DEUTSCHES ELEKTRONEN-SYNCHROTRON
Ein Forschungszentrum der Helmholtz-Gemeinschaft

DESY 21-226
arXiv:2112.11296
December 2021

Conservative Dynamics of Binary Systems at Fourth Post-Minkowskian Order in the Large-Eccentricity Expansion

C. Dlapa, G. Kälin, Z. Liu, R. A. Porto

Deutsches Elektronen-Synchrotron DESY, Hamburg

ISSN 0418–9833

NOTKESTRASSE 85 – 22607 HAMBURG

Herausgeber und Vertrieb:

Verlag Deutsches Elektronen-Synchrotron DESY

DESY Bibliothek
Notkestr. 85
22607 Hamburg
Germany

DESY 21-226

Conservative Dynamics of Binary Systems at Fourth Post-Minkowskian Order in the Large-eccentricity Expansion

Christoph Dlapa,[1] Gregor Kälin,[1] Zhengwen Liu,[1] and Rafael A. Porto[1]
[1] *Deutsches Elektronen-Synchrotron DESY, Notkestr. 85, 22607 Hamburg, Germany*

We compute the conservative dynamics of non-spinning binaries at fourth Post-Minkowskian order in the large-eccentricity limit, including both potential and radiation-reaction tail effects. This is achieved by obtaining the scattering angle in the worldline effective field theory approach and deriving the bound radial action via analytic continuation. The associated integrals are bootstrapped to all orders in velocities through differential equations, with boundary conditions in the potential and radiation regions. The large angular momentum expansion captures all the local-in-time effects as well as the trademark logarithmic corrections for generic bound orbits. Agreement is found in the overlap with the state-of-the-art in Post-Newtonian theory.

arXiv:2112.11296v2 [hep-th] 27 May 2022

Introduction. The era of gravitational wave (GW) science began in spectacular fashion with several detections already reported by the LIGO-Virgo-KAGRA collaboration [1], and many more yet to come with the future planned observatories such as LISA [2] and the Einstein Telescope [3]. Motivated by the initial breakthroughs and the expected scientific output [4–8], a community effort has been established toward constructing high-accurate waveform models for the emission of GWs from binary systems. This includes numerical simulations for the merger phase [9–11] as well as analytic techniques to tackle the inspiral in the Post-Newtonian (PN) regime, using both traditional [12, 13] and modern methodologies such as the effective field theory (EFT) approach [14–17].

These developments, in particular the use of tools from particle physics pioneered in [18], have impacted our understanding of the two-body problem in PN theory, e.g. [19–54], leading to the knowledge of the conservative spin-independent gravitational potential at fourth PN (4PN) order [32–36], in parallel with independent derivations using traditional methods [55–63]. The current state-of-the-art includes reports of contributions at 5PN [38–43], and partial results at 6PN [45, 46, 61–63].

Inspired by the EFT framework in the PN regime [14–17], novel ideas from the theory of scattering amplitudes [64], and the existence of a correspondence for observables in hyperbolic-like and elliptic-like orbits via analytic continuation in the binding energy and angular momentum—dubbed the *Boundary-to-Bound* (B2B) correspondence [65, 66]—significant efforts have been invested in recent years towards studying scattering processes within the Post-Minkowskian (PM) expansion, both with amplitude-based [65–89] and EFT-based [90–101] methodologies. The PM expansion naturally encapsulates an infinite (resummed) series of velocity corrections at each order in Newton's constant, which may result in improved waveform models, e.g. [102].

After the seminal result at third PM (3PM) order for non-spinning binary systems [75, 76, 93], partial (potential-only) corrections at 4PM have been obtained within both approaches [83, 99]. Here we extend the knowledge of the two-body dynamics at $\mathcal{O}(G^4)$, by including both potential and radiation-reaction tail effects—the latter being due to the back-scattering of the outgoing GWs on the background geometry—thus removing spurious divergent terms in previous potential-only computations. Similarly to the Lamb shift [34], yet in a classical setting, mode-factorization into potential (off-shell) and radiation (on-shell) regions led to infrared(IR)/ultraviolet(UV) divergences in PN computations [55–59], which ultimately cancel out in physical observables [32, 33, 36]. As we demonstrate here, the explicit cancelation is also manifest in the PM regime, yielding (ambiguity-free) finite results.

Our derivation proceeds through the scattering angle computed in the EFT approach [92, 93], in conjunction with the B2B map [65, 66] between unbound and bound motion extended to radiative effects [103]. Using multi-loop integration tools from particle physics [104–132], the calculation is reduced to a series of 'three-loop' (massless) integrals which are computed through differential equations. The resulting deflection angle features *logarithm*, *dilogarithm* and *complete elliptic integrals* of the first and second kind, and agrees in the overlap with the state-of-the-art in PN theory [40–43, 61, 62]. For completeness, we reconstruct the center-of-mass momentum.

The EFT formalism. Following the procedure discussed in [92, 93, 99], the effective action ($S_{\rm eff}$) is obtained by *integrating out* the metric perturbation, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, using the (classical) saddle-point approximation of the path integral, schematically

$$\int Dh\, e^{i(S_{\rm EH}+S_{\rm pp})} \to e^{iS_{\rm eff}}\,, \tag{1}$$

in Einstein's gravity ($S_{\rm EH}$) coupled to point-like worldline sources ($S_{\rm pp}$), ignoring quantum effects. The computation is reduced to a series of ('tree-level') Feynman diagrams. The full set of topologies at 4PM are shown in Fig. 1. As before [99], we must include mirror images and *iterations* from lower order solutions to the trajectories.

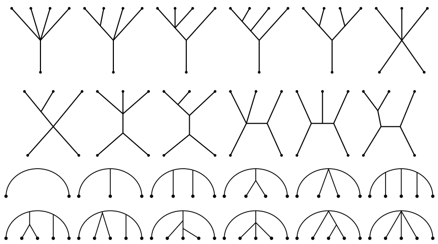

Figure 1. Feynman topologies needed to compute the deflection angle at $\mathcal{O}(G^4)$. The solid lines represent the gravitational field and the dots account for the worldline sources. The ones in the third and fourth row are *self-energy* diagrams (involving only a single particle) needed when radiation fields are included.

We restrict ourselves here to potential modes and conservative tail terms. The latter entail Feynman diagrams with only two radiation modes coupled to a background potential. In this scenario, the conservative contribution is captured by the standard Feynman rules and $i0$-prescription for the propagators of the gravitational field, i.e. $\frac{i}{p_0^2-\boldsymbol{p}^2+i0}$, as long as we consider the real part of the effective action [32, 44], ignoring dissipative effects. This allows us to retain the integration machinery intact.

From the resulting effective action we can then compute the scattering angle, χ, through the impulse, $\Delta p^\mu_{a=1,2}$, evaluated on the classical trajectory [92, 93, 99],

$$2\sin(\chi/2) = \sqrt{-\Delta p_a^2}/p_\infty\,, \tag{2}$$

with p_∞ the incoming momentum in the center-of-mass frame. As we mentioned, spurious IR/UV divergences appear due to mode factorization, e.g. [33]. Hence, we work in dimensional regularization and write the intermediate result for the PM expansion of the angle in $d = 4 - 2\epsilon$ dimensions as [99]

$$\frac{\chi}{2} = \sum_n \left(\left(4\bar{\mu}^2 b^2\right)^\epsilon \frac{GM}{b}\right)^n \chi_b^{(n)}\,, \tag{3}$$

where $b = \sqrt{-b^\mu b_\mu}$ is the impact parameter, $\chi_b^{(n)}$ the associated PM coefficients, $\bar{\mu}^2 \equiv 4\pi\mu^2 e^{\gamma_E}$ the renormalization scale (with γ_E Euler's constant), and $M = m_1 + m_2$ the total mass. As expected, the combined result is devoid of divergences or ambiguities [33, 34].

Integration. The scattering angle can be reduced to the computation of the following set of (three-loop) integrals,

$$\prod_{i=1}^{3} \int \frac{\mathrm{d}^d \ell_i}{\pi^{d/2}} \frac{\delta(\ell_i \cdot u_{a_i})}{(\pm \ell_i \cdot u_{\not{a}_i} - i0)^{n_i}} \frac{1}{\prod_{j=1}^{9} D_j^{\nu_j}}\,, \tag{4}$$

restricted by Dirac-δ functions, where $\ell_{i=1,2,3}$ are the loop momenta, n_i, ν_j are integers, $a_i \in \{1, 2\}$ ($u_{\not{1}} = u_2$, $u_{\not{2}} = u_1$), and D_j are various sets of quadratic propagators: $\{\ell_i^2 + i0, (\ell_i - q)^2 + i0, \ldots\}$. The external data obeys $q \cdot u_a = 0$ and $u_a^2 = 1$, with q the transfer momentum and $u_{a=1,2}$ the incoming velocities. Hence, after factoring out the dependence on q^2 using dimensional analysis, the result of the integrals can only be a function of $\gamma \equiv u_1 \cdot u_2$.

Following our previous derivations [93, 99], introducing the parameter x defined through $\gamma \equiv (x^2+1)/2x$ [79], the value of these integrals is obtained by the method of differential equations [105–111], with boundary conditions computed in the near-static limit $x \simeq 1$. We make extensive use of the integration-by-parts (IBP) relations [112, 113], via `FIRE6` [114, 115] and `LiteRed` [116, 117], as well as the `INITIAL` algorithm [130]. The final result for the deflection angle involves logarithms, dilogarithms ($\mathrm{Li}_2(x)$), as well as complete elliptic integrals of the first ($\mathrm{K}(x)$) and second ($\mathrm{E}(x)$) kind [83, 99]. (See [133] for more details in the integration procedure.)

Potential Region. As a check, we re-evaluated the boundary conditions for the differential equations at $x = 1$ with potential modes. As discussed in [99], these may be reduced into a basis containing only seven integrals via additional IBP identities, which we computed by direct integration. As expected, the self-energy diagrams in Fig. 1 turn into (scaleless) integrals which vanish in dimensional regularization, and therefore do not contribute in the potential region. Adding the pieces together we recover the result in [83, 99]

$$\frac{\chi^{(4)}_{b\,(\mathrm{pot})}}{\pi\Gamma} = \chi_s(x) + \nu\left(\frac{\chi_{2\epsilon}(x)}{2\epsilon} + \chi_p(x)\right)\,, \tag{5}$$

for the potential contribution to the scattering angle at $\mathcal{O}(G^4)$, where $\Gamma \equiv E/M$, with E the total energy, and $\nu = m_1 m_2/M^2$ the symmetric mass-ratio. Expressions for the $(\chi_s, \chi_{2\epsilon}, \chi_p)$ coefficients are given in [99] and the ancillary file, see also (8) and the supplemental material.

Tail Region. The boundary conditions including radiation-reaction effects is more challenging, mainly due to the interplay between potential and radiation modes. We use the *asy2.m* code in the `FIESTA` package to identify the relevant regions of integration [119, 120, 131]. We find several contributions featuring one, two and up to three radiation modes. We keep only regions consistent with conservative radiation-reaction tail effects.

After performing a Laurent expansion around $x \simeq 1$, yielding the anticipated pole in $(1-x)^{-4\epsilon}$ [99], we computed the associated boundary integrals using various consistency relations [132]. Collecting the terms we find

$$\frac{\chi^{(4)}_{b\,(\mathrm{tail})}}{\pi\Gamma} = \nu\left(-\frac{\chi_{2\epsilon}(x)}{2\epsilon}(1-x)^{-4\epsilon} + \chi_t(x)\right)\,, \tag{6}$$

for the (conservative) contribution to the deflection angle due to tail effects at 4PM. The value of $\chi_t(x)$ is given in the supplemental material and ancillary file.

Combined Result. As expected, the divergence and μ-dependence cancel out and the combined result becomes

$$\frac{\chi^{(4)}_{b\,(\mathrm{comb})}}{\pi\Gamma} = \chi_s + \nu\Big(\chi_c(x) + 2\chi_{2\epsilon}(x)\log(1-x)\Big)\,, \tag{7}$$

where

$$\begin{aligned}
\chi_s(x) &= \frac{105 h_1(x)}{128\left(x^2-1\right)^4}\,, \quad \chi_{2\epsilon}(x) = -\frac{3h_2(x)\log(x)}{32x\left(x^2-1\right)^5} + \frac{3h_3(x)\log\left(\frac{x+1}{2}\right)}{32x^2\left(x^2-1\right)^2} + \frac{h_4(x)}{64x^2\left(x^2-1\right)^4}\,,\\
\chi_c(x) &= -\frac{21h_6(x)\mathrm{E}^2\left(1-x^2\right)}{8\left(x^2-1\right)^4} + \frac{3h_7(x)\mathrm{K}\left(1-x^2\right)\mathrm{E}\left(1-x^2\right)}{8\left(x^2-1\right)^4} - \frac{15h_8(x)\mathrm{K}^2\left(1-x^2\right)}{16\left(x^2-1\right)^4} - \frac{h_{16}(x)\log\left(x^2+1\right)}{32x^3\left(x^2-1\right)^4}\\
&\quad + \frac{3h_{19}(x)\mathrm{Li}_2\left(-\frac{(x-1)^2}{(x+1)^2}\right)}{128x^4\left(x^2-1\right)^2} + \frac{\pi^2 h_{35}(x)}{512(x-1)^3x^4(x+1)^5} + \frac{3h_{36}(x)\log^2(2)}{16x^2\left(x^2-1\right)^2} + \frac{3h_{37}(x)\log(2)\log(x)}{8\left(x^2-1\right)^5} - \frac{3h_{38}(x)\log(2)\log(x+1)}{16x^2\left(x^2-1\right)^2}\\
&\quad + \frac{3h_{39}(x)\log(2)}{16x^2\left(x^2-1\right)^4} + \frac{3h_{40}(x)\log^2(x)}{256x^4\left(x^2-1\right)^8} - \frac{3h_{41}(x)\log(x)\log(x+1)}{128x^4\left(x^2-1\right)^5} + \frac{h_{42}(x)\log(x)}{64x^3\left(x^2-1\right)^7} - \frac{3h_{43}(x)\log^2(x+1)}{2x\left(x^2-1\right)^2}\\
&\quad + \frac{h_{44}(x)\log(x+1)}{32x^3\left(x^2-1\right)^4} + \frac{3h_{45}(x)\left(\mathrm{Li}_2\left(\frac{x-1}{x}\right) - \mathrm{Li}_2(-x)\right)}{128(x-1)^3x^4(x+1)^5} - \frac{3h_{46}(x)\mathrm{Li}_2\left(\frac{x-1}{x+1}\right)}{64(x-1)^2x^4} + \frac{h_{47}(x)}{384x^3\left(x^2-1\right)^6\left(x^2+1\right)^7}\,,
\end{aligned} \tag{8}$$

with the explicit value of the $h_i(x)$ polynomials displayed in the ancillary file (see also the supplemental material for the intermediate results leading to (8)). After expanding in small velocities we find perfect agreement with the PN state-of-the-art value reported in [40–43, 61, 62].[1]

Boundary-to-Bound correspondence. As it was shown in [65, 66, 103], the B2B dictionary allows us to derive PM-expanded observables for bound orbits from the scattering angle computed in a large-eccentricity (or large angular momentum) expansion. After analytic continuation in angular momentum, and to negative binding energies, we obtain

$$i_r^{4\mathrm{PM}} = \frac{2(1-\gamma^2)^2}{3(\Gamma j)^3}\left[\chi_s + \nu\left(\chi_c + \chi_{2\epsilon}\log(x-1)^2\right)\right] \tag{9}$$

for the B2B large-eccentricity approximation to the (reduced) 4PM bound radial action, with $j \equiv J/(GM^2\nu)$ the (dimensionless) angular momentum. From the radial action we can then derive all the observables for elliptic-like orbits through differentiation, including the binding energy which is one of the main ingredients needed to compute the GW phase evolution in an adiabatic approximation [12, 102], providing an infinite series of velocity corrections at $\mathcal{O}(G^4)$.

As it is known from the PN literature, e.g. [32, 61, 62], tail terms feature both local- as well as non-local-in-time dynamical effects. As it was discussed in [65, 66, 103], the expression in (9) readily captures all local-in-time contributions to gauge-invariant observables for generic bound orbits (also for aligned-spin effects). Remarkably, the same is true for the trademark (non-local) logarithmic tail corrections, which may be obtained via [99]

$$i^{4\mathrm{PM}}_{r(\log)} = \frac{2\nu}{3}\frac{(1-\gamma^2)^2}{(\Gamma j)^3}\chi_{2\epsilon}(\gamma)\log|\mathcal{E}|\,, \tag{10}$$

to all orders in velocity, with $\mathcal{E} \equiv \frac{E-M}{M\nu}$ the (reduced) binding energy. This is not the case, however, with other non-local-in-time effects for generic orbits, which do not transition smoothly between hyperbolic- and elliptic-like motion and therefore cannot be derived from the knowledge of the scattering angle [103].

From the deflection angle we can also reconstruct the 4PM coefficient, $f_4(\mathcal{E})$, of the PM expansion of an effective (local-in-time) center-of-mass momentum

$$\boldsymbol{p}^2 = p_\infty^2\left[1 + \sum_{n=1}^{\infty} f_n(\mathcal{E})\left(\frac{GM}{r}\right)^n\right]\,, \tag{11}$$

in an isotropic gauge, such that $i_r \propto \int p_r dr$, using the relationship [65, 66]

$$\begin{aligned} f_4 =& \frac{8}{3\pi}\chi_b^{(4)} - 2\chi_b^{(3)}\chi_b^{(1)} - \frac{8}{\pi^2}(\chi_b^{(2)})^2\\ &+ \frac{8}{\pi}(\chi_b^{(1)})^2\chi_b^{(2)} - \frac{2}{3}(\chi_b^{(1)})^4\,, \end{aligned} \tag{12}$$

together with previous results to 3PM [92, 93]. Explicit values are given in the ancillary file. The Hamiltonian can be also reconstructed using the algebraic relations in [65, 66]. Notice, as advertised in [99, 103], that in all cases the factors of $\log r$ in the intermediate results drop out of the final expressions.

[1] There is a mismatch at $\mathcal{O}(\nu^2)$ between the results in [40–43] and those in [61, 62], which can be traced to the definition of conservative terms in [61, 62].

Conclusions. We have computed the contribution from potential and radiation-reaction tail effects to the conservative dynamics of binary compact objects to 4PM order in the large-eccentricity limit. Our (ambiguity-free) result for the deflection angle at 4PM agrees in the overlap with state-of-the-art computations in PN theory for the combined potential and tail contributions [40–43, 61, 62]. As it was already the case in previous derivations in the EFT approach [93, 99], the PM result can be entirely bootstrapped from PN data to all orders in velocities through differential equations—at this order including a sector involving elliptic integrals—without resorting to PN resummations. The boundary conditions (in the near-static limit) were obtained via the method of regions with potential and radiation modes.

There are, however, some important caveats that need to be addressed in order to complete the knowledge of the conservative 4PM dynamics for generic orbits, notably the mapping between unbound and bound motion for all the non-local-in-time effects. Moreover, there is also the issue of conservative non-linear memory terms.[2] The latter arise from the interaction between the outgoing GW radiation and the waves emitted by the binary system at an earlier time. The derivation of memory effects at 4PM, the extension of the B2B map to generic non-local-in-time terms, as well as the computation of higher PM orders, is underway in the EFT approach.

Acknowledgements. We thank Johannes Blümlein, Ekta Chaubey, Gihyuk Cho, Stefano Foffa, Ryusuke Jinno, Francois Larrouturou, Andreas Maier, Massimiliano Riva, Henrique Rubira, Riccardo Sturani, Yang Zhang, and Simone Zoia for useful conversations. This work received support from the ERC-CoG *Precision Gravity: From the LHC to LISA* provided by the European Research Council (ERC) under the European Union's H2020 research and innovation programme (grant No. 817791). The research of C.D. was funded in part by the ERC-CoG grant *Novel structures in scattering amplitudes* (grant No. 725110).

Supplemental Material

The coefficients of the scattering angle for the potential (beyond the test-body limit) and tail contributions to 4PM order displayed in the text are given by:

$$\begin{aligned}
\chi_p(x) =& \frac{\pi^2 h_5(x)}{1024x^4\left(x^2-1\right)^5} - \frac{21h_6(x)\mathrm{E}^2\left(1-x^2\right)}{8\left(x^2-1\right)^4} + \frac{3h_7(x)\mathrm{K}\left(1-x^2\right)\mathrm{E}\left(1-x^2\right)}{8\left(x^2-1\right)^4} - \frac{15h_8(x)\mathrm{K}^2\left(1-x^2\right)}{16\left(x^2-1\right)^4} + \frac{3h_9(x)\log^2\left(\frac{x+1}{2}\right)}{16x^2\left(x^2-1\right)^2} \\
&+ \frac{3h_{10}(x)\log(2)\log(x)}{8\left(x^2-1\right)^5} + \frac{h_{11}(x)\log(2)}{128x^3\left(x^2-1\right)^4} + \frac{3h_{12}(x)\log^2(x)}{512x^4\left(x^2-1\right)^8} - \frac{3h_{13}(x)\log(x)\log(x+1)}{256x^4\left(x^2-1\right)^5} + \frac{h_{14}(x)\log(x)}{128x^3\left(x^2-1\right)^7} \\
&- \frac{h_{15}(x)\log(x+1)}{128x^3\left(x^2-1\right)^4} - \frac{h_{16}(x)\log\left(x^2+1\right)}{64x^3\left(x^2-1\right)^4} + \frac{3h_{17}(x)\mathrm{Li}_2\left(\frac{x-1}{x}\right)}{256x^4\left(x^2-1\right)^5} - \frac{3h_{18}(x)\mathrm{Li}_2(-x)}{256x^4\left(x^2-1\right)^5} + \frac{3h_{19}(x)\mathrm{Li}_2\left(-\frac{(x-1)^2}{(x+1)^2}\right)}{256x^4\left(x^2-1\right)^2} \\
&- \frac{3h_{20}(x)\mathrm{Li}_2\left(\frac{x-1}{x+1}\right)}{128x^4\left(x^2-1\right)^2} + \frac{h_{21}(x)}{1536x^3\left(x^2-1\right)^6\left(x^2+1\right)^7}\,, \\
\chi_t(x) =& -\frac{h_{16}(x)\log\left(x^2+1\right)}{64x^3\left(x^2-1\right)^4} + \frac{3h_{19}(x)\mathrm{Li}_2\left(-\frac{(x-1)^2}{(x+1)^2}\right)}{256x^4\left(x^2-1\right)^2} + \frac{h_{22}(x)\left(12\mathrm{Li}_2(-x)+\pi^2\right)}{1024x^4\left(x^2-1\right)^5} - \frac{24h_{23}(x)\log^2(2)}{\left(x^2-1\right)^2} - \frac{6h_{24}(x)\log(2)\log(x)}{\left(x^2-1\right)^5} \\
&+ \frac{3h_{25}(x)\log(2)\log(x+1)}{16x^2\left(x^2-1\right)^2} - \frac{h_{26}(x)\log(2)}{128x^3\left(x^2-1\right)^4} - \frac{3h_{27}(x)\log^2(x)}{512x^4\left(x^2-1\right)^8} + \frac{3h_{28}(x)\log(x+1)\log(x)}{256x^4\left(x^2-1\right)^5} - \frac{h_{29}(x)\log(x)}{128x^3\left(x^2-1\right)^7} \\
&- \frac{3h_{30}(x)\log^2(x+1)}{16x^2\left(x^2-1\right)^2} + \frac{h_{31}(x)\log(x+1)}{128x^3\left(x^2-1\right)^4} - \frac{3h_{32}(x)\mathrm{Li}_2\left(\frac{x-1}{x}\right)}{256x^4\left(x^2-1\right)^5} + \frac{3h_{33}(x)\mathrm{Li}_2\left(\frac{x-1}{x+1}\right)}{128x^4\left(x^2-1\right)^2} + \frac{h_{34}(x)}{1536x^3\left(x^2-1\right)^6\left(x^2+1\right)^7}\,,
\end{aligned}$$

[2] The result in [43] suggests the appearance of a conservative memory term in the overlap between 4PM and 5PN orders at $\mathcal{O}(\nu^2)$.

where the $h_i(x)$'s are polynomials in x up to degree 32, collected in the ancillary file. We use the following conventions

$$\mathrm{Li}_2(z) \equiv \int_z^0 dt\, \frac{\log(1-t)}{t}\,,$$
$$\mathrm{K}(z) \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-zt^2)}}\,,$$
$$\mathrm{E}(z) \equiv \int_0^1 dt\, \frac{\sqrt{1-zt^2}}{\sqrt{1-t^2}}\,,$$

for the dilogarithm, and complete elliptic integral of the first and second kind, respectively.

[1] R. Abbott *et al.* (LIGO Scientific, VIRGO, KAGRA), GWTC-3: Compact Binary Coalescences Observed by LIGO and Virgo During the Second Part of the Third Observing Run, (2021), arXiv:2111.03606 [gr-qc].

[2] P. Amaro-Seoane *et al.* (LISA), Laser Interferometer Space Antenna, (2017), arXiv:1702.00786 [astro-ph.IM].

[3] M. Punturo *et al.*, The einstein telescope: A third-generation gravitational wave observatory, Class. Quant. Grav. **27**, 194002 (2010).

[4] A. Buonanno and B. Sathyaprakash, Sources of Gravitational Waves: Theory and Observations, (2014), arXiv:1410.7832.

[5] R. A. Porto, The Tune of Love and the Nature(ness) of Spacetime, Fortsch. Phys. **64**, 723 (2016), arXiv:1606.08895.

[6] R. A. Porto, The Music of the Spheres: The Dawn of Gravitational Wave Science, (2017), arXiv:1703.06440 [physics.pop-ph].

[7] M. Maggiore *et al.*, Science Case for the Einstein Telescope, JCAP **03**, 050, arXiv:1912.02622 [astro-ph.CO].

[8] E. Barausse *et al.*, Prospects for Fundamental Physics with LISA, Gen. Rel. Grav. **52**, 81, arXiv:2001.09793 [gr-qc].

[9] P. Ajith *et al.*, The NINJA-2 catalog of hybrid post-Newtonian/numerical-relativity waveforms for non-precessing black-hole binaries, Class. Quant. Grav. **29**, 124001 (2012), [Erratum: Class.Quant.Grav. 30, 199401 (2013)], arXiv:1201.5319 [gr-qc].

[10] B. Szilágyi, J. Blackman, A. Buonanno, A. Taracchini, H. P. Pfeiffer, M. A. Scheel, T. Chu, L. E. Kidder, and Y. Pan, Approaching the Post-Newtonian Regime with Numerical Relativity: A Compact-Object Binary Simulation Spanning 350 Gravitational-Wave Cycles, Phys. Rev. Lett. **115**, 031102 (2015), arXiv:1502.04953 [gr-qc].

[11] T. Dietrich *et al.*, CoRe database of binary neutron star merger waveforms, Class. Quant. Grav. **35**, 24LT01 (2018), arXiv:1806.01625 [gr-qc].

[12] L. Blanchet, Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries, Living Rev.Rel. **17**, 2 (2014), arXiv:1310.1528 [gr-qc].

[13] G. Schäfer and P. Jaranowski, Hamiltonian formulation of general relativity and post-Newtonian dynamics of compact binaries, Living Rev. Rel. **21**, 7 (2018), arXiv:1805.07240.

[14] W. D. Goldberger, Les Houches lectures on effective field theories and gravitational radiation, in *Les Houches Summer School - Session 86* (2007) arXiv:hep-ph/0701129.

[15] I. Rothstein, Progress in Effective Field Theory Approach to the Binary Inspiral Problem, Gen. Rel. Grav. **46**, 1726 (2014).

[16] S. Foffa and R. Sturani, Effective field theory methods to model compact binaries, Class. Quant. Grav. **31**, 043001 (2014), arXiv:1309.3474.

[17] R. A. Porto, The effective field theorist's approach to gravitational dynamics, Phys. Rept. **633**, 1 (2016), arXiv:1601.04914.

[18] W. D. Goldberger and I. Z. Rothstein, An Effective field theory of gravity for extended objects, Phys. Rev. **D73**, 104029 (2006), arXiv:hep-th/0409156.

[19] R. A. Porto, Post-Newtonian Corrections to the Motion of Spinning Bodies in NRGR, Phys. Rev. D **73**, 104031 (2006), arXiv:gr-qc/0511061.

[20] W. Goldberger and I. Rothstein, Dissipative Effects in the Worldline Approach to Black Hole Dynamics, Phys. Rev. D **73**, 104030 (2006), arXiv:hep-th/0511133.

[21] R. A. Porto, Absorption Effects due to Spin in the Worldline Approach to Black Hole Dynamics, Phys. Rev. D **77**, 064026 (2008), arXiv:0710.5150.

[22] R. A. Porto and I. Z. Rothstein, The hyperfine Einstein-Infeld-Hoffmann potential, Phys. Rev. Lett. **97**, 021101 (2006), arXiv:gr-qc/0604099 [gr-qc].

[23] R. A. Porto and I. Z. Rothstein, Spin(1)Spin(2) Effects in the Motion of Inspiralling Compact Binaries at Third Order in the Post-Newtonian Expansion, Phys.Rev. **D78**, 044012 (2008), arXiv:0802.0720.

[24] R. A. Porto and I. Z. Rothstein, Next to Leading Order Spin(1)Spin(1) Effects in the Motion of Inspiralling Compact Binaries, Phys.Rev. **D78**, 044013 (2008), arXiv:0804.0260.

[25] W. D. Goldberger and A. Ross, Gravitational radiative corrections from effective field theory, Phys. Rev. **D81**, 124015 (2010), arXiv:0912.4254.

[26] A. Ross, Multipole expansion at the level of the action, Phys. Rev. **D85**, 125033 (2012), arXiv:1202.4750.

[27] R. A. Porto, A. Ross, and I. Z. Rothstein, Spin induced multipole moments for the gravitational wave amplitude from binary inspirals to 2.5 Post-Newtonian order, JCAP **1209**, 028, arXiv:1203.2962 [gr-qc].

[28] R. A. Porto, A. Ross, and I. Z. Rothstein, Spin induced multipole moments for the gravitational wave flux from binary inspirals to third Post-Newtonian order, JCAP **1103**, 009, arXiv:1007.1312 [gr-qc].

[29] C. R. Galley and M. Tiglio, Radiation reaction and gravitational waves in the effective field theory approach,

Phys. Rev. D **79**, 124027 (2009).
[30] N. T. Maia, C. R. Galley, A. K. Leibovich, and R. A. Porto, Radiation reaction for spinning bodies in effective field theory I: Spin-orbit effects, Phys. Rev. **D96**, 084064 (2017), arXiv:1705.07934.
[31] N. T. Maia, C. R. Galley, A. K. Leibovich, and R. A. Porto, Radiation reaction for spinning bodies in effective field theory II: Spin-spin effects, Phys. Rev. **D96**, 084065 (2017), arXiv:1705.07938.
[32] C. Galley, A. Leibovich, R. A. Porto, and A. Ross, Tail Effect in Gravitational Radiation Reaction: Time Nonlocality and Renormalization Group Evolution, Phys. Rev. D **93**, 124010 (2016), arXiv:1511.07379.
[33] R. A. Porto and I. Rothstein, Apparent Ambiguities in the Post-Newtonian Expansion for Binary Systems, Phys. Rev. D **96**, 024062 (2017), arXiv:1703.06433.
[34] R. A. Porto, Lamb Shift and the Gravitational Binding Energy for Binary Black Holes, Phys. Rev. D **96**, 024063 (2017), arXiv:1703.06434.
[35] S. Foffa and R. Sturani, Dynamics of the gravitational two-body problem at fourth post-Newtonian order and at quadratic order in the Newton constant, Phys. Rev. D **87**, 064011 (2013), arXiv:1206.7087 [gr-qc].
[36] S. Foffa, R. A. Porto, I. Rothstein, and R. Sturani, Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach II: Renormalized Lagrangian, Phys. Rev. **D100**, 024048 (2019), arXiv:1903.05118.
[37] L. Blanchet, S. Foffa, F. Larrouturou, and R. Sturani, Logarithmic tail contributions to the energy function of circular compact binaries, Phys. Rev. D **101**, 084045 (2020), arXiv:1912.12359.
[38] S. Foffa, P. Mastrolia, R. Sturani, C. Sturm, and W. J. Torres Bobadilla, Static two-body potential at fifth post-Newtonian order, Phys. Rev. Lett. **122**, 241605 (2019), arXiv:1902.10571.
[39] J. Blümlein, A. Maier, and P. Marquard, Five-Loop Static Contribution to the Gravitational Interaction Potential of Two Point Masses, Phys. Lett. B **800**, 135100 (2020), arXiv:1902.11180.
[40] S. Foffa and R. Sturani, Hereditary terms at next-to-leading order in two-body gravitational dynamics, Phys. Rev. D **101**, 064033 (2020), arXiv:1907.02869 [gr-qc].
[41] G. L. Almeida, S. Foffa, and R. Sturani, Tail contributions to gravitational conservative dynamics, (2021), arXiv:2110.14146 [gr-qc].
[42] J. Blümlein, A. Maier, P. Marquard, and G. Schäfer, The fifth-order post-Newtonian Hamiltonian dynamics of two-body systems from an effective field theory approach: potential contributions, Nucl. Phys. B **965**, 115352 (2021), arXiv:2010.13672 [gr-qc].
[43] J. Blümlein, A. Maier, P. Marquard, and G. Schäfer, The fifth-order post-Newtonian Hamiltonian dynamics of two-body systems from an effective field theory approach, (2021), arXiv:2110.13822 [gr-qc].
[44] S. Foffa and R. Sturani, Near and far zones in two-body dynamics: An effective field theory perspective, Phys. Rev. D **104**, 024069 (2021), arXiv:2103.03190 [gr-qc].
[45] J. Blümlein, A. Maier, P. Marquard, and G. Schäfer, Testing binary dynamics in gravity at the sixth post-Newtonian level, Phys. Lett. B **807**, 135496 (2020), arXiv:2003.07145 [gr-qc].
[46] J. Blümlein, A. Maier, P. Marquard, and G. Schäfer, The 6th post-Newtonian potential terms at $O(G_N^4)$, Phys. Lett. B **816**, 136260 (2021), arXiv:2101.08630 [gr-qc].
[47] W. D. Goldberger, J. Li, and I. Z. Rothstein, Non-conservative effects on spinning black holes from world-line effective field theory, JHEP **06**, 053, arXiv:2012.14869 [hep-th].
[48] M. Levi, A. J. Mcleod, and M. Von Hippel, N^{3}LO gravitational quadratic-in-spin interactions at G^4, JHEP **07**, 116, arXiv:2003.07890 [hep-th].
[49] M. Levi, A. J. Mcleod, and M. Von Hippel, N^{3}LO gravitational spin-orbit coupling at order G^4, JHEP **07**, 115, arXiv:2003.02827 [hep-th].
[50] C. Galley and R. A. Porto, Gravitational Self-Force in the Ultra-Relativistic Limit: the "Large-N" Expansion, JHEP **11**, 096, arXiv:1302.4486.
[51] A. K. Leibovich, N. T. Maia, I. Z. Rothstein, and Z. Yang, Second post-Newtonian order radiative dynamics of inspiralling compact binaries in the Effective Field Theory approach, Phys. Rev. D **101**, 084058 (2020), arXiv:1912.12546.
[52] B. A. Pardo and N. T. Maia, Next-to-leading order spin-orbit effects in the equations of motion, energy loss and phase evolution of binaries of compact bodies in the effective field theory approach, Phys. Rev. D **102**, 124020 (2020), arXiv:2009.05628 [gr-qc].
[53] G. Cho, B. Pardo, and R. A. Porto, Gravitational radiation from inspiralling compact objects: Spin-spin effects completed at the next-to-leading post-Newtonian order, Phys. Rev. D **104**, 024037 (2021), arXiv:2103.14612 [gr-qc].
[54] G. Cho, R. A. Porto, and Z. Yang, Gravitational radiation from inspiralling compact objects: Spin effects to fourth Post-Newtonian order, (2022), arXiv:2201.05138 [gr-qc].
[55] T. Damour, P. Jaranowski, and G. Schäfer, Nonlocal-In-Time Action for the Fourth Post-Newtonian Conservative Dynamics of Two-Body Systems, Phys. Rev. D **89**, 064058 (2014), arXiv:1401.4548.
[56] P. Jaranowski and G. Schäfer, Derivation of local-in-time fourth post-Newtonian ADM Hamiltonian for spinless compact binaries, Phys. Rev. **D92**, 124043 (2015), arXiv:1508.01016.
[57] L. Bernard, L. Blanchet, A. Bohe, G. Faye, and S. Marsat, Fokker action of nonspinning compact binaries at the fourth post-Newtonian approximation, Phys. Rev. D **93**, 084037 (2016), arXiv:1512.02876.
[58] L. Bernard, L. Blanchet, A. Bohe, G. Faye, and S. Marsat, Dimensional regularization of the IR divergences in the Fokker action of point-particle binaries at the fourth post-Newtonian order, Phys. Rev. **D96**, 104043 (2017), arXiv:1706.08480.
[59] T. Marchand, L. Bernard, L. Blanchet, and G. Faye, Ambiguity-Free Completion of the Equations of Motion of Compact Binary Systems at the Fourth Post-Newtonian Order, Phys. Rev. D **97**, 044023 (2018), arXiv:1707.09289.
[60] D. Bini, T. Damour, and A. Geralico, Novel approach to binary dynamics: application to the fifth post-Newtonian level, Phys. Rev. Lett. **123**, 231104 (2019), arXiv:1909.02375 [gr-qc].
[61] D. Bini, T. Damour, and A. Geralico, Sixth post-Newtonian local-in-time dynamics of binary systems, Phys. Rev. D **102**, 024061 (2020), arXiv:2004.05407 [gr-qc].

[62] D. Bini, T. Damour, and A. Geralico, Sixth post-Newtonian nonlocal-in-time dynamics of binary systems, Phys. Rev. D **102**, 084047 (2020), arXiv:2007.11239 [gr-qc].

[63] D. Bini, T. Damour, and A. Geralico, Radiative contributions to gravitational scattering, Phys. Rev. D **104**, 084031 (2021), arXiv:2107.08896 [gr-qc].

[64] Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson, and R. Roiban, The Duality Between Color and Kinematics and its Applications, (2019), arXiv:1909.01358.

[65] G. Kälin and R. A. Porto, From Boundary Data to Bound States, JHEP **01**, 072, arXiv:1910.03008.

[66] G. Kälin and R. A. Porto, From boundary data to bound states. Part II: Scattering angle to dynamical invariants (with twist), JHEP **02**, 120, arXiv:1911.09130.

[67] D. Neill and I. Z. Rothstein, Classical Space-Times from the S Matrix, Nucl. Phys. **B877**, 177 (2013), arXiv:1304.7263.

[68] V. Vaidya, Gravitational spin Hamiltonians from the S matrix, Phys. Rev. **D91**, 024017 (2015), arXiv:1410.5348.

[69] C. Cheung, I. Z. Rothstein, and M. P. Solon, From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion, Phys. Rev. Lett. **121**, 251101 (2018), arXiv:1808.02489.

[70] A. Guevara, A. Ochirov, and J. Vines, Scattering of Spinning Black Holes from Exponentiated Soft Factors, JHEP **09**, 056, arXiv:1812.06895.

[71] D. A. Kosower, B. Maybee, and D. O'Connell, Amplitudes, Observables, and Classical Scattering, JHEP **02**, 137, arXiv:1811.10950.

[72] B. Maybee, D. O'Connell, and J. Vines, Observables and amplitudes for spinning particles and black holes, JHEP **12**, 156, arXiv:1906.09260.

[73] A. Cristofoli, N. E. J. Bjerrum-Bohr, P. H. Damgaard, and P. Vanhove, Post-Minkowskian Hamiltonians in general relativity, Phys. Rev. D **100**, 084040 (2019), arXiv:1906.01579 [hep-th].

[74] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Plante, and P. Vanhove, General Relativity from Scattering Amplitudes, Phys. Rev. Lett. **121**, 171601 (2018), arXiv:1806.04920.

[75] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order, Phys. Rev. Lett. **122**, 201603 (2019), arXiv:1901.04424.

[76] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Black Hole Binary Dynamics from the Double Copy and Effective Theory, JHEP **10**, 206, arXiv:1908.01493.

[77] K. Haddad and A. Helset, Tidal effects in quantum field theory, JHEP **12**, 024, arXiv:2008.04920 [hep-th].

[78] R. Aoude, K. Haddad, and A. Helset, On-shell heavy particle effective theories, JHEP **05**, 051, arXiv:2001.09164 [hep-th].

[79] J. Parra-Martinez, M. S. Ruf, and M. Zeng, Extremal black hole scattering at $\mathcal{O}(G^3)$: graviton dominance, eikonal exponentiation, and differential equations, JHEP **11**, 023, arXiv:2005.04236 [hep-th].

[80] Z. Bern, A. Luna, R. Roiban, C.-H. Shen, and M. Zeng, Spinning black hole binary dynamics, scattering amplitudes, and effective field theory, Phys. Rev. D **104**, 065014 (2021), arXiv:2005.03071 [hep-th].

[81] C. Cheung and M. P. Solon, Tidal Effects in the Post-Minkowskian Expansion, Phys. Rev. Lett. **125**, 191601 (2020), arXiv:2006.06665 [hep-th].

[82] D. Kosmopoulos and A. Luna, Quadratic-in-Spin Hamiltonian at $\mathcal{O}(G^2)$ from Scattering Amplitudes, (2021), arXiv:2102.10137 [hep-th].

[83] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$, Phys. Rev. Lett. **126**, 171601 (2021), arXiv:2101.07254 [hep-th].

[84] P. A. Kreer and S. Weinzierl, The H-graph with equal masses in terms of multiple polylogarithms, Phys. Lett. B **819**, 136405 (2021), arXiv:2104.07488 [hep-ph].

[85] E. Herrmann, J. Parra-Martinez, M. S. Ruf, and M. Zeng, Gravitational Bremsstrahlung from Reverse Unitarity, Phys. Rev. Lett. **126**, 201602 (2021), arXiv:2101.07255 [hep-th].

[86] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, Radiation Reaction from Soft Theorems, Phys. Lett. B **818**, 136379 (2021), arXiv:2101.05772 [hep-th].

[87] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, The eikonal approach to gravitational scattering and radiation at $\mathcal{O}(\mathrm{G}^3)$, JHEP **07**, 169, arXiv:2104.03256 [hep-th].

[88] A. Cristofoli, R. Gonzo, D. A. Kosower, and D. O'Connell, Waveforms from Amplitudes, (2021), arXiv:2107.10193 [hep-th].

[89] Y. F. Bautista, A. Guevara, C. Kavanagh, and J. Vines, From Scattering in Black Hole Backgrounds to Higher-Spin Amplitudes: Part I, (2021), arXiv:2107.10179 [hep-th].

[90] W. D. Goldberger and A. K. Ridgway, Radiation and the classical double copy for color charges, Phys. Rev. D **95**, 125010 (2017), arXiv:1611.03493.

[91] W. D. Goldberger and A. K. Ridgway, Bound states and the classical double copy, Phys. Rev. **D97**, 085019 (2018), arXiv:1711.09493.

[92] G. Kälin and R. A. Porto, Post-Minkowskian Effective Field Theory for Conservative Binary Dynamics, JHEP **11**, 106, arXiv:2006.01184 [hep-th].

[93] G. Kälin, Z. Liu, and R. A. Porto, Conservative Dynamics of Binary Systems to Third Post-Minkowskian Order from the Effective Field Theory Approach, Phys. Rev. Lett. **125**, 261103 (2020), arXiv:2007.04977 [hep-th].

[94] G. Kälin, Z. Liu, and R. A. Porto, Conservative Tidal Effects in Compact Binary Systems to Next-to-Leading Post-Minkowskian Order, Phys. Rev. D **102**, 124025 (2020), arXiv:2008.06047 [hep-th].

[95] Z. Liu, R. A. Porto, and Z. Yang, Spin Effects in the Effective Field Theory Approach to Post-Minkowskian Conservative Dynamics, JHEP **06**, 012, arXiv:2102.10059 [hep-th].

[96] G. Mogull, J. Plefka, and J. Steinhoff, Classical black hole scattering from a worldline quantum field theory, JHEP **02**, 048, arXiv:2010.02865 [hep-th].

[97] G. U. Jakobsen, G. Mogull, J. Plefka, and J. Steinhoff, Classical Gravitational Bremsstrahlung from a Worldline Quantum Field Theory, Phys. Rev. Lett. **126**, 201103 (2021), arXiv:2101.12688 [gr-qc].

[98] S. Mougiakakos, M. M. Riva, and F. Vernizzi, Gravitational Bremsstrahlung in the post-Minkowskian effective field theory, Phys. Rev. D **104**, 024041 (2021),

arXiv:2102.08339 [gr-qc].

[99] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, Dynamics of Binary Systems to Fourth Post-Minkowskian Order from the Effective Field Theory Approach, (2021), arXiv:2106.08276 [hep-th].

[100] G. U. Jakobsen, G. Mogull, J. Plefka, and J. Steinhoff, Gravitational Bremsstrahlung and Hidden Supersymmetry of Spinning Bodies, (2021), arXiv:2106.10256 [hep-th].

[101] M. M. Riva and F. Vernizzi, Radiated momentum in the post-Minkowskian worldline approach via reverse unitarity, JHEP **11**, 228, arXiv:2110.10140 [hep-th].

[102] A. Antonelli, A. Buonanno, J. Steinhoff, M. van de Meent, and J. Vines, Energetics of two-body Hamiltonians in post-Minkowskian gravity, Phys. Rev. **D99**, 104004 (2019), arXiv:1901.07102.

[103] G. Cho, G. Kälin, and R. A. Porto, From Boundary Data to Bound States III: Radiative Effects, (2021), arXiv:2112.03976 [hep-th].

[104] V. A. Smirnov, *Analytic tools for Feynman integrals* (Springer, 2012).

[105] A. Kotikov, Differential equation method: The Calculation of N point Feynman diagrams, Phys. Lett. B **267**, 123 (1991), [Erratum: Phys.Lett.B 295, 409–409 (1992)].

[106] E. Remiddi, Differential equations for Feynman graph amplitudes, Nuovo Cim. A **110**, 1435 (1997), arXiv:hep-th/9711188.

[107] J. M. Henn, Multiloop integrals in dimensional regularization made simple, Phys. Rev. Lett. **110**, 251601 (2013), arXiv:1304.1806.

[108] M. Prausa, epsilon: A tool to find a canonical basis of master integrals, Comput. Phys. Commun. **219**, 361 (2017), arXiv:1701.00725 [hep-ph].

[109] R. N. Lee, Libra: A package for transformation of differential systems for multiloop integrals, Comput. Phys. Commun. **267**, 108058 (2021), arXiv:2012.00279 [hep-ph].

[110] R. N. Lee, Reducing differential equations for multiloop master integrals, JHEP **04**, 108, arXiv:1411.0911 [hep-ph].

[111] L. Adams and S. Weinzierl, The ε-form of the differential equations for Feynman integrals in the elliptic case, Phys. Lett. B **781**, 270 (2018), arXiv:1802.05020 [hep-ph].

[112] K. Chetyrkin and F. Tkachov, Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops, Nucl. Phys. B **192**, 159 (1981).

[113] F. Tkachov, A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions, Phys. Lett. B **100**, 65 (1981).

[114] A. V. Smirnov and F. S. Chuharev, FIRE6: Feynman Integral REduction with Modular Arithmetic, Comput. Phys. Commun. **247**, 106877 (2020), arXiv:1901.07808 [hep-ph].

[115] A. Smirnov and V. Smirnov, How to choose master integrals, Nucl. Phys. B **960**, 115213 (2020), arXiv:2002.08042 [hep-ph].

[116] R. Lee, Presenting LiteRed: a tool for the Loop InTEgrals REDuction, (2012), arXiv:1212.2685.

[117] R. N. Lee, LiteRed 1.4: a powerful tool for reduction of multiloop integrals, J. Phys. Conf. Ser. **523**, 012059 (2014), arXiv:1310.1145 [hep-ph].

[118] M. Beneke and V. A. Smirnov, Asymptotic expansion of Feynman integrals near threshold, Nucl. Phys. **B522**, 321 (1998), arXiv:hep-ph/9711391.

[119] B. Jantzen, A. V. Smirnov, and V. A. Smirnov, Expansion by regions: revealing potential and Glauber regions automatically, Eur. Phys. J. C **72**, 2139 (2012), arXiv:1206.0546 [hep-ph].

[120] A. V. Smirnov, FIESTA4: Optimized Feynman integral calculations with GPU support, Comput. Phys. Commun. **204**, 189 (2016), arXiv:1511.03614 [hep-ph].

[121] C. Meyer, Evaluating multi-loop Feynman integrals using differential equations: automatizing the transformation to a canonical basis, PoS **LL2016**, 028 (2016).

[122] C. Meyer, Transforming differential equations of multiloop Feynman integrals into canonical form, JHEP **04**, 006, arXiv:1611.01087 [hep-ph].

[123] J. Broedel, C. Duhr, F. Dulat, R. Marzucca, B. Penante, and L. Tancredi, An analytic solution for the equal-mass banana graph, JHEP **09**, 112, arXiv:1907.03787 [hep-th].

[124] A. Primo and L. Tancredi, Maximal cuts and differential equations for Feynman integrals. An application to the three-loop massive banana graph, Nucl. Phys. B **921**, 316 (2017), arXiv:1704.05465 [hep-ph].

[125] M. Hidding, DiffExp, a Mathematica package for computing Feynman integrals in terms of one-dimensional series expansions, Comput. Phys. Commun. **269**, 108125 (2021), arXiv:2006.05510 [hep-ph].

[126] A. Goncharov, Multiple polylogarithms and mixed Tate motives, (2001), arXiv:math/0103059.

[127] K.-T. Chen, Iterated path integrals, Bull. Am. Math. Soc. **83**, 831 (1977).

[128] C. Duhr, Mathematical aspects of scattering amplitudes, (2014), arXiv:1411.7538 [hep-ph].

[129] C. Duhr and F. Dulat, PolyLogTools — polylogs for the masses, JHEP **08**, 135, arXiv:1904.07279 [hep-th].

[130] C. Dlapa, J. Henn, and K. Yan, Deriving canonical differential equations for Feynman integrals from a single uniform weight integral, JHEP **05**, 025, arXiv:2002.02340 [hep-ph].

[131] A. V. Smirnov, N. D. Shapurov, and L. I. Vysotsky, FIESTA5: numerical high-performance Feynman integral evaluation, (2021), arXiv:2110.11660 [hep-ph].

[132] R. N. Lee, A. V. Smirnov, V. A. Smirnov, and M. Steinhauser, Four-loop quark form factor with quartic fundamental colour factor, JHEP **02**, 172, arXiv:1901.02898 [hep-ph].

[133] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, *In preparation.*