$$
$$+e_t^{16}\Bigg[\left(-\frac{39012948117}{573440} + \frac{267733323\nu}{4096}\right)(\ln(a_r) - 2\gamma_E)$$
$$-\frac{8748776901657}{32112640} + \frac{51000706143}{229376}\nu + \left(-\frac{569049681664742359}{5689958400}\right.$$

$$
\begin{aligned}
&+\frac{179163776275618758 29}{21337344000}\nu\Bigg)\ln(2)+\Bigg(\frac{192054917627699907573}{2301834035200}\\
&-\frac{170702078449397459 6637}{2877292544000}\nu\Bigg)\ln(3)+\Bigg(-\frac{813867482269935804 6875}{178990614577152}\\
&+\frac{214230939178466796875}{913217421312}\nu\Bigg)\ln(5)+\Bigg(\frac{289871107442470811 53}{7247757312000}\\
&-\frac{224384811477454209253}{3261490790400}\nu\Bigg)\ln(7)+\Bigg(\frac{8140274938683976111 3321}{4474765364428800}-\\
&\frac{8140274938683976111 3321}{1118691341107200}\nu\Bigg)\ln(11)\Bigg]+e_t^{18}\Bigg[-\frac{30986035007243}{72253440}\\
&+\frac{945865602119}{2752512}\nu+\Bigg(-\frac{4426707571}{43008}+\frac{1565114265}{16384}\nu\Bigg)(\ln(a_r)-2\gamma_E)\\
&+\Bigg(\frac{328393145609363098807}{864162432000}-\frac{862890935509435128773}{329204736000}\nu\Bigg)\ln(2)\\
&+\Bigg(-\frac{203719103345458519569}{1438646272000}+\frac{1666270795169318918 9427}{11509170176000}\nu\Bigg)\ln(3)\\
&+\Bigg(\frac{1061079118186925166 40625}{604093324197888}-\frac{772369312821197509765625}{690392370511872}\nu\Bigg)\ln(5)\\
&+\Bigg(-\frac{414569765004538677 4577}{660451885056000}+\frac{1441862096462156526 73877}{1761205026816000}\nu\Bigg)\ln(7)\\
&-\Bigg(\frac{35656966913567027996954261}{226534996574208000}-\frac{47026368320777329995165541 7}{604093324197888000}\\
&\nu\Bigg)\ln(11)\Bigg]+e_t^{20}\Bigg[-\frac{177047266722689}{275251200}+\frac{11982611767181}{23592960}\nu\\
&+\Bigg(-\frac{48797431397}{327680}+\frac{132236197093}{983040}\nu\Bigg)(\ln(a_r)-2\gamma_E)\\
&+\Bigg(-\frac{1113890589789636783281623}{691329945600000}+\frac{6591498923973012595615229}{691329945600000}\nu\Bigg)\ln(2)\\
&+\Bigg(-\frac{2196423646340482773996 2163}{147317378252800000}-\frac{2797890027543253456292 1069}{368293445632000000}\nu\Bigg)\ln(3)\\
&-\Bigg(\frac{420710016768367311936953125}{927887345967955968}-\frac{7609603377677345275878 90625}{231971836491988992}\nu\Bigg)\ln(5)\\
&+\Bigg(\frac{3170132399737851136840 5793}{422689206435840000 0}-\frac{7773788715591302571385 9691}{1056723016089600000}\nu\Bigg)\ln(7)
\end{aligned}
$$

$$+\left(\frac{785440570888415469567164881 3}{11835297780203520000}-\frac{112327084059654478481943854653}{28996479561498624000}\nu\right)\ln(11)$$
$$+\left(\frac{917333301932686166583996160 09}{579929591229972480000}-\frac{917333301932686166583996160 09}{144982397807493120000}\nu\right)\ln(13)\Bigg]\Bigg\}$$
$$+O(e_t^{22}). \tag{94}$$

C The contour integral for the Delaunay variable i_r

The integral J_1, describing effect of the Newton dynamics, is given by

$$J_1=\frac{1}{2\pi i}\oint dx\sqrt{A+\frac{2B}{x}+\frac{C}{x^2}} \;=\; \frac{B}{\sqrt{-A}}-\sqrt{-C}. \tag{95}$$

All the remaining integrals are directly obtained from the residue at $x=0$.

The integral in (56) reads to 5PN

$$\begin{aligned}
i_r=&\ \frac{B}{\sqrt{-A}}-\sqrt{-C}\Bigg\{1-\eta^2\frac{BD_1}{2C^2}+\eta^4\Bigg[\frac{3D_1^2\big(-5B^2+AC\big)}{16C^4}-\frac{D_2\big(-3B^2+AC\big)}{4C^3}+\frac{BD_3}{4C^4}\\
&\times\big(-5B^2+3AC\big)\Bigg]+\eta^6\Bigg[-\frac{BD_5\big(63B^4-70AB^2C+15A^2C^2\big)}{16C^6}+D_1\Bigg(\frac{5BD_2\big(7B^2-3AC\big)}{8C^5}\\
&-\frac{15D_3\big(21B^4-14AB^2C+A^2C^2\big)}{32C^6}\Bigg)+\frac{35BD_1^3\big(-3B^2+AC\big)}{32C^6}+\frac{D_4}{16C^5}\big(35B^4-30AB^2C\\
&+3A^2C^2\big)\Bigg]+\eta^8\Bigg[\frac{7D_3^2}{128C^8}\big(-429B^6+495AB^4C-135A^2B^2C^2\\
&+5A^3C^3\big)-\frac{D_6}{32C^7}\big(-231B^6+315AB^4C-105A^2B^2C^2+5A^3C^3\big)+\frac{BD_7}{32C^8}\big(-429B^6\\
&+693AB^4C-315A^2B^2C^2+35A^3C^3\big)+D_1\Bigg(\frac{7D_5}{64C^8}\big(-429B^6+495AB^4C\\
&-135A^2B^2C^2+5A^3C^3\big)+\frac{21BD_4\big(33B^4-30AB^2C+5A^2C^2\big)}{32C^7}\Bigg)\\
&+D_1^2\Bigg(\frac{105D_2\big(33B^4-18AB^2C+A^2C^2\big)}{128C^7}-\frac{63BD_3\big(143B^4-110AB^2C+15A^2C^2\big)}{128C^8}\Bigg)\\
&-\frac{105D_1^4\big(143B^4-66AB^2C+3A^2C^2\big)}{1024C^8}-\frac{15D_2^2\big(21B^4-14AB^2C+A^2C^2\big)}{64C^6}\\
&+\frac{21BD_2D_3\big(33B^4-30AB^2C+5A^2C^2\big)}{32C^7}\Bigg]+\eta^{10}\Bigg[-\frac{D_8}{256C^9}\big(6435B^8-12012AB^6C\\
&+6930A^2B^4C^2-1260A^3B^2C^3+35A^4C^4\big)+\frac{BD_9}{256C^{10}}\big(12155B^8-25740AB^6C\\
&+18018A^2B^4C^2-4620A^3B^2C^3+315A^4C^4\big)+D_2\Bigg(-\frac{9BD_5}{64C^9}\big(715B^6-1001AB^4C
\end{aligned}$$

$$
+385A^2B^2C^2 - 35A^3C^3\big) + \frac{7D_4}{64C^8}\big(429B^6 - 495AB^4C + 135A^2B^2C^2 - 5A^3C^3\big)\Bigg) + D_1^2\Bigg(
$$

$$
\frac{495BD_5}{256C^{10}}\big(221B^6 - 273AB^4C + 91A^2B^2C^2 - 7A^3C^3\big) - \frac{315D_4}{256C^9}\big(143B^6 - 143AB^4C
$$

$$
+33A^2B^2C^2 - A^3C^3\big)\Bigg) + D_3\Bigg(-\frac{9BD_4}{64C^9}\big(715B^6 - 1001AB^4C + 385A^2B^2C^2 - 35A^3C^3\big)
$$

$$
+\frac{45D_5}{512C^{10}}\big(2431B^8 - 4004AB^6C + 2002A^2B^4C^2 - 308A^3B^2C^3 + 7A^4C^4\big)\Bigg) + D_1^3\Bigg(\frac{1155D_3}{512C^{10}}
$$

$$
\big(221B^6 - 195AB^4C + 39A^2B^2C^2 - A^3C^3\big) - \frac{1155B\big(39B^4 - 26AB^2C + 3A^2C^2\big)D_2}{256C^9}\Bigg)
$$

$$
+D_1\Bigg(-\frac{9BD_6}{64C^9}\big(715B^6 - 1001AB^4C + 385A^2B^2C^2 - 35A^3C^3\big) + \frac{495BD_3^2}{256C^{10}}\big(221B^6
$$

$$
-273AB^4C + 91A^2B^2C^2 - 7A^3C^3\big) - \frac{315D_2D_3}{128C^9}\big(143B^6 - 143AB^4C + 33A^2B^2C^2 - A^3C^3\big)
$$

$$
+\frac{45D_7}{512C^{10}}\big(2431B^8 - 4004AB^6C + 2002A^2B^4C^2 - 308A^3B^2C^3 + 7A^4C^4\big)
$$

$$
+\frac{63B\big(143B^4 - 110AB^2C + 15A^2C^2\big)D_2^2}{128C^8}\Bigg) + \frac{9009B\big(17B^4 - 10AB^2C + A^2C^2\big)D_1^5}{2048C^{10}}\Bigg]\Bigg\}. \quad (96)
$$

The coefficients A to D_9 are determined iteratively expanding (57) in powers of η^2. They depend on the respective Hamiltonian for which one may choose a pole- and log–free form.

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