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Matching the Standard Model to HQET and NRQCD

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Abstract: We find the leading electro-weak corrections to the HQET/NRQCD Lagrangian. These corrections appear in the Wilson coefficients of the two and four quark operators and are considered here up to $\mathcal{O}(1/m^3)$ at one-loop order. The two quark operators up to this order will include new CP-violating terms, which we derived analogously to the CP preserving QCD result at one-loop order.

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1 Introduction

This paper is mainly concerned with extending heavy quark effective theory (HQET), and non-relativistic QCD (NRQCD) from pure QCD to the full Standard Model (SM). Originally, EFTs were developed to take advantage of the fact that the masses of the heavy quarks (top, bottom and charm) are much larger than the remaining dynamical scales being considered. More specifically, HQET has mainly been employed to study systems with one heavy quark, [1–3]. In these studies, when considering heavy-light systems, the authors reduce the problem down to one with two dynamical scales; the heavy quark mass, m, and the rest which is chosen to be the quark confinement scale, Λ_{QCD}, the scale of all processes in pure QCD - i.e. independent of quark mass. One then constructs the HQET Lagrangian as a power series in the inverse heavy quark pole mass. One can then estimate the size of each term by assigning the scale Λ_{QCD} to every parameter present other than the heavy quark mass. One is then left with operators exhibiting two distinct structures; terms containing light degrees of freedom describing gluons and light quarks; or terms that are bi-linear in the heavy quark fields.

On the other hand, we have NRQCD which is mostly employed to study systems with a heavy quark and anti-quark, $Q\bar{Q}$, bound state [4, 5]. In NRQCD one usually takes into account two additional dynamical scales, the relative momentum, $\boldsymbol{q} \sim mv \sim \Lambda_{QCD}$, such that v is the relative velocity of the $Q\bar{Q}$ combination, and binding energy, $E \sim mv^2$, of the $Q\bar{Q}$ bound state. These extra scales add increased complexity to the power counting rules. Thus the size of each term in the NRQCD Lagrangian is no longer unique but dependent on the system under consideration. One can, however, still provide reasonable estimates of the leading size of each term is estimable with velocity counting rules [4, 6]. The difference between HQET and NRQCD is immediately clear by considering the first two bi-linear terms in the effective Lagrangian,

$$\mathcal{L} = \bar{Q}\left(iD^0 + \frac{\boldsymbol{D}^2}{2m}\right)Q. \tag{1.1}$$

To compare the two theories, one can note that the first term and second term is $\mathcal{O}(\Lambda_{QCD})$ $\mathcal{O}(\Lambda_{QCD}^2/m)$, respectively, in HQET while both terms are of order $mv^2 \sim \mathcal{O}(\Lambda_{QCD}^2/m)$. Thus one can immediately note that the heavy quark propagator in HQET is $i/(k^0 + i\epsilon)$ and in NRQCD it is $i/(k^0 - \boldsymbol{k}^2/2m + i\epsilon)$. The NRQCD Lagrangian mimics the HQET Lagrangian in that it consists of terms in a power series expansion in heavy quark mass. It contains two and four fermion operators, i.e. terms bi-linear in the heavy (anti)-quark fields and terms bi-linear in both heavy quark and anti-quark fields, respectively.

Our work is focused on calculating the primary building block of an effective theory, the EFT Lagrangian and its matching to the full theory Lagrangian. The matching process is achievable by making sure that the full theory and EFT S-matrix elements are equal. Both the NRQCD and HQET matching conditions are computed in the same way, and the Lagrangians are thus identical [7]. The parameters that are modified by the matching procedure are called the matching (or Wilson) coefficients, which factor each operator in the EFT. The matching in NRQCD is then achieved order by order in the strong coupling, α_s, and inverse heavy quark mass [8].

This study will focus on extending the NRQCD Lagrangian and considering the leading electro-weak (EW) corrections to one loop order with terms up to and including $\mathcal{O}(\alpha\alpha_s/m^3, \alpha^2/m^2)$, for the two and four fermion operators of NRQCD. Although the Wilson coefficients are known in the EFT up to $\mathcal{O}(\alpha_s^2/m^4)$, the EW corrections have not yet been considered. They must be incorporated since at leading order they start altering the matching coefficients at the same order as the higher-order QCD terms. Whence, we study the effect at leading order of incorporating the EW contributions and noticing how the matching coefficients are improved. Moreover, the Lagrangian itself must be extended to include CP-violating operators for the matching procedure to hold with the SM as CP symmetry holds for QCD but not the full SM. The utility of our efforts lies in the prolific use of heavy quark effective theories for high precision observable predictions at threshold energies which would be the primary purpose of a future collider [9]. For instance, with regards to the top quark mass determination, which is crucial for understanding the stability of the EW vacuum [10]. Many so-called threshold quark mass definitions [11–13] have arisen from the HQ EFT frameworks and we know that the EW sector plays a crucial role in determining the $\overline{\text{MS}}$ mass of the top quark [14, 15] thus it stands to reason that the same is true for the threshold mass definitions.

2 The Lagrangian

The continuum NRQCD Lagrangian up to the same order we are considering have previously been computed [7, 8] using dimensional regularisation for the IR and UV divergences taking the external states to be on shell. To express the NRQCD effective Lagrangian, one must consider heavy fermions and anti-fermions coupled to non-Abelian gauge fields. Enforcing Hermicity, parity, time-reversal and rotational invariance. One can further perform heavy field re-definitions to eliminate time derivatives acting on the heavy fermions at higher orders in $1/m$, this is known as the canonical form of the heavy particle Lagrangian [16]. Note that when employing the NRQCD Lagrangian which we define below, NRQCD has a UV cut-off, $\nu_{\text{NR}} = \{\nu_p, \nu_s\}$, where $mv \ll \nu_{\text{NR}} \ll m$, which corresponds to integrating out the hard modes of QCD to obtain NRQCD [17]. More specifically, ν_p is the UV cut-off of the relative three momentum between the heavy quark and anti-quark while ν_s is the UV cut-off of the three-momentum of the gluons and light quarks. The NRQCD Lagrangian including light fermions reads (up to field redefinitions) is [5, 18, 19],

$$\mathcal{L} = (\mathcal{L}_\psi + \mathcal{L}_\chi + \mathcal{L}_{\psi\chi}) + \mathcal{L}_g + \mathcal{L}_l, \tag{2.1}$$

such that ψ and χ are the Pauli spinors that annihilate a fermion and create an anti-fermion, respectively. We are mainly interested in the bracketed parts of the Lagrangian as these terms will attain the leading EW corrections to their matching coefficients. More explicitly, the Lagrangian for heavy quarks of masses $m_{1,2} \gg \Lambda_{qcd}$ and velocity, v, in a frame where

$v = (1, \mathbf{0})$ has bi-linear terms (up to the order we are considering) [5, 7, 20],

$$
\begin{aligned}
\mathcal{L}_{\psi,\chi} =& \psi^\dagger \Bigg\{ i c_0 D_t + c_2 \frac{\boldsymbol{D}^2}{2m} + c_4 \frac{\boldsymbol{D}^4}{8m^3} + c_{\rm F} g_s \frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2m} + c_{\rm D} g_s \frac{[\boldsymbol{D}\cdot\boldsymbol{E}]}{8m^2} + i c_{\rm S} g_s \frac{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{D})}{8m^2} \\
& + c_{\rm W1} g_s \frac{\{\boldsymbol{D}^2, \boldsymbol{\sigma}\cdot\boldsymbol{B}\}}{8m^3} - 2 c_{\rm W2} g_s \frac{\boldsymbol{D}_i \boldsymbol{\sigma}\cdot\boldsymbol{B}\boldsymbol{D}_i}{8m^3} + c_q g_s \frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}\boldsymbol{B}\cdot\boldsymbol{D} + \boldsymbol{D}\cdot\boldsymbol{B}\boldsymbol{\sigma}\cdot\boldsymbol{D}}{8m^3} \\
& + i c_{\rm M} g_s \frac{\boldsymbol{D}\cdot[\boldsymbol{D}\times\boldsymbol{B}] + [\boldsymbol{D}\times\boldsymbol{B}]\cdot\boldsymbol{D}}{8m^3} \Bigg\} \psi + (\text{h.c}, \psi \leftrightarrow \chi) + \mathcal{O}(1/m^4, g_s^2/m^3),
\end{aligned}
\tag{2.2}
$$

and four quark operators given by [21],

$$
\begin{aligned}
\mathcal{L}_{\psi\chi} =& \frac{d_{ss}}{m_1 m_2} \psi_1^\dagger \psi_1 \chi_2^\dagger \chi_2 + \frac{d_{sv}}{m_1 m_2} \psi_1^\dagger \boldsymbol{\sigma} \psi_1 \chi_2^\dagger \boldsymbol{\sigma} \chi_2 \\
& + \frac{d_{vs}}{m_1 m_2} \psi_1^\dagger T^a \psi_1 \chi_2^\dagger T^a \chi_2 + \frac{d_{vv}}{m_1 m_2} \psi_1^\dagger T^a \boldsymbol{\sigma} \psi_1 \chi_2^\dagger T^a \boldsymbol{\sigma} \chi_2,
\end{aligned}
\tag{2.3}
$$

The terms in this Lagrangian require some unpacking; the covariant derivative is $D^\mu = \partial^\mu + i g_s A_a^\mu T^a \equiv (D^0, -\boldsymbol{D})$ defined in the usual way, $iD_t = i\partial_t - g_s A_0$ and $i\boldsymbol{D} = i\boldsymbol{\partial} + g_s \boldsymbol{A}$, with combinations thereof, $\boldsymbol{B}^i = \frac{i}{2g_s}\epsilon_{ijk}[\boldsymbol{D}_j, \boldsymbol{D}_k]$ and $\boldsymbol{E} = -\frac{i}{g_s}[D_t, \boldsymbol{D}]$. Moreover, covariant derivatives in square brackets act only on the fields within the brackets. The subscripts F,S and D on the Wilson coefficients stand for Fermi, spin-orbit and Darwin, respectively. We use the common summation convention, $X^i Y^i \equiv \sum_{i=1}^3 X^i Y^i$, and define $[X, Y] \equiv XY - YX$, $\{X, Y\} \equiv XY + YX$ to denote commutators and anti-commutators, respectively. The QCD analogues of the electric and magnetic fields are defined as usual by $\boldsymbol{E} = -[\partial_t \boldsymbol{A}] - [\boldsymbol{\partial} A^0]$ and $\boldsymbol{B} = [\boldsymbol{\partial} \times \boldsymbol{A}]$. The most general term we obtained in (2.2) and (2.3) are constructed from all possible rotationally invariant, Hermitian combinations of iD_t, $\boldsymbol{D}$, $\boldsymbol{E}$, $i\boldsymbol{B}$, $i\boldsymbol{\sigma}$, with parity requiring even numbers of factors of $\boldsymbol{D}$ and $\boldsymbol{E}$.

On the other hand, the four quark operators in the Lagrangian represented by 2.3 have sub-indices, $\{1, 2\}$, which distinguishes for the case of distinct heavy quarks with unequal masses. Moreover, one can re-write these terms by applying a Fiertz transformation,

$$
\begin{aligned}
\mathcal{L}_{\psi\chi} =& \frac{d_{ss}^c}{m_1 m_2} \psi_1^\dagger \chi_2 \chi_2^\dagger \psi_1 + \frac{d_{sv}^c}{m_1 m_2} \psi_1^\dagger \boldsymbol{\sigma} \chi_2 \chi_2^\dagger \boldsymbol{\sigma} \psi_1 \\
& + \frac{d_{vs}^c}{m_1 m_2} \psi_1^\dagger T^a \chi_2 \chi_2^\dagger T^a \psi_1 + \frac{d_{vs}^c}{m_1 m_2} \psi_1^\dagger T^a \boldsymbol{\sigma} \chi_2 \chi_2^\dagger T^a \boldsymbol{\sigma} \psi_1,
\end{aligned}
\tag{2.4}
$$

where one can transform between the two bases with the relations,

$$
\begin{aligned}
d_{ss} &= -\frac{d_{ss}^c}{2N_c} - \frac{3 d_{sv}^c}{2N_c} - \frac{N_c^2 - 1}{4N_c^2} d_{vs}^c - 3\frac{N_c^2 - 1}{4N_c^2} d_{vv}^c, \\
d_{sv} &= -\frac{d_{ss}^c}{2N_c} + \frac{d_{sv}^c}{2N_c} - \frac{N_c^2 - 1}{4N_c^2} d_{vs}^c + \frac{N_c^2 - 1}{4N_c^2} d_{vv}^c, \\
d_{vs} &= -d_{ss}^c - 3 d_{sv}^c + \frac{d_{vs}^c}{2N_c} + \frac{3 d_{vv}^c}{2N_c}, \\
d_{vv} &= -d_{ss}^c + d_{sv}^c + \frac{d_{cs}^c}{2N_c} - \frac{d_{vv}^c}{2N_c}.
\end{aligned}
\tag{2.5}
$$

Both versions of $\mathcal{L}_{\psi\chi}$ are employed, the Lagrangian in 2.4 is more convenient for matching, when one is studying the equal mass case with annihilation processes. On the other hand, 2.3 is preferable when considering a bound state calculation. We employ 2.3 for matching in the unequal mass case.

3 Form Factors and Matching

Any loop diagram in an integrable QFT can be written as a function, $F(\{p\},\{m\},\mu,\epsilon)$, such that $\{p\}$ are the external momenta, $\{m\}$, the external and internal masses, μ the scale parameter in dimensional regularisation where the calculation is done in $d = 4-2\epsilon$ dimensions. Let us then consider, for instance, the radiative corrections to the quark-gluon three point vertex. This vertex can be expressed fully in terms of two form factors in QCD, $F_{1,2}(q^2)$, defined by the irreducible three point function,

$$\Gamma_3^{\rm QCD} = -ig_s T^a \bar{u}(p')\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q_\nu}{2m}\right]A^a_\mu(q)u(p), \tag{3.1}$$

where $q = p'-p$, m is the mass of the heavy quark, $\sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu,\gamma^\nu]$. We only have two form factors as $\{\gamma_\mu, \sigma^{\mu\nu}q_\nu\}$ are the only Lorentz structures that appear in QCD due to the non-chiral nature of the theory. On the other hand, if one considers Γ_3 in the full SM, two additional chiral Lorentz structures emerge, and their corresponding form factors have the following form,

$$\Gamma_3^{\rm SM} = \Gamma_3^{\rm QCD} - ig_s T^a \bar{u}(p')\left[F_3(q^2)\gamma^\mu\gamma_5 + F_4(q^2)\frac{q^\mu\gamma_5}{2m}\right]A^a_\mu(q)u(p) \tag{3.2}$$

Employing dimensional regularisation on the diagrams one finds that the form factors $F_{1,3}(q^2)$ are UV and IR divergent [7]. We can always expand our two form factors, $F_i(q^2/m^2, \mu/m, \epsilon)$, as a power series in q^2/m^2 at fixed ϵ, then take the limit $\epsilon \to 0$ to obtain an expression of the form,

$$\begin{aligned} F_i =& F_i(0)\left(\frac{A_0}{\epsilon_{\rm UV}} + \frac{B_0}{\epsilon_{\rm IR}} + (A_0+B_0)\log\frac{\mu}{m} + D_0\right) + \\ & q^2\partial_{q^2}F_i(0)\left(\frac{A_1}{\epsilon_{\rm UV}} + \frac{B_1}{\epsilon_{\rm IR}} + (A_1+B_1)\log\frac{\mu}{m} + D_1\right), \end{aligned} \tag{3.3}$$

Conventionally, we label ϵ with the subscripts, $\epsilon_{\rm UV}$ and $\epsilon_{\rm IR}$ to indicate whether the divergence is ultraviolet or infrared, respectively. UV divergences are cancelled by renormalisation counter-terms while IR divergences cancel when a physical observable is considered. The coefficients of the effective Lagrangian are determinable from the difference between the form factors in the full theory versus the effective theory of interest. More specifically, the non-analytic terms in the form factors cancel in the difference while the analytic ones determine the Wilson coefficients of the Lagrangian. By inspection of the terms in the effective Lagrangian in (2.2), all terms contain at least one power of A^μ, the gauge field. Thus all form factors at one-loop are attainable by computing the three-point on-shell scattering amplitude, which have been previously calculated [22].

To find the relationship between the full theory form factors and the Wilson coefficients for a low-momentum heavy quark scattering off a background vector potential, we expand (3.2) in the non-relativistic (NR) limit and multiply by a factor of $\sqrt{m/E}$ for both the incoming and outgoing quark. If we take $\boldsymbol{p}$ and $\boldsymbol{p'}$ to be the three-momentum of the incoming and outgoing quark, respectively, then $\boldsymbol{q} = \boldsymbol{p'} - \boldsymbol{p}$ is the transfer momentum of the background vector potential. We are then left with the following effective interaction operator,

$$- ig_s T^a u^\dagger_{\text{NR}}(\boldsymbol{p'})[A^a_0 j^0 - \boldsymbol{A}^a \cdot \boldsymbol{j}]u_{\text{NR}}(\boldsymbol{p}), \tag{3.4}$$

which can then be compared to the scattering amplitude in the effective theory Lagrangian to relate the Wilson coefficients to the form factors. We re-computed the NR expansion of (3.4) in QCD and confirmed the previous result [7, 20], i.e. we found for the time component of the current,

$$j^0 = F_1(q^2)\left\{1 - \frac{1}{8m^2}\boldsymbol{q}^2 + \frac{i}{4m^2}\boldsymbol{\sigma}\cdot(\boldsymbol{p'}\times\boldsymbol{p})\right\} + F_2(q^2)\left\{-\frac{1}{4m^2}\boldsymbol{q}^2 + \frac{1}{2m^2}\boldsymbol{\sigma}\cdot(\boldsymbol{p'}\times\boldsymbol{p})\right\} \tag{3.5}$$

and the spatial component of the current,

$$\begin{aligned}
\boldsymbol{j} =& F_1(q^2)\left\{\frac{1}{2m}(\boldsymbol{p}+\boldsymbol{p'}) + \frac{i}{2m}\boldsymbol{\sigma}\times\boldsymbol{q} - \frac{i}{8m^3}(\boldsymbol{p}^2+\boldsymbol{p'}^2)\boldsymbol{\sigma}\times\boldsymbol{q} - \frac{1}{16m^3}(\boldsymbol{p'}^2-\boldsymbol{p}^2)\boldsymbol{q}\right.\\
&\left.-\frac{i}{16m^3}(\boldsymbol{p}^2-\boldsymbol{p'}^2)\boldsymbol{\sigma}\times(\boldsymbol{p}+\boldsymbol{p'}) - \frac{1}{8m^3}(\boldsymbol{p'}^2+\boldsymbol{p}^2)(\boldsymbol{p'}+\boldsymbol{p})\right\}\\
&+F_2(q^2)\left\{\frac{i}{2m}\boldsymbol{\sigma}\times\boldsymbol{q} - \frac{i}{16m^3}\boldsymbol{q}^2\boldsymbol{\sigma}\times\boldsymbol{q} - \frac{1}{16m^3}\boldsymbol{q}^2(\boldsymbol{p}+\boldsymbol{p'}) - \frac{1}{16m^3}(\boldsymbol{p'}^2-\boldsymbol{p}^2)\boldsymbol{q}\right.\\
&\left.-\frac{i}{8m^3}(\boldsymbol{p'}^2-\boldsymbol{p}^2)\boldsymbol{\sigma}\times(\boldsymbol{p'}+\boldsymbol{p}) + \frac{i}{8m^3}\boldsymbol{\sigma}(\boldsymbol{p'}+\boldsymbol{p})(\boldsymbol{p'}\times\boldsymbol{p})\right\}.
\end{aligned} \tag{3.6}$$

This can then be compared to the relevant subset of the Hamiltonian of (2.2),

$$\begin{aligned}
\mathcal{H}_{\psi,\chi} \supset \psi^\dagger&\left\{g_s A^0 - c_2\frac{g_s}{2m}\boldsymbol{A}\cdot(\boldsymbol{p'}+\boldsymbol{p}) - ic_{\text{F}}\frac{g_s}{2m}\boldsymbol{A}\cdot(\boldsymbol{\sigma}\times\boldsymbol{q}) - c_{\text{D}}\frac{g_s}{16m^3}\boldsymbol{q}\cdot\boldsymbol{A}\right.\\
&+ic_{\text{S}}\frac{g_s}{4m^2}\boldsymbol{\sigma}\cdot(\boldsymbol{p'}\times\boldsymbol{p})A^0 + ic_{\text{S}}\frac{g_s}{16m^3}(\boldsymbol{p}'^2-\boldsymbol{p}^2)\boldsymbol{A}\cdot\boldsymbol{\sigma}\times(\boldsymbol{p'}+\boldsymbol{p})\\
&+i(c_{\text{W}_1}-c_{\text{W}_2})\frac{g_s}{8m^3}(\boldsymbol{p}'^2+\boldsymbol{p}^2)\boldsymbol{A}\cdot(\boldsymbol{\sigma}\times\boldsymbol{q}) + ic_{\text{W}_2}\frac{g_s}{8m^3}\boldsymbol{q}^2\boldsymbol{A}\cdot(\boldsymbol{\sigma}\times\boldsymbol{q})\\
&-ic_q\frac{g_s}{8m^3}\boldsymbol{\sigma}\cdot(\boldsymbol{p'}+\boldsymbol{p})\boldsymbol{A}\cdot(\boldsymbol{p'}\times\boldsymbol{p}) - c_{\text{M}}\frac{g_s}{8m^3}(\boldsymbol{p}'^2-\boldsymbol{p}^2)\boldsymbol{A}\cdot\boldsymbol{q}\\
&\left.+c_{\text{M}}\frac{g_s}{8m^3}\boldsymbol{q}^2\boldsymbol{A}\cdot(\boldsymbol{p'}+\boldsymbol{p})\right\}\psi + (\text{h.c}, \psi\leftrightarrow\chi) \qquad (3.7\text{a})\\
&\equiv g_s\psi^\dagger\{A^0 j^0 - \boldsymbol{A}\cdot\boldsymbol{j}\}\psi + (\text{h.c}, \psi\leftrightarrow\chi) \qquad (3.7\text{b})
\end{aligned}$$

and matching the Lorentz structures provides one with the following relations between the

Wilson coefficients and form factors,

$$c_0 = c_2 = c_4 = F_1, \tag{3.8a}$$
$$c_{\rm F} = F_1 + F_2, \tag{3.8b}$$
$$c_{\rm D} = F_1 + 2F_2 + 8F_1', \tag{3.8c}$$
$$c_{\rm S} = F_1 + 2F_2, \tag{3.8d}$$
$$c_{\rm W_1} = F_1 + \frac{1}{2}F_2 + 4F_1' + 4F_2', \tag{3.8e}$$
$$c_{\rm W_2} = \frac{1}{2}F_2 + 4F_1' + 4F_2', \tag{3.8f}$$
$$c_{\rm q} = F_2, \tag{3.8g}$$
$$c_{\rm M} = \frac{1}{2}F_2 + 4F_1', \tag{3.8h}$$

such that,

$$F_i = F_i(0) \quad \text{and} \quad F_i' = \left.\frac{dF_i}{d(q^2/m^2)}\right|_{q^2=0}. \tag{3.9}$$

These relations between the form factors and Wilson coefficients remain unchanged by the allowance of further interactions from the standard model. This can be seen by taking the NR limit of (3.2), the 4-current $j \mapsto j + j'$ where j' includes the new form factors and their expanded Lorentz structures, for the time component of the current one obtains,

$$\begin{aligned} j'^0 = F_3(q^2)&\left\{\frac{1}{2m}\boldsymbol{\sigma}\cdot(\boldsymbol{p'}+\boldsymbol{p}) - \frac{1}{8m^3}(\boldsymbol{\sigma}\cdot\boldsymbol{p'}\boldsymbol{p'}^2 + \boldsymbol{\sigma}\cdot\boldsymbol{p}\boldsymbol{p}^2) - \frac{1}{16m^3}\boldsymbol{\sigma}\cdot(\boldsymbol{p'}+\boldsymbol{p})(\boldsymbol{p'^2}+\boldsymbol{p^2})\right\} \\ &+ F_4(q^2)\left\{-\frac{1}{4m^3}\boldsymbol{\sigma}\cdot\boldsymbol{q}(\boldsymbol{p}'^2 - \boldsymbol{p}^2)\right\} \end{aligned} \tag{3.10}$$

and the spatial component of the current,

$$\begin{aligned} \boldsymbol{j}' = F_3(q^2)&\left\{\boldsymbol{\sigma} - \frac{1}{4m^2}\boldsymbol{\sigma}(\boldsymbol{p'}^2+\boldsymbol{p}^2) + \frac{1}{8m^2}\boldsymbol{\sigma}\boldsymbol{q}^2 + \frac{1}{4m^2}(\boldsymbol{\sigma}\cdot\boldsymbol{p}\boldsymbol{p}' + \boldsymbol{\sigma}\cdot\boldsymbol{p'}\boldsymbol{p}) - \frac{i}{4m^2}\boldsymbol{p}'\times\boldsymbol{p}\right\} \\ &+ F_4(q^2)\left\{-\frac{1}{4m^2}\boldsymbol{q}\boldsymbol{\sigma}\cdot\boldsymbol{q}\right\}. \end{aligned} \tag{3.11}$$

By comparison, one can see that $F_{3,4}$ are factors of entirely different Lorentz structures. In fact, one can count nine independent structures and thus one requires nine new linearly independent terms in the effective Lagrangian that result in the same Lorentz structures upon inspection of the Hamiltonian. Due to the fact that the SM is chiral and exhibits less symmetry than QCD there is more freedom in selecting the possible terms to include in the effective Lagrangian, we thus select a set that provides us with the correct Lorentz structures without claiming uniqueness,

$$\begin{aligned}\mathcal{L}_{\rm Ch} =&\psi^\dagger(p')\Big\{b_0 i\boldsymbol{\sigma}\cdot\boldsymbol{D} - ib_1\frac{g_s}{2m}\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{E}} + ib_2\frac{g_s}{8m^2}(\boldsymbol{D}\cdot\boldsymbol{B}+\boldsymbol{B}\cdot\boldsymbol{D})\\ &+b_3\frac{g_s}{8m^2}\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{B}+\boldsymbol{B}\times\boldsymbol{D}) + ib_4\frac{1}{2m^2}\{\boldsymbol{\sigma}\cdot\boldsymbol{\partial},\boldsymbol{D}^2\} + ib_5\frac{1}{4m^2}[\boldsymbol{D}^2\boldsymbol{\sigma}\cdot\boldsymbol{D}]\\ &+b_6\frac{g_s}{2m^2}[D_t,\boldsymbol{\sigma}\cdot\boldsymbol{E}] + ib_7\frac{g_s}{16m^3}\{\boldsymbol{D}^2,\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{E}}\} + ib_8\frac{g_s}{8m^3}\boldsymbol{D}_i\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{E}}\boldsymbol{D}_i\\ &+ib_9\frac{g_s}{8m^3}(\boldsymbol{\sigma}\cdot\boldsymbol{D}\tilde{\boldsymbol{E}}\cdot\boldsymbol{D}+\boldsymbol{D}\cdot\tilde{\boldsymbol{E}}\boldsymbol{\sigma}\cdot\boldsymbol{D})\Big\}\psi(p) + (\text{h.c}, \psi\leftrightarrow\chi) + \mathcal{O}(1/m^4, g_s^2/m^3),\end{aligned} \tag{3.12}$$

with the operator, $\tilde{\boldsymbol{E}} = -\frac{i}{g_s}\{D_t,\boldsymbol{D}\}$. Upon employing the free field Schrodinger equation (up to $\mathcal{O}(1/m)$),

$$i\frac{\partial\psi}{\partial t} + \frac{\nabla^2\psi}{2m} = 0, \quad \psi(t,\boldsymbol{x}) = e^{ip\cdot x} \Rightarrow \{\partial_t\psi = -ip^0\psi, \boldsymbol{\partial}\psi = i\boldsymbol{p}\psi\} \tag{3.13}$$

and similarly for the vector field, $A(x^\mu)$. Therefore, after Legendre transforming the Lagrangian in (3.12) to its Hamiltonian one can then match the relevant terms by inspection of Lorentz structures. This can then be compared to the chiral Hamiltonian and the Lorentz structures matched to provide the following relations between the new Wilson coefficients and form factors,

$$b_0 = b_1 = b_2 = b_4 = b_9 = -F_3, \tag{3.14a}$$
$$b_3 = F_3 + 2F_4, \tag{3.14b}$$
$$b_5 = 4F_3' + F_4, \tag{3.14c}$$
$$b_6 = -F_4, \tag{3.14d}$$
$$b_7 = 8F_3', \tag{3.14e}$$
$$b_8 = F_3 - 8F_3. \tag{3.14f}$$

Note that we have written HQET Lagrangians in the special frame, $v = (1,\boldsymbol{0})$, and the notation of [20] was employed. However, one can re-write (2.2) in an arbitrary frame as follows,

$$\begin{aligned}\mathcal{L}_v =&\bar{Q}_v\Big\{c_0 iD\cdot v - c_2\frac{D_\perp^2}{2m} + c_4\frac{D_\perp^4}{8m^3} - g_s c_{\rm F}\frac{\sigma_{\mu\nu}G^{\mu\nu}}{4m} - g_s c_{\rm D}\frac{v^\mu[D_\perp^\nu G_{\mu\nu}]}{8m^2}\\ &+ig_s c_{\rm S}\frac{v_\lambda\sigma_{\mu\nu}\{D_\perp^\mu, G^{\nu\lambda}\}}{8m^2} + g_s c_{\rm W_1}\frac{\{D_\perp^2,\sigma_{\mu\nu}G^{\mu\nu}\}}{16m^3} - g_s c_{\rm W_2}\frac{D_\perp^\lambda\sigma_{\mu\nu}G^{\mu\nu}D_{\perp\lambda}}{8m^3}\\ &+g_s c_{\rm q}\frac{\sigma^{\mu\nu}(D_\perp^\lambda G_{\lambda\mu}D_{\perp\nu} + D_{\perp\nu}G_{\lambda\mu}D_\perp^\lambda - D_\perp^\lambda G_{\mu\nu}D_{\perp\lambda})}{8m^3}\\ &-ig_s c_{\rm M}\frac{D_{\perp\mu}[D_{\perp\nu}G^{\mu\nu}] + [D_{\perp\nu}G^{\mu\nu}]D_{\perp\mu}}{8m^3}\Big\}Q_v,\end{aligned} \tag{3.15}$$

such that,

$$D_\perp^\mu = D^\mu - v^\mu v\cdot D, \tag{3.16}$$

and $\sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ and $G^{\mu\nu} = \frac{1}{ig_s}[D^\mu, D^\nu]$. We can also write the chiral Lagrangian in (3.12) in the same covariant form,

$$\begin{aligned}\mathcal{L}_v^{\rm Ch} =& \bar{Q}_v \Big\{ -2b_0\gamma_5 v^\mu \sigma_{\mu\nu} D^\nu + ib_1 \frac{1}{m}\gamma_5\{v^\mu D_\mu, v^\nu\sigma_{\nu\lambda}D^\lambda\} - b_2\frac{g_s}{4m^2}\gamma_5 v_\mu \sigma^{\mu\nu}[D^\lambda, G_{\nu\lambda}] \\ & -b_3\frac{g_s}{16m^2}\gamma_5\{\sigma_{\mu\nu},\gamma_\lambda\}\{D^\mu, G^{\nu\lambda}\} + b_4\frac{1}{m^2}\gamma_5\{v^\mu\sigma_{\mu\nu}\partial^\nu, D_\perp^2\} \\ & +b_5\frac{1}{2m^2}\gamma_5[v^\mu\sigma_{\mu\nu}D^\nu D_\perp^2] + ib_6\frac{g_s}{2m^2}\gamma_5[v^\mu D_\mu, \sigma_{\nu\lambda}G^{\nu\lambda}] \\ & +ib_7\frac{1}{8m^3}\gamma_5\{D_\perp^2, \{v^\mu D_\mu, v^\nu\sigma_{\nu\lambda}D^\lambda\}\} + ib_8\frac{1}{4m^3}\gamma_5 D_\perp^\alpha\{v^\mu D_\mu, v^\nu\sigma_{\nu\lambda}D^\lambda\}D_{\perp\alpha} \\ & +ib_9\frac{1}{4m^3}\gamma_5(v^\mu\sigma_{\mu\nu}D^\nu\{v^\lambda D_\lambda, D_\perp^\alpha\}D_{\perp\alpha} + D_{\perp\alpha}\{v^\lambda D_\lambda, D_\perp^\alpha\}v^\mu\sigma_{\mu\nu}D^\nu)\Big\} Q_v, \end{aligned} \tag{3.17}$$

in which the chirality is made explicit by the appearance of γ_5 factoring each term.

4 Two Quark Matching

The self energy contributions which contribute to the wave function renormalisation (WFR), represented in figure 1, can be split into left/right and scalar components, respectively,

$$\Sigma(p) = \Sigma_L + \Sigma_R - \Sigma_S/2 \tag{4.1}$$

$$= P_L\omega_L + P_R\omega_R - \Sigma_S/2, \tag{4.2}$$

such that $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$ are the usual left/right chiral projection operators, from this expression one can obtain the on-shell wave-function renormalisation correction,

$$\delta Z = \delta Z_L + \delta Z_R, \tag{4.3}$$

such that,

$$\delta Z_{L/R} = -\{\Sigma_{L/R} + m^2(\Sigma'_L + \Sigma'_R - 2\Sigma'_S)\}|_{q^2=m^2}, \tag{4.4}$$

and therefore,

$$\delta Z = -\{[\omega_L + \omega_R + 2m^2(\omega'_L + \omega'_R + \Sigma'_S)] - \gamma_5[\omega_R - \omega_L + 2m^2(\omega'_R - \omega'_L)]\}|_{q^2=m^2} \tag{4.5}$$

$$= \delta Z_1 + \gamma_5\delta Z_3. \tag{4.6}$$

The total on-shell form factors at one loop can then be calculated from the amplitudes

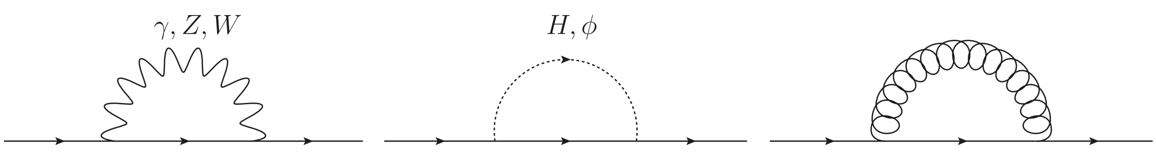

Figure 1: Self-energy diagrams contributing to the one-loop WFRC.

present in figure 2. We present the result in the large external on-shell quark mass, $m \equiv m_1$, limit and small new internal mass appearing from flavour changing, m_2, along with small transfer momentum, q,

$$F_1 = 1 - \delta Z_1 + F_1^{(a)} + F_1^{(b)} = 1 + \frac{\alpha_s}{\pi}\frac{q^2}{m_1^2}\left[\left(-\frac{1}{8} + \frac{1}{3}\log\frac{m_1}{\mu}\right)C_F + \left(-\frac{1}{16} + \frac{5}{24}\log\frac{m_1}{\mu}\right)C_A\right]$$
$$+ \frac{\alpha}{\pi}\frac{q^2}{m_1^2}\left[\frac{4}{27}\log\frac{m_{\rm Z}}{\mu} + \left(-\frac{13}{288} + \frac{17}{216}\log\frac{m_1}{m_{\rm Z}}\right)c_{\rm W} + \left(-\frac{5}{96} + \frac{3}{64}y_{\rm H_1} - \frac{1}{192}y_{\rm W_1} + \frac{1}{192}y_{\rm H}\right.\right.$$
$$\left.\left. - \frac{1}{8}y_{\rm H}\log\frac{m_1}{m_{\rm H}} + \frac{1}{24}y_{\rm W_1}\log\frac{m_2}{m_{\rm H}} + 2\log\frac{m_2}{m_1} + \log\frac{m_2}{m_{\rm Z}} - \frac{i\pi}{48}\left(1 + \frac{1}{3}y_{\rm W_1}\right)\right)s_{\rm W}\right], \tag{4.7}$$

$$F_2 = F_2^{(a)} + F_2^{(b)} = \frac{\alpha_s}{\pi}\left[\frac{1}{2}C_F + \left(\frac{1}{2} - \frac{1}{2}\log\frac{m_1}{\mu}\right)C_F\right] + \frac{\alpha_s}{\pi}\frac{q^2}{m_1^2}\left[\frac{1}{12}C_F + \left(\frac{1}{12} - \frac{1}{2}\log\frac{m_1}{\mu}\right)C_A\right]$$
$$+ \frac{\alpha}{\pi}\left[\left(\frac{35}{144} - \frac{1}{8}\log\frac{m_1}{m_{\rm Z}}\right)c_{\rm W} + \left(\frac{7}{16} + \frac{1}{16}y_{\rm W_1} - \frac{1}{8}y_{\rm H_1} - \frac{3}{8}y_{\rm H}\log\frac{m_1}{m_{\rm H}} - \frac{1}{8}\log\frac{m_1}{m_{\rm Z}}\right)s_{\rm W}\right]$$
$$+ \frac{\alpha}{\pi}\frac{q^2}{m_1^2}\left[\frac{11}{432}c_{\rm W} + \left(\frac{1}{48} + \frac{1}{96}y_{\rm W_1} - \frac{1}{32}y_{\rm H_1} - \frac{1}{32}y_{\rm H} + \frac{1}{24}y_{\rm W_1}\log\frac{m_1}{m_2} + \frac{1}{8}y_{\rm H}\log\frac{m_1}{m_{\rm H}} + \frac{i\pi}{48}y_{\rm W_1}\right)s_{\rm W}\right], \tag{4.8}$$

$$F_3 = 1 - \delta Z_3 + F_3^{(a)} + F_3^{(b)} = 1 + \frac{\alpha}{\pi}\left[\left(\frac{5}{16} - \frac{5}{24}\log\frac{m_1}{m_{\rm Z}}\right)c_{\rm W} + \left(-\frac{7}{16} + \frac{1}{16}y_{\rm W_1} + \frac{1}{4}\log\frac{m_1}{m_{\rm W}}\right.\right.$$
$$\left.\left. + \frac{1}{8}\log\frac{m_1}{m_{\rm Z}} + \frac{i\pi}{8}\right)s_{\rm W}\right] + \frac{\alpha}{\pi}\frac{q^2}{m_1^2}\left[\left(-\frac{35}{576} + \frac{5}{48}\log\frac{m_1}{m_{\rm Z}}\right)c_{\rm W} + \left(\frac{9}{64} + \frac{1}{64}y_{\rm W_1} - \frac{1}{24}y_{\rm W_1}\log\frac{m_1}{m_2}\right.\right.$$
$$\left.\left. - \frac{1}{24}\log\frac{m_2}{m_{\rm Z}} - \frac{1}{48}\log\frac{m_1}{m_{\rm Z}} + \frac{i\pi}{48}(1 - y_{\rm W_1})\right)s_{\rm W}\right], \tag{4.9}$$

$$F_4 = F_4^{(a)} + F_4^{(b)} = \frac{\alpha}{\pi}\left[\left(-\frac{35}{144} + \frac{5}{12}\log\frac{m_1}{m_{\rm Z}}\right)c_{\rm W} + \left(\frac{9}{16} + \frac{1}{16}y_{\rm W_1} - \frac{1}{6}y_{\rm W_1}\log\frac{m_1}{m_{\rm Z}} - \frac{1}{6}\log\frac{m_2}{m_{\rm Z}}\right.\right.$$
$$\left.\left. - \frac{1}{12}\log\frac{m_1}{m_{\rm Z}} + \frac{i\pi}{12}(1 - y_{\rm W_1})\right)s_{\rm W}\right] + \frac{\alpha}{\pi}\frac{q^2}{m_1^2}\left[\left(-\frac{7}{96} + \frac{5}{72}\log\frac{m_1}{m_{\rm Z}}\right)c_{\rm W} + \left(\frac{17}{96} - \frac{1}{60}y_{\rm W_1}\frac{m_1^2}{m_2^2}\right.\right.$$
$$\left.\left. + \frac{19}{480}y_{\rm W_1} + \frac{1}{60}y_{\rm W_2} + \frac{1}{60}\frac{m_1^2}{m_2^2} - \frac{7}{120}y_{\rm W_1}\log\frac{m_1}{m_2} - \frac{1}{24}\log\frac{m_1}{m_{\rm Z}} - \frac{1}{10}\log\frac{m_1}{m_2} - \frac{i\pi}{20} - \frac{7i\pi}{240}y_{\rm W_1}\right)s_{\rm W}\right], \tag{4.10}$$

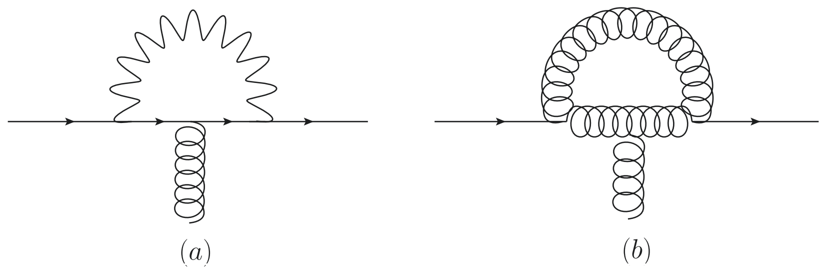

Figure 2: Diagrams that contribute to three-point matching coefficients in the SM. The Abelian and non-Abelian contributions are given by diagrams (a) and (b), respectively.

where $y_{\mathrm{W}_{1,2}} \equiv \frac{m_{1,2}^2}{m_{\mathrm{W}}^2}$, $y_{\mathrm{H}} \equiv \frac{m_{\mathrm{H}}^2}{m_{\mathrm{W}}^2}$ and $y_{\mathrm{H}_{1,2}} \equiv \frac{\pi m_{1,2} m_{\mathrm{H}}}{m_{\mathrm{W}}^2}$, we fix the Yukawa coupling to the EW coupling, α, and quark masses in the standard way [23]. We also define the square inverse of mixing angles, $s_{\mathrm{W}} = 1/\sin^2\theta_{\mathrm{W}}$ and $c_{\mathrm{W}} = 1-1/s_{\mathrm{W}}$. Moreover, we leave out IR divergences, ϵ_{IR}, to reduce the size of the expressions and they are conventionally not included in the matching coefficients. Although the form factors in the limit presented above provides an adequate approximation for $m_1 \gg m_{\mathrm{W,Z}}, m_{\mathrm{H}} \gg m_2, q^2$, in the SM the correct limit is $m_1 \sim m_{\mathrm{W,Z}}, m_{\mathrm{H}} \gg m_2, q^2$ and thus we recommend the latter for precision calculations. We leave a limit comparison to future numerical studies and the full expression with no approximations is included with an ancillary file.

5 Four Quark Matching

To achieve the matching we follow the procedure originally outlined in [8] reproducing there results and extending them. One begins by expanding the dimensionally regulated matrix elements about zero residual momentum. This expansion is done to zeroth order since there are no derivative terms in the four fermion portion of our effective Lagrangian, by inspection of (2.3) and (2.4) - i.e. we solely require the matrix elements for the four heavy quarks at rest. Diagrammatically, this means the amputated legs in a given diagram can be multiplied by a projector, P_+ and P_-, to the particle and anti-particle sub-spaces, respectively. The kinematic factor which relates the relativistic and non-relativistic expansions, $\sqrt{m/E}$ may also be set to unity WLOG.

The calculation of such matrix elements in QCD and HQET have been achieved in previous studies [8, 24, 25]. In the S-matrix elements of such heavy-heavy systems, one can see a unique IR behaviour appearing, which gives rise to the Coulomb pole and hence to the standard non-relativistic weak coupling bound states. This behaviour in the IR appears expectantly in both the effective and full theory. Expanding the dimensionally regulated matrix elements of QCD about the residual momentum, one would expect an IR singularity - reflecting the Coulomb pole - to emerge. This odd power-like IR divergence is set to zero in dimensional regularisation; the EFT has identical IR behaviour which is consistently put to zero by dimensional regularisation. Crucially, we are taking into account all the non-analytic behaviour in the heavy quark masses coming from high momenta such as in QCD logarithms, for instance.

The $\overline{\mathrm{MS}}$ scheme is employed throughout for both UV and IR divergences. As was done previously, we avoid on-shell wave-function renormalisation (WFR) and stick to $\overline{\mathrm{MS}}$ [8]. The scheme is followed to avoid identifying the UV divergences in the on-shell (OS) scheme which correspond to a WFR constant and subtracting them accordingly, this is less straightforward than employing $\overline{\mathrm{MS}}$ throughout. The price to be paid for this choice is that the heavy quark fields cease to be adequately normalised - hence one requires the proper wave function renormalisation (WFR) factor, Z, to be included when calculating the on-shell matrix elements, for instance, in QCD one has,

$$Z^{QCD} = 1 + C_F \frac{\alpha_s}{\pi}\left(\frac{3}{4}\log\frac{m^2}{\mu^2} - 1\right) + \mathcal{O}(\alpha_s^2), \quad Z^{NRQCD} = 1. \tag{5.1}$$

To be clear, Z only contribute at one loop order in the equal mass case, the amplitudes of which are illustrated in figure 4. Lastly, we note that in our calculation, the Wilson coefficients in (2.3) and (2.4) are invariant under local field re-definitions as discussed in detail previously [7].

5.1 Unequal Mass Case

In the unequal fermion mass case, annihilation diagrams do not contribute, and thus we are left with the box diagrams present in figure 3. The aforementioned Coulomb singularity and the mechanism by which it vanishes is identifiable. The upshot is that a suitable dimensionful parameter - the relative momentum of the heavy quarks - is not present in the calculation. Thus dimensional regularisation has no way to reproduce the Coulomb pole which was pointed out and discussed in detail in Refs. [8, 24].

We re-calculate the following known QCD matching coefficients in the large $m_{1,2}$ limit and confirm the result of [8],

$$d_{ss} = -C_F\left(\frac{C_A}{2} - C_F\right)\frac{\alpha_s^2}{m_1^2 - m_2^2}\left\{m_1^2\left(\log\frac{m_2^2}{\mu^2} + \frac{1}{3}\right) - m_2^2\left(\log\frac{m_1^2}{\mu^2} + \frac{1}{3}\right)\right\}, \tag{5.2a}$$

$$d_{sv} = C_F\left(\frac{C_A}{2} - C_F\right)\frac{\alpha_s^2}{m_1^2 - m_2^2}m_1 m_2\log\frac{m_1^2}{m_2^2}, \tag{5.2b}$$

$$\begin{aligned} d_{vs} = {} & \left(\frac{3}{4}C_A - 2C_F\right)\frac{\alpha_s^2}{m_1^2 - m_2^2}\left\{m_1^2\left(\log\frac{m_2^2}{\mu^2} + \frac{1}{3}\right) - m_2^2\left(\log\frac{m_1^2}{\mu^2} + \frac{1}{3}\right)\right\} \\ & + \frac{C_A\alpha_s^2}{4(m_1^2 - m_2^2)m_1 m_2}\left\{m_1^4\left(\log\frac{m_2^2}{\mu^2} + \frac{10}{3}\right) - m_2^4\left(\log\frac{m_1^2}{\mu^2} + \frac{10}{3}\right)\right\}, \end{aligned} \tag{5.2c}$$

$$\begin{aligned} d_{vv} = {} & \frac{2C_F\alpha_s^2}{m_1^2 - m_2^2}m_1 m_2\log\frac{m_1^2}{m_2^2} + \frac{C_A\alpha_s^2}{4(m_1^2 - m_2^2)}\left\{m_1^2\left(\log\frac{m_2^2}{\mu^2} + \frac{10}{3}\right)\right. \\ & \left. - m_2^2\left(\log\frac{m_1^2}{\mu^2} + 3\right) - 3m_1 m_2\log\frac{m_1^2}{m_2^2}\right\}. \end{aligned} \tag{5.2d}$$

Note that imaginary parts appear in Wilson coefficients, this occurs often and are qualitatively related to the inelastic cross sections which are unattainable with non-relativistic theory alone. Moreover, the decay width of heavy quarkonium states into light hardons are also implicated in the imaginary parts, which has been previously calculated [5], which agrees with our results. The $\mathcal{O}(\alpha\alpha_s)$ real EW corrections, which we define as d'_{ij}, to these coefficients will be presented in the following limit, $m_1 \gg m_2 \gg m_{\text{W,Z,H}}$. We choose this limit for compactness mainly but the full result up to $\mathcal{O}(\alpha^2)$ in the analogous limit to the QCD result is included as an ancillary file. We note that at $\mathcal{O}(\alpha\alpha_s)$, $d'_{ss} = d'_{sv} = 0$, and what remains to display are the following coefficients,

$$\begin{aligned} d'_{vs} = \alpha\alpha_s & \left\{\frac{1}{24}\left(1 + 3\log\frac{m_2^2}{\mu^2} + 12i\pi - 4y_{12}i\pi - y_{12}\frac{y_{\text{H}}}{36}\left[6\log\frac{m_2^2}{\mu^2} + 19\right]\right.\right. \\ & \left.\left. - y_{\text{W}_2}\left[\frac{9}{4} + 4i\pi\right] - \frac{1}{2}y_{\text{W}_1}i\pi\right)s_{\text{W}} + \frac{5}{216}\left(3\log\frac{m_2^2}{\mu^2} + 1\right)c_{\text{W}}\right\}, \end{aligned} \tag{5.3a}$$

$$\begin{aligned} d'_{vv} = \alpha\alpha_s & \left\{\frac{5}{72}\left(11 + 3\log\frac{m_2^2}{\mu^2} + \frac{y_{\text{H}}}{36}\left[6\log\frac{m_2^2}{\mu^2} + 11\right] + y_{\text{W}_2}\log\frac{m_1}{m_2}\right)s_{\text{W}}\right. \\ & \left. + \frac{5}{72}\left(3\log\frac{m_2^2}{\mu^2} + 11\right)c_{\text{W}}\right\}. \end{aligned} \tag{5.3b}$$

Where $y_{12} \equiv \frac{m_1}{m_2}$ and the rest of the EW parameters present in this expression mimic the definitions present in section 4.

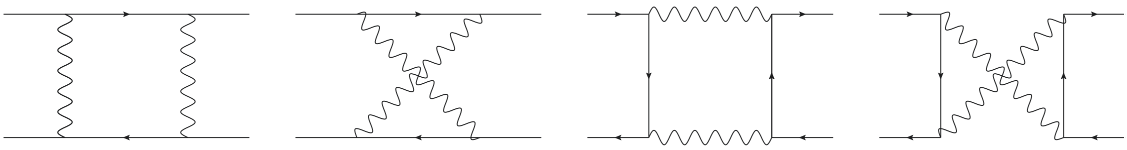

Figure 3: Relevant diagrams for the matching of the four-fermion operators at one-loop order and $\mathcal{O}(1/m^2)$ in the unequal mass case. The incoming and outcoming particles are on-shell and exactly at rest.

5.2 Equal Mass Case

When considering the equal particle case more amplitudes are involved since annihilation processes are now allowed and must be taken into account (see figure 4). The inclusion of annihilation processes, most significantly, includes, at leading order, the tree level contributions. We confirm the previously calculated matching coefficients in pure QCD,

$$d_{ss}^c = \alpha_s^2 C_F \left(\frac{C_A}{2} - C_F\right)(2 - 2\log 2 + i\pi), \tag{5.4a}$$

$$d_{sv}^c = 0, \tag{5.4b}$$

$$d_{vs}^c = \frac{\alpha_s^2}{2}\left(-\frac{3}{2}C_A + 4C_F\right)(2 - 2\log 2 + i\pi), \tag{5.4c}$$

$$\begin{aligned} d_{vv}^c = (-\pi\alpha_s)\Bigg\{1 + \frac{\alpha_s}{\pi}\left[\frac{N_f}{6}\left(\log\frac{m_1^2}{\mu^2} + 2\log 2 - \frac{5}{3}\right) - \frac{8}{9} + \frac{1}{3}\log\frac{m_1^2}{\mu^2}\right] \\ + C_A\left[-\frac{11}{12}\log\frac{m_1^2}{\mu^2} + \frac{109}{36}\right] - 4C_F\Bigg\}. \end{aligned} \tag{5.4d}$$

The $\mathcal{O}(\alpha, \alpha\alpha_s)$ EW corrections to these coefficients, defined as $d_{ij}^{c'}$, will be presented in the following limit, $m_1 \gg m_{\mathrm{W,Z,H}} \gg m_2$. This limit is again chosen for compactness but the full result up to $\mathcal{O}(\alpha^2)$ is included as an ancillary file,

$$\begin{aligned} d_{ss}^{c'} = \alpha\Bigg\{\alpha_s\left(-\frac{1}{8}C_F + \frac{3}{32}\log\frac{m_1^2}{\mu^2}\right)c_{\mathrm{W}} - \frac{1}{4}\pi y_{\mathrm{W}_1} c_{\mathrm{W}} \\ + \alpha_s\left(-\frac{1}{8}C_F + \frac{3}{32}\log\frac{m_1^2}{\mu^2} - 4C_F y_{\mathrm{W}_1} + \frac{25}{8}C_F y_{\mathrm{W}_1}\log\frac{m_1^2}{\mu^2}\right)c_{\mathrm{W}}\Bigg\}, \end{aligned} \tag{5.5a}$$

$$d_{sv}^{c'} = \alpha\alpha_s\left\{\left(\frac{1}{8}C_F + \frac{3}{32}C_F\log\frac{m_1^2}{\mu^2}\right)s_{\mathrm{W}} + \left(\frac{25}{72}C_F + \frac{25}{96}C_F\log\frac{m_1^2}{\mu^2}\right)c_{\mathrm{W}}\right\}, \tag{5.5b}$$

$$\begin{aligned} d_{vs}^{c'} = \alpha\Bigg\{\left(-\frac{\pi}{16} + \alpha_s\left[\frac{1}{4} - \frac{15}{16}C_F + \frac{19}{32}\log\frac{m_1^2}{\mu^2} - \frac{1}{4}\log 2\right] - \frac{1}{8}i\pi\right)s_{\mathrm{W}} \\ + \left(-\frac{25\pi}{144} + \alpha_s\left[\frac{25}{36} - \frac{125}{48}C_F + \frac{475}{288}\log\frac{m_1^2}{\mu^2} - \frac{25}{36}\log 2\right] - \frac{25}{72}i\pi\right)c_{\mathrm{W}} + \frac{16}{9}i\pi\alpha_s\Bigg\}, \end{aligned} \tag{5.5c}$$

$$\begin{aligned} d_{vv}^{c'} = \alpha\alpha_s\Bigg\{\left(\frac{88}{9} + \frac{44}{9}\log\frac{m_1^2}{\mu^2} - \frac{152}{9}\log\frac{m_1}{m_{\mathrm{Z}}} + \frac{8\pi m_1\cos\theta_{\mathrm{W}}}{9m_{\mathrm{W}}} - \frac{2\pi m_1}{9m_{\mathrm{W}}\cos\theta_{\mathrm{W}}}\right) \\ + \left(-\frac{361}{72} - \frac{89}{36}\log\frac{m_1}{m_{\mathrm{Z}}}\right)c_{\mathrm{W}} + \Bigg(-\frac{13}{12} - \frac{1}{6}y_{\mathrm{W}_1} + \frac{7}{8}y_{\mathrm{H}} - \frac{19}{48}y_{\mathrm{H}_1} - \frac{1}{2}\frac{m_1^2}{m_{\mathrm{H}}^2}y_{\mathrm{H}_1} + \frac{1}{3}\left(\log 2 - \frac{7}{4}i\pi\right) \\ - \frac{1}{3}y_{\mathrm{W}_1}(\log 2 + 2i\pi) + \frac{1}{12}y_{\mathrm{H}}(\log 2 + 2i\pi) + \frac{1}{16}y_{\mathrm{W}_1}\log\frac{m_1^2}{\mu^2} + y_{\mathrm{W}_1}\log\frac{m_{\mathrm{H}}}{m_{\mathrm{Z}}} - \frac{1}{2}y_{\mathrm{W}_1}\log\frac{m_1}{m_{\mathrm{Z}}}\Bigg)s_{\mathrm{W}}\Bigg\} \end{aligned} \tag{5.5d}$$

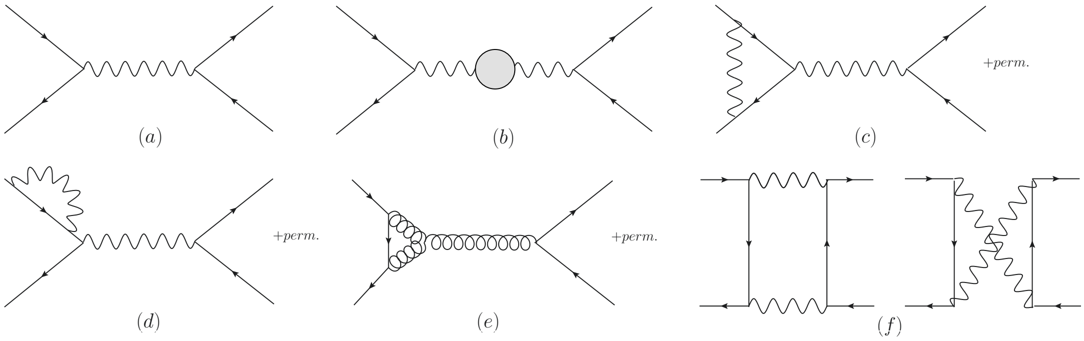

Figure 4: Relevant diagrams to the matching for four-fermion operators at one-loop order and $\mathcal{O}(1/m^2)$ in the equal mass case. The incoming and outgoing particles are on-shell and exactly at rest.

6 Discussion

To underline our discussion the full set of EW corrections to the two and four quark matching coefficients is presented in tables 1 and 2, respectively. We avoid taking any limits and plug in the latest SM parameters to compare with the known QCD result. The reason we choose the full expression up to the order we are considering is to maximise accuracy and we focus on comparing the real parts of the Wilson coefficients. For our comparison we choose for our renormalisation scale, $\mu = m_Z$, $m_1 = m_t(m_Z)$, $m_2 = m_b(m_Z)$ and the coupling, $\alpha_s = \alpha_s(m_Z)$ and the parameters were taken from the latest PDG review [26]. We will begin by considering the b_i and c_i Wilson coefficients factoring the two quark operators.

Coeff.	$c_{0,2,4}$	c_F	c_D	c_S	c_{W_1}	c_{W_2}	c_q	c_M
QCD	1	1.04	1.192	1.08	0.996	-0.004	0.04	0.076
EW corr.	0	0.0006	-0.1012	0.0012	-0.0639	-0.0629	0.0006	-0.0509
Coeff.	$b_{0,1,2}$	b_3	b_4	b_5	b_6	b_7	b_8	b_9
EW corr.	-1.002	1.001	-0.501	-0.04	0.02	-0.04	1.042	-1.002

Table 1: Three point matching coefficients with $\mu = m_Z$ and SM parameters taken from PDG.

By inspection of table 1, at the renormalisation scale we are inspecting, it is clear that the EW corrections alter the Wilson coefficients significantly. Moreover, the size of these corrections varies widely depending on the coefficient under consideration and this provides further credence to the lack of reliability of naive order of magnitude estimates. As for the new CP violating operators, they come equipped with non-negligible matching coefficients of similar order of magnitude to the ones factoring the CP preserving operators. On the other hand, the matching coefficients of the four quark operators vary even more strongly in both the QCD and EW sectors.

Coeff.	d_{ss}	d_{sv}	d_{vs}	d_{vv}	d^c_{ss}	d^c_{sv}	d^c_{vs}	d^c_{vv}
QCD	0.02	0.0004	-2.269	-0.038	0.0018	0	0.003	-0.366
EW corr.	0.093	-0.077	-0.2734	2.145	-0.134	0.002	-0.014	-0.034

Table 2: Four point matching coefficients with the equal and unequal mass cases distinguished by the superscript, c, with $\mu = m_Z$ and SM parameters taken from PDG.

If we now consider table 2, we may focus on the largest Wilson coefficients in QCD which are d_{vs} and d^c_{vv} in the unequal and equal mass cases, respectively. The EW corrections to these coefficients are an order of magnitude smaller which align well with naive estimates, i.e. $\mathcal{O}(\alpha\alpha_s)$. However, the largest EW contributions which arise in d_{vv} and d^c_{ss} are of the same order as the largest QCD coefficients and further justify the necessity of including them in precision calculations.

We end by noting that these results were achieved with the help of `Mathematica` accompanied by the package, `FeynCalc` [27], to compute the necessary amplitudes and deal with the algebra. We employed further sub-packages of `FeynCalc` such as `FeynHelpers` [28] which reduces and provides explicit expressions for one-loop scalar integrals by connecting the reduction package, `fire` [29], with the analytic scalar integrals program, `Package-X` [30]. Lastly, we employed the `FeynOnium` sub-package, which comes equipped with functions for dealing with calculations in the non-relativistic limit [31].

7 Conclusion

The matching coefficients of the NRQCD Lagrangian has been computed at one-loop up to and including terms of order $\mathcal{O}(1/m^3)$ with QCD as the full theory, confirming previous results. The Lagrangian was then extended to include the leading QCD+EW and EW corrections at one loop, of which various limits were presented and discussed. New CP violating operators were found to be necessary for the two quark terms in the effective Lagrangian, and we showed them to be frame independent. The new terms arose due to the SM being CP-violating and new Lorentz structures emerged that are not present in the non-relativistic limit of QCD; thus, the matching coefficients accompanying the CP-violating terms exhibited EW corrections purely. When studying the four quark operators, we considered both the equal and unequal external heavy quark mass cases. We rounded off by comparing all the matching coefficients for a particular renormalisation scale with and without EW corrections and found the contributions from the EW regime to be relevant. Therefore, we recommend that these contributions be included in future high precision studies that employ heavy quark effective theories.

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