Matching the Heavy-Quark Fields in QCD and HQET at Four Loops

A. G. Grozin

Budker Institute of Nuclear Physics, Novosibirsk, Russia

and

Novosibirsk State University, Russia

P. Marquard

Deutsches Elektronen-Synchrotron DESY, Zeuthen

A. V. Smirnov

Research Computing Center, Moscow State University, Russia

V. A. Smirnov

Skobeltsyn Institute of Nuclear Physics, Moscow State University, Russia

M. Steinhauser

Institut für Theoretische Teilchenphysik,
Karlsruher Institut für Technologie (KIT)

Matching the heavy-quark fields in QCD and HQET at four loops

Andrey G. Grozin,[1, 2, *] Peter Marquard,[3, †]
Alexander V. Smirnov,[4, ‡] Vladimir A. Smirnov,[5, §] and Matthias Steinhauser[6, ¶]

[1]*Budker Institute of Nuclear Physics, Lavrentyev St. 11, Novosibirsk 630090, Russia*
[2]*Novosibirsk State University, Pirogov St. 1, Novosibirsk 630090, Russia*
[3]*Deutsches Elektronen-Synchrotron, DESY, 15738 Zeuthen, Germany*
[4]*Research Computing Center, Moscow State University, 119991 Moscow, Russia*
[5]*Skobeltsyn Institute of Nuclear Physics of Moscow State University, 119991 Moscow, Russia*
[6]*Institut für Theoretische Teilchenphysik, Karlsruher Institut für Technologie (KIT), 76128 Karlsruhe, Germany*

The QCD/HQET matching coefficient for the heavy-quark field is calculated up to four loops. It must be finite; this requirement produces analytical results for some terms in the four-loop on-shell heavy-quark field renormalization constant which were previously only known numerically. The effect of a non-zero lighter-flavor mass is calculated up to three loops. A class of on-shell integrals with two masses is analyzed in detail. By specifying our result to QED, we obtain the relation between the electron field and the Bloch–Nordsieck field with four-loop accuracy.

I. INTRODUCTION

Some classes of QCD problems with a single heavy quark can be examined in a simpler effective theory, the so-called heavy quark effective theory (HQET, see, e. g., [1–3]). Let us consider QCD with a single heavy flavor Q and n_l light flavors ($n_f = n_l + n_h$, $n_h = 1$). The heavy-quark momentum can be decomposed as $p = Mv + k$, where M is the on-shell Q mass, and v is some reference 4-velocity ($v^2 = 1$). In the case of QED, it is called Bloch–Nordsieck effective theory [4].

In the effective theory, the heavy quark (respectively lepton) is represented by the field h_v. The $\overline{\text{MS}}$ renormalized fields $Q(\mu)$ and $h_v(\mu)$ are related by [5]

$$Q(\mu) = e^{-iMvx} \left[\sqrt{z(\mu)} \left(1 + \frac{D_\perp}{2M} \right) h_v(\mu) + \mathcal{O}\left(\frac{1}{M^2} \right) \right],$$
(1)

where $D_\perp^\mu = D^\mu - v^\mu v \cdot D$, and the matching coefficient is given by

$$z(\mu) = \frac{Z_h(\alpha_s^{(n_l)}(\mu), \xi^{(n_l)}(\mu)) Z_Q^{\text{os}}(g_0^{(n_f)}, \xi_0^{(n_f)})}{Z_Q(\alpha_s^{(n_f)}(\mu), \xi^{(n_f)}(\mu)) Z_h^{\text{os}}(g_0^{(n_l)}, \xi_0^{(n_l)})}.$$
(2)

Here Z_Q^{os} and Z_h^{os} are the on-shell field renormalization constants (they depend on the corresponding bare couplings and bare gauge-fixing parameters), and Z_Q and Z_h are the $\overline{\text{MS}}$ renormalization constants. The covariant-gauge fixing parameter is defined in such a way that the bare gluon propagator is given by $(g_{\mu\nu} - \xi_0 p_\mu p_\nu/p^2)/p^2$; it is renormalized by the gluon-field renormalization constant: $1 - \xi_0 = Z_A(\alpha_s(\mu), \xi(\mu))(1 - \xi(\mu))$. The $1/M$ correction in (1) is fixed by reparametrization invariance [6].

The $\overline{\text{MS}}$ renormalized matching coefficient is obviously finite at $\varepsilon \to 0$, because it relates the off-shell renormalized propagators in the two theories, which are both finite. The ultraviolet divergences cancel in the ratios Z_Q/Z_Q^{os} and Z_h/Z_h^{os}, because they relate renormalized fields; the infrared divergences cancel in $Z_Q^{\text{os}}/Z_h^{\text{os}}$, because HQET is constructed to reproduce the infrared behavior of QCD; the $\overline{\text{MS}}$ renormalization constants Z_Q and Z_h (purely off-shell quantities) are infrared finite. If we assume that all light flavors are massless we have $Z_h^{\text{os}} = 1$: all loop corrections vanish because they contain no scale, ultraviolet and infrared divergences of Z_h^{os} mutually cancel. Taking light-quark masses m_i into account produces corrections suppressed by powers of m_i/M, see Sect. III.

The matching coefficient satisfies the renormalization-group equation

$$\frac{d \log z(\mu)}{d \log \mu} =$$
$$\gamma_h(\alpha_s^{(n_l)}(\mu), \xi^{(n_l)}(\mu)) - \gamma_Q(\alpha_s^{(n_f)}(\mu), \xi^{(n_f)}(\mu)), \quad (3)$$

where the anomalous dimensions are defined as $\gamma_i = d \log Z_i/d \log \mu$ ($i = Q, h$). It is sufficient to obtain the initial condition $z(\mu_0)$ for some scale $\mu_0 \sim M$; $z(\mu)$ for other renormalization scales μ can be found by solving Eq. (3). We choose to present the result for $\mu_0 = M$.

The heavy-quark field matching coefficient $z(\mu)$ has been calculated up to three loops [5]. When the matching coefficient is used within a quantity containing $1/\varepsilon$ divergences, terms with positive powers of ε in $z(\mu)$ are needed; such terms were not given in [5]. We present the four-loop result in Sect. II. Power corrections due to lighter-flavor masses up to three loops are obtained in Sect. III. The QED result, i. e. the four-loop relation between the lepton field and the Bloch–Nordsieck field, is discussed in Sect. IV. In Appendix A we provide analytic results for the decoupling coefficients for the strong coupling constant and the gluon field up to three-loop order including linear ε terms. Appendix B contains a detailed analysis of a class of on-shell integrals with two masses. It al-

* A.G.Grozin@inp.nsk.su
† peter.marquard@desy.de
‡ asmirnov80@gmail.com
§ smirnov@theory.sinp.msu.ru
¶ matthias.steinhauser@kit.edu

lows us, in particular, to obtain exact results for the three-loop term in the $\overline{\text{MS}}$–on-shell mass relation with a closed massless and a closed lighter-flavor massive fermion loop (previously this term was only known as a truncated series in this mass ratio).

II. THE QCD AND HQET HEAVY-QUARK FIELDS

If we assume that all light flavors are massless, then (2) gives

$$
\log z(\mu) = \log Z_Q^{\text{os}}(g_0^{(n_f)}, \xi_0^{(n_f)}) \tag{4}
$$
$$
- \log Z_Q(\alpha_s^{(n_f)}(\mu), \xi^{(n_f)}(\mu)) + \log Z_h(\alpha_s^{(n_l)}(\mu), \xi^{(n_l)}(\mu)) \, .
$$

The on-shell heavy-quark field renormalization constant Z_Q^{os} depends on the bare coupling $g_0^{(n_f)}$, the bare gauge parameter $\xi_0^{(n_f)}$ and the on-shell mass M:

$$
Z_Q^{\text{os}} = 1 + \sum_{L=1}^{\infty} \left(4 \frac{(g_0^{(n_f)})^2 M^{-2\varepsilon}}{(4\pi)^{d/2}} e^{-\gamma_E \varepsilon} \right)^L Z_L \, ,
$$
$$
Z_L = \sum_{n=0}^{\infty} Z_{L,n}(\xi_0^{(n_f)}) \varepsilon^{n-L} \, . \tag{5}
$$

The two-loop expression is known exactly in ε [7]; it contains a single non-trivial master integral, further terms of its ε expansion are presented in [8, 9]. The three-loop term has been calculated in [10, 11]. At four loops, the terms with n_l^3 and n_l^2 are known analytically [12], and the remaining ones numerically [13]. Recently the QED-like color structures C_F^4, $C_F^3 T_F n_h$, $C_F^2(T_F n_h)^2$, $C_F^3(T_F n_h)^3$, $d_{FF} n_h$ have been calculated analytically [14]. Here and below we use the notation

$$
d_{FF} = \frac{d_F^{abcd} d_F^{abcd}}{N_F} \, , \quad d_{FA} = \frac{d_F^{abcd} d_A^{abcd}}{N_F} \, , \tag{6}
$$

where $N_R = \text{Tr} \, \mathbf{1}_R$ (with $R = F$), $d_R^{abcd} = \text{Tr} \, t_R^{(a} t_R^b t_R^c t_R^{d)}$ (with $R = F$ or A), and the round brackets mean symmetrization (for $SU(N_c)$ gauge group $d_{FF} = (N_c^2 - 1)(N_c^4 - 6N_c^2 + 18)/(96 N_c^3)$, $d_{FA} = (N_c^2 - 1)(N_c^2 + 6)/48$). This result contains the same master integrals as the electron $g - 2$ [15, 16].

In [15] they have been calculated numerically to 1100 digits, and analytical expressions have been reconstructed using PSLQ. In the case of the light-by-light contribution $d_{FF} n_h$ the results contain ε^0 terms of 6 master integrals (known numerically to 1100 digits); all the remaining constants are completely expressed via known transcendental numbers (Note that the definition of the constant t_{63} is missing in the journal article [14]; it is included in the version v3 of the arXiv publication.).

The $\overline{\text{MS}}$ quark-field anomalous dimension γ_q (and hence $\log Z_Q$) is well known [17–20]. The HQET field anomalous dimension γ_h (and hence $\log Z_h$) is known at three loops [10, 21]. At four loops, some color structures are known analytically: $C_F(T_F n_l)^3$ [22], $C_F^2(T_F n_l)^2$ [23, 24], $C_F C_A(T_F n_l)^2$ [13], $C_F^3 T_F n_l$ [25], $d_F^{abcd} d_F^{abcd} n_l$ [26], $C_F^2 C_A T_F n_l$ and $C_F C_A^2 T_F n_l$ [27]; $C_F C_A^3$ and $d_F^{abcd} d_A^{abcd}$ are known numerically [13].

We need to express the three terms in (4) in terms of the same set of variables, for which we choose $\alpha_s^{(n_f)}(\mu)$ and $\xi^{(n_f)}(\mu)$. Expressing $g_0^{(n_f)}$ and $\xi_0^{(n_f)}$ via these variables is straightforward, since the three-loop renormalization constants in QCD are well known. Expressing $\alpha_s^{(n_l)}(\mu)$ and $\xi^{(n_l)}(\mu)$ via the n_f-flavor quantities requires decoupling relations up to $\mathcal{O}(\varepsilon)$ at three loops. For convenience we present explicit results in Appendix A.

The resulting matching coefficient $z(M)$ must be finite at $\varepsilon \to 0$. This requirement together with the known results for Z_Q and Z_h leads to analytical expressions for the four-loop coefficients $Z_{4,0}$, $Z_{4,1}$, and $Z_{4,2}$ in (5) as well as for $Z_{4,3}$, except two color structures $C_F C_A^3$ and d_{FA} where the corresponding terms in γ_h are not known analytically. The analytic results are presented in the tables I and II. We refrain from showing results for the n_l^2 and n_l^3 terms, which are already known since a few years [12]. Furthermore, we have introduced $a_n = \text{Li}_n(1/2)$ (in particular $a_1 = \log 2$); ζ_n denotes the Riemann zeta function and $\xi_0 = \xi_0^{(n_f)}$. Analytical results for the color structures C_F^4, $C_F^3 T_F n_h$, $C_F^2(T_F n_h)^2$, $C_F^3(T_F n_h)^3$, $d_{FF} n_h$ were recently obtained [14]. They agree with the expressions given in tables I and II. Numerical results for these coefficients are given in the tables V, VI, and VII of Ref. [13]. Good agreement is found.

Using the matching coefficient $z(\mu)$ together with quantities which contains $1/\varepsilon$ divergences, terms with positive powers of ε are needed. In order to get the finite four-loop contribution, we need the α_s^L term in $z(\mu)$ expanded up to ε^{4-L}. Our result for $\mu = M$ is given by

$$
z(M) = 1 - \frac{\alpha_s}{\pi} C_F \left[1 + \varepsilon \left(\frac{\pi^2}{16} + 2 \right) - \varepsilon^2 \left(\frac{\zeta_3}{4} - \frac{\pi^2}{12} - 4 \right) - \varepsilon^3 \left(\frac{\zeta_3}{3} - \frac{3}{640}\pi^4 - \frac{\pi^2}{6} - 8 \right) + \mathcal{O}(\varepsilon^4) \right]
$$
$$
+ \left(\frac{\alpha_s}{\pi} \right)^2 C_F \left\{ C_F \left(\pi^2 a_1 - \frac{3}{2}\zeta_3 - \frac{13}{16}\pi^2 + \frac{241}{128} \right) - \frac{C_A}{2} \left(\pi^2 a_1 - \frac{3}{2}\zeta_3 - \frac{5}{8}\pi^2 + \frac{1705}{192} \right) \right.
$$
$$
- \frac{T_F n_h}{3} \left(\pi^2 - \frac{947}{96} \right) + \frac{T_F n_l}{12} \left(\pi^2 + \frac{113}{8} \right)
$$
$$
+ \varepsilon \left[-C_F \left(24 a_4 + a_1^4 + 2\pi^2 a_1^2 - \frac{23}{4}\pi^2 a_1 + \frac{147}{8}\zeta_3 - \frac{7}{20}\pi^4 + \frac{347}{128}\pi^2 + \frac{557}{256} \right) \right.
$$

TABLE I. Coefficients $Z_{4,n}$ of the $1/\varepsilon^{4,3,2}$ terms entering the four-loop result Z_4 in Eq. (5). Note that the color structures $d_{FF}n_l$, $d_{FF}n_h$, d_{FA} have zero coefficients.

Color	ε^{-4}	ε^{-3}	ε^{-2}
C_F^4	$\frac{27}{2048}$	$\frac{171}{2048}$	$\frac{3}{32}\left(3\pi^2 a_1 - \frac{9}{2}\zeta_3 - \frac{153}{64}\pi^2 + \frac{1945}{256}\right)$
$C_F^3 C_A$	$-\frac{99}{1024}$	$-\frac{779}{1024}$	$\frac{1}{64}\left(-119\pi^2 a_1 + \frac{363}{2}\zeta_3 + \frac{1487}{16}\pi^2 - \frac{77405}{192}\right)$
$C_F^2 C_A^2$	$\frac{1331}{6144}$	$\frac{203}{96}$	$\frac{1}{32}\left(\frac{649}{6}\pi^2 a_1 - \frac{2653}{16}\zeta_3 + \frac{\pi^4}{45} - \frac{46321}{576}\pi^2 + \frac{431051}{768}\right) + \frac{\xi_0}{128}\left(\frac{9}{8}\zeta_3 - \frac{\pi^4}{45} + \frac{1}{8}\right)$
$C_F C_A^3$	$-\frac{1331}{9216}$	$-\frac{97669}{55296}$	$\frac{1}{32}\left(-\frac{121}{3}\pi^2 a_1 + \frac{7531}{128}\zeta_3 - \frac{127}{2160}\pi^4 + \frac{20449}{864}\pi^2 - \frac{419083}{864}\right)$ $- \frac{59}{1024}\xi_0\left(\frac{3}{8}\zeta_3 - \frac{\pi^4}{135} + \frac{1}{24}\right) + \frac{\xi_0^2}{1024}\left(\frac{9}{8}\zeta_3 - \frac{\pi^4}{45} + \frac{1}{8}\right)$
$C_F^3 T_F n_h$	$\frac{9}{128}$	$\frac{131}{512}$	$\frac{1}{4}\left(3\pi^2 a_1 - \frac{9}{2}\zeta_3 - \frac{87}{32}\pi^2 + \frac{1747}{192}\right)$
$C_F^2 C_A T_F n_h$	$\frac{-67+\xi_0}{256}$	$-\frac{1619+\frac{29}{3}\xi_0}{1024}$	$\frac{1}{8}\left(-\frac{53}{3}\pi^2 a_1 + \frac{433}{16}\zeta_3 + \frac{1697}{96}\pi^2 - \frac{257689}{2304}\right) - \frac{\xi_0}{128}\left(3\zeta_3 - \frac{\pi^2}{6} - \frac{137}{48}\right)$
$C_F C_A^2 T_F n_h$	$\frac{441-\frac{97}{9}\xi_0}{2048}$	$\frac{\frac{216101}{36}+\frac{413}{12}\xi_0+\xi_0^2}{3072}$	$\frac{1}{12}\left(11\pi^2 a_1 - \frac{2081}{128}\zeta_3 + \frac{11}{1080}\pi^4 - \frac{24725}{1536}\pi^2 + \frac{13306637}{55296}\right)$ $+ \frac{\xi_0}{16}\left(\frac{27}{64}\zeta_3 - \frac{\pi^4}{405} - \frac{97}{3456}\pi^2 - \frac{6251}{4608}\right) - \frac{\xi_0^2}{6144}(\zeta_3 + 5)$
$C_F^2 (T_F n_h)^2$	$\frac{3}{32}$	$\frac{27}{128}$	$\frac{\pi^2}{3}a_1 - \frac{\zeta_3}{2} - \frac{47}{96}\pi^2 + \frac{4337}{1536}$
$C_F C_A (T_F n_h)^2$	$\frac{-15+\frac{\xi_0}{3}}{128}$	$-\frac{\frac{1571}{18}+\xi_0}{128}$	$\frac{1}{2}\left(-\frac{\pi^2}{3}a_1 + \frac{\zeta_3}{2} + \frac{779}{576}\pi^2 - \frac{14449}{864}\right) + \frac{\xi_0}{144}\left(\frac{\pi^2}{8} + \frac{17}{3}\right)$
$C_F (T_F n_h)^3$	$\frac{1}{36}$	$\frac{41}{864}$	$\frac{1}{36}\left(-\frac{11}{3}\pi^2 + \frac{679}{16}\right)$
$C_F^3 T_F n_l$	$\frac{9}{256}$	$\frac{1}{4}$	$\frac{1}{4}\left(\frac{5}{2}\pi^2 a_1 - 3\zeta_3 - \frac{121}{64}\pi^2 + \frac{5491}{768}\right)$
$C_F^2 C_A T_F n_l$	$-\frac{121}{768}$	$-\frac{373}{256}$	$\frac{1}{16}\left(-\frac{103}{3}\pi^2 a_1 + 41\zeta_3 + \frac{3431}{144}\pi^2 - \frac{32869}{192}\right)$
$C_F C_A^2 T_F n_l$	$\frac{121}{768}$	$\frac{2903}{1536}$	$\frac{1}{4}\left(\frac{11}{3}\pi^2 a_1 - \frac{859}{256}\zeta_3 + \frac{11}{4320}\pi^4 - \frac{715}{576}\pi^2 + \frac{410389}{6912}\right) + \frac{\xi_0}{64}\left(\frac{3}{8}\zeta_3 - \frac{\pi^4}{135} + \frac{1}{24}\right)$
$C_F^2 T_F^2 n_h n_l$	$\frac{3}{32}$	$\frac{65}{128}$	$\frac{2}{3}\pi^2 a_1 - \frac{3}{4}\zeta_3 - \frac{21}{32}\pi^2 + \frac{2425}{576}$
$C_F C_A T_F^2 n_h n_l$	$\frac{-89+\xi_0}{576}$	$-\frac{\frac{4555}{18}+\xi_0}{192}$	$-\frac{\pi^2}{3}a_1 + \frac{\zeta_3}{4} + \frac{1063}{1728}\pi^2 - \frac{68323}{5184} + \frac{\xi_0}{288}\left(\frac{\pi^2}{6} + 11\right)$
$C_F T_F^3 n_h^2 n_l$	$\frac{1}{24}$	$\frac{19}{96}$	$\frac{1}{72}\left(-13\pi^2 + \frac{4895}{24}\right)$

$$+ C_A\left(12a_4 + \frac{a_1^4}{2} + \pi^2 a_1^2 - \frac{23}{8}\pi^2 a_1 + \frac{129}{16}\zeta_3 - \frac{7}{40}\pi^4 + \frac{769}{1152}\pi^2 - \frac{9907}{768}\right)$$

$$+ T_F n_h\left(2\pi^2 a_1 - 7\zeta_3 - \frac{445}{288}\pi^2 + \frac{17971}{1728}\right) + T_F n_l\left(\zeta_3 + \frac{127}{288}\pi^2 + \frac{851}{192}\right)\Bigg]$$

$$+ \varepsilon^2\Bigg[-C_F\left(144a_5 + 138a_4 - \frac{6}{5}a_1^5 + \frac{23}{4}a_1^4 - 4\pi^2 a_1^3 + \frac{23}{2}\pi^2 a_1^2 + \frac{13}{15}\pi^4 a_1 - \frac{41}{2}\pi^2 a_1\right.$$

$$\left. - \frac{609}{4}\zeta_5 - \frac{11}{4}\pi^2\zeta_3 + \frac{2061}{32}\zeta_3 - \frac{1555}{1536}\pi^4 + \frac{8947}{768}\pi^2 - \frac{1817}{512}\right)$$

$$+ C_A\left(72a_5 + 69a_4 - \frac{3}{5}a_1^5 + \frac{23}{8}a_1^4 - 2\pi^2 a_1^3 + \frac{23}{4}\pi^2 a_1^2 + \frac{13}{30}\pi^4 a_1 - \frac{41}{4}\pi^2 a_1\right.$$

$$\left. - \frac{609}{8}\zeta_5 - \frac{11}{8}\pi^2\zeta_3 + \frac{7595}{288}\zeta_3 - \frac{14359}{23040}\pi^4 + \frac{6367}{2304}\pi^2 - \frac{79225}{1536}\right)$$

$$- T_F n_h\left(48a_4 + 2a_1^4 + 4\pi^2 a_1^2 - \frac{19}{2}\pi^2 a_1 + \frac{2405}{72}\zeta_3 - \frac{93}{320}\pi^4 + \frac{8605}{1728}\pi^2 - \frac{422747}{10368}\right)$$

$$+ \frac{T_F n_l}{24}\left(\frac{305}{3}\zeta_3 + \frac{199}{80}\pi^4 + \frac{853}{24}\pi^2 + \frac{5753}{16}\right)\Bigg] + \mathcal{O}(\varepsilon^3)\Bigg\}$$

$$+ \left(\frac{\alpha_s}{\pi}\right)^3 C_F\Bigg\{-C_F^2\left(28a_4 + \frac{7}{6}a_1^4 - \frac{3}{2}\pi^2 a_1^2 - \frac{223}{12}\pi^2 a_1 + \frac{5}{16}\zeta_5 - \frac{\pi^2}{8}\zeta_3 + \frac{157}{8}\zeta_3 + \frac{19}{240}\pi^4 + \frac{4801}{576}\pi^2 + \frac{3023}{768}\right)$$

$$- C_F C_A\left(\frac{a_4}{6} + \frac{a_1^4}{144} + \frac{181}{72}\pi^2 a_1^2 + \frac{43}{9}\pi^2 a_1 - \frac{145}{16}\zeta_5 + \frac{45}{16}\pi^2\zeta_3 + \frac{289}{24}\zeta_3 - \frac{6697}{17280}\pi^4 - \frac{2137}{576}\pi^2 - \frac{24131}{4608}\right)$$

$$+ \frac{C_A^2}{2}\left[\frac{1}{3}\left(\frac{85}{2}a_4 + \frac{85}{48}a_1^4 + \frac{127}{24}\pi^2 a_1^2 - \frac{325}{24}\pi^2 a_1 - 37\zeta_5 + \frac{127}{12}\pi^2\zeta_3 + \frac{5857}{96}\zeta_3 - \frac{3419}{3840}\pi^4 - \frac{4339}{576}\pi^2 - \frac{1654711}{20736}\right)\right.$$

TABLE II. Coefficients $Z_{4,3}$ of the $1/\varepsilon$ term entering the four-loop result Z_4 in Eq. (5). Note that the color structures $C_F C_A^3$ and d_{FA} are not known analytically.

Color	ε^{-1}
C_F^4	$\frac{1}{4}\left(57a_4 + \frac{19}{8}a_1^4 - \frac{27}{4}\pi^2 a_1^2 - \frac{1571}{32}\pi^2 a_1 - \frac{25}{16}\zeta_5 - \frac{3}{8}\pi^2\zeta_3 + \frac{5045}{128}\zeta_3 + \frac{101}{160}\pi^4 + \frac{33539}{1536}\pi^2 + \frac{23865}{2048}\right)$
$C_F^3 C_A$	$-\frac{1}{4}\left(129a_4 + \frac{43}{8}a_1^4 - \frac{467}{12}\pi^2 a_1^2 - \frac{33263}{192}\pi^2 a_1 + 25\zeta_5 - \frac{157}{16}\pi^2\zeta_3 + \frac{5477}{128}\zeta_3 + \frac{26707}{5760}\pi^4 + \frac{46967}{576}\pi^2 + \frac{245183}{3072}\right)$
$C_F^2 C_A^2$	$-\frac{1}{16}\left(\frac{4171}{3}a_4 + \frac{4171}{72}a_1^4 + \frac{8257}{36}\pi^2 a_1^2 - \frac{20803}{144}\pi^2 a_1 - \frac{14733}{32}\zeta_5 + \frac{1739}{12}\pi^2\zeta_3 + \frac{472475}{288}\zeta_3 - \frac{217663}{5760}\pi^4 - \frac{23849}{576}\pi^2 - \frac{2018473}{2304}\right)$ $-\frac{\xi_0}{64}\left(\frac{9}{16}\zeta_5 + \frac{\pi^2}{3}\zeta_3 - \frac{95}{32}\zeta_3 + \frac{383}{8640}\pi^4 - \frac{3}{16}\pi^2 - \frac{13}{12}\right)$
$C_F^3 T_F n_h$	$a_4 + \frac{a_1^4}{24} - \frac{21}{8}\pi^2 a_1^2 - \frac{249}{16}\pi^2 a_1 + \frac{5}{16}\zeta_5 - \frac{\pi^2}{8}\zeta_3 + \frac{2705}{768}\zeta_3 + \frac{99}{320}\pi^4 + \frac{103157}{13824}\pi^2 + \frac{142385}{18432}$
$C_F^2 C_A T_F n_h$	$\frac{278}{3}a_4 + \frac{139}{36}a_1^4 + \frac{761}{144}\pi^2 a_1^2 - \frac{469}{36}\pi^2 a_1 - \frac{545}{64}\zeta_5 + \frac{169}{64}\pi^2\zeta_3 + \frac{661373}{9216}\zeta_3 - \frac{69871}{69120}\pi^4 + \frac{434299}{82944}\pi^2 - \frac{6467663}{110592}$ $-\frac{\xi_0}{2}\left(a_4 + \frac{a_1^4}{24} - \frac{\pi^2}{24}a_1^2 + \frac{59}{192}\zeta_3 - \frac{91}{11520}\pi^4 + \frac{29}{4608}\pi^2 - \frac{407}{6144}\right)$
$C_F C_A^2 T_F n_h$	$-\frac{1}{3}\left(154a_4 + \frac{77}{12}a_1^4 + \frac{515}{96}\pi^2 a_1^2 - \frac{335}{3}\pi^2 a_1 - \frac{4217}{384}\zeta_5 + \frac{1963}{576}\pi^2\zeta_3 + \frac{1044785}{6144}\zeta_3 - \frac{283447}{276480}\pi^4 + \frac{857267}{12288}\pi^2 - \frac{498708329}{1327104}\right)$ $+\frac{\xi_0}{4}\left(a_4 + \frac{a_1^4}{24} - \frac{\pi^2}{24}a_1^2 - \frac{7}{144}\zeta_5 - \frac{55}{54}\zeta_3 + \frac{55}{3456}\zeta_3 - \frac{4631}{414720}\pi^4 + \frac{701}{27648}\pi^2 + \frac{59923}{36864}\right) + \frac{\xi_0^2}{3072}\left(\frac{7}{2}\zeta_3 - \frac{\pi^4}{120} + \frac{\pi^2}{3} + \frac{167}{8}\right)$
$C_F^2 (T_F n_h)^2$	$-20a_4 - \frac{5}{6}a_1^4 - \frac{\pi^2}{6}a_1^2 + \frac{55}{36}\pi^2 a_1 - \frac{32131}{2304}\zeta_3 + \frac{53}{720}\pi^4 - \frac{11663}{51840}\pi^2 + \frac{167545}{13824}$
$C_F C_A (T_F n_h)^2$	$12a_4 + \frac{a_1^4}{2} - \frac{973}{72}\pi^2 a_1 + \frac{15}{16}\zeta_5 - \frac{11}{48}\pi^2\zeta_3 + \frac{124621}{4608}\zeta_3 - \frac{\pi^4}{90} + \frac{1059347}{103680}\pi^2 - \frac{4538573}{82944} - \frac{\xi_0}{24}\left(\frac{11}{24}\zeta_3 + \frac{\pi^2}{16} + \frac{29}{9}\right)$
$C_F (T_F n_h)^3$	$\frac{1}{3}\left(2\pi^2 a_1 - \frac{127}{9}\zeta_3 - \frac{7211}{4320}\pi^2 + \frac{71143}{3456}\right)$
$d_{FF} n_h$	$-\frac{1}{8}$
$C_F^3 T_F n_l$	$9a_4 + \frac{3}{8}a_1^4 - \frac{37}{12}\pi^2 a_1^2 - \frac{1253}{96}\pi^2 a_1 + \frac{25}{32}\zeta_5 - \frac{\pi^2}{8}\zeta_3 + \frac{531}{128}\zeta_3 + \frac{1087}{2880}\pi^4 + \frac{6485}{1152}\pi^2 + \frac{2991}{1024}$
$C_F^2 C_A T_F n_l$	$\frac{205}{3}a_4 + \frac{205}{72}a_1^4 + \frac{295}{36}\pi^2 a_1^2 - \frac{9289}{576}\pi^2 a_1 - \frac{605}{64}\zeta_5 + \frac{45}{16}\pi^2\zeta_3 + \frac{75143}{1152}\zeta_3 - \frac{389}{270}\pi^4 + \frac{995}{192}\pi^2 - \frac{107579}{4608}$
$C_F C_A^2 T_F n_l$	$-\frac{1}{4}\left(\frac{437}{3}a_4 + \frac{437}{72}a_1^4 + \frac{479}{36}\pi^2 a_1^2 - \frac{1631}{36}\pi^2 a_1 - \frac{6133}{256}\zeta_5 + \frac{3}{16}\zeta_3^2 + \frac{4057}{576}\pi^2\zeta_3 + \frac{430895}{4608}\zeta_3 - \frac{854171}{414720}\pi^4 - \frac{26779}{3456}\pi^2\right)$ $-\frac{1583779}{5184}\right) - \frac{\xi_0}{384}\left(\frac{19}{4}\zeta_5 + \frac{5}{3}\pi^2\zeta_3 - \frac{29}{2}\zeta_3 + \frac{439}{2160}\pi^4 - \pi^2 - \frac{53}{12}\right)$
$C_F^2 T_F^2 n_h n_l$	$-\frac{1}{3}\left(100a_4 + \frac{25}{6}a_1^4 + \frac{23}{6}\pi^2 a_1^2 - \frac{209}{12}\pi^2 a_1 + \frac{6467}{96}\zeta_3 - \frac{8}{9}\pi^4 + \frac{26719}{3456}\pi^2 - \frac{184019}{4608}\right)$
$C_F C_A T_F^2 n_h n_l$	$\frac{56}{3}a_4 + \frac{7}{9}a_1^4 + \frac{5}{9}\pi^2 a_1^2 - \frac{142}{9}\pi^2 a_1 + \frac{15}{16}\zeta_5 - \frac{11}{48}\pi^2\zeta_3 + \frac{37781}{1728}\zeta_3 - \frac{47}{540}\pi^4 + \frac{105421}{10368}\pi^2 - \frac{9614047}{124416} + \frac{\xi_0}{24}\left(\frac{55}{36}\zeta_3 - \frac{\pi^2}{24} - 5\right)$
$C_F T_F^3 n_h^2 n_l$	$\frac{1}{3}\left(4\pi^2 a_1 - \frac{121}{6}\zeta_3 - \frac{1829}{480}\pi^2 + \frac{54391}{1152}\right)$
$d_{FF} n_l$	$-\frac{1}{4}\left(\frac{5}{8}\zeta_5 - \frac{\pi^2}{3}\zeta_3 - \frac{\zeta_3}{2} + \frac{\pi^2}{3} + \frac{1}{2}\right)$

$$+ \frac{\xi}{8}\left(\frac{7}{24}\zeta_5 + \frac{\pi^2}{9}\zeta_3 - \frac{13}{16}\zeta_3 + \frac{17}{1728}\pi^4 - \frac{\pi^2}{16} - \frac{13}{48}\right)\Bigg]$$

$$+ C_F T_F n_h\left(12a_4 + \frac{a_1^4}{2} - \frac{\pi^2}{2}a_1^2 + \frac{17}{9}\pi^2 a_1 + \frac{233}{288}\zeta_3 + \frac{31}{720}\pi^4 - \frac{553}{324}\pi^2 - \frac{13571}{3456}\right)$$

$$- C_A T_F n_h\left[\left(8a_4 + \frac{a_1^4}{3} - \frac{\pi^2}{3}a_1^2 - \frac{80}{9}\pi^2 a_1 + \frac{15}{16}\zeta_5 - \frac{11}{48}\pi^2\zeta_3 + \frac{2813}{576}\zeta_3 + \frac{17}{360}\pi^4 + \frac{9067}{1296}\pi^2 - \frac{788639}{41472}\right)\right.$$

$$\left.+ \frac{\xi}{24}\left(\zeta_3 - \frac{2387}{576}\right)\right]$$

$$+ \frac{C_F T_F n_l}{3}\left(16a_4 + \frac{2}{3}a_1^4 + \frac{4}{3}\pi^2 a_1^2 - \frac{47}{6}\pi^2 a_1 + \frac{137}{8}\zeta_3 - \frac{229}{720}\pi^4 + \frac{113}{24}\pi^2 + \frac{35}{6}\right)$$

$$- \frac{C_A T_F n_l}{3}\left(8a_4 + \frac{a_1^4}{3} + \frac{2}{3}\pi^2 a_1^2 - \frac{47}{12}\pi^2 a_1 + \frac{35}{24}\zeta_3 - \frac{19}{360}\pi^4 - \frac{13}{16}\pi^2 - \frac{111791}{5184}\right)$$

$$+ \frac{(T_F n_h)^2}{3}\left(7\zeta_3 + \frac{2}{15}\pi^2 - \frac{8425}{864}\right) + \frac{T_F^2 n_h n_l}{36}\left(13\pi^2 - \frac{4721}{36}\right) - \frac{(T_F n_l)^2}{18}\left(7\zeta_3 + \frac{19}{6}\pi^2 + \frac{5767}{432}\right)$$

$$+ \varepsilon\left[-C_F^2\left(\frac{440}{3}a_5 - 16\pi^2 a_4 + \frac{2444}{3}a_4 - \frac{11}{9}a_1^5 - \frac{2}{3}\pi^2 a_1^4 + \frac{611}{18}a_1^4 + \frac{115}{27}\pi^2 a_1^3 + \frac{2}{3}\pi^4 a_1^2 + \frac{2309}{36}\pi^2 a_1^2 - 14\pi^2\zeta_3 a_1\right.\right.$$

$$\left.\left.+ \frac{751}{432}\pi^4 a_1 - \frac{367}{2}\pi^2 a_1 - \frac{53}{2}\zeta_5 - \frac{29}{32}\zeta_3^2 - \frac{5861}{288}\pi^2\zeta_3 + \frac{5119}{16}\zeta_3 + \frac{899}{5670}\pi^6 - \frac{54467}{34560}\pi^4 + \frac{74245}{2048}\pi^2 + \frac{19337}{1536}\right)\right.$$

$$\left.- C_F C_A\left(\frac{487}{3}a_5 - 6\pi^2 a_4 - \frac{1796}{9}a_4 - \frac{487}{360}a_1^5 - \frac{\pi^2}{4}a_1^4 - \frac{449}{54}a_1^4 - \frac{1135}{108}\pi^2 a_1^3 + \frac{\pi^4}{4}a_1^2 + \frac{7235}{216}\pi^2 a_1^2 - \frac{21}{4}\pi^2\zeta_3 a_1\right.\right.$$

$$
\begin{aligned}
&- \frac{949}{1080}\pi^4 a_1 + \frac{30803}{432}\pi^2 a_1 - \frac{125473}{384}\zeta_5 + \frac{143}{4}\zeta_3^2 + \frac{2703}{128}\pi^2\zeta_3 - \frac{16339}{288}\zeta_3 + \frac{27331}{181440}\pi^6 - \frac{496741}{103680}\pi^4 - \frac{17665}{55296}\pi^2 \\
&\qquad - \frac{861659}{27648}\bigg) \\
&+ C_A^2\bigg[\frac{707}{6}a_5 - 7\pi^2 a_4 + \frac{935}{9}a_4 - \frac{707}{720}a_1^5 - \frac{7}{24}\pi^2 a_1^4 + \frac{935}{216}a_1^4 - \frac{905}{216}\pi^2 a_1^3 + \frac{7}{24}\pi^4 a_1^2 + \frac{7081}{216}\pi^2 a_1^2 - \frac{49}{8}\pi^2\zeta_3 a_1 \\
&\qquad - \frac{41}{8640}\pi^4 a_1 - \frac{8833}{864}\pi^2 a_1 - \frac{41569}{256}\zeta_5 + \frac{7451}{384}\zeta_3^2 + \frac{14915}{4608}\pi^2\zeta_3 + \frac{67807}{3456}\zeta_3 + \frac{45047}{362880}\pi^6 - \frac{126391}{51840}\pi^4 - \frac{150229}{41472}\pi^2 \\
&\qquad - \frac{72476083}{746496} - \frac{\xi}{128}\bigg(\frac{149}{6}\zeta_5 - \frac{25}{3}\zeta_3^2 - \frac{77}{72}\pi^2\zeta_3 + \frac{63}{2}\zeta_3 - \frac{49}{405}\pi^6 - \frac{383}{1080}\pi^4 + \frac{35}{8}\pi^2 + \frac{35}{2}\bigg)\bigg] \\
&+ C_F T_F n_h\bigg(72a_5 - \frac{229}{6}a_4 - \frac{3}{5}a_1^5 - \frac{229}{144}a_1^4 + \pi^2 a_1^3 - \frac{2219}{144}\pi^2 a_1^2 + \frac{143}{180}\pi^4 a_1 + \frac{293}{6}\pi^2 a_1 - \frac{87}{8}\zeta_5 - \frac{81}{8}\pi^2\zeta_3 \\
&\qquad - \frac{10913}{192}\zeta_3 + \frac{3649}{8640}\pi^4 - \frac{818609}{41472}\pi^2 + \frac{164069}{6912}\bigg) \\
&- C_A T_F n_h\bigg[48a_5 - 8\pi^2 a_4 + \frac{4247}{12}a_4 - \frac{2}{5}a_1^5 - \frac{\pi^2}{3}a_1^4 + \frac{4247}{288}a_1^4 + \frac{2}{3}\pi^2 a_1^3 + \frac{\pi^4}{3}a_1^2 + \frac{18133}{288}\pi^2 a_1^2 - 7\pi^2\zeta_3 a_1 \\
&\qquad + \frac{97}{180}\pi^4 a_1 - \frac{775}{9}\pi^2 a_1 + \frac{551}{64}\zeta_5 - \frac{181}{32}\zeta_3^2 - \frac{549}{64}\pi^2\zeta_3 + \frac{88855}{384}\zeta_3 + \frac{1501}{15120}\pi^6 - \frac{12607}{5760}\pi^4 + \frac{286961}{13824}\pi^2 \\
&\qquad - \frac{35801821}{248832} - \frac{\xi}{8}\bigg(\zeta_3 + \frac{\pi^4}{60} - \frac{121}{1728}\pi^2 - \frac{7367}{1152}\bigg)\bigg] \\
&+ \frac{C_F T_F n_l}{3}\bigg(224a_5 + \frac{1028}{3}a_4 - \frac{28}{15}a_1^5 + \frac{257}{18}a_1^4 - \frac{56}{9}\pi^2 a_1^3 + \frac{257}{9}\pi^2 a_1^2 - \frac{17}{90}\pi^4 a_1 - \frac{539}{9}\pi^2 a_1 - \frac{1027}{4}\zeta_5 \\
&\qquad - \frac{119}{16}\pi^2\zeta_3 + \frac{1081}{6}\zeta_3 - \frac{18599}{8640}\pi^4 + \frac{160081}{4608}\pi^2 + \frac{3103}{72}\bigg) \\
&- C_A T_F n_l\bigg(\frac{112}{3}a_5 + \frac{514}{9}a_4 - \frac{14}{45}a_1^5 + \frac{257}{108}a_1^4 - \frac{28}{27}\pi^2 a_1^3 + \frac{257}{54}\pi^2 a_1^2 - \frac{17}{540}\pi^4 a_1 - \frac{539}{54}\pi^2 a_1 - \frac{859}{24}\zeta_5 \\
&\qquad - \frac{11}{16}\pi^2\zeta_3 + \frac{1229}{432}\zeta_3 - \frac{3691}{6480}\pi^4 - \frac{1991}{648}\pi^2 - \frac{4500377}{93312}\bigg) \\
&+ \frac{(T_F n_h)^2}{3}\bigg(56a_4 + \frac{7}{3}a_1^4 - \frac{7}{3}\pi^2 a_1^2 - \frac{4}{5}\pi^2 a_1 + \frac{3221}{80}\zeta_3 - \frac{31}{72}\pi^4 + \frac{39661}{7200}\pi^2 - \frac{636911}{8640}\bigg) \\
&+ T_F^2 n_h n_l\bigg(\frac{32}{3}a_4 + \frac{4}{9}a_1^4 + \frac{8}{9}\pi^2 a_1^2 - \frac{35}{9}\pi^2 a_1 + \frac{27}{2}\zeta_3 + \frac{179}{1080}\pi^4 + \frac{2245}{1296}\pi^2 - \frac{264817}{7776}\bigg) \\
&- \frac{(T_F n_l)^2}{54}\bigg(275\zeta_3 + \frac{23}{5}\pi^4 + \frac{1081}{16}\pi^2 + \frac{253783}{864}\bigg)\bigg] + \mathcal{O}(\varepsilon^2)\bigg\} \\
&+ \bigg(\frac{\alpha_s}{\pi}\bigg)^4\bigg\{C_F^4\bigg[L_0 - \frac{139}{2}a_5 + 12\pi^2 a_4 - \frac{9137}{16}a_4 + \frac{139}{240}a_1^5 + \frac{\pi^2}{2}a_1^4 - \frac{9137}{384}a_1^4 - \frac{311}{72}\pi^2 a_1^3 - \frac{\pi^4}{2}a_1^2 - \frac{8597}{192}\pi^2 a_1^2 \\
&\qquad + \frac{21}{2}\pi^2\zeta_3 a_1 - \frac{2783}{2880}\pi^4 a_1 + \frac{33687}{256}\pi^2 a_1 - \frac{2937}{128}\zeta_5 + \frac{87}{128}\zeta_3^2 + \frac{2755}{192}\pi^2\zeta_3 - \frac{113181}{512}\zeta_3 - \frac{899}{7560}\pi^6 + \frac{18553}{23040}\pi^4 \\
&\qquad - \frac{24129}{1024}\pi^2 - \frac{90577}{8192}\bigg] \\
&+ C_F^3 C_A(14.12 \pm 3.6) - C_F^2 C_A^2\big[8.75607 \pm 2.9 - (0.00269 \pm 0.0012)\xi\big] \\
&- C_F C_A^3\big[142.552 \pm 0.82 - (0.43649 \pm 0.00076)\xi + (0.0205278 \pm 0.00012)\xi^2\big] \\
&+ d_{FA}\big[9.4 \pm 2.1 + (0.147 \pm 0.013)\xi - (0.0748 \pm 0.0028)\xi^2\big] \\
&+ C_F^3 T_F n_h\bigg[L_1 + \frac{46}{3}a_5 + 16\pi^2 a_4 - \frac{35189}{48}a_4 - \frac{23}{180}a_1^5 + \frac{2}{3}\pi^2 a_1^4 - \frac{35189}{1152}a_1^4 - \frac{703}{108}\pi^2 a_1^3 - \frac{2}{3}\pi^4 a_1^2 - \frac{77155}{1152}\pi^2 a_1^2 \\
&\qquad + 14\pi^2\zeta_3 a_1 - \frac{569}{2160}\pi^4 a_1 + \frac{3273}{16}\pi^2 a_1 - \frac{3067}{32}\zeta_5 + \frac{29}{32}\zeta_3^2 + \frac{2981}{288}\pi^2\zeta_3 - \frac{119743}{384}\zeta_3 - \frac{899}{5670}\pi^6 + \frac{65953}{69120}\pi^4 \\
&\qquad - \frac{572525}{13824}\pi^2 - \frac{305411}{36864}\bigg]
\end{aligned}
$$

$$+ C_F^2 C_A T_F n_h \left[14.893 \pm 0.083 - (0.657352 \pm 0.00024)\xi\right]$$

$$- C_F C_A^2 T_F n_h \left[3.1601 \pm 0.056 - (0.198984 \pm 0.00013)\xi + 0.0244254\xi^2\right]$$

$$+ C_F^2 (T_F n_h)^2 \left[L_2 + 120a_5 + \frac{2749}{48}a_4 - a_1^5 + \frac{2749}{1152}a_1^4 - \frac{\pi^2}{3}a_1^3 - \frac{10525}{1152}\pi^2 a_1^2 + \frac{43}{36}\pi^4 a_1 + \frac{711}{20}\pi^2 a_1 - \frac{493}{8}\zeta_5 \right.$$

$$\left. - \frac{269}{24}\pi^2\zeta_3 - \frac{10127}{2560}\zeta_3 - \frac{5513}{13824}\pi^4 - \frac{678719}{64800}\pi^2 - \frac{8452817}{414720}\right]$$

$$- C_F C_A (T_F n_h)^2 \left[0.01995 \pm 0.0062 - 0.10436\xi\right]$$

$$+ C_F (T_F n_h)^3 \left[L_3 + \frac{1}{3}\left(104a_4 + \frac{13}{3}a_1^4 + \frac{5}{3}\pi^2 a_1^2 - \frac{103}{10}\pi^2 a_1 + \frac{5881}{80}\zeta_3 - \frac{299}{360}\pi^4 + \frac{31451}{2700}\pi^2 - \frac{5981281}{51840}\right)\right]$$

$$+ d_{FF} n_h L_l - C_F^3 T_F n_l (4.92605 \pm 0.0067) + C_F^2 C_A T_F n_l (15.0599 \pm 0.012)$$

$$+ C_F C_A^2 T_F n_l \left[166.421 \pm 0.031 - 0.134051\xi\right] - C_F^2 T_F^2 n_h n_l (5.08715 \pm 0.000074)$$

$$+ C_F C_A T_F^2 n_h n_l \left[0.53235 \pm 0.0015 + 0.0910988\xi\right] + 0.0138079 C_F T_F^3 n_h^2 n_l - d_{FF} n_l (2.18 \pm 0.8)$$

$$- C_F^2 (T_F n_l)^2 \left(\frac{32}{3}a_5 + \frac{188}{9}a_4 - \frac{4}{45}a_1^5 + \frac{47}{54}a_1^4 - \frac{8}{27}\pi^2 a_1^3 + \frac{47}{27}\pi^2 a_1^2 - \frac{31}{270}\pi^4 a_1 - \frac{239}{54}\pi^2 a_1 - \frac{601}{48}\zeta_5 - \frac{\pi^2}{2}\zeta_3 \right.$$

$$\left. + \frac{6925}{576}\zeta_3 - \frac{1181}{10368}\pi^4 + \frac{1043}{384}\pi^2 + \frac{3146969}{497664}\right)$$

$$+ \frac{C_F C_A (T_F n_l)^2}{3}\left(16a_5 + \frac{94}{3}a_4 - \frac{2}{15}a_1^5 + \frac{47}{36}a_1^4 - \frac{4}{9}\pi^2 a_1^3 + \frac{47}{18}\pi^2 a_1^2 - \frac{31}{180}\pi^4 a_1 - \frac{239}{36}\pi^2 a_1 - \frac{365}{32}\zeta_5 - \frac{11}{12}\pi^2\zeta_3 \right.$$

$$\left. - \frac{1111}{64}\zeta_3 - \frac{4333}{17280}\pi^4 - \frac{6815}{1152}\pi^2 - \frac{4767085}{165888}\right)$$

$$+ \frac{C_F T_F^3 n_h n_l^2}{3}\left(\frac{\zeta_3}{16} - \frac{4}{15}\pi^4 + \frac{19}{27}\pi^2 + \frac{399325}{20736}\right) + \frac{C_F (T_F n_l)^3}{216}\left(\frac{467}{2}\zeta_3 + \frac{71}{20}\pi^4 + \frac{167}{3}\pi^2 + \frac{103933}{864}\right) + \mathcal{O}(\varepsilon)\Bigg\}$$

$$+ \mathcal{O}(\alpha_s^5), \tag{7}$$

where $\alpha_s = \alpha_s^{(n_f)}(M)$, $\xi = \xi^{(n_f)}(M)$. $L_{0,l,1,2,3}$ are the ε^0 parts of the quantities $Z_2^{(4,0)}$, $Z_2^{(4,l)}$, $Z_2^{(4,1)}$, $Z_2^{(4,2)}$, $Z_2^{(4,3)}$ given in Eqs. (28–32) of [14]. Their numerical values are given in Eqs. (5–9) of that paper. The finite four-loop terms of Eq. (7) are equal to the corresponding finite four-loop terms in Z_Q^{os} plus products of lower-loop quantities which are all known analytically. For 14 out of 23 color structures these coefficients in Z_Q^{os} are only known numerically [13]. We use these numerical values, together with their uncertainty estimates, from the tables V, VI, and VII of that paper. Note that in Ref. [13] Z_Q^{os} has been computed in an expansion in ξ up to the second order; 9 out of these 19 color structures are obviously gauge invariant, and 7 more seem to be either gauge-invariant or have at most linear ξ terms (though we know no explicit proof). The remaining 3 structures ($C_F C_A^3$, d_{FA}, $C_F C_A^2 T_F n_h$) may contain terms with higher powers of ξ, which are not known. The same is true for the corresponding terms in $z(\mu)$ in Eq. (7).

If we re-express $z(M)$ in Eq. (7) via $\alpha_s^{(n_l)}(M)$, the terms up to three loops agree with [5]. (Note that positive powers of ε are not presented [5].) The $\alpha_s^4 n_l^3$ term also agrees with [5].

After specifying the color factors to QCD with $N_c = 3$ we obtain for $\varepsilon = 0$

$$z(M) = 1 - \frac{4}{3}\frac{\alpha_s}{\pi} - \left(\frac{\alpha_s}{\pi}\right)^2 (17.45 - 1.33 n_l)$$

$$- \left(\frac{\alpha_s}{\pi}\right)^3 (262.42 - 0.78\xi - 35.81 n_l + 0.98 n_l^2)$$

$$- \left(\frac{\alpha_s}{\pi}\right)^4 \left[5137.52 - 15.67\xi + 1.07\xi^2\right.$$

$$\left. - (1030.82 - 0.71\xi) n_l + 60.30 n_l^2 - 1.00 n_l^3\right]$$

$$+ \mathcal{O}(\alpha_s^5). \tag{8}$$

In Landau gauge ($\xi^{(n_f)} = 1$) at $n_l = 4$ this gives

$$z(M) = 1 - \frac{4}{3}\frac{\alpha_s}{\pi} - 12.12\left(\frac{\alpha_s}{\pi}\right)^2 - 134.11\left(\frac{\alpha_s}{\pi}\right)^3$$

$$- 1903.22\left(\frac{\alpha_s}{\pi}\right)^4 + \mathcal{O}(\alpha_s^5), \tag{9}$$

while the naive nonabelianization [22] (large β_0 limit) predicts [5]

$$1 - \frac{4}{3}\frac{\alpha_s}{\pi} - 16.66\left(\frac{\alpha_s}{\pi}\right)^2 - 153.41\left(\frac{\alpha_s}{\pi}\right)^3$$

$$- 1953.40\left(\frac{\alpha_s}{\pi}\right)^4 + \mathcal{O}(\alpha_s^5). \tag{10}$$

The comparison to Eq. (9) shows that up to four loops these predictions are rather good. The coefficients are all negative and grow very fast, which can be explained by the infrared renormalon at $u = 1/2$ [5]. This is the closest possible position of a renormalon singularity in the Borel plane u to the

origin, and it leads to the fastest possible growth of perturbative terms $(L-1)!\,(\beta_0/2)^L(\alpha_s/\pi)^L$. The coefficients of powers of ξ are much smaller than the ξ-independent terms.

III. EFFECT OF A LIGHTER-FLAVOR MASS

Now we suppose that n_m light flavors have a non-zero mass m, while the remaining $n_0 = n_l - n_m$ light flavors are massless. In practice, $n_m = 1$, e. g. c in b-quark HQET. In this case the massless result (7) for the matching coefficient should be multiplied by the additional factor

$$
\begin{aligned}
z' = {} & \frac{Z_Q^{\rm os}(g_0^{(n_f)}, \xi_0^{(n_f)}, m_0^{(n_f)})}{Z_Q^{\rm os}(g_0^{(n_f)}, \xi_0^{(n_f)}, 0)} \\
& \times \frac{Z_h^{\rm os}(g_0^{(n_l)}, \xi_0^{(n_l)}, 0)}{Z_h^{\rm os}(g_0^{(n_l)}, \xi_0^{(n_l)}, m_0^{(n_l)})}\,,
\end{aligned}
\tag{11}
$$

where $Z_{Q,h}^{\rm os}(\ldots,0) \equiv Z_{Q,h}^{\rm os}(\ldots)$ in Eq. (2) and $Z_h^{\rm os}(g_0^{(n_l)}, \xi_0^{(n_l)}, 0) = 1$. This factor does not depend on the renormalization scale μ. In the expression

$$
\begin{aligned}
\log z' = {} & \log Z_Q^{\rm os}(g_0^{(n_f)}, \xi_0^{(n_f)}, m_0^{(n_f)}) \\
& - \log Z_Q^{\rm os}(g_0^{(n_f)}, \xi_0^{(n_f)}, 0) - \log Z_h^{\rm os}(g_0^{(n_l)}, \xi_0^{(n_l)}, m_0^{(n_l)})
\end{aligned}
\tag{12}
$$

we re-express all terms via $\alpha_s^{(n_f)}(M)$, $\xi^{(n_f)}(M)$ and the on-shell lighter-flavor mass m (it is the same in both n_f and n_l flavor theories). The result depends on the dimensionless ratio

$$
x = \frac{m}{M}\,.
\tag{13}
$$

If we express z' via $\alpha_s^{(n_f)}(\mu)$, $\xi^{(n_f)}(\mu)$, the coefficients will depend on μ. This dependence is determined by the renormalization-group equation

$$
\frac{d\log z'}{d\log \mu} = 0
\tag{14}
$$

together with

$$
\begin{aligned}
\frac{d\log \alpha_s^{(n_f)}(\mu)}{d\log \mu} &= -2\varepsilon - 2\beta^{(n_f)}(\alpha_s^{(n_f)}(\mu))\,, \\
\frac{d\log(1-\xi^{(n_f)}(\mu))}{d\log \mu} &= -\gamma_A^{(n_f)}(\alpha_s^{(n_f)}(\mu), \xi^{(n_f)}(\mu))\,.
\end{aligned}
$$

Ultraviolet divergences cancel in each fraction in (11). On the other hand, the on-shell wave-function renormalization factors have extra infrared divergences at $m = 0$. However, z' in Eq. (11) has a smooth limit for $x \to 0$. In the following we illustrate the cancellation for infrared divergences at two-loop order. Similar mechanisms are also at work at higher loop

orders. For dimensional reasons the two-loop corrections in Fig. 1a lead to $\log Z_h^{\rm os}(m) \sim g_0^4 m^{-4\varepsilon}$. Furthermore, we have $\log Z_h^{\rm os}(0) = 0$. Thus, the limit $x \to 0$ is discontinuous. In QCD (Fig. 1b) we have $\log Z_Q^{\rm os}(0) \sim g_0^4 M^{-4\varepsilon}$ for dimensional reasons. For $m \ll M$ there are 3 regions (see [28, 29]):

- Hard (all momenta $\sim M$): a regular series in m^2, $\log Z_Q^{\rm os}(m)\big|_{\rm hard} = \log Z_Q^{\rm os}(0)\left[1 + \mathcal{O}(x^2)\right].$

- Soft-hard (momentum of one m-line is $\sim m$, all the remaining momenta are $\sim M$). If we take the term m from the numerator $k + m$ of the soft propagator, there is another factor m in the numerator of the hard mass-m propagator, and the soft-loop integral is $\sim m^{2-2\varepsilon}$; if we take k instead, we have to expand the hard subdiagram in k up to the linear term, and the soft loop is $\sim m^{4-2\varepsilon}$. We obtain $\log Z_Q^{\rm os}(m)\big|_{\rm soft-hard} \sim g_0^4 M^{-2\varepsilon} m^{-2\varepsilon} x^4.$

- Soft (all momenta $\sim m$): the leading term is the HQET one, the Taylor series is in x (not in x^2), $\log Z_Q^{\rm os}(m)\big|_{\rm soft} = \log Z_h^{\rm os}(m)\left[1 + \mathcal{O}(x)\right].$

As a result, $\log Z_Q^{\rm os}(m)\big|_{\rm hard} - \log Z_Q^{\rm os}(0)$ is smooth at $x \to 0$; $\log Z_Q^{\rm os}(m)\big|_{\rm soft-hard}$ is subleading and hence smooth; $\log Z_Q^{\rm os}(m)\big|_{\rm soft}$ has the same discontinuity as $\log Z_h^{\rm os}$; hence $\log z'$ (12) has a smooth limit 1 at $x \to 0$.
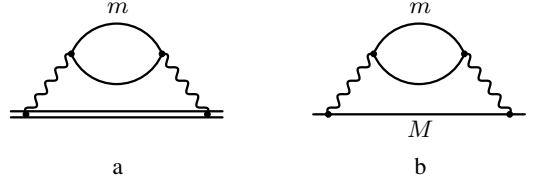

FIG. 1. Two-loop contributions to the on-shell wave-function renormalization constants: (a) in HQET; (b) in QCD.

The two-loop term in $Z_Q^{\rm os}(g_0^{(n_f)}, \xi_0^{(n_f)}, m_0^{(n_f)})$ has been calculated up to ε^0 in [7]; the result exact in ε has been obtained in [30]. The three-loop term has been calculated up to ε^0 in [31]. Some master integrals are only known as truncated series in x or as numerical interpolations, see [32] for detailed discussion of these master integrals. Exact results in x for the coefficient of $C_F T_F^2 n_m n_0 \alpha_s^3$ can be obtained using the formulas of Appendix B.

The HQET renormalization constant $Z_h^{\rm os}(g_0^{(n_l)}, \xi_0^{(n_l)}, m_0^{(n_l)})$ at two loops has been calculated in [22], and at three loops in [33] (one of the master integrals is discussed in [34]; note that there are some typos in formulas in the journal version of [33] fixed later in arXiv).

Altogether we are now in the position to obtain z' up to three loops. The expansion of z' in terms of $\alpha_s^{(n_f)}(M)$ and its decomposition into color factors is given by

$$
z' = 1 + C_F T_F \left(\frac{\alpha_s^{(n_f)}(M)}{\pi}\right)^2 \left(A_0 + A_1\varepsilon + \mathcal{O}(\varepsilon^2)\right)
$$

$$+ C_F T_F \left(\frac{\alpha_s^{(n_f)}(M)}{\pi} \right)^3 \left(C_F A_F + C_A A_A + T_F n_0 A_l + T_F n_m A_m + T_F n_h A_h + \mathcal{O}(\varepsilon) \right) + \mathcal{O}(\alpha_s^4), \tag{15}$$

where

$$A_0 = \frac{1}{4} \Bigg[(1-x)(2-x-x^2-6x^3)H_{1,0}(x) - (1+x)(2+x-x^2+6x^3)H_{-1,0}(x)$$
$$- \frac{3}{2}\pi^2 x + (4\log x + 7)x^2 - \frac{5}{2}\pi^2 x^3 + (6\log^2 x + \pi^2)x^4 \Bigg]. \tag{16}$$

The expansion of this function in x reads

$$A_0 = \frac{1}{4} \Bigg[-\frac{3}{2}\pi^2 x + 12x^2 - \frac{5}{2}\pi^2 x^3 + \left(6\log^2 x - 11\log x + \pi^2 + \frac{125}{12} \right) x^4 + \sum_{n=3}^{\infty} \left(2g(2n)\log x + \frac{d\,g(2n)}{d\,n} \right) x^{2n} \Bigg],$$
$$g(x) = \frac{2}{x} - \frac{3}{x-1} - \frac{5}{x-3} + \frac{6}{x-4}. \tag{17}$$

Note that the only terms with odd powers of x are x^1 and x^3. The expansion in x^{-1} is given by

$$A_0 = \frac{1}{4} \Bigg[-2\log^2 x^{-1} + \frac{19}{3}\log x^{-1} - \frac{\pi^2}{3} - \frac{229}{36} + \sum_{n=1}^{\infty} \left(2g(-2n)\log x^{-1} + \frac{d\,g(-2n)}{d\,n} \right) x^{-2n} \Bigg]. \tag{18}$$

For illustration we show in Fig. 2 $A_0(x)$ for $x \in [0,1]$. The $\mathcal{O}(\varepsilon)$ term at two loops reads

$$A_1 = \frac{1}{4} \Bigg[(1-x)(2-x-x^2-6x^3)\big(2H_{1,1,0}(x) - 4H_{1,-1,0}(x)\big)$$
$$+ (1+x)(2+x-x^2+6x^3)\big(2H_{-1,-1,0}(x) - 4H_{-1,1,0}(x) - \pi^2 H_{-1}(x)\big)$$
$$+ (1-x)(9 - 6x + 6x^2 - 17x^3)H_{1,0}(x) - (1+x)(9 + 6x + 6x^2 + 17x^3)H_{-1,0}(x)$$
$$+ 4x(3 + 5x^2)\big(H_{0,1,0}(x) + H_{0,-1,0}(x)\big) + 6\pi^2 \left(L + 2a_1 - \frac{5}{4} \right) x + \left(L + 2\pi^2 + \frac{53}{2} \right) x^2$$
$$+ 10\pi^2 \left(L + 2a_1 - \frac{23}{20} \right) x^3 - 12 \left(L^3 - \frac{17}{12}L^2 - \zeta_3 - \frac{17}{72}\pi^2 \right) x^4 \Bigg]$$
$$= \pi^2 \left(\frac{3}{2}L + 3a_1 - \frac{19}{8} \right) x + \frac{5}{2}x^2 + \pi^2 \left(\frac{5}{2}L + 5a_1 - \frac{8}{3} \right) x^3 - \left(3L^3 - \frac{17}{4}L^2 - \frac{3}{8}L - 3\zeta_3 + \frac{2}{3}\pi^2 + \frac{2827}{288} \right) x^4$$
$$- \frac{63}{80}\pi^2 x^5 - \frac{2}{15} \left(\frac{61}{5}L - 2\pi^2 - \frac{4243}{225} \right) x^6 - \frac{15}{112}\pi^2 x^7 - \frac{3}{56} \left(\frac{53}{35}L - \frac{3}{2}\pi^2 - \frac{5909}{1960} \right) x^8 + \mathcal{O}(x^9), \tag{19}$$

where $L = \log x$.

At three-loop order the $C_F T_F^2 n_m n_0 \alpha_s^3$ term is known exactly via harmonic polylogarithms of x:

$$A_l = \frac{1}{3} \Bigg[(1-x)(2-x-x^2-6x^3) \left(H_{1,-1,0}(x) + \frac{\pi^2}{12}H_1(x) \right) + (1+x)(2+x-x^2+6x^3) \left(H_{-1,1,0}(x) + \frac{5}{12}\pi^2 H_{-1}(x) \right)$$
$$- \frac{1}{6}(1-x)(19 - 11x + x^2 - 39x^3)H_{1,0}(x) + \frac{1}{6}(1+x)(19 + 11x + x^2 + 39x^3)H_{-1,0}(x)$$
$$- x(3 + 5x^2)\big(H_{0,1,0}(x) + H_{0,-1,0}(x)\big) - \pi^2 \left(\frac{3}{2}L + 3a_1 - \frac{5}{2} \right) x - \left(\frac{17}{2}L + 2\pi^2 + \frac{91}{4} \right) \frac{x^2}{3} - 5\pi^2 \left(\frac{L}{2} + a_1 - \frac{2}{3} \right) x^3$$
$$+ \left(2L^3 - \frac{13}{2}L^2 - \pi^2 L - 9\zeta_3 - \frac{13}{12}\pi^2 \right) x^4 \Bigg]$$
$$= -\pi^2 \left(\frac{L}{2} + a_1 - \frac{7}{6} \right) x - \frac{7}{3}x^2 - \frac{5}{3}\pi^2 \left(\frac{L}{2} + a_1 - \frac{7}{12} \right) x^3 + \left[\frac{2}{3}L^3 - \frac{13}{6}L^2 - \left(\frac{\pi^2}{3} - \frac{3}{4} \right) L - 3\zeta_3 + \frac{\pi^2}{4} + \frac{1175}{432} \right] x^4$$
$$+ \frac{21}{40}\pi^2 x^5 + \frac{4}{45} \left(\frac{13}{5}L - \frac{4}{3}\pi^2 - \frac{2414}{225} \right) x^6 + \frac{5}{56}\pi^2 x^7 + \left(\frac{4}{35}L - \frac{\pi^2}{4} - \frac{40489}{29400} \right) \frac{x^8}{7} + \mathcal{O}(x^9), \tag{20}$$

where after the second equality sign we show the expansion in x. In principle, it is straightforward to obtain exact results in x also the four-loop $C_F T_F^3 n_n n_0^2 \alpha_s^4$ term. However, we refrain from presenting such results because the remaining four-loop color structures are not known.

The remaining three-loop terms can be obtained in a series expansion in x with the help of the result from [31]. Including terms up to order x^8 gives

$$
\begin{aligned}
A_F =\ & \frac{\pi^2}{3}\left(8a_1 + \frac{13}{4}\pi - \frac{343}{24}\right)x - \left(L^2 - \frac{67}{6}L - \frac{17}{8}\pi^2 + \frac{229}{18}\right)x^2 + \frac{\pi^2}{3}\left(\frac{11}{3}L + \frac{44}{3}a_1 + \frac{35}{8}\pi - \frac{157}{8}\right)x^3 \\
& + \left[\frac{19}{6}L^3 - \frac{911}{120}L^2 - \left(3\pi^2 a_1 - \frac{3}{2}\zeta_3 - \frac{45}{16}\pi^2 - \frac{40567}{3600}\right)L + 20a_4 + \frac{5}{6}a_1^4 + \frac{2}{3}\pi^2 a_1^2 + \frac{11}{16}\pi^2 a_1 + \frac{387}{32}\zeta_3 \right. \\
& \left. - \frac{43}{144}\pi^4 - \frac{155}{64}\pi^2 - \frac{2534579}{216000}\right]x^4 + \frac{7}{5}\pi^2\left(\frac{3}{32}\pi + \frac{1}{5}\right)x^5 \\
& + \left[\frac{1579}{70}L^2 + \left(\frac{77}{16}\pi^2 - \frac{328067}{11025}\right)L - \frac{1}{16}\left(77\pi^2 a_1 - \frac{539}{2}\zeta_3 - \frac{83}{15}\pi^2 - \frac{126231437}{1157625}\right)\right]\frac{x^6}{9} - \frac{\pi^2}{28}\left(\frac{25}{16}\pi + \frac{1}{7}\right)x^7 \\
& + \left[\frac{2843}{105}L^2 + \left(\frac{21}{2}\pi^2 - \frac{718639}{33075}\right)\frac{L}{2} - \frac{1}{4}\left(21\pi^2 a_1 - \frac{147}{2}\zeta_3 - \frac{4379}{240}\pi^2 + \frac{1213332979}{83349000}\right)\right]\frac{x^8}{32} + \mathcal{O}(x^9)\,, \\[6pt]
A_A =\ & \frac{\pi^2}{8}\left(\frac{25}{2}L + \frac{313}{3}a_1 - \frac{13}{3}\pi - \frac{2473}{36}\right)x \\
& + \left[\frac{L^2}{2} + \left(\frac{3}{2}\zeta_3 - \frac{31}{90}\pi^4 + 7\pi^2 - \frac{7}{3}\right)L - 5\zeta_5 - \frac{7}{2}\pi^2\zeta_3 + \frac{79}{4}\zeta_3 - \frac{17}{180}\pi^4 + \frac{35}{18}\pi^2 + \frac{517}{9}\right]\frac{x^2}{4} \\
& + \frac{\pi^2}{24}\left(\frac{269}{6}L + \frac{1291}{3}a_1 - \frac{35}{2}\pi - \frac{865}{3}\right)x^3 - \left[\frac{83}{48}L^3 + \left(3\pi^2 - \frac{3977}{60}\right)\frac{L^2}{8} - \left(\frac{3}{2}\pi^2 a_1 - 3\zeta_3 - \frac{13}{24}\pi^2 - \frac{230293}{28800}\right)L \right. \\
& \left. + 10a_4 + \frac{5}{12}a_1^4 + \frac{\pi^2}{3}a_1^2 + \frac{11}{32}\pi^2 a_1 + \frac{111}{64}\zeta_3 - \frac{161}{1440}\pi^4 - \frac{631}{1152}\pi^2 - \frac{452033}{864000}\right]x^4 + \frac{\pi^2}{20}\left(\frac{79}{9}L - \frac{21}{16}\pi - \frac{2671}{432}\right)x^5 \\
& + \left[\frac{5}{3}L^3 + \frac{9911}{840}L^2 - \left(\pi^2 + \frac{8394157}{529200}\right)L + \frac{1}{12}\left(77\pi^2 a_1 - \frac{509}{2}\zeta_3 - \frac{3607}{60}\pi^2 + \frac{8471770063}{18522000}\right)\right]\frac{x^6}{24} \\
& + \frac{\pi^2}{28}\left(\frac{57}{25}L + \frac{25}{32}\pi - \frac{11549}{14000}\right)x^7 + \left[\frac{43}{27}L^3 + \frac{209}{20}L^2 + \left(125\pi^2 - \frac{12327647}{14700}\right)\frac{L}{216}\right. \\
& \left. + \frac{1}{8}\left(21\pi^2 a_1 - \frac{1435}{18}\zeta_3 - \frac{1213519}{45360}\pi^2 + \frac{103012097}{2058000}\right)\right]\frac{x^8}{32} + \mathcal{O}(x^9)\,, \\[6pt]
A_h =\ & -\left(2L + \frac{13}{5}\right)\frac{x^2}{5} + \frac{2}{15}\pi^2 x^3 + \left[\frac{3}{70}L^2 + \left(\pi^2 - \frac{35887}{4900}\right)\frac{L}{3} - \frac{1}{36}\left(13\pi^2 - \frac{59985349}{514500}\right)\right]x^4 \\
& - \left(244L^2 - \frac{92779}{315}L + \frac{353877541}{793800}\right)\frac{x^6}{945} - \left(47L^2 + \frac{925823}{13860}L - \frac{4543985839}{384199200}\right)\frac{x^8}{770} + \mathcal{O}(x^9)\,, \\[6pt]
A_m =\ & -\pi^2\left(\frac{L}{2} - \frac{2}{15}\right)x - \frac{7}{3}x^2 - \frac{5}{6}\pi^2 L x^3 + \left[\frac{2}{3}L^3 - \frac{13}{6}L^2 - \left(\frac{\pi^2}{3} - \frac{15}{4}\right)L + \frac{1}{4}\left(\pi^2 + \frac{203}{108}\right)\right]x^4 \\
& - \left(\frac{308}{5}L + \frac{16}{3}\pi^2 - \frac{13159}{225}\right)\frac{x^6}{45} + \left(3L^2 - \frac{751}{70}L + \frac{2095}{336}\right)\frac{x^8}{14} + \mathcal{O}(x^9)\,.
\end{aligned}
\tag{21}
$$

Starting from three loops the individual terms in Eq. (12) are gauge parameter dependent. However, ξ cancels in the three-loop expression for z'. It might be that z' is gauge invariant to all orders, but we have no proof of this conjecture.

IV. THE QED AND BLOCH–NORDSIECK HEAVY-LEPTON FIELDS

In QED the matching coefficient $z(\mu)$ is gauge invariant to all orders in α [5]. The proof given in this paper is literally valid only for $n_f = 1$ lepton flavor, but can be easily generalized for any n_f, as we demonstrate in the following.

The QED on-shell renormalization constant Z_ψ^{os} is gauge invariant to all orders [10, 35, 36]. Gauge dependence of the $\overline{\mathrm{MS}}$ Z_ψ can be found using the so-called LKF transformation [37, 38] for arbitrary n_f. In the gauge where the free photon propagator is

$$
D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \Delta(k)k_\mu k_\nu\,,
$$

the full bare lepton propagator reads

$$
\begin{aligned}
S(x) &= S_L(x)e^{-ie_0^2(\tilde{\Delta}(x) - \tilde{\Delta}(0))}\,, \\
\tilde{\Delta}(x) &= \int \frac{d^d k}{(2\pi)^d}\Delta(k)e^{-ikx}\,,
\end{aligned}
\tag{22}
$$

where $S_L(x)$ is the Landau-gauge propagator. In the covariant
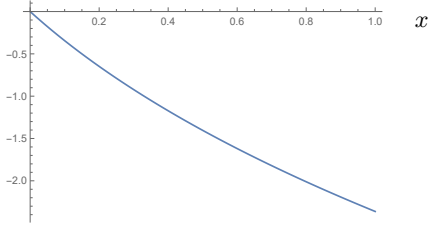
FIG. 2. The function $A_0(x)$.

gauge $\Delta(k) = (1 - \xi_0)/(k^2)^2$, and $\tilde\Delta(0) = 0$ in dimensional regularization. The lepton fields renormalization does not depend on their masses, so, let us assume that all n_f flavors are massless. The propagator has a single Dirac structure

$$S(x) = S_0(x)e^{\sigma(x)},$$

where $S_0(x)$ is the d-dimensional free propagator. Then

$$\sigma(x) = \sigma_L(x) + (1 - \xi_0)\frac{e_0^2}{(4\pi)^{d/2}}\left(-\frac{x^2}{4}\right)^\varepsilon \Gamma(-\varepsilon);$$

re-expressing this result via the renormalized quantities, we obtain

$$\log Z_\psi(\alpha, \xi) = \log Z_L(\alpha) - (1 - \xi)\frac{\alpha}{4\pi\varepsilon}. \qquad (23)$$

In QED $Z_A Z_\alpha = 1$ due to Ward identities, hence

$$\frac{d\log((1 - \xi(\mu))\alpha(\mu))}{d\log\mu} = -2\varepsilon$$

exactly, and the anomalous dimension

$$\gamma_\psi(\alpha, \xi) = \gamma_L(\alpha) + 2(1 - \xi)\frac{\alpha}{4\pi} \qquad (24)$$

contains ξ only in the one-loop term.

In the Bloch-Nordsieck EFT with n_l light lepton flavors Z_h^{os} is gauge-invariant (even if some of these flavors have non-zero masses). Gauge dependence of the $\overline{\mathrm{MS}}$ Z_h can be found using exponentiation. The full bare propagator is

$$S_h(t) = S_{h0}(t)\exp\left(\sum_i w_i\right),$$

where w_i are webs [39, 40]. In QED all webs have even numbers of photon legs; all webs with > 2 legs are gauge invariant; all 2-leg webs except the trivial one (the free photon propagator) are gauge invariant, too. Therefore,

$$\log\frac{S_h(t)}{S_{hL}(t)} = (1 - \xi_0)\frac{e_0^2}{(4\pi)^{d/2}}\left(\frac{it}{2}\right)^{2\varepsilon}\Gamma(-\varepsilon);$$

re-expressing this result via the renormalized quantities, we obtain

$$\log Z_h(\alpha, \xi) = \log Z_{hL}(\alpha) - (1 - \xi)\frac{\alpha}{4\pi\varepsilon}, \qquad (25)$$

$$\gamma_h(\alpha, \xi) = \gamma_{hL}(\alpha) + 2(1 - \xi)\frac{\alpha}{4\pi}. \qquad (26)$$

Finally, in the abelian case $\zeta_\alpha(\mu) = \zeta_A(\mu)^{-1}$ due to Ward identities, hence $(1 - \xi^{(n_f)}(\mu))\alpha^{(n_f)}(\mu)) = (1 - \xi^{(n_l)}(\mu))\alpha^{(n_l)}(\mu))$, and we arrive at the conclusion that $z(\mu)$ is gauge invariant (some light flavors may be massive, this does not matter).

Let us in the following specify $z(M)$ from Eq (7) to QED. Setting $C_F = T_F = d_{FF} = 1$ and $C_A = d_{FA} = 0$ we see that our four-loop result is indeed gauge invariant and is given by

$$
\begin{aligned}
z(M) =\ & 1 - \frac{\alpha}{\pi}\left[1 + \varepsilon\left(\frac{\pi^2}{16} + 2\right) - \varepsilon^2\left(\frac{\zeta_3}{4} - \frac{\pi^2}{12} - 4\right) - \varepsilon^3\left(\frac{\zeta_3}{3} - \frac{3}{640}\pi^4 - \frac{\pi^2}{6} - 8\right) + \mathcal{O}(\varepsilon^4)\right] \\
& + \left(\frac{\alpha}{\pi}\right)^2\left\{\pi^2 a_1 - \frac{3}{2}\zeta_3 - \frac{55}{48}\pi^2 + \frac{5957}{1152} + \frac{n_l}{12}\left(\pi^2 + \frac{113}{8}\right)\right. \\
& + \varepsilon\left[-24a_4 - a_1^4 - 2\pi^2 a_1^2 + \frac{31}{4}\pi^2 a_1 - \frac{203}{8}\zeta_3 + \frac{7}{20}\pi^4 - \frac{4903}{1152}\pi^2 + \frac{56845}{6912} + n_l\left(\zeta_3 + \frac{127}{288}\pi^2 + \frac{851}{192}\right)\right] \\
& + \varepsilon^2\left[-144a_5 - 186a_4 + \frac{6}{5}a_1^5 - \frac{31}{4}a_1^4 + 4\pi^2 a_1^3 - \frac{31}{2}\pi^2 a_1^2 - \frac{13}{15}\pi^4 a_1 + 30\pi^2 a_1 + \frac{609}{4}\zeta_5 + \frac{11}{4}\pi^2\zeta_3 - \frac{28169}{288}\zeta_3\right. \\
& \left.\left. + \frac{10007}{7680}\pi^4 - \frac{114943}{6912}\pi^2 + \frac{1838165}{41472} + \frac{n_l}{24}\left(\frac{305}{3}\zeta_3 + \frac{199}{80}\pi^4 + \frac{853}{24}\pi^2 + \frac{5753}{16}\right)\right] + \mathcal{O}(\varepsilon^3)\right\} \\
& + \left(\frac{\alpha}{\pi}\right)^3\left\{-16a_4 - \frac{2}{3}a_1^4 + \pi^2 a_1^2 + \frac{737}{36}\pi^2 a_1 - \frac{5}{16}\zeta_5 + \frac{\pi^2}{8}\zeta_3 - \frac{4747}{288}\zeta_3 - \frac{13}{360}\pi^4 - \frac{259133}{25920}\pi^2 - \frac{230447}{20736}\right. \\
& + \frac{n_l}{3}\left(16a_4 + \frac{2}{3}a_1^4 + \frac{4}{3}\pi^2 a_1^2 - \frac{47}{6}\pi^2 a_1 + \frac{137}{8}\zeta_3 - \frac{229}{720}\pi^4 + \frac{139}{24}\pi^2 - \frac{2201}{432}\right) - \frac{n_l^2}{18}\left(7\zeta_3 + \frac{19}{6}\pi^2 + \frac{5767}{432}\right) \\
& + \varepsilon\left[-\frac{224}{3}a_5 + 16\pi^2 a_4 - \frac{5005}{6}a_4 + \frac{28}{45}a_1^5 + \frac{2}{3}\pi^2 a_1^4 - \frac{5005}{144}a_1^4 - \frac{88}{27}\pi^2 a_1^3 - \frac{2}{3}\pi^4 a_1^2 - \frac{11567}{144}\pi^2 a_1^2 + 14\pi^2\zeta_3 a_1\right.
\end{aligned}
$$

$$-\frac{2039}{2160}\pi^4 a_1 + \frac{3481}{15}\pi^2 a_1 + \frac{125}{8}\zeta_5 + \frac{29}{32}\zeta_3^2 + \frac{2945}{288}\pi^2\zeta_3 - \frac{348821}{960}\zeta_3 - \frac{899}{5670}\pi^6 + \frac{64103}{34560}\pi^4 - \frac{224592113}{4147200}\pi^2$$

$$-\frac{2783713}{207360} + \frac{n_l}{3}\left(224 a_5 + \frac{1124}{3}a_4 - \frac{28}{15}a_1^5 + \frac{281}{18}a_1^4 - \frac{56}{9}\pi^2 a_1^3 + \frac{281}{9}\pi^2 a_1^2 - \frac{17}{90}\pi^4 a_1 - \frac{644}{9}\pi^2 a_1 - \frac{1027}{4}\zeta_5\right.$$

$$\left.-\frac{119}{16}\pi^2\zeta_3 + \frac{662}{3}\zeta_3 - \frac{14303}{8640}\pi^4 + \frac{552083}{13824}\pi^2 - \frac{153109}{2592}\right)$$

$$-\frac{n_l^2}{54}\left(275\zeta_3 + \frac{23}{5}\pi^4 + \frac{1081}{16}\pi^2 + \frac{253783}{864}\right)\right] + \mathcal{O}(\varepsilon^2)\bigg\}$$

$$+\left(\frac{\alpha}{\pi}\right)^4\left[L_{\text{QED}} + \frac{395}{6}a_5 + 28\pi^2 a_4 - \frac{58187}{48}a_4 - \frac{79}{144}a_1^5 + \frac{7}{6}\pi^2 a_1^4 - \frac{58187}{1152}a_1^4 - \frac{2411}{216}\pi^2 a_1^3 - \frac{7}{6}\pi^4 a_1^2\right.$$

$$-\frac{69311}{576}\pi^2 a_1^2 + \frac{49}{2}\pi^2\zeta_3 a_1 - \frac{61}{1728}\pi^4 a_1 + \frac{1414153}{3840}\pi^2 a_1 - \frac{23093}{128}\zeta_5 + \frac{203}{128}\zeta_3^2 + \frac{7771}{576}\pi^2\zeta_3 - \frac{327897}{640}\zeta_3 - \frac{899}{3240}\pi^6$$

$$+\frac{74911}{69120}\pi^4 - \frac{148407527}{2073600}\pi^2 - \frac{778181617}{9953280} - n_l(12.18 \pm 0.8)$$

$$-n_l^2\left(\frac{32}{3}a_5 + \frac{188}{9}a_4 - \frac{4}{45}a_1^5 + \frac{47}{54}a_1^4 - \frac{8}{27}\pi^2 a_1^3 + \frac{47}{27}\pi^2 a_1^2 - \frac{31}{270}\pi^4 a_1 - \frac{239}{54}\pi^2 a_1 - \frac{601}{48}\zeta_5 - \frac{\pi^2}{2}\zeta_3 + \frac{6913}{576}\zeta_3\right.$$

$$\left.-\frac{1297}{51840}\pi^4 + \frac{25729}{10368}\pi^2 - \frac{15877}{165888}\right) + \frac{n_l^3}{216}\left(\frac{467}{2}\zeta_3 + \frac{71}{20}\pi^4 + \frac{167}{3}\pi^2 + \frac{103933}{864}\right) + \mathcal{O}(\varepsilon)\right] + \mathcal{O}(\alpha^5)\,, \tag{27}$$

where $\alpha = \alpha^{(n_f)}(M)$; $L_{\text{QED}} = \sum_{i=0,1,2,3,l} L_i$ is the ε^0 term in $Z_2^{(4)}$ of Eq. (26) in [14]. Its numerical value is given in Eq. (15) in this paper.

Numerically, in pure QED ($n_l = 0$) at $\varepsilon = 0$ we have

$$z(M) = 1 - \frac{\alpha}{\pi} - 1.09991\left(\frac{\alpha}{\pi}\right)^2 + 4.40502\left(\frac{\alpha}{\pi}\right)^3$$
$$- 2.16215\left(\frac{\alpha}{\pi}\right)^4 + \mathcal{O}(\alpha^5)\,, \tag{28}$$

where $\alpha = \alpha^{(1)}(M)$, the $\overline{\text{MS}}$ QED coupling with one active flavor at $\mu = M$, the on-shell electron mass. In contrast to the QCD case (7) the coefficients are numerically smaller and have different signs.

V. CONCLUSION

We have calculated the (finite) matching coefficient between the QCD heavy-quark field Q and the corresponding HQET field h_v up to four loops. Explicit results are presented for $\mu = M$; results for different values of μ can be obtained with the help of (known) renormalization group equations. The effect of a non-zero light-flavor mass (e. g., c in b-quark HQET) is calculated up to three loops. We also present results for the matching constant in QED.

As a possible application of our results we want to mention the possibility to obtain the QCD heavy-quark propagator (say, in Landau gauge) from lattice QCD results for the HQET propagator. A heavy-quark field can be put onto the lattice only if $Ma \ll 1$, where a is the lattice spacing. On the other hand, in HQET simulations there is no lattice h_v field at all. The HQET propagator is just a straight Wilson line, i. e. a product of lattice gauge links. It is therefore much easier to obtain the HQET propagator from lattice simulations.

After taking the continuum limit, one can get the continuum coordinate-space HQET propagator. Then the QCD heavy-quark propagator can be obtained with the help of the matching coefficient $z(\mu)$, provided that $1/M^n$ corrections can be neglected. Note that this can be done for arbitrarily heavy QCD quark, including the case when the use of the dynamic heavy-quark field on the lattice is impossible.

Appendix A: The coupling and gluon-field decoupling coefficients

The n_l-flavor QCD strong coupling constant and gauge parameter are related to the corresponding quantities in the n_f-flavor theory by the decoupling relations

$$\alpha_s^{(n_l)}(\mu) = \zeta_\alpha(\mu)\alpha_s^{(n_f)}(\mu)\,, \tag{A1}$$
$$1 - \xi^{(n_l)}(\mu) = \zeta_A(\mu)\left[1 - \xi^{(n_f)}(\mu)\right]\,.$$

The decoupling coefficients satisfy the renormalization group equations

$$\frac{d\log\zeta_\alpha(\mu)}{d\log\mu} = 2\left[\beta^{(n_f)}(\alpha_s^{(n_f)}(\mu)) - \beta^{(n_l)}(\alpha_s^{(n_l)}(\mu))\right],$$

$$\frac{d\log\zeta_A(\mu)}{d\log\mu} = \gamma_A^{(n_f)}(\alpha_s^{(n_f)}(\mu), \xi^{(n_f)}(\mu))$$

$$- \gamma_A^{(n_l)}(\alpha_s^{(n_l)}(\mu), \xi^{(n_l)}(\mu)). \tag{A2}$$

It is sufficient to have initial conditions, say, at $\mu = M$ for solving these equations. For the computation of $z(M)$ we need the decoupling coefficients up to $\alpha_s^3\varepsilon$. Up to the order α_s^2 expression exact in ε can be found in [41]. The finite three-loop results have been obtained in [42] in term of N_c and in [43] for an arbitrary color group. The $\alpha_s^3\varepsilon$ terms were derived in the course of four-loop calculations [43–45]. However, results for an arbitrary color group, including positive powers of ε, are not explicitly presented in these publications. Therefore, we present them here:

$$\zeta_\alpha(M) = 1 - \frac{\alpha_s}{\pi}T_F n_h \frac{\varepsilon}{9}\left(\frac{\pi^2}{4} - \zeta_3\varepsilon + \frac{3}{160}\pi^4\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right)$$

$$- \left(\frac{\alpha_s}{\pi}\right)^2 T_F n_h \left\{\frac{15}{16}C_F - \frac{2}{9}C_A + \frac{\varepsilon}{4}\left[C_F\left(\frac{\pi^2}{4} + \frac{31}{2}\right) + \frac{C_A}{9}\left(\frac{5}{4}\pi^2 + \frac{43}{3}\right)\right]\right.$$

$$\left. - \varepsilon^2\left[\frac{C_F}{4}\left(\frac{\zeta_3}{3} - \frac{5}{8}\pi^2 - \frac{223}{16}\right) + \frac{C_A}{9}\left(\frac{5}{4}\zeta_3 + \frac{\pi^2}{3} + \frac{523}{72}\right) + \frac{\pi^4}{1296}T_F n_h\right] + \mathcal{O}(\varepsilon^3)\right\}$$

$$+ \left(\frac{\alpha_s}{\pi}\right)^3 T_F n_h \left\{\frac{C_F^2}{3}\left(\pi^2 a_1 - \frac{\zeta_3}{64} - \frac{5}{8}\pi^2 - \frac{77}{192}\right) - \frac{C_F C_A}{6}\left(\pi^2 a_1 + \frac{1081}{128}\zeta_3 - \frac{\pi^2}{3} + \frac{8321}{864}\right)\right.$$

$$- \frac{C_A^2}{768}\left(\frac{5}{2}\zeta_3 - \frac{11347}{27}\right)$$

$$- C_F T_F n_h\left(\frac{7}{64}\zeta_3 + \frac{\pi^2}{9} - \frac{695}{648}\right) - \frac{7}{64}C_A T_F n_h\left(\frac{\zeta_3}{2} - \frac{35}{81}\right) + \frac{C_F T_F n_l}{18}\left(\pi^2 + \frac{311}{72}\right) - \frac{C_A T_F n_l}{2592}$$

$$- \varepsilon\left[C_F^2\left(\frac{37}{12}a_4 + \frac{37}{288}a_1^4 + \frac{251}{288}\pi^2 a_1^2 - 2\pi^2 a_1 + \frac{2759}{576}\zeta_3 - \frac{241}{3456}\pi^4 + \frac{439}{384}\pi^2 + \frac{3329}{3456}\right)\right.$$

$$+ C_F C_A\left(\frac{63}{16}a_4 + \frac{21}{128}a_1^4 - \frac{85}{128}\pi^2 a_1^2 + \pi^2 a_1 + \frac{2413}{512}\zeta_3 - \frac{1391}{23040}\pi^4 - \frac{281}{1728}\pi^2 + \frac{451831}{62208}\right)$$

$$- \frac{C_A^2}{96}\left(263 a_4 + \frac{263}{24}a_1^4 - \frac{263}{24}\pi^2 a_1^2 + \frac{27347}{288}\zeta_3 - \frac{1687}{1440}\pi^4 - \frac{1063}{216}\pi^2 - \frac{345115}{1944}\right)$$

$$+ C_F T_F n_h\left(\frac{3}{4}a_4 + \frac{a_1^4}{32} - \frac{\pi^2}{32}a_1^2 - \frac{2}{3}\pi^2 a_1 + \frac{3353}{1152}\zeta_3 - \frac{17}{1920}\pi^4 + \frac{407}{864}\pi^2 - \frac{67037}{15552}\right)$$

$$+ \frac{C_A T_F n_h}{8}\left(3 a_4 + \frac{a_1^4}{8} - \frac{\pi^2}{8}a_1^2 + \frac{1799}{864}\zeta_3 - \frac{17}{480}\pi^4 + \frac{113}{1296}\pi^2 + \frac{1165}{11664}\right)$$

$$\left.\left. - \frac{C_F T_F n_l}{9}\left(\zeta_3 + \frac{403}{192}\pi^2 + \frac{24911}{864}\right) - \frac{C_A T_F n_l}{27}\left(5\zeta_3 + \frac{47}{192}\pi^2 - \frac{6553}{1728}\right)\right] + \mathcal{O}(\varepsilon^2)\right\} + \mathcal{O}(\alpha_s^4), \tag{A3}$$

$$\zeta_A(M) = 1 + \frac{\alpha_s}{\pi}T_F n_h \frac{\varepsilon}{9}\left(\frac{\pi^2}{4} - \zeta_3\varepsilon + \frac{3}{160}\pi^4\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right)$$

$$+ \left(\frac{\alpha_s}{\pi}\right)^2 T_F n_h\left\{\frac{1}{16}\left(15 C_F - \frac{13}{12}C_A\right) + \frac{\varepsilon}{16}\left[C_F\left(\frac{\pi^2}{3} + \frac{31}{2}\right) + \frac{C_A}{12}\left(5\pi^2 + \frac{169}{6}\right)\right]\right.$$

$$\left. - \frac{\varepsilon^2}{4}\left[C_F\left(\frac{\zeta_3}{3} - \frac{5}{8}\pi^2 - \frac{223}{16}\right) + \frac{C_A}{12}\left(5\zeta_3 - \frac{\pi^4}{48} + \frac{13}{24}\pi^2 + \frac{1765}{144}\right)\right] + \mathcal{O}(\varepsilon^3)\right\}$$

$$- \left(\frac{\alpha_s}{\pi}\right)^3 T_F n_h\left\{\frac{C_F^2}{3}\left(\pi^2 a_1 - \frac{\zeta_3}{64} - \frac{5}{8}\pi^2 - \frac{77}{192}\right)\right.$$

$$- C_F C_A\left(2 a_4 + \frac{a_1^4}{12} - \frac{\pi^2}{12}a_1^2 + \frac{\pi^2}{6}a_1 + \frac{1765}{768}\zeta_3 - \frac{11}{720}\pi^4 - \frac{\pi^2}{18} + \frac{15977}{20736}\right)$$

$$+ C_A^2\left(a_4 + \frac{a_1^4}{24} - \frac{\pi^2}{24}a_1^2 + \frac{1805}{4608}\zeta_3 - \frac{53}{5760}\pi^4 + \frac{7985}{31104} - \frac{\xi^{(n_f)}(M)}{48}\left(\zeta_3 - \frac{677}{144}\right)\right)$$

$$
- C_F T_F n_h \left(\frac{7}{64}\zeta_3 + \frac{\pi^2}{9} - \frac{695}{648} \right) - \frac{C_A T_F n_h}{144} \left(\frac{287}{8}\zeta_3 - \frac{605}{27} \right) + \frac{C_F T_F n_l}{18} \left(\pi^2 + \frac{311}{72} \right) + \frac{C_A T_F n_l}{9} \left(\zeta_3 - \frac{665}{432} \right)
$$

$$
- \varepsilon \Bigg[C_F^2 \left(\frac{37}{12}a_4 + \frac{37}{288}a_1^4 + \frac{251}{288}\pi^2 a_1^2 - 2\pi^2 a_1 + \frac{2759}{576}\zeta_3 - \frac{241}{3456}\pi^4 + \frac{439}{384}\pi^2 + \frac{3329}{3456} \right)
$$

$$
+ C_F C_A \Bigg(12 a_5 + \frac{179}{16}a_4 - \frac{a_1^5}{10} + \frac{179}{384}a_1^4 + \frac{\pi^2}{6}a_1^3 - \frac{371}{384}\pi^2 a_1^2 + \frac{17}{120}\pi^4 a_1 + \pi^2 a_1 - \frac{203}{16}\zeta_5 + \frac{\pi^2}{32}\zeta_3 + \frac{3141}{512}\zeta_3
$$

$$
- \frac{1057}{7680}\pi^4 - \frac{281}{1728}\pi^2 + \frac{1199393}{124416} \Bigg)
$$

$$
- C_A^2 \Bigg(6 a_5 + \frac{611}{96}a_4 - \frac{a_1^5}{20} + \frac{611}{2304}a_1^4 + \frac{\pi^2}{12}a_1^3 - \frac{611}{2304}\pi^2 a_1^2 + \frac{17}{240}\pi^4 a_1 - \frac{185}{32}\zeta_5 + \frac{3}{128}\pi^2 \zeta_3 + \frac{59395}{27648}\zeta_3
$$

$$
- \frac{6679}{138240}\pi^4 - \frac{10181}{165888}\pi^2 - \frac{886909}{373248} + \frac{\xi^{(n_f)}(M)}{96} \left(7\zeta_3 + \frac{\pi^4}{10} - \frac{233}{576}\pi^2 - \frac{5737}{144} \right) \Bigg)
$$

$$
+ C_F T_F n_h \left(\frac{3}{4}a_4 + \frac{a_1^4}{32} - \frac{\pi^2}{32}a_1^2 - \frac{2}{3}\pi^2 a_1 + \frac{3353}{1152}\zeta_3 - \frac{17}{1920}\pi^4 + \frac{113}{216}\pi^2 - \frac{67037}{15552} \right)
$$

$$
+ \frac{C_A T_F n_h}{24} \left(41 a_4 + \frac{41}{24}a_1^4 - \frac{41}{24}\pi^2 a_1^2 + \frac{5551}{288}\zeta_3 - \frac{697}{1440}\pi^4 - \frac{7}{32}\pi^2 - \frac{4415}{1944} \right)
$$

$$
- \frac{C_F T_F n_l}{9} \left(\zeta_3 + \frac{403}{192}\pi^2 + \frac{24911}{864} \right) - \frac{C_A T_F n_l}{18} \left(\frac{5}{3}\zeta_3 - \frac{\pi^4}{10} + \frac{253}{576}\pi^2 + \frac{27845}{1296} \right) \Bigg] + \mathcal{O}(\varepsilon^2) \Bigg\} + \mathcal{O}(\alpha_s^4) , \tag{A4}
$$

where $\alpha_s = \alpha_s^{(n_f)}(M)$.

Appendix B: On-shell diagrams with two masses

Light-quark mass effects in the heavy-quark on-shell propagator diagrams arise for the first time at two loops, see Fig. 1b. The corresponding integral family can be defined as

$$
= \frac{C}{(i\pi^{d/2})^2} \int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4}} , \quad C = \frac{1}{\Gamma^2(1+\varepsilon)} ,
$$

$$
D_1 = M^2 - (p+k_1)^2 , \quad D_2 = -k_1^2 ,
$$

$$
D_3 = m^2 - k_2^2 , \quad D_4 = m^2 - (k_1-k_2)^2 , \tag{B1}
$$

with $p^2 = M^2$. If there are insertions to gluon lines in Fig. 1b containing only massless lines, such diagrams are expressed via the integrals (B1) with $n_2 = n + l\varepsilon$, where l is the total number of loops in these insertions and n is integer ($n_{1,3,4}$ are always integer). These integrals have been studied in [30]. The IBP algorithm obtained there reduces them to four master integrals

We set $M = 1$ and $m = x$.

It is more convenient to use the column vector

$$
j = \left(I_{0,l\varepsilon,2,2}, I_{2,l\varepsilon,2,0}, I_{2,l\varepsilon,2,1}, I_{1,l\varepsilon,2,2} \right)^T \tag{B3}
$$

as master integrals instead of (B2). Differentiating them in m and reducing the results back to j [46], we obtain the differential equations

$$
\frac{dj}{dx} = M(\varepsilon, x) j . \tag{B4}
$$

In many cases such equations can be reduced to an ε-form [47]

$$
j = T(\varepsilon, x) J , \quad \frac{dJ}{dx} = \varepsilon M(x) J . \tag{B5}
$$

This makes their iterative solution to any order in ε almost trivial.

Several terms of small-x and large-x expansions of these integrals (with $l = 0$) were obtained in [48] using the method of regions (though expressed in a somewhat different language). Differential equations for on-shell sunsets $I_{n_1,0,n_3,n_4}$ were considered in [33, 49], but they were not in ε-form. Several terms of small-x expansions were obtained from differential equations in [50]. However, the easiest way to obtain any finite number of terms in the small-x and large-x expansions is neither the method of regions nor differential equations, but calculating the corresponding residues in the Mellin–Barnes representation [30].

We use the Mathematica package Libra [51] which implements the algorithm of [52] to reduce the master integrals j in

Eq. (B3) to a canonical basis J:

$$j_1 = I_{0,l\varepsilon,2,2} = CV_{2,2,l\varepsilon}x^{-2(l+2)\varepsilon} = \frac{2(1-(l+1)\varepsilon)}{(l+2)(1-\varepsilon)}J_1,$$

$$j_2 = I_{0,l\varepsilon,2,0} = CV_2 M_{2,l\varepsilon}x^{-2\varepsilon} = \frac{1-2(l+1)\varepsilon}{1-(l+2)\varepsilon}J_2,$$

$$j_3 = -\frac{1}{2}(J_3+J_4),$$

$$j_4 = I_{1,l\varepsilon,2,2} = \frac{1}{2x}\left\{-\left[1-2x-\frac{2l\varepsilon(1-x)}{1-2\varepsilon}\right]J_3\right.$$
$$\left.+\left[1+2x-\frac{2l\varepsilon(1+x)}{1-2\varepsilon}\right]J_4\right\}, \tag{B6}$$

where

$$V_{n_1} = \bigcirc = \frac{\Gamma\left(\frac{d}{2}-n_1\right)}{\Gamma(n_1)}, \tag{B7}$$

$$V_{n_1 n_2 n_3} = \underset{1}{\overset{2}{\langle 3 \rangle}} = H_{0,n_3,n_1,n_2}, \tag{B8}$$

$$M_{n_1 n_2} = \underset{1}{\overset{2}{\frown}}$$
$$= \frac{\Gamma\left(n_1+n_2-\frac{d}{2}\right)\Gamma(d-n_1-2n_2)}{\Gamma(n_1)\Gamma(d-n_1-n_2)}, \tag{B9}$$

and [33]

$$H_{n_1 n_2 n_3 n_4} = \underset{1}{\overset{2}{\underset{4}{\langle 3 \rangle}}} =$$
$$\frac{\Gamma(n_1/2)\Gamma((n_1-d)/2+n_2+n_3)\Gamma((n_1-d)/2+n_2+n_4)}{2\Gamma(n_1)\Gamma(n_3)\Gamma(n_4)}$$
$$\times \frac{\Gamma(n_1/2+n_2+n_3+n_4-d)\Gamma((d-n_1)/2-n_2)}{\Gamma(n_1+2n_2+n_3+n_4-d)\Gamma((d-n_1)/2)}. \tag{B10}$$

The integrals J satisfy the ε-form differential equations

$$\frac{dJ}{dx} = \varepsilon\left(\frac{M_0}{x}+\frac{M_{+1}}{1-x}+\frac{M_{-1}}{1+x}\right)J, \tag{B11}$$

where

$$M_0 = \begin{pmatrix} -2(l+2) & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & -1 & -(l+2) & l+2 \\ 1 & -1 & l+2 & -(l+2) \end{pmatrix},$$

$$M_{+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 2(l+2) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -2(l+2) & -2 \end{pmatrix}. \tag{B12}$$

The first two are, of course, known exactly:

$$J_1 = \frac{x^{-2(l+2)\varepsilon}}{(l+1)\varepsilon^2}\times$$
$$\frac{\Gamma(1-(l+1)\varepsilon)\Gamma^2(1+(l+1)\varepsilon)\Gamma(1+(l+2)\varepsilon)}{\Gamma(1-\varepsilon)\Gamma^2(1+\varepsilon)\Gamma(1+2(l+1)\varepsilon)},$$

$$J_2 = \frac{x^{-2\varepsilon}}{(l+1)\varepsilon^2}\frac{\Gamma(1-2(l+1)\varepsilon)\Gamma(1+(l+1)\varepsilon)}{\Gamma(1-(l+2)\varepsilon)\Gamma(1+\varepsilon)}. \tag{B13}$$

The equations for $J_{3,4}$ can be solved iteratively in terms of harmonic polylogarithms [53] of x. However, we need initial conditions. They can be fixed using the asymptotics of $I_{n_1 n_2 n_3 n_4}$ at $x\to 0$. It is given by contributions of three regions (Sect. III) corresponding to residues of the Mellin–Barnes representation [30] at three series of poles:

- Hard: the poles $s = -n-n_3-n_4+d/2$ ($n\geq 0$ is integer), the result is a regular series in x^2. The leading term is $CG_{n_3 n_4}M_{n_1,n_2+n_3+n_4-d/2}$, where

$$G_{n_1 n_2} = \underset{1}{\overset{2}{\frown}}$$
$$= \frac{\Gamma\left(n_1+n_2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-n_1\right)\Gamma\left(\frac{d}{2}-n_2\right)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2)}. \tag{B14}$$

- Sort-hard: $s = -n-n_{3,4}$. All these poles are double except the first $|n_3-n_4|$ ones (and hence, the representation of $I_{n_1 n_2 n_3 n_4}$ via hypergeometric functions of x is awkward). We assume $n_3\geq n_4$, then the result is x^{d-2n_3} times a regular series in x^2. If $n_3 > n_4$ then the leading term is $CV_{n_3}M_{n_1,n_2+n_4}x^{d-2n_3}$; if $n_3 = n_4$ there is an extra factor 2 because each of the lines 3, 4 can be soft.

- Soft: $s = (n_1-d-n)/2+n_2$, the result is $x^{2(d-n_2-n_3-n_4)-n_1}$ times a regular series in x. The leading term is $H_{n_1 n_2 n_3 n_4}x^{2(d-n_2-n_3-n_4)-n_1}$ (B10).

For example,

$$I_{2,l\varepsilon,2,1}\to\frac{1}{2\varepsilon^2}\times$$
$$\left[\frac{1}{l+2}\frac{\Gamma^2(1-\varepsilon)\Gamma(1-2(l+2)\varepsilon)\Gamma(1+(l+2)\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(1-(l+3)\varepsilon)\Gamma(1+\varepsilon)}\right.$$
$$-\frac{x^{-2\varepsilon}}{l+1}\frac{\Gamma(1-2(l+1)\varepsilon)\Gamma(1+(l+1)\varepsilon)}{\Gamma(1-(l+2)\varepsilon)\Gamma(1+\varepsilon)}$$
$$+\frac{x^{-2(l+2)\varepsilon}}{(l+1)(l+2)}\times$$
$$\left.\frac{\Gamma(1-(l+1)\varepsilon)\Gamma^2(1+(l+1)\varepsilon)\Gamma(1+(l+2)\varepsilon)}{\Gamma(1-\varepsilon)\Gamma^2(1+\varepsilon)\Gamma(1+2(l+1)\varepsilon)}\right], \tag{B15}$$

where the 3 contributions are the hard one $CG_{21}M_{2,1+(l+1)\varepsilon}$ (the pole $s = -1-\varepsilon$), the soft-3 one $CV_2 M_{2,1+l\varepsilon}$ (the pole $s = -1$), and the soft one $CH_{2,l\varepsilon,2,1}x^{-2(l+2)\varepsilon}$ (the pole $s =$

$-1 - (l+1)\varepsilon)$. The leading asymptotics of $I_{1,l\varepsilon,2,2}$ is given by the soft contribution $CH_{1,l\varepsilon,2,2}x^{-1-2(l+2)\varepsilon}$ (the pole $s = -3/2 + (l+1)\varepsilon$):

$$I_{1,l\varepsilon,2,2} \to 2^{-1-4(l+2)\varepsilon}\pi^2 x^{-1-2(l+2)\varepsilon}\frac{1-2(l+1)\varepsilon}{1-2\varepsilon}$$

$$\times \frac{\Gamma(1-\varepsilon)\Gamma(1-2(l+1)\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(1-(l+1)\varepsilon)}$$
$$\times \frac{\Gamma(1+2(l+1)\varepsilon)\Gamma(1+2(l+2)\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma^2(1+(l+1)\varepsilon)\Gamma(1+(l+2)\varepsilon)}. \quad \text{(B16)}$$

Now we can easily obtain any number of expansion terms of $J_{3,4}$ in ε for $x < 1$ using Libra [51]:

$$J_3 = -2\Bigg\{H_{1,0}(x) + H_{0,0}(x) + \frac{\pi^2}{3} + \Bigg[-(l+2)\big(2H_{1,-1,0}(x) + H_{0,1,0}(x) + H_{0,-1,0}(x)\big) + 2H_{1,1,0}(x)$$

$$- 2(l+3)H_{0,0,0}(x) - l\frac{\pi^2}{6}H_1(x) - (l+3)\frac{\pi^2}{3}H_0(x) + \frac{1}{2}(3l+2)\zeta_3 - (l+2)\pi^2 a_1\Bigg]\varepsilon$$

$$+ 2\Bigg[(l+2)^2\big(-2H_{1,-1,1,0}(x) + H_{0,1,-1,0}(x) - H_{0,-1,1,0}(x) + H_{0,0,1,0}(x) + H_{0,0,-1,0}(x)\big)$$

$$+ (l+1)(l+2)\big(H_{1,0,1,0}(x) + H_{1,0,-1,0}(x)\big)$$

$$+ (l+2)\big(-2H_{1,1,-1,0}(x) + 2H_{1,-1,-1,0}(x) - H_{0,1,1,0}(x) + H_{0,-1,-1,0}(x) - 2H_{1,0,0,0}(x)\big) + 2H_{1,1,1,0}(x)$$

$$+ 2(l^2+5l+7)H_{0,0,0,0}(x) + \frac{\pi^2}{12}\Big[-2lH_{1,1}(x) - (l+2)(5l+6)\big(2H_{1,-1}(x) + H_{0,-1}(x)\big)$$

$$+ (8l^2+23l+12)H_{1,0}(x) + l(l+2)H_{0,1}(x) + (6l^2+23l+24)H_{0,0}(x)\Big] + \frac{1}{2}(l+3)(3l+2)\zeta_3 H_1(x)$$

$$+ (l+2)\pi^2 a_1\big[(l+1)H_1(x) + (l+2)H_0(x)\big] + (l+2)^2\pi^2 a_1^2 + (84l^2+227l+122)\frac{\pi^4}{720}\Bigg]\varepsilon^2\Bigg\} + \mathcal{O}(\varepsilon^3),$$

$$J_4 = -2\Bigg\{-H_{-1,0}(x) + H_{0,0}(x) - \frac{\pi^2}{6} + \Bigg[-(l+2)\big(2H_{-1,1,0}(x) - H_{0,-1,0}(x) - H_{0,1,0}(x)\big) + 2H_{-1,-1,0}(x)$$

$$- 2(l+3)H_{0,0,0}(x) - (5l+6)\frac{\pi^2}{6}H_{-1}(x) + (2l+3)\frac{\pi^2}{3}H_0(x) + \frac{1}{2}(3l+2)\zeta_3 + (l+2)\pi^2 a_1\Bigg]\varepsilon$$

$$+ 2\Bigg[(l+2)^2\big(2H_{-1,1,-1,0}(x) + H_{0,-1,1,0}(x) - H_{0,1,-1,0}(x) - H_{0,0,-1,0}(x) - H_{0,0,1,0}(x)\big)$$

$$+ (l+1)(l+2)\big(H_{-1,0,-1,0}(x) + H_{-1,0,1,0}(x)\big)$$

$$+ (l+2)\big(2H_{-1,-1,1,0}(x) - 2H_{-1,1,1,0}(x) - H_{0,-1,-1,0}(x) + H_{0,1,1,0}(x) + 2H_{-1,0,0,0}(x)\big) - 2H_{-1,-1,-1,0}(x)$$

$$+ 2(l^2+5l+7)H_{0,0,0,0}(x) + \frac{\pi^2}{12}\Big[2(5l+6)H_{-1,-1}(x) + l(l+2)\big(2H_{-1,1}(x) - H_{0,1}(x)\big)$$

$$+ (4l^2+13l+12)H_{-1,0}(x) + (l+2)(5l+6)H_{0,-1}(x) - (2l+3)(3l+8)H_{0,0}(x)\Big] - \frac{1}{2}(l+3)(3l+2)\zeta_3 H_{-1}(x)$$

$$+ (l+2)\pi^2 a_1\big[(l+1)H_{-1}(x) - (l+2)H_0(x)\big] - (l+2)^2\pi^2 a_1^2 - (36l^2+103l+58)\frac{\pi^4}{720}\Bigg]\varepsilon^2\Bigg\} + \mathcal{O}(\varepsilon^3). \quad \text{(B17)}$$

Up to order ε^1 all harmonic polylogarithms can be transformed to logarithms and ordinary polylogarithms up to Li_3, e.g., using the Mathematica package HPL [54, 55].

Next we consider the case $x > 1$. We can re-write the differential equation (B11) in the form

$$\frac{dJ}{dx^{-1}} = \varepsilon \times \quad \text{(B18)}$$
$$\left(\frac{-M_0 + M_{+1} - M_{-1}}{x^{-1}} + \frac{M_{+1}}{1 - x^{-1}} + \frac{M_{-1}}{1 + x^{-1}}\right)J.$$

It can be solved in terms of harmonic polylogarithms of x^{-1}, this is convenient for $x > 1$. We use the asymptotics $x \to$ $+\infty$ for boundary conditions. There are 2 regions:

- All lines in (B1) are hard (momenta of order m). This corresponds to the series of right poles in the Mellin–Barnes representation $s = n + n_1 + n_2 - d/2$, i.e., to the first term in the hypergeometric representation (A1) in [30], and gives $x^{2(d-n_1-n_2-n_3-n_4)}$ times a regular series in x^{-2}. The leading contribution to $I_{n_1 n_2 n_3 n_4}$ is $CV_{n_3,n_4,n_1+n_2}x^{2(d-n_1-n_2-n_3-n_4)}$.

- Lines 1, 2 are soft (momenta of order M). This corresponds to right poles at $s = n$, i.e., to the second hypergeometric term, and gives $x^{d-2(n_3+n_4)}$ times

a regular series in x^{-2}. The leading asymptotics is $CM_{n_1n_2}V_{n_3+n_4}x^{d-2(n_3+n_4)}$.

For $I_{2,l\varepsilon,2,1}$ these two contributions are $\sim x^{-2-2\varepsilon}$ and $\sim$

$x^{-2-(l+2)\varepsilon}$; for $I_{1,l\varepsilon,2,2}$ the leading contribution is hard, $\sim x^{-2-(l+1)\varepsilon}$. This information is sufficient for solving the differential equations for $x > 1$ using Libra [51]:

$$
\begin{aligned}
J_3 = -2\Bigg\{ &-H_{1,0}(x^{-1}) + \Bigg[-(l+4)H_{0,1,0}(x^{-1}) - 2(l+3)H_{1,0,0}(x^{-1}) + (l+2)\big(2H_{1,-1,0}(x^{-1}) + H_{0,-1,0}(x^{-1})\big) \\
&- 2H_{1,1,0}(x^{-1}) + l\frac{\pi^2}{6}H_1(x^{-1}) \Bigg]\varepsilon \\
&+ 2\Bigg[(l+2)^2\big(2H_{1,-1,1,0}(x^{-1}) + H_{0,-1,1,0}(x^{-1})\big) + (l+2)(l+5)H_{1,0,-1,0}(x^{-1}) \\
&+ (l+2)(l+4)\big(H_{0,1,-1,0}(x^{-1}) + H_{0,0,-1,0}(x^{-1})\big) + (l+2)(l+3)\big(2H_{1,-1,0,0}(x^{-1}) + H_{0,-1,0,0}(x^{-1})\big) \\
&- (l+3)(l+4)H_{0,1,0,0}(x^{-1}) - (l^2+6l+10)H_{0,0,1,0}(x^{-1}) - (l^2+5l+8)H_{1,0,1,0}(x^{-1}) \\
&- 2(l^2+5l+7)H_{1,0,0,0}(x^{-1}) - (l+4)H_{0,1,1,0}(x^{-1}) - 2(l+3)H_{1,1,0,0}(x^{-1}) \\
&+ (l+2)\big(2H_{1,1,-1,0}(x^{-1}) - 2H_{1,-1,-1,0}(x^{-1}) - H_{0,-1,-1,0}(x^{-1})\big) - 2H_{1,1,1,0}(x^{-1}) \\
&+ l\frac{\pi^2}{12}\big[(l+4)H_{0,1}(x^{-1}) - (l+2)\big(2H_{1,-1}(x^{-1}) + H_{0,-1}(x^{-1})\big) + 2H_{1,1}(x^{-1}) + H_{1,0}(x^{-1})\big]\Bigg]\varepsilon^2 \Bigg\} + \mathcal{O}(\varepsilon^3)\,, \\[6pt]
J_4 = -2\Bigg\{ &H_{-1,0}(x^{-1}) + \Bigg[(l+4)H_{0,-1,0}(x^{-1}) + 2(l+3)H_{-1,0,0}(x^{-1}) + (l+2)\big(2H_{-1,1,0}(x^{-1}) - H_{0,1,0}(x^{-1})\big) \\
&- 2H_{-1,-1,0}(x^{-1}) - l\frac{\pi^2}{6}H_{-1}(x^{-1}) \Bigg]\varepsilon \\
&+ 2\Bigg[(l+2)^2\big(-2H_{-1,1,-1,0}(x^{-1}) + H_{0,1,-1,0}(x^{-1})\big) + (l+2)(l+5)H_{-1,0,1,0}(x^{-1}) \\
&+ (l+2)(l+4)\big(H_{0,-1,1,0}(x^{-1}) - H_{0,0,1,0}(x^{-1})\big) + (l+2)(l+3)\big(2H_{-1,1,0,0}(x^{-1}) - H_{0,1,0,0}(x^{-1})\big) \\
&+ (l+3)(l+4)H_{0,-1,0,0}(x^{-1}) + (l^2+6l+10)H_{0,0,-1,0}(x^{-1}) - (l^2+5l+8)H_{-1,0,-1,0}(x^{-1}) \\
&+ 2(l^2+5l+7)H_{-1,0,0,0}(x^{-1}) - (l+4)H_{0,-1,-1,0}(x^{-1}) - 2(l+3)H_{-1,-1,0,0}(x^{-1}) \\
&- (l+2)\big(2H_{-1,-1,1,0}(x^{-1}) - 2H_{-1,1,1,0}(x^{-1}) + H_{0,1,1,0}(x^{-1})\big) + 2H_{-1,-1,-1,0}(x^{-1}) \\
&+ l\frac{\pi^2}{12}\big[-(l+4)H_{0,-1}(x^{-1}) - (l+2)\big(2H_{-1,1}(x^{-1}) - H_{0,1}(x^{-1})\big) + 2H_{-1,-1}(x^{-1}) - H_{-1,0}(x^{-1})\big]\Bigg]\varepsilon^2 \Bigg\} \\
&+ \mathcal{O}(\varepsilon^3)\,.
\end{aligned}
\tag{B19}
$$

This is, of course, the analytical continuation of (B17) to $x > 1$. The same results (B19) can be obtained if we express $J_{3,4}$ via $I_{2,l\varepsilon,2,1}$ and $I_{1,l\varepsilon,2,2}$ using (B6) and expand the hypergeometric representations (see Eq. (A1) in [30]) of these two integrals in ε using the Mathematica package HypExp [56, 57]. However, solving the differential equations (B18) up to higher orders in ε is simpler than expanding hypergeometric functions.

Both (B17) and (B19) lead to identical results at $x = 1$:

$$
\begin{aligned}
J_3(1) = &-\frac{\pi^2}{3} + \frac{1}{2}(l+2)\big(2\pi^2 a_1 - 7\zeta_3\big)\varepsilon \\
&- \Bigg[(l+2)(l+3)\Big(8a_4 + \frac{1}{3}a_1^4 + \frac{2}{3}\pi^2 a_1^2\Big) \\
&+ (17l^2 - 36l - 124)\frac{\pi^4}{360}\Bigg]\varepsilon^2 + \mathcal{O}(\varepsilon^3)\,,
\end{aligned}
$$

$$
\begin{aligned}
J_4(1) = &\frac{\pi^2}{6} - \frac{1}{2}\big(2\pi^2 a_1 - 7\zeta_3\big)\varepsilon \\
&+ \Bigg[(l+3)\Big(8a_4 + \frac{1}{3}a_1^4 + \frac{2}{3}\pi^2 a_1^2\Big) \\
&+ (24l^2 + 27l - 62)\frac{\pi^4}{360}\Bigg]\varepsilon^2 + \mathcal{O}(\varepsilon^3)\,.
\end{aligned}
\tag{B20}
$$

If $l = 0$ and $x = 1$, we obviously have $I_{1022}(1) = I_{2021}(1)$, and hence

$$
J_3(1) = -2J_4(1) = -4I_{2021}(1)\,.
\tag{B21}
$$

Expanding the hypergeometric representation [30] of I_{2021} (or I_{1022}) at $x = 1$ in ε we get (B20) with $l = 0$. Alternatively, we can use another hypergeometric representation [8, 9]. Using integration by parts we obtain

$$
I_{2021}(1) = \frac{7}{32\varepsilon^2}\left[\frac{\Gamma(1-\varepsilon)\Gamma^2(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1+4\varepsilon)} - 1 \right]
$$

$$+ \frac{2^{-2-6\varepsilon}\pi^2}{3} \frac{\Gamma^3(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma^5(1+\varepsilon)\Gamma^2(1+2\varepsilon)} + \frac{3}{4}\varepsilon^2 B_4(\varepsilon)\,, \quad \text{(B22)}$$

where $B_4(\varepsilon)$ is given by the formulas (41), (43) in [9]. This leads to the same result.

The functions $L_{\mp}(x) = -\frac{1}{2}J_{3,4}(l=0, \varepsilon=0)$ were used in [7, 8, 22, 30]. In addition to the two expressions for these functions in (B17) and (B19), several additional representations can be found in [30].

The results (B17) and (B19) are expansions in ε where the coefficients are exact functions of x. On the other hand, it is straightforward to obtain expansions of $J_{3,4}$ in x (or x^{-1}) to any finite order using residues of left (or right) poles in the Mellin–Barnes representations of the integrals $j_{3,4}$ (B3), the coefficients being exact functions of ε. If we expand them in ε, they should agree with expansions of (B17) in x and of (B19) in x^{-1}. We have checked this up to rather high degrees of x

and x^{-1}.

Now we can find all contributions to Z_j^{os} ($j = M, Q$) with the maximum number of quark loops, at most one of which is massive, to all orders exactly in ε:

$$Z_j^{\mathrm{os}} = 1 + C_F \sum_l T_F^{l-1}(n_0 P)^{l-2}\Big[n_0 P B_{j0}^{(l)} \qquad \text{(B23)}$$
$$+ (l-1)\sum_i B_j^{(l)}(x_i)\Big] \left(\frac{g_0^2 M^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^l + \cdots,$$

where $g_0 \equiv g_0^{(n_f)}$, n_0 is the number of massless flavors, the sum runs over all massive flavors with $x_i = m_i/M$ (including the external flavor with $x = 1$) and dots refer to other color structures. Here

$$P = -4\frac{1-\varepsilon}{(1-2\varepsilon)(3-2\varepsilon)}\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\,,$$

$$B_{M0}^{(l)} = -2\frac{(3-2\varepsilon)(1-l\varepsilon)}{l(1-(l+1)\varepsilon)(2-(l+1)\varepsilon)}\frac{\Gamma(1+l\varepsilon)\Gamma(1-2l\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1-(l+1)\varepsilon)}\,, \quad B_{Q0}^{(l)} = B_{M0}^{(l)}(1+(l-1)\varepsilon)\,,$$

$$B_M^{(l)}(x) = 2p_0\left\{-2\frac{1-\varepsilon}{l}\left[1-l\varepsilon+l\varepsilon\frac{1-(l-1)\varepsilon}{1+(l-1)\varepsilon}x^2\right]J_1^{(l-2)}(x) + \left[p_1 + 2\varepsilon\frac{1+(2l-3)\varepsilon - (l-1)(l+2)\varepsilon^2}{1+(l-1)\varepsilon}x^2\right]J_2^{(l-2)}(x)\right.$$

$$\left. - (p_1(1+x^2) + p_2 x)(1-x)^2 J_3^{(l-2)}(x) - (p_1(1+x^2) - p_2 x)(1+x)^2 J_4^{(l-2)}(x)\right\}\,,$$

$$B_Q^{(l)}(x) = p_0\left\{-\frac{2\varepsilon}{l(1-\varepsilon)(1+2(l-1)\varepsilon)(3+2(l-1)\varepsilon)}\right.$$

$$\times \left[(1+(l-1)\varepsilon)(19l - 3 - (11l^2 + 50l - 11)\varepsilon - 2(4l^3 - 20l^2 - 15l + 6)\varepsilon^2 + 4(4l^3 - 11l^2 + 2l + 1)\varepsilon^3 - 8l(l-1)^2\varepsilon^4)\right.$$

$$\left. + \frac{l(1-\varepsilon)(1+2(l-1)\varepsilon)(3+2(l-1)\varepsilon)}{1+(l-1)\varepsilon}(4 - (3l+1)\varepsilon - (l-1)(l-7)\varepsilon^2 - 4(l-1)\varepsilon^3)x^2\right]J_1^{(l-2)}(x)$$

$$+ 2\left[2(1-\varepsilon)(1+(l-1)\varepsilon)(1-l\varepsilon) + \varepsilon\frac{4+(11l-15)\varepsilon - (l-1)(l+17)\varepsilon^2 - 2(l-1)(l^2-3)\varepsilon^3}{1+(l-1)\varepsilon}x^2\right]J_2^{(l-2)}(x)$$

$$\left. + (p_3 + p_4 x + p_5 x^2 + p_6 x^3)(1-x)J_3^{(l-2)}(x) + (p_3 - p_4 x + p_5 x^2 - p_6 x^3)(1+x)J_4^{(l-2)}(x)\right\}\,, \qquad \text{(B24)}$$

where

$$p_0 = \frac{2\varepsilon^2}{(1-2\varepsilon)(1-(l+1)\varepsilon)(2-(l+1)\varepsilon)}\,,$$

$$p_1 = 2(1-\varepsilon)(1-l\varepsilon)\,,$$

$$p_2 = \frac{(1-(l+1)\varepsilon)(2+(l-3)\varepsilon - 2(l-1)\varepsilon^2)}{1+(l-1)\varepsilon}\,,$$

$$p_3 = -4(1-\varepsilon)(1+(l-1)\varepsilon)(1-l\varepsilon)\,,$$

$$p_4 = \frac{2+(3l-5)\varepsilon - (l-1)(5l-1)\varepsilon^2 + 4(l-1)^2\varepsilon^3(2-\varepsilon)}{1+(l-1)\varepsilon}\,,$$

$$p_5 = [2 - (5l+13)\varepsilon + (l^2 - 6l + 29)\varepsilon^2$$
$$+ 2(l^3 - l^2 + 9l - 13)\varepsilon^3 - 8(l-1)\varepsilon^4]/[1+(l-1)\varepsilon]\,,$$

$$p_6 = 4(1-\varepsilon)(3-2\varepsilon)(1-l\varepsilon)\,.$$

The results (B24) at $l = 2$ agree with [30] exactly in ε. Note that

$$\lim_{x\to 0} B_M^{(l)}(x) = B_{M0}^{(l)} P\,, \qquad \text{(B25)}$$

so that Z_M^{os} has a smooth limit $x \to 0$; this is not so for Z_Q^{os}.

The contribution of these color structures to the ratio of the $\overline{\mathrm{MS}}$ mass and the on-shell one $z_m(\mu) = M(\mu)/M$ can be written as

$$z_m(M) = z_m^{(\beta_0)} + \sum_i \Delta_m(x_i) + \cdots, \qquad \text{(B26)}$$

where $z_m^{(\beta_0)}$ is the well-known large-β_0 result [58]

$$z_m^{(\beta_0)} = 1 + \frac{1}{2}\int_0^b \frac{db}{b}\left(\frac{\gamma_m(b)}{b} - \frac{\gamma_{m0}}{\beta_0}\right)$$

$$+ \frac{1}{\beta_0} \int_0^\infty du\, S(u)\, e^{-u/b},$$

$$\gamma_m(b) = \frac{2}{3} C_F \frac{b}{\beta_0} \frac{(3+2b)\Gamma(4+2b)}{\Gamma(3+b)\Gamma^2(2+b)\Gamma(1-b)},$$

$$\gamma_{m0} = 6C_F,$$

$$S(u) = -6C_F \left[e^{(5/3)u} \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)}(1-u) - \frac{1}{2u} \right],$$

$$b = \beta_0 \frac{\alpha_s}{4\pi} \quad (\alpha_s \equiv \alpha_s^{(n_f)}(M)). \tag{B27}$$

Note that we first expand $S(u)$ in u, then integrate term-by-term assuming $\beta_0 > 0$, and at the very end substitute $\beta_0 \to -(4/3)T_F n_f$. $\Delta_m(x)$ comes from the differences of diagrams with a single massive quark loop and corresponding diagrams with all quark loops being massless and is given by

$$\Delta_m(x) = C_F T_F \left(\frac{\alpha_s}{\pi}\right)^2 \left\{ \frac{1}{2} \left[(1-x)^2(1+x+x^2)H_{1,0}(x) - (1+x)^2(1-x+x^2)H_{-1,0}(x) + 2x^4 H_{0,0}(x) \right. \right.$$

$$\left. + x^2 H_0(x) - x(3+3x^2-x^3)\frac{\pi^2}{6} + \frac{3}{2}x^2 \right]$$

$$+ T_F n_0 \frac{\alpha_s}{\pi} \frac{2}{3} \left[(1-x)^2(1+x+x^2)\left(H_{1,-1,0}(x) + \frac{\pi^2}{12}H_1(x)\right) \right.$$

$$+ (1+x)^2(1-x+x^2)\left(H_{-1,1,0}(x) + \frac{5\pi^2}{12}H_{-1}(x)\right) - x(1+x^2)\left(H_{0,1,0}(x) + H_{0,-1,0}(x) + \pi^2 a_1\right)$$

$$+ x^4\left(2H_{0,0,0}(x) - \frac{13}{6}H_{0,0}(x) - \frac{3}{2}\zeta_3\right) - x(3+3x^2+x^3)\frac{\pi^2}{6}H_0(x) - \frac{1}{12}(1-x)^2(13+10x+13x^2)H_{1,0}(x)$$

$$+ \frac{1}{12}(1+x)^2(13-10x+13x^2)H_{-1,0}(x) + x(48-12x+48x^2-13x^3)\frac{\pi^2}{72} - \frac{7}{12}x^2 H_0(x) - \frac{11}{8}x^2 \right]$$

$$+ \left(T_F n_0 \frac{\alpha_s}{\pi}\right)^2 \frac{2}{3} \left[(1-x)^2(1+x+x^2) \right.$$

$$\times \left(-2H_{1,-1,1,0}(x) + H_{1,0,1,0}(x) + H_{1,0,-1,0}(x) - \frac{\pi^2}{6}(5H_{1,-1}(x) - 4H_{1,0}(x)) + \left(\pi^2 a_1 + \frac{3}{2}\zeta_3\right)H_1(x) \right)$$

$$+ (1+x)^2(1-x+x^2)$$

$$\times \left(2H_{-1,1,-1,0}(x) + H_{-1,0,-1,0}(x) + H_{-1,0,1,0}(x) + \frac{\pi^2}{6}(H_{-1,1}(x) + 2H_{-1,0}(x)) + \left(\pi^2 a_1 - \frac{3}{2}\zeta_3\right)H_{-1}(x) \right)$$

$$+ 2x(1+x^2)\left(-H_{0,1,-1,0}(x) + H_{0,-1,1,0}(x) - H_{0,0,1,0}(x) - H_{0,0,-1,0}(x) + \frac{4}{3}\left(H_{0,1,0}(x) + H_{0,-1,0}(x) + \pi^2 a_1\right) \right.$$

$$\left. - \frac{\pi^2}{12}(H_{0,1}(x) - 5H_{0,-1}(x) + 6H_{0,0}(x)) - \pi^2 a_1 H_0(x) - \pi^2 a_1^2 \right)$$

$$+ x^4\left(4H_{0,0,0,0}(x) - \frac{13}{3}H_{0,0,0}(x) + \frac{89}{36}H_{0,0}(x) \right)$$

$$- \frac{1}{6}(1-x)^2(13+10x+13x^2)\left(H_{1,-1,0}(x) + \frac{\pi^2}{12}H_1(x)\right) - \frac{1}{6}(1+x)^2(13-10x+13x^2)\left(H_{-1,1,0}(x) + \frac{5\pi^2}{12}H_{-1}(x)\right)$$

$$+ \frac{1}{72}(1-x)^2(89+68x+89x^2)H_{1,0}(x) - \frac{1}{72}(1+x)^2(89-68x+89x^2)H_{-1,0}(x)$$

$$+ x(48+6x+48x^2+13x^3)\frac{\pi^2}{36}H_0(x) + \frac{47}{72}x^2 H_0(x) + \frac{1}{4}x^2(6+13x^2)\zeta_3 - x(5+5x^2-2x^3)\frac{\pi^4}{30}$$

$$- x(330-192x+330x^2-89x^3)\frac{\pi^2}{432} + \frac{33}{16}x^2 \right] + \mathcal{O}(\alpha_s^3) \bigg\}. \tag{B28}$$

Note that $\Delta_m(0) = 0$. Expanding the three-loop term in x we reproduce the series (up to x^8) obtained in [31]. The three-loop coefficient exact in x (Fig. 3) and well as the four-loop one are new. The contribution of the external flavor ($m = M$)

is given by

$$\Delta_m(1) = C_F T_F \left(\frac{\alpha_s}{\pi}\right)^2 \left[-\frac{\pi^2-3}{4}\right.$$

$$+ T_F n_0 \frac{\alpha_s}{\pi}\left(\zeta_3 + \frac{13}{36}\pi^2 - \frac{11}{12}\right)$$

$$- \left(T_F n_0 \frac{\alpha_s}{\pi}\right)^2\left(\frac{13}{6}\zeta_3 + \frac{4}{45}\pi^4 + \frac{53}{216}\pi^2 - \frac{11}{8}\right)$$

$$\left. + \mathcal{O}(\alpha_s^3)\right]. \tag{B29}$$

The three- and four-loop terms here agree with [11] and [12]. We do not present lower-loop terms of z_m with positive powers of ε which may be needed when this ratio is used within calculations containing $1/\varepsilon$ divergences; these terms can be easily obtained from Eqs. (B23) and (B24).
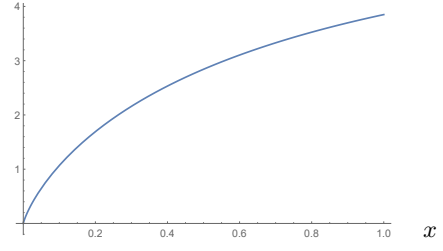

FIG. 3. The coefficient of $(\alpha_s/\pi)^3 C_F T_F^2 n_0$ in $\Delta_m(x)$.

Contributions to Z_h^{os} with the color structures $C_F T_F^{l-1} n_0^{l-2}$ (i.e., the maximum number of quark loops, one of them is massive with mass m_i) can be calculated using Eq. (B10). The results read

$$Z_h^{\text{os}} = 1 + C_F \sum_{l=2}^{\infty} T_F^{l-1}(n_0 P)^{l-2}(l-1)B_h^{(l)}$$

$$\times \sum_i \left(\frac{g_0^2 m_i^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\right)^l + \cdots,$$

$$B_h^{(l)} = 4\frac{(3-2\varepsilon)(1+(l-1)\varepsilon)\Gamma^2(1+(l-1)\varepsilon)}{l(l-1)(3+2(l-1)\varepsilon)\Gamma(2-\varepsilon)\Gamma^2(1+\varepsilon)}$$

$$\times \frac{\Gamma(1-(l-1)\varepsilon)\Gamma(1+l\varepsilon)}{\Gamma(2+2(l-1)\varepsilon)}, \tag{B30}$$

where $g_0 \equiv g_0^{(n_l)}$ and dots denote other color structures. The $(l=2)$-loop term agrees with [22], and the three-loop one with the corresponding color structure in [33]. According to the regions-based argument in Sect. III,

$$\lim_{x\to 0}\left[B_Q^{(l)}(x) - B_{Q0}^{(l)}P - B_h^{(l)}x^{-2l\varepsilon}\right] = 0. \tag{B31}$$

[1] M. Neubert, Phys. Rept. **245**, 259 (1994), arXiv:hep-ph/9306320 [hep-ph].
[2] A. V. Manohar and M. B. Wise, *Heavy quark physics*, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol., Vol. 10 (Cambridge university press, Cambridge, 2000).
[3] A. G. Grozin, *Heavy quark effective theory*, Springer Tracts Mod. Phys., Vol. 201 (Springer, Berlin, 2004).
[4] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).
[5] A. G. Grozin, Phys. Lett. **B692**, 161 (2010), arXiv:1004.2662 [hep-ph].
[6] M. E. Luke and A. V. Manohar, Phys. Lett. **B286**, 348 (1992), arXiv:hep-ph/9205228 [hep-ph].
[7] D. J. Broadhurst, N. Gray, and K. Schilcher, Z. Phys. **C52**, 111 (1991).
[8] D. J. Broadhurst, Z. Phys. **C54**, 599 (1992).
[9] D. J. Broadhurst, (1996), arXiv:hep-th/9604128 [hep-th].
[10] K. Melnikov and T. van Ritbergen, Nucl. Phys. **B591**, 515 (2000), arXiv:hep-ph/0005131 [hep-ph].
[11] P. Marquard, L. Mihaila, J. H. Piclum, and M. Steinhauser, Nucl. Phys. **B773**, 1 (2007), arXiv:hep-ph/0702185 [hep-ph].
[12] R. Lee, P. Marquard, A. V. Smirnov, V. A. Smirnov, and M. Steinhauser, JHEP **03**, 162 (2013), arXiv:1301.6481 [hep-ph].
[13] P. Marquard, A. V. Smirnov, V. A. Smirnov, and M. Steinhauser, Phys. Rev. **D97**, 054032 (2018), arXiv:1801.08292 [hep-ph].
[14] S. Laporta, Phys. Lett. **B802**, 135264 (2020), arXiv:2001.02739 [hep-ph].
[15] S. Laporta, Phys. Lett. **B772**, 232 (2017), arXiv:1704.06996 [hep-ph].
[16] P. Marquard, A. V. Smirnov, V. A. Smirnov, M. Steinhauser, and D. Wellmann, EPJ Web Conf. **218**, 01004 (2019), arXiv:1708.07138 [hep-ph].
[17] K. G. Chetyrkin and A. Retey, Nucl. Phys. **B583**, 3 (2000), arXiv:hep-ph/9910332 [hep-ph].
[18] M. Czakon, Nucl. Phys. **B710**, 485 (2005), arXiv:hep-ph/0411261 [hep-ph].
[19] T. Luthe, A. Maier, P. Marquard, and Y. Schröder, JHEP **01**, 081 (2017), arXiv:1612.05512 [hep-ph].
[20] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, JHEP **04**, 119 (2017), arXiv:1702.01458 [hep-ph].
[21] K. G. Chetyrkin and A. G. Grozin, Nucl. Phys. **B666**, 289 (2003), arXiv:hep-ph/0303113 [hep-ph].
[22] D. J. Broadhurst and A. G. Grozin, Phys. Rev. **D52**, 4082

(1995), arXiv:hep-ph/9410240 [hep-ph].

[23] A. Grozin, J. M. Henn, G. P. Korchemsky, and P. Marquard, JHEP **01**, 140 (2016), arXiv:1510.07803 [hep-ph].

[24] A. Grozin, *Proceedings, 13th DESY Workshop on Elementary Particle Physics: Loops and Legs in Quantum Field Theory (LL2016): Leipzig, Germany, April 24-29, 2016*, PoS **LL2016**, 053 (2016), arXiv:1605.03886 [hep-ph].

[25] A. Grozin, JHEP **06**, 073 (2018), [Addendum: JHEP **01**, 134 (2019)], arXiv:1805.05050 [hep-ph].

[26] A. Grozin, J. Henn, and M. Stahlhofen, JHEP **10**, 052 (2017), arXiv:1708.01221 [hep-ph].

[27] R. Brüser, A. Grozin, J. M. Henn, and M. Stahlhofen, JHEP **05**, 186 (2019), arXiv:1902.05076 [hep-ph].

[28] M. Beneke and V. A. Smirnov, Nucl. Phys. B **522**, 321 (1998), arXiv:hep-ph/9711391.

[29] V. A. Smirnov, *Applied asymptotic expansions in momenta and masses*, Springer Tracts Mod. Phys. **177**, 1 (2002).

[30] A. I. Davydychev and A. G. Grozin, Phys. Rev. **D59**, 054023 (1999), arXiv:hep-ph/9809589 [hep-ph].

[31] S. Bekavac, A. Grozin, D. Seidel, and M. Steinhauser, JHEP **10**, 006 (2007), arXiv:0708.1729 [hep-ph].

[32] S. Bekavac, A. G. Grozin, D. Seidel, and V. A. Smirnov, Nucl. Phys. **B819**, 183 (2009), arXiv:0903.4760 [hep-ph].

[33] A. G. Grozin, A. V. Smirnov, and V. A. Smirnov, JHEP **11**, 022 (2006), arXiv:hep-ph/0609280 [hep-ph].

[34] A. G. Grozin, T. Huber, and D. Maître, JHEP **07**, 033 (2007), arXiv:0705.2609 [hep-ph].

[35] K. Johnson and B. Zumino, Phys. Rev. Lett. **3**, 351 (1959).

[36] B. Zumino, J. Math. Phys. **1**, 1 (1960).

[37] L. Landau and I. Khalatnikov, Sov. Phys. JETP **2**, 69 (1956).

[38] E. Fradkin, Zh. Eksp. Teor. Fiz. **29**, 258 (1955).

[39] J. Gatheral, Phys. Lett. B **133**, 90 (1983).

[40] J. Frenkel and J. Taylor, Nucl. Phys. B **246**, 231 (1984).

[41] A. Grozin, Int. J. Mod. Phy. **A28**, 1350015 (2013),

arXiv:1212.5144 [hep-ph].

[42] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, Nucl. Phys. **B510**, 61 (1998), arXiv:hep-ph/9708255 [hep-ph].

[43] M. Gerlach, F. Herren, and M. Steinhauser, JHEP **11**, 141 (2018), arXiv:1809.06787 [hep-ph].

[44] Y. Schröder and M. Steinhauser, JHEP **01**, 051 (2006), arXiv:hep-ph/0512058 [hep-ph].

[45] K. G. Chetyrkin, J. H. Kühn, and C. Sturm, Nucl. Phys. **B744**, 121 (2006), arXiv:hep-ph/0512060 [hep-ph].

[46] A. V. Kotikov, Phys. Lett. **B254**, 158 (1991).

[47] J. M. Henn, Phys. Rev. Lett. **110**, 251601 (2013), arXiv:1304.1806 [hep-th].

[48] L. Avdeev and M. Kalmykov, Nucl. Phys. B **502**, 419 (1997), arXiv:hep-ph/9701308.

[49] M. Argeri, P. Mastrolia, and E. Remiddi, Nucl. Phys. **B631**, 388 (2002), arXiv:hep-ph/0202123 [hep-ph].

[50] A. Onishchenko and O. Veretin, Phys. Atom. Nucl. **68**, 1405 (2005), [Yad. Fiz.68,1461(2005)], arXiv:hep-ph/0207091 [hep-ph].

[51] R. N. Lee, "Libra," Unpublished, available from the author by request.

[52] R. N. Lee, JHEP **04**, 108 (2015), arXiv:1411.0911 [hep-ph].

[53] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. **A15**, 725 (2000), arXiv:hep-ph/9905237 [hep-ph].

[54] D. Maître, Comput. Phys. Commun. **174**, 222 (2006), arXiv:hep-ph/0507152 [hep-ph].

[55] D. Maître, Comput. Phys. Commun. **183**, 846 (2012), arXiv:hep-ph/0703052 [hep-ph].

[56] T. Huber and D. Maître, Comput. Phys. Commun. **175**, 122 (2006), arXiv:hep-ph/0507094 [hep-ph].

[57] T. Huber and D. Maître, Comput. Phys. Commun. **178**, 755 (2008), arXiv:0708.2443 [hep-ph].

[58] M. Beneke and V. M. Braun, Nucl. Phys. B **426**, 301 (1994), arXiv:hep-ph/9402364.