DEUTSCHES ELEKTRONEN-SYNCHROTRON
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DESY 20-199
DO-TH 20/06
SAGEX-20-08
arXiv:2101.08630
January 2021

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ISSN 0418–9833

PLATANENALLEE 6 — 15738 ZEUTHEN

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arXiv:2101.08630 [gr-qc]

The 6th Post-Newtonian Potential Terms at $O(G_N^4)$

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Abstract

We calculate the potential contributions of the Hamiltonian in harmonic coordinates up 6PN for binary mass systems to $O(G_N^4)$ and perform comparisons to recent results in the literature [1] and [2].

Gravitational wave signals from merging black holes and neutron stars [3,4] provide tests of precision predictions within Einstein gravity for the dynamics of binary systems of massive objects. Their dynamics is calculated using post-Newtonian [2, 5–11] and the post-Minkowskian [1, 12][1] methods[2]. To match present and future measurements, high order computations are necessary. In the post-Newtonian (PN) case the level of 6PN [2, 10, 11] has now been reached for the conservative part of the two-body motion, while in the post-Minkowskian approach the calculation of the $O(G_N^4)$ potential terms is the most far reaching result [1]. Here G_N denotes Newton's constant. The different calculations are often performed using different gauges (harmonic, ADM, isotropic and EOB) in deriving the Lagrangian or Hamiltonian. It is important to cross check the results between the two methods, and to compare different representations in the most general way possible. One either can perform special comparisons in calculating the same observable or one performs a canonical transformation [14] between the different Hamiltonians obtained. The latter result implies that the results for all observables are the same because of the invariance of the action.

In this paper we report about the calculation of the $O(G_N^4)$ potential terms to 6PN, extending previous work covering the terms up to $O(G_N^3)$. We use the effective field theory approach [15], for details see [10]. We work in the harmonic gauge and $D = 4 - 2\varepsilon$ dimensions. The Feynman diagrams are generated using `QGRAF` [16]. The Lorentz algebra is carried out using `Form` [17] and we perform the integration by parts (IBP) reduction to master integrals using the code `Crusher` [18]. Table 1 gives an overview on the present calculation.

#loops	QGRAF	source irred	no source loops	no tadpoles	symmetrised	masters
0	3	3	3	3	3	
1	72	72	72	72	24	1
2	4322	4322	4322	3512	485	1
3	111752	86900	85467	61863	5553	2

Table 1: Numbers of contributing diagrams at the different loop levels and master integrals.

Redundant diagrams are eliminated in a series of steps outlined in Ref. [7]. 6065 diagrams do finally contribute to the present result. The computation time amounted to a few days on an `Intel(R) Xeon(R) CPU E5-2643 v4`. The Lagrange function of mth order, still containing the accelerations a_i and time derivatives thereof, are converted into a first order Lagrange density by applying double zero insertions [19, 20] together with partial integration and the remaining linear accelerations by a shift [20, 21], cf. [6]. By this operation we leave harmonic coordinates. A Legendre transformation leads then to the potential contributions of the Hamiltonian, which still contains pole terms in the dimensional parameter ε.

We define $H = (H_{\rm real} - Mc^2)/(\mu c^2)$ with $M = m_1 + m_2, \mu = m_1 m_2/M, \nu = \mu/M$, with $m_{1,2}$ the two masses of the binary system, c the velocity of light, and obtain the following result for the potential contributions[3] to the 6PN Hamiltonian to $O(G_N^4)$

$$H_{\rm pot} = \frac{p^2}{2} + \frac{1}{8}p^4(-1+3\nu) + \frac{1}{16}p^6\left(1 - 5\nu + 5\nu^2\right) + p^8\left(-\frac{5}{128} + \frac{35\nu}{128} - \frac{35\nu^2}{64} + \frac{35\nu^3}{128}\right)$$

[1] See also Ref. [12] in [10].

[2] For surveys see [13].

[3] including the kinetic terms.

$$+p^{10}\left(\frac{7}{256}-\frac{63\nu}{256}+\frac{189\nu^2}{256}-\frac{105\nu^3}{128}+\frac{63\nu^4}{256}\right)+p^{12}\left(-\frac{21}{1024}+\frac{231\nu}{1024}-\frac{231\nu^2}{256}\right.$$
$$\left.+\frac{1617\nu^3}{1024}-\frac{1155\nu^4}{1024}+\frac{231\nu^5}{1024}\right)+p^{14}\left(\frac{33}{2048}-\frac{429\nu}{2048}+\frac{2145\nu^2}{2048}-\frac{1287\nu^3}{512}\right.$$
$$\left.+\frac{3003\nu^4}{1024}-\frac{3003\nu^5}{2048}+\frac{429\nu^6}{2048}\right)$$
$$+\frac{1}{r}\left[-1+\frac{1}{2}p^2(-3-\nu)-\frac{(p.n)^2\nu}{2}-\frac{1}{4}p^2(p.n)^2(-1+\nu)\nu-\frac{3}{8}(p.n)^4\nu^2\right.$$
$$-\frac{3}{16}p^2(p.n)^4(-1+\nu)\nu^2-\frac{5}{16}(p.n)^6\nu^3-\frac{5}{32}p^2(p.n)^6(-2+\nu)\nu^3-\frac{35}{128}(p.n)^8\nu^4$$
$$-\frac{63}{256}(p.n)^{10}\nu^5-\frac{231(p.n)^{12}\nu^6}{1024}+\frac{1}{8}p^4\left(5-22\nu-3\nu^2\right)+p^6\left(-\frac{7}{16}+\frac{45\nu}{16}\right.$$
$$\left.-\frac{31\nu^2}{8}-\frac{5\nu^3}{16}\right)+p^4(p.n)^2\left(-\frac{3\nu}{16}+\frac{11\nu^2}{16}-\frac{3\nu^3}{16}\right)+p^8\left(\frac{45}{128}-\frac{95\nu}{32}+\frac{475\nu^2}{64}-\frac{267\nu^3}{64}\right.$$
$$\left.-\frac{35\nu^4}{128}\right)+p^6(p.n)^2\left(\frac{5\nu}{32}-\frac{29\nu^2}{32}+\frac{11\nu^3}{32}-\frac{5\nu^4}{32}\right)+p^4(p.n)^4\left(-\frac{9\nu^2}{64}+\frac{33\nu^3}{64}-\frac{9\nu^4}{64}\right)$$
$$+p^{10}\left(-\frac{77}{256}+\frac{805\nu}{256}-\frac{2865\nu^2}{256}+\frac{3995\nu^3}{256}-\frac{1615\nu^4}{256}-\frac{63\nu^5}{256}\right)$$
$$+p^2(p.n)^8\left(-\frac{35\nu^3}{256}+\frac{105\nu^4}{128}-\frac{35\nu^5}{256}\right)+p^8(p.n)^2\left(-\frac{35\nu}{256}+\frac{275\nu^2}{256}-\frac{221\nu^3}{64}+\frac{867\nu^4}{256}\right.$$
$$\left.-\frac{35\nu^5}{256}\right)+p^4(p.n)^6\left(-\frac{15\nu^3}{128}-\frac{45\nu^4}{128}-\frac{15\nu^5}{128}\right)+p^6(p.n)^4\left(\frac{15\nu^2}{128}-\frac{3\nu^3}{128}-\frac{3\nu^4}{64}-\frac{15\nu^5}{128}\right)$$
$$+p^2(p.n)^{10}\left(\frac{63\nu^3}{512}-\frac{441\nu^4}{512}+\frac{945\nu^5}{512}-\frac{63\nu^6}{512}\right)+p^4(p.n)^8\left(-\frac{105\nu^3}{512}+\frac{1505\nu^4}{1024}-\frac{1785\nu^5}{512}\right.$$
$$\left.-\frac{105\nu^6}{1024}\right)+p^8(p.n)^4\left(-\frac{105\nu^2}{1024}+\frac{2169\nu^3}{1024}-\frac{6891\nu^4}{1024}+\frac{3003\nu^5}{1024}-\frac{105\nu^6}{1024}\right)$$
$$+p^6(p.n)^6\left(-\frac{5\nu^3}{64}-\frac{5\nu^4}{8}+\frac{1025\nu^5}{256}-\frac{25\nu^6}{256}\right)$$
$$+p^{12}\left(\frac{273}{1024}-\frac{1701\nu}{512}+\frac{15631\nu^2}{1024}-\frac{30277\nu^3}{1024}+\frac{5007\nu^4}{256}+\frac{597\nu^5}{1024}-\frac{231\nu^6}{1024}\right)$$
$$\left.+p^{10}(p.n)^2\left(\frac{63\nu}{512}-\frac{623\nu^2}{512}+\frac{87\nu^3}{64}+\frac{95\nu^4}{16}-\frac{1889\nu^5}{256}-\frac{63\nu^6}{512}\right)\right]$$
$$+\frac{1}{r^2}\left[\frac{1}{2}+\frac{1}{2}(p.n)^2(-1+6\nu)+\frac{1}{16}p^4(-1+8\nu)(29+12\nu)+\frac{1}{4}p^2(11+15\nu)\right.$$
$$+\frac{1}{3}(p.n)^4\nu(7+69\nu)+\frac{1}{4}p^2(p.n)^2\left(1-36\nu-36\nu^2\right)+p^2(p.n)^4\left(-\frac{79\nu}{192}-\frac{4737\nu^2}{64}\right.$$

$$-\frac{7511\nu^3}{96}\Bigg)+p^6\left(\frac{55}{32}-\frac{667\nu}{64}+\frac{1217\nu^2}{64}-\frac{89\nu^3}{64}\right)+p^4(p.n)^2\left(-\frac{3}{16}-\frac{99\nu}{16}+\frac{733\nu^2}{16}\right.$$
$$+\frac{3189\nu^3}{64}\Bigg)+(p.n)^6\left(\frac{487\nu}{160}+\frac{543\nu^2}{32}+\frac{4609\nu^3}{80}\right)+p^2(p.n)^6\left(-\frac{5117\nu}{320}-\frac{8331\nu^2}{160}\right.$$
$$+\frac{136977\nu^3}{1280}-\frac{1178329\nu^4}{1280}\Bigg)+p^6(p.n)^2\left(\frac{5}{32}-\frac{589\nu}{16}+\frac{4969\nu^2}{64}+\frac{177\nu^3}{256}-\frac{62143\nu^4}{256}\right)$$
$$+p^8\left(-\frac{445}{256}+\frac{937\nu}{32}-\frac{11535\nu^2}{128}+\frac{16283\nu^3}{256}+\frac{6649\nu^4}{256}\right)+(p.n)^8\left(\frac{159\nu}{28}+\frac{751\nu^2}{28}\right.$$
$$-\frac{289839\nu^3}{4480}+\frac{1443091\nu^4}{4480}\Bigg)+p^4(p.n)^4\left(\frac{8951\nu}{384}+\frac{925\nu^2}{24}-\frac{125225\nu^3}{768}+\frac{652381\nu^4}{768}\right)$$
$$+p^4(p.n)^6\left(\frac{9647\nu}{320}+\frac{42947\nu^2}{320}-\frac{4799\nu^3}{8}+\frac{4064307\nu^4}{2560}-\frac{98191\nu^5}{80}\right)$$
$$+p^8(p.n)^2\left(-\frac{35}{256}-\frac{6747\nu}{128}+\frac{14485\nu^2}{32}-\frac{1169335\nu^3}{1024}+\frac{1033543\nu^4}{1024}-\frac{289599\nu^5}{1024}\right)$$
$$+(p.n)^{10}\left(\frac{5333\nu}{1152}+\frac{40343\nu^2}{1152}-\frac{1815535\nu^3}{16128}+\frac{2108389\nu^4}{8064}+\frac{87329\nu^5}{16128}\right)$$
$$+p^{10}\left(\frac{917}{512}+\frac{315\nu}{256}-\frac{26259\nu^2}{512}+\frac{125919\nu^3}{1024}-\frac{87953\nu^4}{1024}+\frac{16495\nu^5}{1024}\right)$$
$$+p^2(p.n)^8\left(-\frac{25007\nu}{1792}-\frac{1226481\nu^2}{8960}+\frac{7963357\nu^3}{17920}-\frac{9471627\nu^4}{8960}+\frac{7130709\nu^5}{17920}\right)$$
$$+p^6(p.n)^4\left(-\frac{7201\nu}{1536}-\frac{222197\nu^2}{768}+\frac{53039\nu^3}{48}-\frac{1277989\nu^4}{768}+\frac{390901\nu^5}{384}\right)\Bigg]$$
$$+\frac{1}{r^3}\Bigg[-\frac{1}{2}-\frac{\nu}{4}+p^2\left(-\frac{17}{4}+\frac{643\nu}{72}-\frac{7\pi^2\nu}{8}-\frac{3\nu^2}{2}\right)+(p.n)^2\left(\frac{3}{2}-\frac{1013\nu}{12}+\frac{21\pi^2\nu}{8}\right.$$
$$+\frac{49\nu^2}{4}\Bigg)+(p.n)^4\left(-\frac{6695\nu}{32}+\frac{4395\pi^2\nu}{1024}-\frac{200369\nu^2}{320}+\frac{345\pi^2\nu^2}{128}-\frac{333\nu^3}{32}\right)$$
$$+p^2(p.n)^2\left(-\frac{5}{4}+\frac{294477\nu}{800}-\frac{2955\pi^2\nu}{512}+\frac{167173\nu^2}{1200}+\frac{1095\pi^2\nu^2}{128}+\frac{11\nu^3}{16}\right)$$
$$+p^4\left(\frac{65}{16}-\frac{94439\nu}{800}+\frac{1091\pi^2\nu}{1024}+\frac{319789\nu^2}{14400}-\frac{217\pi^2\nu^2}{64}+\frac{205\nu^3}{32}\right)+(p.n)^6\left(-\frac{627281\nu}{960}\right.$$
$$+\frac{42105\pi^2\nu}{4096}+\frac{18031\nu^2}{3360}+\frac{14175\pi^2\nu^2}{4096}-\frac{14830647\nu^3}{4480}-\frac{65625\pi^2\nu^3}{1024}-\frac{17623\nu^4}{240}\Bigg)$$
$$+p^2(p.n)^4\left(\frac{31715507\nu}{23520}-\frac{89625\pi^2\nu}{4096}+\frac{848889\nu^2}{1568}+\frac{127125\pi^2\nu^2}{4096}+\frac{373945981\nu^3}{94080}\right.$$

$$-\frac{30075\pi^2\nu^3}{1024}-\frac{5749\nu^4}{96}\Bigg)+p^6\Bigg(-\frac{161}{32}+\frac{11206267\nu}{141120}-\frac{7719\pi^2\nu}{4096}+\frac{3605263\nu^2}{29400}$$
$$+\frac{29987\pi^2\nu^2}{4096}+\frac{108551131\nu^3}{4233600}-\frac{20259\pi^2\nu^3}{1024}-\frac{593\nu^4}{32}\Bigg)+p^4(p.n)^2\Bigg(\frac{21}{16}-\frac{162949463\nu}{235200}$$
$$+\frac{58887\pi^2\nu}{4096}-\frac{1945067\nu^2}{2450}-\frac{172311\pi^2\nu^2}{4096}-\frac{2369976949\nu^3}{1411200}+\frac{106947\pi^2\nu^3}{1024}+\frac{549\nu^4}{32}\Bigg)$$
$$+p^4(p.n)^4\Bigg(-\frac{132847139\nu}{141120}+\frac{3302175\pi^2\nu}{65536}-\frac{22546873057\nu^2}{1693440}-\frac{413055\pi^2\nu^2}{4096}$$
$$+\frac{17629672339\nu^3}{1128960}+\frac{1825635\pi^2\nu^3}{4096}-\frac{47772068147\nu^4}{846720}-\frac{12653055\pi^2\nu^4}{8192}-\frac{1431581\nu^5}{768}\Bigg)$$
$$+(p.n)^8\Bigg(\frac{4756417\nu}{26880}+\frac{2925405\pi^2\nu}{131072}-\frac{192771863\nu^2}{40320}+\frac{23625\pi^2\nu^2}{4096}+\frac{23044829\nu^3}{10752}$$
$$-\frac{784665\pi^2\nu^3}{4096}-\frac{1282851793\nu^4}{32256}-\frac{9135\pi^2\nu^4}{32}-\frac{1816323\nu^5}{2560}\Bigg)+p^8\Bigg(\frac{1605}{256}-\frac{18459883\nu}{268800}$$
$$+\frac{310029\pi^2\nu}{131072}+\frac{1275787309\nu^2}{3175200}-\frac{23771\pi^2\nu^2}{2048}-\frac{38805398273\nu^3}{25401600}+\frac{73075\pi^2\nu^3}{2048}$$
$$+\frac{174300143\nu^4}{268800}-\frac{446081\pi^2\nu^4}{8192}-\frac{7797\nu^5}{512}\Bigg)+p^6(p.n)^2\Bigg(-\frac{45}{32}-\frac{204352217\nu}{2822400}$$
$$-\frac{685233\pi^2\nu}{32768}+\frac{11064793483\nu^2}{2116800}+\frac{291459\pi^2\nu^2}{4096}-\frac{6068679041\nu^3}{1693440}-\frac{1182561\pi^2\nu^3}{4096}$$
$$+\frac{26379298087\nu^4}{2822400}+\frac{5051661\pi^2\nu^4}{8192}+\frac{54337\nu^5}{128}\Bigg)+p^2(p.n)^6\Bigg(\frac{257519\nu}{336}-\frac{1824025\pi^2\nu}{32768}$$
$$+\frac{8649135391\nu^2}{725760}+\frac{212625\pi^2\nu^2}{4096}-\frac{81220205\nu^3}{8064}-\frac{209335\pi^2\nu^3}{4096}+\frac{8667605527\nu^4}{103680}$$
$$+\frac{10868515\pi^2\nu^4}{8192}+\frac{1100997\nu^5}{640}\Bigg)\Bigg]$$
$$+\frac{1}{r^4}\Bigg[\frac{3}{8}-\frac{1279\nu}{72}+\frac{15\pi^2\nu}{64}+(p.n)^2\Bigg(-\frac{11}{4}+\frac{73801\nu}{1600}-\frac{4429\pi^2\nu}{192}+\frac{953891\nu^2}{7200}$$
$$-\frac{4033\pi^2\nu^2}{128}\Bigg)+p^2\Bigg(\frac{95}{16}+\frac{115733\nu}{2880}+\frac{643\pi^2\nu}{128}-\frac{1223723\nu^2}{7200}+\frac{1419\pi^2\nu^2}{128}\Bigg)$$
$$+(p.n)^4\Bigg(-\frac{1}{8}+\frac{1895797259\nu}{235200}-\frac{3293913\pi^2\nu}{4096}-\frac{1742633989\nu^2}{117600}+\frac{2617363\pi^2\nu^2}{4096}$$
$$+\frac{14035555739\nu^3}{705600}-\frac{361499\pi^2\nu^3}{1024}\Bigg)+p^4\Bigg(-\frac{499}{64}+\frac{2128837091\nu}{1411200}-\frac{1328147\pi^2\nu}{12288}$$
$$-\frac{420686323\nu^2}{132300}+\frac{2076041\pi^2\nu^2}{12288}+\frac{617770201\nu^3}{423360}+\frac{98447\pi^2\nu^3}{3072}\Bigg)+p^2(p.n)^2\Bigg(\frac{29}{8}$$

$$-\frac{2385014243\nu}{282240}+\frac{5042575\pi^2\nu}{6144}+\frac{35606467999\nu^2}{2116800}-\frac{5962205\pi^2\nu^2}{6144}-\frac{3656476457\nu^3}{235200}$$
$$+\frac{131231\pi^2\nu^3}{1536}\Bigg)+p^2(p.n)^4\Bigg(\frac{3}{16}+\frac{175079560811\nu}{940800}-\frac{93462353\pi^2\nu}{8192}-\frac{3559323922849\nu^2}{2822400}$$
$$+\frac{166850287\pi^2\nu^2}{4096}+\frac{1804730974343\nu^3}{1881600}+\frac{219605317\pi^2\nu^3}{8192}+\frac{906978233137\nu^4}{1693440}$$
$$-\frac{631940135\pi^2\nu^4}{8192}\Bigg)+p^6\Bigg(\frac{1567}{128}+\frac{331187219953\nu}{50803200}-\frac{9597775\pi^2\nu}{24576}-\frac{242295730217\nu^2}{8467200}$$
$$+\frac{343433\pi^2\nu^2}{1536}-\frac{10130224103\nu^3}{10160640}+\frac{72402467\pi^2\nu^3}{24576}-\frac{30642112157\nu^4}{4233600}+\frac{22396811\pi^2\nu^4}{24576}\Bigg)$$
$$+p^4(p.n)^2\Bigg(-\frac{165}{32}-\frac{1488934043759\nu}{16934400}+\frac{32454227\pi^2\nu}{6144}+\frac{135138293977\nu^2}{282240}$$
$$-\frac{266286257\pi^2\nu^2}{24576}-\frac{574199075573\nu^3}{3386880}-\frac{110076807\pi^2\nu^3}{4096}-\frac{1067026232959\nu^4}{8467200}$$
$$+\frac{41375245\pi^2\nu^4}{3072}\Bigg)+(p.n)^6\Bigg(-\frac{313940389879\nu}{3175200}+\frac{78229555\pi^2\nu}{12288}+\frac{332644084181\nu^2}{403200}$$
$$-\frac{274433707\pi^2\nu^2}{8192}-\frac{44775952005941\nu^3}{50803200}+\frac{17322247\pi^2\nu^3}{3072}-\frac{5330869608031\nu^4}{12700800}$$
$$+\frac{846593393\pi^2\nu^4}{12288}\Bigg)\Bigg]$$
$$+\frac{1}{\varepsilon}\Bigg[\frac{1}{r^3}\Bigg[-\frac{17p^2\nu}{6}+\frac{17(p.n)^2\nu}{2}+\frac{5}{3}(p.n)^4\nu(12+37\nu)-\frac{11}{60}p^2(p.n)^2\nu(195+133\nu)$$
$$-\frac{1}{180}p^4\nu(-1425+757\nu)-\frac{1}{56}p^2(p.n)^4\nu\Bigg(4536+7280\nu+18979\nu^2\Bigg)+p^6\Bigg(\frac{1173\nu}{80}$$
$$-\frac{13583\nu^2}{336}-\frac{6889\nu^3}{360}\Bigg)+p^4(p.n)^2\Bigg(-\frac{2271\nu}{80}+\frac{23047\nu^2}{112}+\frac{16538\nu^3}{105}\Bigg)$$
$$+(p.n)^6\Bigg(77\nu-\frac{91\nu^2}{6}+\frac{2891\nu^3}{12}\Bigg)+p^4(p.n)^4\Bigg(\frac{8469\nu}{56}+\frac{80489\nu^2}{168}+\frac{243865\nu^3}{672}$$
$$-\frac{25805\nu^4}{28}\Bigg)+(p.n)^8\Bigg(\frac{333\nu}{2}+106\nu^2+290\nu^3-\frac{1789\nu^4}{2}\Bigg)+p^8\Bigg(\frac{23587\nu}{672}$$
$$-\frac{2158799\nu^2}{15120}+\frac{823993\nu^3}{3780}-\frac{758113\nu^4}{6048}\Bigg)+p^6(p.n)^2\Bigg(-\frac{149963\nu}{1120}+\frac{1677317\nu^2}{5040}$$
$$-\frac{1954571\nu^3}{2520}+\frac{607031\nu^4}{1008}\Bigg)+p^2(p.n)^6\Bigg(-\frac{549\nu}{2}-\frac{9545\nu^2}{18}-\frac{129523\nu^3}{288}+\frac{419977\nu^4}{288}\Bigg)\Bigg]$$
$$+\frac{1}{r^4}\Bigg[\frac{17\nu}{6}-\frac{1}{180}p^2\nu(-1215+827\nu)+\frac{1}{180}(p.n)^2\nu(-3354+10685\nu)+(p.n)^4\Bigg(-\frac{13059\nu}{70}$$

$$+\frac{16223\nu^2}{28}-\frac{304669\nu^3}{240}\Bigg)+p^4\Bigg(-\frac{49023\nu}{560}+\frac{592957\nu^2}{2520}-\frac{297509\nu^3}{2520}\Bigg)+p^2(p.n)^2\Bigg(\frac{25169\nu}{105}$$
$$-\frac{1895597\nu^2}{2520}+\frac{352834\nu^3}{315}\Bigg)+(p.n)^6\Bigg(\frac{952841\nu}{630}-\frac{7199557\nu^2}{210}+\frac{76415687\nu^3}{1260}$$
$$-\frac{214306523\nu^4}{10080}\Bigg)+p^4(p.n)^2\Bigg(\frac{1324003\nu}{560}-\frac{98849389\nu^2}{3360}+\frac{102404527\nu^3}{3360}$$
$$-\frac{2476553\nu^4}{1120}\Bigg)+p^6\Bigg(-\frac{652717\nu}{2016}+\frac{19320373\nu^2}{7560}-\frac{19664101\nu^3}{15120}-\frac{18212479\nu^4}{30240}\Bigg)$$
$$+p^2(p.n)^4\Bigg(-\frac{448223\nu}{105}+\frac{13413467\nu^2}{210}-\frac{155079209\nu^3}{1680}+\frac{10165237\nu^4}{420}\Bigg)\Bigg]\Bigg]$$
$$+\ln\left(\frac{r}{r_0}\right)\Bigg[\frac{1}{r^3}\Bigg[-17p^2\nu+51(p.n)^2\nu+10(p.n)^4\nu(12+37\nu)-\frac{11}{10}p^2(p.n)^2\nu(195+133\nu)$$
$$-\frac{1}{30}p^4\nu(-1425+757\nu)+\frac{7}{2}(p.n)^6\nu\Bigg(132-26\nu+413\nu^2\Bigg)-\frac{3}{28}p^2(p.n)^4\nu\Bigg(4536+7280\nu$$
$$+18979\nu^2\Bigg)+p^6\Bigg(\frac{3519\nu}{40}-\frac{13583\nu^2}{56}-\frac{6889\nu^3}{60}\Bigg)+p^4(p.n)^2\Bigg(-\frac{6813\nu}{40}+\frac{69141\nu^2}{56}$$
$$+\frac{33076\nu^3}{35}\Bigg)+p^4(p.n)^4\Bigg(\frac{25407\nu}{28}+\frac{80489\nu^2}{28}+\frac{243865\nu^3}{112}-\frac{77415\nu^4}{14}\Bigg)$$
$$+(p.n)^8\Bigg(999\nu+636\nu^2+1740\nu^3-5367\nu^4\Bigg)+p^8\Bigg(\frac{23587\nu}{112}-\frac{2158799\nu^2}{2520}+\frac{823993\nu^3}{630}$$
$$-\frac{758113\nu^4}{1008}\Bigg)+p^6(p.n)^2\Bigg(-\frac{449889\nu}{560}+\frac{1677317\nu^2}{840}-\frac{1954571\nu^3}{420}+\frac{607031\nu^4}{168}\Bigg)$$
$$+p^2(p.n)^6\Bigg(-1647\nu-\frac{9545\nu^2}{3}-\frac{129523\nu^3}{48}+\frac{419977\nu^4}{48}\Bigg)\Bigg]$$
$$+\frac{1}{r^4}\Bigg[\frac{68\nu}{3}-\frac{2}{45}p^2\nu(-1215+827\nu)+\frac{2}{45}(p.n)^2\nu(-3354+10685\nu)+(p.n)^4\Bigg(-\frac{52236\nu}{35}$$
$$+\frac{32446\nu^2}{7}-\frac{304669\nu^3}{30}\Bigg)+p^4\Bigg(-\frac{49023\nu}{70}+\frac{592957\nu^2}{315}-\frac{297509\nu^3}{315}\Bigg)$$
$$+p^2(p.n)^2\Bigg(\frac{201352\nu}{105}-\frac{1895597\nu^2}{315}+\frac{2822672\nu^3}{315}\Bigg)+(p.n)^6\Bigg(\frac{3811364\nu}{315}-\frac{28798228\nu^2}{105}$$
$$+\frac{152831374\nu^3}{315}-\frac{214306523\nu^4}{1260}\Bigg)+p^4(p.n)^2\Bigg(\frac{1324003\nu}{70}-\frac{98849389\nu^2}{420}+\frac{102404527\nu^3}{420}$$
$$-\frac{2476553\nu^4}{140}\Bigg)+p^6\Bigg(-\frac{652717\nu}{252}+\frac{19320373\nu^2}{945}-\frac{19664101\nu^3}{1890}-\frac{18212479\nu^4}{3780}\Bigg)$$

$$+p^2(p.n)^4\left(-\frac{3585784\nu}{105}+\frac{53653868\nu^2}{105}-\frac{155079209\nu^3}{210}+\frac{20330474\nu^4}{105}\right)\Bigg]\Bigg], \tag{1}$$

working in cms coordinates and using the rescaling defined in [9], Eq. (7). In dot-products the vectors are 3–vectors. Otherwise the same symbol denotes their modulus and $p.n = p.r/r$. The implicit counting of the powers in $\eta^2 \equiv 1/c^2$ is defined in [6], Eq. (54).

In the limit $\nu \to 0$ we agree with the Schwarzschild solution in harmonic coordinates [22].

To compare our result with the post–Newtonian expansion of the result of Ref. [1] to 6PN we perform the following canonical transformation

$$\begin{aligned}\bar{H} &= H + \{H,g\} + \frac{1}{2!}\{\{H,g\},g\} + \frac{1}{3!}\{\{\{H,g\},g\},g\} \\ &\quad + \frac{1}{4!}\{\{\{\{H,g\},g\},g\},g\} + \frac{1}{5!}\{\{\{\{\{H,g\},g\},g\},g\},g\} \\ &\quad + \frac{1}{6!}\{\{\{\{\{\{H,g\},g\},g\},g\},g\},g\}\,,\end{aligned} \tag{2}$$

where $\{\cdot,\cdot\}$ denotes the Lie bracket and H and $\bar{H}$ are the Hamiltonians for which the transformation is performed. The corresponding expressions have to be expand to the respective post-Newtonian order. In the logarithmic terms the scale $r_0 = e^{-\gamma_E/2}/(2\sqrt{\pi}\mu_1)$ appears. Here γ_E denotes the Euler–Mascheroni constant and μ_1 is the rescaled mass scale appearing in G_N in D dimensions.

The function g inducing the canonical transformation is in general given by

$$g = p.r \sum_{i=-1}^{0} \sum_{j,k,l=0} \sum_{m=0}^{1} \alpha_{ijklm}\, \varepsilon^i\, r^{-j}\, p^{2k}\, (p.n)^{2l} \ln^m(r/r_0)\,, \tag{3}$$

with ν-dependent coefficients α_{ijklm}. Using this ansatz and evaluating Eq. (2) the corresponding explicit transformation can be found. The generating function, $g_{\rm harm}^{\rm isotr}$, is given in Appendix A, Eq. (4), mapping our result to that of [1]. The potential contributions to the scattering angle have been already found to be the same up to 5PN, cf. [1], referring to the Hamiltonian derived in [9]. Here we proved that this applies to the potential contributions to all observables to 6PN.

Next we compare to the part of the local contributions in Ref. [2], Eq. (7.29) to $O(G_N^4)$ and the lower order terms in G_N, which stem from the potential terms. These are all contributions with the exception of the purely rational terms of order ν^1, ν^2 and ν^3, which contain also local tail contributions [11, 23]. We determine the generating function $\tilde{g}_{\rm harm}^{\rm EOB}$, given in Appendix A, Eq. (5). Again we find full agreement in all these terms, which also will imply agreement for the corresponding contributions to the scattering angle. In this way a thorough test of all results up to 6PN stemming from the potential contributions to $O(G_N^4)$ has been be obtained.

Acknowledgment. We thank Z. Bern for providing a computer-readable version of the Hamiltonian obtained in [1]. This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska–Curie grant agreement No. 764850, SAGEX and KMP Berlin.

A The generators of the canonical transformations

The generator of the canonical transformation from harmonic coordinates to the isotropic coordinates used in [1] is given by

$g_{\rm harm}^{\rm isotr} =$

$$p.r\Bigg\{\frac{1}{r}\Bigg[\frac{\nu}{2}-\frac{1}{8}p^2\nu^2+\frac{(p.n)^2\nu^2}{8}-\frac{1}{16}p^4\nu^3-\frac{1}{48}p^2(p.n)^2\nu^3+\frac{(p.n)^4\nu^3}{16}+\frac{5(p.n)^6\nu^4}{128}$$
$$+\frac{7(p.n)^8\nu^5}{256}+\frac{21(p.n)^{10}\nu^6}{1024}-\frac{5}{384}p^4(p.n)^2\nu^3(4+\nu)-\frac{1}{128}p^2(p.n)^4\nu^3(4+\nu)$$
$$-\frac{1}{128}p^6\nu^3(-96+5\nu)+p^8\left(\frac{91\nu^3}{128}-\frac{97\nu^4}{64}-\frac{7\nu^5}{256}\right)+p^6(p.n)^2\left(-\frac{51\nu^3}{256}+\frac{337\nu^4}{768}-\frac{7\nu^5}{768}\right)$$
$$+p^4(p.n)^4\left(\frac{3\nu^3}{256}+\frac{17\nu^4}{256}-\frac{7\nu^5}{1280}\right)+p^2(p.n)^6\left(\frac{5\nu^3}{256}-\frac{25\nu^4}{256}-\frac{\nu^5}{256}\right)+p^{10}\left(\frac{59\nu^3}{32}-\frac{4101\nu^4}{512}\right.$$
$$\left.+\frac{2133\nu^5}{256}-\frac{21\nu^6}{1024}\right)+p^8(p.n)^2\left(-\frac{221\nu^3}{512}+\frac{115\nu^4}{64}-\frac{1091\nu^5}{768}-\frac{7\nu^6}{1024}\right)+p^6(p.n)^4\left(\frac{177\nu^3}{2560}\right.$$
$$\left.-\frac{111\nu^4}{640}-\frac{1319\nu^5}{5120}-\frac{21\nu^6}{5120}\right)+p^4(p.n)^6\left(\frac{11\nu^3}{512}-\frac{23\nu^4}{128}+\frac{463\nu^5}{1024}-\frac{3\nu^6}{1024}\right)$$
$$+p^2(p.n)^8\left(-\frac{7\nu^3}{512}+\frac{49\nu^4}{512}-\frac{49\nu^5}{256}-\frac{7\nu^6}{3072}\right)\Bigg]$$
$$+\frac{1}{r^2}\Bigg[\frac{1}{4}-\frac{3\nu}{4}+\frac{\nu^2}{4}+p^2\left(\frac{9\nu}{4}-\frac{5\nu^2}{8}+\frac{\nu^3}{16}\right)+(p.n)^2\left(-\frac{7\nu}{12}-\frac{11\nu^2}{2}+\frac{5\nu^3}{48}\right)$$
$$+p^2(p.n)^2\left(-\frac{79\nu}{96}+\frac{413\nu^2}{32}+\frac{797\nu^3}{48}+\frac{\nu^4}{48}\right)$$
$$+p^4\left(\frac{309\nu}{64}-\frac{635\nu^2}{64}-\frac{35\nu^3}{4}+\frac{\nu^4}{32}\right)+(p.n)^4\left(-\frac{487\nu}{960}-\frac{181\nu^2}{64}-\frac{142\nu^3}{15}+\frac{7\nu^4}{96}\right)$$
$$+p^4(p.n)^2\left(-\frac{715\nu}{192}+\frac{31\nu^2}{6}+\frac{439\nu^3}{48}-\frac{9001\nu^4}{96}+\frac{7\nu^5}{768}\right)+p^2(p.n)^4\left(\frac{19\nu}{12}+\frac{3949\nu^2}{960}-\frac{8711\nu^3}{960}\right.$$
$$\left.+\frac{116237\nu^4}{960}+\frac{49\nu^5}{3840}\right)+p^6\left(\frac{1689\nu}{128}-\frac{5601\nu^2}{128}+\frac{475\nu^3}{16}+\frac{3543\nu^4}{128}+\frac{5\nu^5}{256}\right)$$
$$+(p.n)^6\left(-\frac{159\nu}{224}-\frac{751\nu^2}{224}+\frac{2263\nu^3}{280}-\frac{89981\nu^4}{2240}+\frac{15\nu^5}{256}\right)$$
$$+p^4(p.n)^4\left(-\frac{5021\nu}{1920}-\frac{27691\nu^2}{3840}+\frac{9845\nu^3}{256}-\frac{127735\nu^4}{1536}-\frac{140303\nu^5}{7680}+\frac{19\nu^6}{3840}\right)$$
$$+p^6(p.n)^2\left(-\frac{1943\nu}{384}+\frac{49709\nu^2}{768}-\frac{71357\nu^3}{384}+\frac{233293\nu^4}{1536}+\frac{83461\nu^5}{1536}+\frac{\nu^6}{192}\right)$$
$$+p^2(p.n)^6\left(\frac{7783\nu}{8960}+\frac{11253\nu^2}{896}-\frac{75627\nu^3}{2240}+\frac{1258391\nu^4}{17920}+\frac{181733\nu^5}{17920}+\frac{3\nu^6}{320}\right)$$
$$+p^8\left(\frac{14557\nu}{512}-\frac{90189\nu^2}{512}+\frac{21797\nu^3}{64}-\frac{97743\nu^4}{512}-\frac{2917\nu^5}{256}+\frac{7\nu^6}{512}\right)$$
$$+(p.n)^8\left(-\frac{5333\nu}{11520}-\frac{40343\nu^2}{11520}+\frac{226997\nu^3}{20160}-\frac{843847\nu^4}{32256}-\frac{76493\nu^5}{161280}+\frac{77\nu^6}{1536}\right)\Bigg]$$

$$+\frac{1}{r^3}\left[\frac{2789\nu}{144}-\frac{7\pi^2\nu}{8}+\frac{5\nu^2}{16}+\frac{\nu^3}{16}+(p.n)^2\left(\frac{34973\nu}{960}-\frac{879\pi^2\nu}{1024}+\frac{151089\nu^2}{1600}-\frac{69\pi^2\nu^2}{128}\right.\right.$$
$$\left.+\frac{239\nu^3}{192}+\frac{\nu^4}{32}\right)+p^2\left(-\frac{824117\nu}{14400}+\frac{643\pi^2\nu}{1024}+\frac{341089\nu^2}{14400}-\frac{133\pi^2\nu^2}{64}-\frac{3\nu^3}{64}+\frac{\nu^4}{16}\right)$$
$$+p^2(p.n)^2\left(-\frac{12459777\nu}{78400}+\frac{1269\pi^2\nu}{512}-\frac{25111447\nu^2}{470400}-\frac{1455\pi^2\nu^2}{256}-\frac{6705133\nu^3}{14700}\right.$$
$$\left.+\frac{8109\pi^2\nu^3}{512}-\frac{3875\nu^4}{192}+\frac{\nu^5}{64}\right)+(p.n)^4\left(\frac{15911\nu}{192}-\frac{6015\pi^2\nu}{4096}-\frac{775711\nu^2}{188160}-\frac{2025\pi^2\nu^2}{4096}\right.$$
$$\left.+\frac{5196367\nu^3}{12544}+\frac{9375\pi^2\nu^3}{1024}+\frac{5913\nu^4}{640}+\frac{47\nu^5}{1280}\right)+p^4\left(\frac{12576721\nu}{705600}-\frac{5089\pi^2\nu}{4096}+\frac{912076073\nu^2}{2822400}\right.$$
$$\left.+\frac{15153\pi^2\nu^2}{4096}+\frac{281619239\nu^3}{8467200}-\frac{15387\pi^2\nu^3}{1024}+\frac{41\nu^4}{4}+\frac{15\nu^5}{256}\right)+p^4(p.n)^2\left(-\frac{159648631\nu}{4233600}\right.$$
$$-\frac{398991\pi^2\nu}{131072}+\frac{1978082033\nu^2}{1209600}+\frac{33957\pi^2\nu^2}{8192}-\frac{55736243399\nu^3}{33868800}-\frac{356457\pi^2\nu^3}{8192}$$
$$\left.+\frac{66817696627\nu^4}{16934400}+\frac{525735\pi^2\nu^4}{4096}+\frac{1544101\nu^5}{9216}+\frac{23\nu^6}{2304}\right)+p^2(p.n)^4\left(-\frac{128052787\nu}{1693440}\right.$$
$$+\frac{621015\pi^2\nu}{131072}-\frac{263026853\nu^2}{211680}-\frac{12495\pi^2\nu^2}{2048}+\frac{1871276137\nu^3}{1354752}+\frac{277755\pi^2\nu^3}{8192}$$
$$\left.-\frac{13422417097\nu^4}{1693440}-\frac{1405305\pi^2\nu^4}{8192}-\frac{8877829\nu^5}{46080}+\frac{181\nu^6}{11520}\right)+(p.n)^6\left(-\frac{4066361\nu}{120960}\right.$$
$$-\frac{325045\pi^2\nu}{131072}+\frac{53428687\nu^2}{103680}-\frac{2625\pi^2\nu^2}{4096}-\frac{233917561\nu^3}{967680}+\frac{87185\pi^2\nu^3}{4096}+\frac{1274653087\nu^4}{290304}$$
$$\left.+\frac{1015\pi^2\nu^4}{32}+\frac{4666447\nu^5}{64512}+\frac{469\nu^6}{11520}\right)+p^6\left(\frac{2207017\nu}{17640}+\frac{161901\pi^2\nu}{131072}-\frac{11876924429\nu^2}{50803200}\right.$$
$$-\frac{32955\pi^2\nu^2}{8192}-\frac{3077092201\nu^3}{2903040}+\frac{7023\pi^2\nu^3}{512}+\frac{1347718537\nu^4}{2822400}-\frac{213837\pi^2\nu^4}{8192}-\frac{53671\nu^5}{1024}$$
$$\left.\left.+\frac{7\nu^6}{128}\right)\right]+\frac{1}{r^4}\left[\frac{1}{32}+\frac{7799\nu}{180}+\frac{50725\pi^2\nu}{12288}-\frac{62411\nu^2}{450}\right.$$
$$+\frac{1123\pi^2\nu^2}{128}+\frac{5\nu^3}{32}-\frac{\nu^4}{48}+p^2\left(\frac{120067861\nu}{117600}-\frac{811375\pi^2\nu}{8192}-\frac{5554269127\nu^2}{2116800}\right.$$
$$+\frac{3786257\pi^2\nu^2}{24576}+\frac{130885177\nu^3}{75600}+\frac{3805\pi^2\nu^3}{1536}+\frac{\nu^4}{2}-\frac{\nu^5}{96}\left.\right)+(p.n)^2\left(-\frac{62677409\nu}{50400}\right.$$
$$\left.+\frac{4233\pi^2\nu}{32}+\frac{559728613\nu^2}{211680}-\frac{649303\pi^2\nu^2}{6144}-\frac{956636399\nu^3}{302400}+\frac{34639\pi^2\nu^3}{768}-\frac{79\nu^4}{64}-\frac{5\nu^5}{576}\right)$$
$$+p^2(p.n)^2\left(-\frac{402835634819\nu}{19051200}+\frac{24771080291\pi^2\nu}{18874368}+\frac{9455970781\nu^2}{77760}-\frac{87600553\pi^2\nu^2}{24576}\right.$$

$$-\frac{2626094395199\nu^3}{38102400}-\frac{359797489\pi^2\nu^3}{73728}-\frac{233531135\nu^4}{6048}+\frac{52259255\pi^2\nu^4}{9216}+\frac{219\nu^5}{256}-\frac{11\nu^6}{1152}\Bigg)$$
$$+p^4\Bigg(\frac{18919792957\nu}{2540160}-\frac{3779149295\pi^2\nu}{8388608}-\frac{795747395203\nu^2}{25401600}+\frac{16564239\pi^2\nu^2}{32768}+\frac{1916499983\nu^3}{907200}$$
$$+\frac{257894995\pi^2\nu^3}{98304}-\frac{5258571887\nu^4}{604800}+\frac{5306093\pi^2\nu^4}{6144}-\frac{303\nu^5}{512}-\frac{\nu^6}{768}\Bigg)$$
$$+(p.n)^4\Bigg(\frac{157481938247\nu}{12700800}-\frac{2512372145\pi^2\nu}{3145728}-\frac{2514012457207\nu^2}{25401600}+\frac{34311647\pi^2\nu^2}{8192}$$
$$+\frac{2630001389207\nu^3}{25401600}-\frac{8715005\pi^2\nu^3}{12288}+\frac{525737859\nu^4}{9800}-\frac{52792535\pi^2\nu^4}{6144}+\frac{1031\nu^5}{1536}+\frac{11\nu^6}{3840}\Bigg)\Bigg]$$
$$+\frac{1}{\varepsilon}\Bigg[\frac{1}{r^3}\Bigg[-\frac{17\nu}{6}+\frac{1}{90}p^2\nu(585+4\nu)-\frac{1}{3}(p.n)^2\nu(12+37\nu)-(p.n)^4\Bigg(11\nu-\frac{13\nu^2}{6}$$
$$+\frac{413\nu^3}{12}\Bigg)+p^4\Bigg(\frac{759\nu}{40}-\frac{69887\nu^2}{1260}-\frac{10001\nu^3}{720}\Bigg)+p^2(p.n)^2\Bigg(\frac{16\nu}{5}+28\nu^2+\frac{43567\nu^3}{840}\Bigg)$$
$$+p^6\Bigg(\frac{5777\nu}{140}-\frac{780211\nu^2}{4320}+\frac{1631395\nu^3}{6048}-\frac{2975597\nu^4}{30240}\Bigg)+p^2(p.n)^4\Bigg(\frac{213\nu}{14}+\frac{6851\nu^2}{84}$$
$$+\frac{2591\nu^3}{224}-\frac{12837\nu^4}{224}\Bigg)+p^4(p.n)^2\Bigg(-\frac{1801\nu}{280}-\frac{5359\nu^2}{210}-\frac{17339\nu^3}{240}+\frac{23201\nu^4}{1120}\Bigg)$$
$$+(p.n)^6\Bigg(-\frac{37\nu}{2}-\frac{106\nu^2}{9}-\frac{290\nu^3}{9}+\frac{1789\nu^4}{18}\Bigg)\Bigg]+\frac{1}{r^4}\Bigg[-\frac{59\nu}{30}-\frac{443\nu^2}{60}+p^2\Bigg(-\frac{5347\nu}{168}$$
$$+\frac{4828\nu^2}{35}-\frac{15629\nu^3}{126}\Bigg)+(p.n)^2\Bigg(\frac{4087\nu}{140}-\frac{35153\nu^2}{252}+\frac{9067\nu^3}{42}\Bigg)+p^2(p.n)^2\Bigg(\frac{2276941\nu}{3780}$$
$$-\frac{13614733\nu^2}{1890}+\frac{10245317\nu^3}{1080}-\frac{2189263\nu^4}{1008}\Bigg)+p^4\Bigg(-\frac{2461261\nu}{10080}+\frac{23907\nu^2}{10}$$
$$-\frac{1833827\nu^3}{1260}-\frac{846893\nu^4}{1680}\Bigg)+(p.n)^4\Bigg(-\frac{258989\nu}{1260}+\frac{10783751\nu^2}{2520}-\frac{38512301\nu^3}{5040}$$
$$+\frac{616781\nu^4}{240}\Bigg)\Bigg]\Bigg]+\ln\left(\frac{r}{r_0}\right)\Bigg[\frac{1}{r^3}\Bigg[-17\nu+\frac{1}{15}p^2\nu(585+4\nu)-2(p.n)^2\nu(12+37\nu)$$
$$-\frac{1}{2}(p.n)^4\nu\Bigg(132-26\nu+413\nu^2\Bigg)+p^4\Bigg(\frac{2277\nu}{20}-\frac{69887\nu^2}{210}-\frac{10001\nu^3}{120}\Bigg)$$
$$+p^2(p.n)^2\Bigg(\frac{96\nu}{5}+168\nu^2+\frac{43567\nu^3}{140}\Bigg)+p^6\Bigg(\frac{17331\nu}{70}-\frac{780211\nu^2}{720}+\frac{1631395\nu^3}{1008}$$
$$-\frac{2975597\nu^4}{5040}\Bigg)+p^2(p.n)^4\Bigg(\frac{639\nu}{7}+\frac{6851\nu^2}{14}+\frac{7773\nu^3}{112}-\frac{38511\nu^4}{112}\Bigg)+p^4(p.n)^2\Bigg(-\frac{5403\nu}{140}$$

$$-\frac{5359\nu^2}{35}-\frac{17339\nu^3}{40}+\frac{69603\nu^4}{560}\Big)+(p.n)^6\Big(-111\nu-\frac{212\nu^2}{3}-\frac{580\nu^3}{3}+\frac{1789\nu^4}{3}\Big)\Big]$$
$$+\frac{1}{r^4}\Big(-\frac{236\nu}{15}-\frac{619\nu^2}{10}+p^2\Big(-\frac{5347\nu}{21}+\frac{39079\nu^2}{35}-\frac{34728\nu^3}{35}\Big)+(p.n)^2\Big(\frac{8174\nu}{35}$$
$$-\frac{141935\nu^2}{126}+\frac{540121\nu^3}{315}\Big)+p^2(p.n)^2\Big(\frac{4553882\nu}{945}-\frac{217954987\nu^2}{3780}+\frac{71879222\nu^3}{945}$$
$$-\frac{6209047\nu^4}{360}\Big)+p^4\Big(-\frac{2461261\nu}{1260}+\frac{767301\nu^2}{40}-\frac{29764649\nu^3}{2520}-\frac{10268647\nu^4}{2520}\Big)$$
$$+(p.n)^4\Big(-\frac{517978\nu}{315}+\frac{10779278\nu^2}{315}-\frac{154121549\nu^3}{2520}+\frac{17201353\nu^4}{840}\Big)\Big]\Big]\Big\}. \tag{4}$$

The generator of the canonical transformation to the 6PN EOB Hamiltonian reads

$$g^{\rm EOB}_{\rm harm}=$$
$$p.r\Big\{\frac{p^2\nu}{2}-\frac{p^4\nu}{8}+\frac{5p^{10}\nu^3}{96}+p^6\Big(\frac{\nu}{16}-\frac{\nu^2}{16}\Big)+p^8\Big(\frac{5\nu^2}{64}-\frac{\nu^3}{48}\Big)+p^{12}\Big(\frac{\nu^4}{96}+\frac{\nu^5}{240}\Big)$$
$$+\frac{1}{r^2}\Big\{\frac{5\nu}{4}-\frac{\nu^2}{4}+p^8\Big(-\frac{62541\nu^4}{256}-\frac{102821\nu^5}{1536}+\frac{\nu^6}{192}\Big)+\Big(\frac{4\nu}{3}-\frac{83\nu^2}{24}-\frac{\nu^3}{8}\Big)(p.n)^2$$
$$+\Big(\frac{1333\nu^2}{960}+\frac{31\nu^3}{80}+\frac{\nu^4}{48}\Big)(p.n)^4+\Big(\frac{21565\nu^3}{1344}-\frac{63677\nu^4}{2688}+\frac{13\nu^5}{128}\Big)(p.n)^6$$
$$+\Big(-\frac{2351599\nu^4}{32256}+\frac{3989339\nu^5}{32256}-\frac{9\nu^6}{1280}\Big)(p.n)^8+p^6\Big[\frac{16783\nu^3}{384}+\frac{14675\nu^4}{384}-\frac{\nu^5}{384}$$
$$+\Big(\frac{54537\nu^4}{512}+\frac{113807\nu^5}{1440}-\frac{23\nu^6}{11520}\Big)(p.n)^2\Big]+p^4\Big[-\frac{1681\nu^2}{192}-\frac{143\nu^3}{16}-\frac{\nu^4}{24}$$
$$+\Big(\frac{6157\nu^3}{384}-\frac{100025\nu^4}{1152}+\frac{13\nu^5}{576}\Big)(p.n)^2+\Big(-\frac{804613\nu^4}{7680}+\frac{1252559\nu^5}{5760}+\frac{41\nu^6}{1152}\Big)(p.n)^4\Big]$$
$$+p^2\Big[\frac{29\nu}{6}-\frac{19\nu^2}{24}+\frac{5\nu^3}{24}+\Big(\frac{817\nu^2}{96}+\frac{493\nu^3}{32}+\frac{\nu^4}{48}\Big)(p.n)^2+\Big(-\frac{793\nu^3}{20}+\frac{484729\nu^4}{5760}$$
$$-\frac{199\nu^5}{1440}\Big)(p.n)^4+\Big(\frac{7679449\nu^4}{53760}-\frac{481625\nu^5}{1344}-\frac{61\nu^6}{1920}\Big)(p.n)^6\Big]\Big\}+\frac{1}{r}\Big\{-1-\frac{\nu}{2}$$
$$+p^{10}\Big(-\frac{274391\nu^4}{30720}+\frac{118049\nu^5}{10240}\Big)+\Big(-\frac{\nu}{2}-\frac{\nu^2}{8}\Big)(p.n)^2-\frac{\nu^2(p.n)^4}{4}+\frac{5\nu^4(p.n)^6}{128}+\frac{7\nu^4(p.n)^8}{48}$$
$$-\frac{63\nu^6(p.n)^{10}}{1024}+p^8\Big[\frac{569\nu^3}{512}-\frac{1927\nu^4}{768}+\frac{\nu^5}{768}+\Big(\frac{14231\nu^4}{10240}-\frac{64907\nu^5}{92160}-\frac{\nu^6}{7680}\Big)(p.n)^2\Big]$$
$$+p^6\Big[\frac{103\nu^2}{192}+\frac{229\nu^3}{384}+\Big(-\frac{205\nu^3}{384}+\frac{145\nu^4}{768}+\frac{\nu^5}{4608}\Big)(p.n)^2+\Big(\frac{629\nu^4}{5120}-\frac{20833\nu^5}{30720}$$
$$+\frac{19\nu^6}{10240}\Big)(p.n)^4\Big]+p^4\Big[\frac{7\nu}{16}-\frac{37\nu^2}{96}-\frac{\nu^3}{48}+\Big(-\frac{7\nu^2}{96}+\frac{61\nu^3}{384}+\frac{\nu^4}{96}\Big)(p.n)^2+\Big(\frac{65\nu^3}{256}+\frac{29\nu^4}{128}$$
$$+\frac{37\nu^5}{7680}\Big)(p.n)^4+\Big(\frac{191\nu^4}{1024}+\frac{241\nu^5}{6144}-\frac{265\nu^6}{6144}\Big)(p.n)^6\Big]+p^2\Big[-\frac{5\nu}{4}+\frac{\nu^2}{4}+\Big(\frac{\nu}{4}+\frac{7\nu^2}{24}$$
$$+\frac{\nu^3}{96}\Big)(p.n)^2+\Big(\frac{\nu^2}{4}-\frac{\nu^3}{16}-\frac{7\nu^4}{128}\Big)(p.n)^4+\Big(\frac{5\nu^3}{256}-\frac{203\nu^4}{768}-\frac{7\nu^5}{1536}\Big)(p.n)^6+\Big(-\frac{203\nu^4}{768}$$

$$+\frac{35\nu^5}{384}+\frac{161\nu^6}{1536}\Big)(p.n)^8\Big]\Big\}+\frac{1}{\varepsilon}\Big\{\frac{1}{r^4}\Big[-\frac{613\nu^2}{60}-\frac{12805747p^4\nu^4}{30240}+\frac{112745\nu^3(p.n)^2}{504}$$
$$+\frac{2677891\nu^4(p.n)^4}{1440}+p^2\Big(-\frac{16671\nu^3}{140}-\frac{27275749\nu^4(p.n)^2}{15120}\Big)\Big]+\frac{1}{r^3}\Big[-\frac{17\nu}{6}-\frac{4900249p^6\nu^4}{60480}$$
$$-\frac{97\nu^2(p.n)^2}{12}-\frac{57\nu^3(p.n)^4}{8}+\frac{1471\nu^4(p.n)^6}{8}+p^4\Big(-\frac{4061\nu^3}{288}+\frac{32327\nu^4(p.n)^2}{1120}\Big)$$
$$+p^2\Big(\frac{271\nu^2}{360}+\frac{40921\nu^3(p.n)^2}{840}-\frac{45349\nu^4(p.n)^4}{224}\Big)\Big]\Big\}+\frac{1}{r^4}\Big[\frac{1187\pi^2\nu}{1024}+\frac{7\nu^3}{96}-\frac{\nu^4}{6}$$
$$+p^4\Big\{-\frac{59689843\pi^2\nu}{131072}+\frac{14986753\pi^2\nu^2}{24576}+\frac{14964907\pi^2\nu^3}{6144}-\frac{77051\nu^5}{7680}-\frac{139\nu^6}{1920}$$
$$+\nu^4\Big[-\frac{60034957159}{5644800}+\frac{5048789\pi^2}{6144}-\frac{5621293}{1680}\ln\Big(\frac{r}{r_0}\Big)\Big]\Big\}+p^2\Big(-\frac{624073\pi^2\nu}{6144}$$
$$+\frac{920315\pi^2\nu^2}{6144}-\frac{7\nu^4}{40}+\frac{107\nu^5}{480}+(p.n)^2\Big\{\frac{1555146239\pi^2\nu}{1179648}-\frac{6894431\pi^2\nu^2}{2048}-\frac{47430287\pi^2\nu^3}{9216}$$
$$+\frac{174347\nu^5}{11520}+\frac{17\nu^6}{384}+\nu^4\Big[-\frac{73238156053}{1693440}+\frac{52850153\pi^2}{9216}-\frac{7333009}{504}\ln\Big(\frac{r}{r_0}\Big)\Big]\Big\}$$
$$+\nu^3\Big[\frac{183707899}{100800}-\frac{185\pi^2}{32}-\frac{200269}{210}\ln\Big(\frac{r}{r_0}\Big)\Big]\Big)+\nu^2\Big[-\frac{902701}{7200}+\frac{263\pi^2}{32}-\frac{789}{10}\ln\Big(\frac{r}{r_0}\Big)\Big]$$
$$+(p.n)^2\Big\{\frac{2317781\pi^2\nu}{18432}-\frac{89525\pi^2\nu^2}{768}+\frac{3829\nu^4}{2880}-\frac{377\nu^5}{2880}+\nu^3\Big[-\frac{32236479}{11200}+\frac{23879\pi^2}{768}$$
$$+\frac{2263951}{1260}\ln\Big(\frac{r}{r_0}\Big)\Big]\Big\}+(p.n)^4\Big\{-\frac{1581642941\pi^2\nu}{1966080}+\frac{464633167\pi^2\nu^2}{122880}-\frac{1145483\pi^2\nu^3}{3072}$$
$$-\frac{167053\nu^5}{23040}+\frac{1403\nu^6}{11520}+\nu^4\Big[\frac{7419092299}{115200}-\frac{53579429\pi^2}{6144}+\frac{25070309}{1680}\ln\Big(\frac{r}{r_0}\Big)\Big]\Big\}\Big]$$
$$+\frac{1}{r^3}\Big[-\frac{3\nu^2}{16}-\frac{3\nu^3}{16}+p^6\Big\{\frac{161901\pi^2\nu}{131072}-\frac{20583\pi^2\nu^2}{8192}+\frac{226655\pi^2\nu^3}{24576}+\frac{981061\nu^5}{15360}-\frac{3\nu^6}{256}$$
$$+\nu^4\Big[\frac{22497180479}{33868800}-\frac{65287\pi^2}{8192}-\frac{4900249}{10080}\ln\Big(\frac{r}{r_0}\Big)\Big]\Big\}+(p.n)^2\Big\{-\frac{879\pi^2\nu}{1024}-\frac{15\nu^3}{64}-\frac{\nu^4}{8}$$
$$+\nu^2\Big[\frac{109921}{2400}+\frac{99\pi^2}{128}-\frac{97}{2}\ln\Big(\frac{r}{r_0}\Big)\Big]\Big\}+(p.n)^4\Big\{-\frac{6015\pi^2\nu}{4096}+\frac{6765\pi^2\nu^2}{4096}-\frac{7073\nu^4}{1440}$$
$$+\frac{697\nu^5}{11520}+\nu^3\Big[\frac{67938979}{564480}+\frac{9635\pi^2}{1024}-\frac{171}{4}\ln\Big(\frac{r}{r_0}\Big)\Big]\Big\}+\nu\Big[\frac{1795}{72}-\frac{7\pi^2}{8}-17\ln\Big(\frac{r}{r_0}\Big)\Big]$$
$$+(p.n)^6\Big\{-\frac{325045\pi^2\nu}{131072}+\frac{36855\pi^2\nu^2}{8192}+\frac{168035\pi^2\nu^3}{8192}-\frac{17410529\nu^5}{322560}+\frac{1489\nu^6}{7680}$$
$$+\nu^4\Big[\frac{1289317591}{414720}-\frac{3885\pi^2}{2048}+\frac{4413}{4}\ln\Big(\frac{r}{r_0}\Big)\Big]\Big\}+p^4\Big(-\frac{5089\pi^2\nu}{4096}+\frac{1681\pi^2\nu^2}{512}-\frac{7121\nu^4}{640}$$
$$-\frac{77\nu^5}{1280}+\nu^3\Big[-\frac{52415933}{1693440}-\frac{13847\pi^2}{1024}-\frac{4061}{48}\ln\Big(\frac{r}{r_0}\Big)\Big]+(p.n)^2\Big\{-\frac{398991\pi^2\nu}{131072}$$
$$+\frac{110229\pi^2\nu^2}{16384}-\frac{806643\pi^2\nu^3}{16384}-\frac{7441013\nu^5}{46080}+\frac{287\nu^6}{7680}+\nu^4\Big[\frac{17537219339}{4838400}+\frac{597629\pi^2}{4096}$$

$$+\frac{96981}{560}\ln\left(\frac{r}{r_0}\right)\Big]\Big\}\Big)+p^2\Bigg(\frac{643\pi^2\nu}{1024}-\frac{15\nu^3}{64}+\frac{5\nu^4}{24}+(p.n)^4\Big\{\frac{621015\pi^2\nu}{131072}-\frac{237105\pi^2\nu^2}{16384}$$
$$+\frac{813695\pi^2\nu^3}{16384}+\frac{7072291\nu^5}{46080}-\frac{2017\nu^6}{7680}+\nu^4\Big[-\frac{41208333631}{6773760}-\frac{1691015\pi^2}{8192}$$
$$-\frac{136047}{112}\ln\left(\frac{r}{r_0}\right)\Big]\Big\}+\nu^2\Big[\frac{48191}{3600}-\frac{119\pi^2}{64}+\frac{271}{60}\ln\left(\frac{r}{r_0}\right)\Big]+(p.n)^2\Big\{\frac{1269\pi^2\nu}{512}$$
$$-\frac{30705\pi^2\nu^2}{4096}+\frac{105479\nu^4}{5760}+\frac{23\nu^5}{720}+\nu^3\Big[-\frac{279857647}{705600}+\frac{2409\pi^2}{128}+\frac{40921}{140}\ln\left(\frac{r}{r_0}\right)\Big]\Big\}\Big)\Big]\Big\}. \quad (5)$$

Here we excluded the purely rational terms at orders ν, ν^2 and ν^3, to which local parts of the tail terms contribute.

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