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J. Blümlein, P. Marquard

Deutsches Elektronen-Synchrotron DESY, Zeuthen

C. Schneider

RISC, Johannes Kepler Universität Linz, Austria

K. Schönwald

Institut für Theoretische Teilchenphysik, Karlsruher Institut für Technologie (KIT), Karlsruhe

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The massless three-loop Wilson coefficients for the deep-inelastic structure functions F_2, F_L, xF_3 and g_1

J. Blümlein[a], P. Marquard[a], C. Schneider[b] and K. Schönwald[c]

[a] *Deutsches Elektronen–Synchrotron DESY, Platanenallee 6, 15738 Zeuthen, Germany*

[b] *Johannes Kepler University Linz, Research Institute for Symbolic Computation (RISC), Altenberger Straße 69, A–4040 Linz, Austria*

[c] *Institut für Theoretische Teilchenphysik, Karlsruher Institut für Technologie (KIT) D–76128 Karlsruhe, Germany*

Abstract

We calculate the massless unpolarized Wilson coefficients for deeply inelastic scattering for the structure functions $F_2(x,Q^2), F_L(x,Q^2), xF_3(x,Q^2)$ in the $\overline{\mathsf{MS}}$ scheme and the polarized Wilson coefficients of the structure function $g_1(x,Q^2)$ in the Larin scheme up to three–loop order in QCD in a fully automated way based on the method of arbitrary high Mellin moments. We work in the Larin scheme in the case of contributing axial–vector couplings or polarized nucleons. For the unpolarized structure functions we compare to results given in the literature. The polarized three–loop Wilson coefficients are calculated for the first time. As a by–product we also obtain the quarkonic three–loop anomalous dimensions from the $O(1/\varepsilon)$ terms of the unrenormalized forward Compton amplitude. Expansions for small and large values of the Bjorken variable x are provided.

1 Introduction

The scaling violations of the deep–inelastic structure functions $F_k(x,Q^2)$ [1–4] in the massless limit and twist 2 approximation are described by those of the twist 2 parton distribution functions f_i and massless Wilson coefficients $\mathbb{C}_j$ in a perturbative expansion in the strong coupling constant $a_s(Q^2) = \alpha_s(Q^2)/(4\pi)$, where $Q^2 = -q^2$ denotes the space–like virtuality of the scattering process and $x = Q^2/(2p.q)$ is the Bjorken variable, with p the nucleon momentum. In the present paper we calculate the Wilson coefficients in the case of virtual photon exchange in the unpolarized and polarized case to three–loop order, where the latter results are new. For the structure function xF_3 we consider the exchange of both the weak $W^\pm$ bosons. The present results are of importance for the experimental data analysis of the deep–inelastic scattering data at HERA and future lepton–nucleon colliders, such as the EIC [5] and the LHeC–project [6] for precision measurements in the unpolarized and polarized case, to extract the parton distribution functions [7] and to measure the strong coupling constant at highest possible precision [8].

The calculation is based on massless virtual forward Compton amplitudes for on–shell partonic states, which are gauge–invariant quantities. The structure functions are described by

$$F_k(x,Q^2) = \sum_i \mathbb{C}_{k,i}\left(x, \frac{Q^2}{\mu^2}\right) \otimes f_i(x,\mu^2) \tag{1}$$

with

$$\mathbb{C}_{k,i}\left(x, \frac{Q^2}{\mu^2}\right) = \delta_{iq} + \sum_{l=1}^{\infty} a_s^l \sum_{n=0}^{l} \ln^n\left(\frac{Q^2}{\mu^2}\right) C_{k,i}^{(l,n)}(x), \tag{2}$$

where μ^2 denotes the factorization scale and $\otimes$ the Mellin convolution

$$A(x) \otimes B(x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2). \tag{3}$$

The Mellin transform

$$\mathbf{M}[A(x)](N) = \int_0^1 dx x^{N-1} A(x) \tag{4}$$

diagonalizes the expression in Eq. (3) into the product $\mathbf{M}[A](N) \cdot \mathbf{M}[B](N)$ and will therefore be often used in the subsequent calculation. The dependence on the factorization scale diminishes for higher and higher orders in the coupling constant. One may also derive representations in Mellin N space which are scheme–invariant order by order, see e.g. [9–11]. Here the preferred scale of the deep–inelastic process is $\mu^2 = Q^2$.

The massless Wilson coefficients for the structure functions $F_2(x,Q^2), F_L(x,Q^2)$ and $xF_3(x,Q^2)$ have been calculated at one– [12], two– [13, 14] and three–loop order [15–17]. Furthermore, also their Mellin moments have been computed at 2– [18] and 3–loop order [19]. In the polarized massless case the corresponding 1– and 2–loop Wilson coefficients for the structure function $g_1(x,Q^2)$ have been obtained in [20–24].

In the present paper we re–calculate the massless Wilson coefficients for the structure functions $F_2(x,Q^2)$ and $F_L(x,Q^2)$ as well as for the structure function $xF_3(x,Q^2) \equiv xF_3^{W^+}(x,Q^2) + xF_3^{W^-}(x,Q^2)$ and calculate newly those for the polarized structure function $g_1(x,Q^2)$ in the Larin scheme [25,26]. The Wilson coefficients are calculated using the method of the forward Compton amplitude. Here the parameter $x = Q^2/(2p.q)$ is the variable on which the master integrals and the associated differential equations depend. The Mellin moments of the structure functions correspond to the expansion coefficients of the forward Compton amplitude in the unphysical limit $\omega = 1/x \ll 1$ which is frequently exploited to compute low Mellin moments. Therefore, the variable ω corresponds to the parameter t used in Refs. [27–29] and the subsequent technical steps are quite similar to those in the calculation of the three–loop anomalous dimensions.

The paper is organized as follows. In Section 2 we discuss the basic formalism, including the renormalization, and summarize the technical details of the calculation in Section 3. In Section 4 we calculate the one– and two–loop Wilson coefficients to the order in the dimensional parameter $\varepsilon = D - 4$, needed for the present calculation at three–loop order and present the 2–loop Wilson coefficients. In Section 5 the three–loop Wilson coefficients for the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ are given and in Section 6 those for the structure function $xF_3(x, Q^2)$. Finally, we present in Section 7 the three–loop Wilson coefficients for the polarized structure function $g_1(x, Q^2)$ and also remark that the quarkonic anomalous dimensions being obtained as a by–product of the present calculation from the $O(1/\varepsilon)$ terms of the unrenormalized Compton amplitude, agree with the results in the literature. In Section 8 we present the small–x and large–x expansions of the respective Wilson coefficients and compare to the large N_F behaviour predicted in the literature. Here N_F denotes the number of massless flavors. Section 9 contains the conclusions. To shorten the presentation we will identify the factorization scale $\mu^2 = Q^2$ and only print the Mellin N space representation for the complete expressions. In an ancillary file to this paper we present all Wilson coefficients in Mellin N and z–space, in full form, including their scale dependence. Here z denotes the parton momentum fraction in the nucleon, $z \in [0, 1]$, and is, for twist–2, identical to the Bjorken variable x.

2 Basic Formalism

We consider the forward Compton amplitude for neutral current photon exchange in the unpolarized case for the deep–inelastic structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$, as well as for the structure functions $g_1(x, Q^2)$ in the polarized case in the twist–2 approximation to three–loop order. Furthermore, we study the charged current structure function $xF_3(x, Q^2)$.

The hadronic tensor has the principal structure [4, 30]

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x, Q^2) + \frac{\hat{p}_\mu \hat{p}_\nu}{p.q} F_2(x, Q^2) - i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda p^\sigma}{2p.q} F_3(x, Q^2) + i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda S^\sigma}{p.q} g_1(x, Q^2) + \cdots \ . \tag{5}$$

Here the ellipses denote contributions from other structure functions not considered in the present paper, p denotes the nucleon 4–momentum and S the polarization vector of the nucleon, which can be taken in the longitudinal case in the nucleon rest frame

$$S_L = (M, 0, 0, 0) \ , \tag{6}$$

where only the energy component is non–vanishing and M denotes the nucleon mass. The vector $\hat{p}$ is given by

$$\hat{p} = p_\mu - \frac{p.q}{q^2} q_\mu \ . \tag{7}$$

All structure functions can be isolated by using corresponding projectors in $D = 4 + \varepsilon$ space–time dimensions. In the polarized case one furthermore considers the polarization asymmetry, i.e. the difference for $S = S_L$ and $-S_L$. In the following we will consider the structure functions $F_{2,L}(x,Q^2)$ and $g_1(x,Q^2)$ in the case of pure photon exchange, while $xF_3(x,Q^2)$ is measured for the arithmetic mean of $W^\pm$ boson exchange.

The associated forward Compton amplitudes are given by the Fourier transform of the time–ordered product of current operators

$$T_{\mu\nu} = i \int dz \, \mathrm{e}^{iqz} \langle p| T \left(J_\mu^\dagger(z) J_\nu(0) \right) |p\rangle \ , \tag{8}$$

and

$$W_{\mu\nu} = \frac{1}{\pi} \mathsf{Im} T_{\mu\nu} \ . \tag{9}$$

We consider the deep–inelastic scattering cross section for pure photon exchange [31]

$$\begin{aligned} \frac{d^2\sigma^{\gamma^*}(\lambda, \pm S_L)}{dxdy} &= 2\pi S \frac{\alpha^2}{Q^4} \Biggl\{ y^2 2xF_1(x,Q^2) + 2\left(1 - y - \frac{xyM^2}{S}\right) F_2(x,Q^2) \\ &\pm \left[-2\lambda y \left(2 - y - \frac{2xyM^2}{S} \right) x g_1(x,Q^2) + 8\lambda \frac{yx^2M^2}{S} g_2(x,Q^2) \right] \Biggr\} \ , \end{aligned} \tag{10}$$

with λ the longitudinal charged lepton polarization, M the nucleon mass, $y = Q^2/Sx$ a Bjorken variable, and S the cms energy squared. We will use the collinear parton model [32] in the present calculation, where the parton momentum is given by zp, with p the nucleon momentum and $z \in [0,1]$. Due to this the Wilson coefficients of the structure function $g_2(x,Q^2)$ cannot be calculated. It requires the use of the covariant parton model [33], already at tree level.

One furthermore has

$$F_L(x,Q^2) = F_2(x,Q^2) - 2xF_1(x,Q^2) \ . \tag{11}$$

The structure functions are related to the parton densities for pure photon exchange at tree level as [4]

$$F_2(x,Q^2) = x \sum_{f=1}^{N_F} e_f^2 [q_f(x,Q^2) + \bar{q}_f(x,Q^2)] \ , \tag{12}$$

$$g_1(x,Q^2) = \frac{1}{2} \sum_{f=1}^{N_F} e_f^2 [\Delta q_f(x,Q^2) + \Delta \bar{q}_f(x,Q^2)] \ , \tag{13}$$

for pure photon exchange, with $q_i(\bar{q}_i)$ the unpolarized and $\Delta q_i(\Delta \bar{q}_i)$ the polarized parton densities and $F_L = 0$.

In the charged current case we consider the massless Wilson coefficients of the structure function $xF_3(x,Q^2)$. One should notice, that here also strange–charm transitions occur, which imply heavy flavor corrections [34–37]. They are of importance in quantitative analyses. Here we consider all contributing quark flavors as massless. For incoming charged leptons the scattering cross section reads, cf. [38–40],

$$\frac{d^2\sigma^{\mathrm{cc}}}{dxdy} = \frac{G_\mu^2 S}{8\pi} \left[\frac{M_W^2}{Q^2 + M_W^2} \right]^2 \Biggl\{ Y_+ W_2(x,Q^2) + Y_- W_3(x,Q^2) - y^2 W_L(x,Q^2) \Biggr\} \ , \tag{14}$$

where G_μ denotes the Fermi constant and $Y_\pm = 1 \pm (1-y)^2$ and both unpolarized leptons and nucleons are considered. The structure functions are given by

$$
\begin{aligned}
W_2(x,Q^2) &= F_2^{\mathrm{cc},Q_l}(x,Q^2)\ , && (15)\\
W_3(x,Q^2) &= -\mathrm{sign}(Q_l) x F_3^{\mathrm{cc},Q_l}(x,Q^2)\ , && (16)
\end{aligned}
$$

where Q_l is the charge of the incoming charged lepton. At tree level these structure functions have the following quark flavor decomposition [39]

$$
\begin{aligned}
F_2^{\mathrm{cc},+}(x,Q^2) &= 2x\sum_i \left[d_i(x,Q^2)+\bar{u}_i(x,Q^2)\right]\ , && (17)\\
F_2^{\mathrm{cc},-}(x,Q^2) &= 2x\sum_i \left[u_i(x,Q^2)+\bar{d}_i(x,Q^2)\right]\ , && (18)\\
xF_3^{\mathrm{cc},+}(x,Q^2) &= 2x\sum_i \left[d_i(x,Q^2)-\bar{u}_i(x,Q^2)\right]\ , && (19)\\
xF_3^{\mathrm{cc},-}(x,Q^2) &= 2x\sum_i \left[u_i(x,Q^2)-\bar{d}_i(x,Q^2)\right]\ , && (20)
\end{aligned}
$$

e.g. for four flavors[1], and $W_L = F_L^{\mathrm{cc},Q_l}$ denotes the longitudinal structure function with $F_L^{\mathrm{cc},\pm} = F_2^{\mathrm{cc},\pm} - 2xF_1^{\mathrm{cc},\pm}$.

In the present paper we consider in the charged current case the flavor non–singlet structure function combination $xF_3^{\mathrm{cc},+}(x,Q^2)+xF_3^{\mathrm{cc},-}(x,Q^2)$. At the experimental side one might consider the scattering off deuteron targets, obeying the $SU(2)$ isospin flavor symmetry

$$
u(x,Q^2) = d(x,Q^2)\ , \qquad \bar{u}(x,Q^2) = \bar{d}(x,Q^2)\ , \qquad (21)
$$

after deuteron wave function corrections. Through this one obtains at tree level

$$
\begin{aligned}
F_2^{\mathrm{cc},d,+} &= x[u+d+2s+\bar{u}+\bar{d}+2\bar{c}]\ , && (22)\\
F_2^{\mathrm{cc},d,-} &= x[u+d+2c+\bar{u}+\bar{d}+2\bar{s}]\ , && (23)\\
F_2^{\mathrm{cc},d,+}+F_2^{\mathrm{cc},d,-} &= 2x\Sigma\ , && (24)\\
xF_3^{\mathrm{cc},d,+} &= x[u+d+2s-\bar{u}-\bar{d}-2\bar{c}]\ , && (25)\\
xF_3^{\mathrm{cc},d,-} &= x[u+d+2sc-\bar{u}-\bar{d}-2\bar{s}]\ , && (26)\\
xF_3^{\mathrm{cc},d,+}+xF_3^{\mathrm{cc},d,-} &= 2x[u+d-\bar{u}-\bar{d}+s-\bar{s}+c-\bar{c}]\ ,\\
&\equiv 2x[u_v+d_v], && (27)
\end{aligned}
$$

and

$$
xF_3 = -\frac{8\pi}{Y_- G_\mu^2 S}\left[\frac{d^2\sigma^{\mathrm{cc},d,+}}{dxdy}-\frac{d^2\sigma^{\mathrm{cc},d,-}}{dxdy}\right]+\frac{Y_+}{Y_-}\left(F_2^{\mathrm{cc},d,+}-F_2^{\mathrm{cc},d,+}\right)-\frac{y^2}{Y_-}\left(W_L^{+,d}-W_L^{-d}\right) \qquad (28)
$$

Here the non–singlet combinations for F_2^d and $W_L^d \propto Y_+, y^2$ correspond to the quark flavor combination

$$
f^{\mathrm{NS},+} = 2x[(s-\bar{s})-(c-\bar{c})]\ , \qquad (29)
$$

[1]More generally, one has to account for the Cabibbo–Kobayashi–Maskawa mixing, cf. e.g. [35], effectively redefining the down–quark densities in the charged current case.

which vanishes under the assumption of symmetric sea–quarks of the same flavor and one obtains the direct projection on xF_3^d by the weighted differential cross section difference. In other cases one has to perform fits in Bjorken y to separate the Y_- from the Y_+ and y^2 contributions.

The projectors of the hadronic tensor allowing to isolate the Wilson coefficients of the different structure functions are given by

$$P_L^{\mu\nu} = \frac{8x^3}{Q^2} p^\mu p^\nu \ , \tag{30}$$

$$P_2^{\mu\nu} = -\frac{2x}{D-2}\left(g^{\mu\nu} - (D-1)\frac{4x^2}{Q^2} p^\mu p^\nu\right) \ , \tag{31}$$

$$P_3^{\mu\nu} = \frac{-i}{(D-2)(D-3)}\frac{4x}{Q^2}\varepsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma \ , \tag{32}$$

$$P_{g_1}^{\mu\nu} = \frac{i}{(D-2)(D-3)}\frac{2x}{Q^2}\varepsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma \ , \tag{33}$$

applied to $W_{\mu\nu}$, cf. also [41]. The twist–2 contributions to the structure functions in the massless case are given by Eq. (1).

The perturbative expansion of the structure functions

$$F_i(x,Q^2) = \sum_{k=0}^{\infty} a_s^k(Q^2) F_{i,k}(x) \tag{34}$$

obey the following renormalization group equation (RGE) in the massless limit [42, 43], see also [9],

$$\left[\mathcal{D} + \gamma_{J_1} + \gamma_{J_2} - n_\psi \gamma_\psi - n_A \gamma_A\right] F_i(N,Q^2) = 0 \tag{35}$$

with

$$\mathcal{D} = \mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s)\frac{\partial}{\partial a_s} \tag{36}$$

in Mellin space with $a_s = a_s(\mu^2)$, $\beta(a_s)$ the β function of Quantum Chromodynamics (QCD), and setting $\mu \equiv \mu_F = \mu_R$, where $\mu_{F,R}$ are the factorization and renormalization scale. γ_{J_i} are the anomalous dimensions of the currents forming the forward Compton amplitude and $n_{\psi(A)}$ and $\gamma_{\psi(A)}$ are the numbers of external quark (gluon fields) and their anomalous dimensions. One may now split (35) for those of the renormalized Wilson coefficients and renormalized parton densities,

$$\sum_j \left[\mathcal{D}(\mu^2)\delta_{ij} + \gamma_{ij}^{\rm S,NS} - n_\psi \gamma_\psi - n_A \gamma_A\right] f_j(N,\mu^2) = 0 \ , \tag{37}$$

$$\sum_j \left[\mathcal{D}(\mu^2)\delta_{ij} + \gamma_{J_1} + \gamma_{J_2} - \gamma_{ij}^{\rm S,NS}\right] C_i\left(N, \frac{Q^2}{\mu^2}\right) = 0 \ . \tag{38}$$

In this way the Z factors of the massless Wilson coefficients and the massless parton densities are related.[2]

Before we turn to the renormalized Wilson coefficients we will study the unrenormalized ones, as those emerge in the calculation. They have the following representation. Here and in the following we work in Mellin N space.

$$\hat{C}_{i,q}^{\rm NS} = 1 + \hat{a}_s\left\{\frac{1}{\varepsilon} c_{i,q}^{{\rm NS},(1,-1)} + c_{i,q}^{{\rm NS},(1,0)} + \varepsilon c_{i,q}^{{\rm NS},(1,1)} + \varepsilon^2 c_{i,q}^{{\rm NS},(1,2)} + O(\varepsilon^3)\right\}$$

[2] This is different in the massive case, cf. [44], and even in treating heavy quarks as light at asymptotic scales.

$$
\begin{aligned}
&+\hat{a}_s^2\Biggl\{\frac{1}{\varepsilon^2}c_{i,q}^{\mathrm{NS},(2,-2)}+\frac{1}{\varepsilon}c_{i,q}^{\mathrm{NS},(2,-1)}+c_{i,q}^{\mathrm{NS},(2,0)}+\varepsilon c_{i,q}^{\mathrm{NS},(2,1)}+O(\varepsilon^2)\Biggr\}\\
&+\hat{a}_s^3\Biggl\{\frac{1}{\varepsilon^3}c_{i,q}^{\mathrm{NS},(3,-3)}+\frac{1}{\varepsilon^2}c_{i,q}^{\mathrm{NS},(3,-2)}+\frac{1}{\varepsilon}c_{i,q}^{\mathrm{NS},(3,-1)}+c_{i,q}^{\mathrm{NS},(3,0)}+O(\varepsilon)\Biggr\}+O\left(\hat{a}_s^4\right)\ , \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
\hat{C}_{i,q}^{\mathrm{PS}} &= \hat{a}_s^2\Biggl\{\frac{1}{\varepsilon^2}c_{i,q}^{\mathrm{PS},(2,-2)}+\frac{1}{\varepsilon}c_{i,q}^{\mathrm{PS},(2,-1)}+c_{i,q}^{\mathrm{PS},(2,0)}+\varepsilon c_{i,q}^{\mathrm{PS},(2,1)}+O(\varepsilon^2)\Biggr\}\\
&+\hat{a}_s^3\Biggl\{\frac{1}{\varepsilon^3}c_{i,q}^{\mathrm{PS},(3,-3)}+\frac{1}{\varepsilon^2}c_{i,q}^{\mathrm{PS},(3,-2)}+\frac{1}{\varepsilon}c_{i,q}^{\mathrm{PS},(3,-1)}+c_{i,q}^{\mathrm{PS},(3,0)}+O(\varepsilon)\Biggr\}+O\left(\hat{a}_s^4\right)\ , \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
\hat{C}_{i,g} &= \hat{a}_s\Biggl\{\frac{1}{\varepsilon}c_{i,g}^{(1,-1)}+c_{i,g}^{(1,0)}+\varepsilon c_{i,g}^{(1,1)}+\varepsilon^2 c_{i,g}^{(1,2)}+O(\varepsilon^3)\Biggr\}\\
&+\hat{a}_s^2\Biggl\{\frac{1}{\varepsilon^2}c_{i,g}^{(2,-2)}+\frac{1}{\varepsilon}c_{i,g}^{(2,-1)}+c_{i,g}^{(2,0)}+\varepsilon c_{i,g}^{(2,1)}+O(\varepsilon^2)\Biggr\}\\
&+\hat{a}_s^3\Biggl\{\frac{1}{\varepsilon^3}c_{i,g}^{(3,-3)}+\frac{1}{\varepsilon^2}c_{i,g}^{(3,-2)}+\frac{1}{\varepsilon}c_{i,g}^{(3,-1)}+c_{i,g}^{(3,0)}+O(\varepsilon)\Biggr\}+O\left(\hat{a}_s^4\right)\ , \qquad (41)
\end{aligned}
$$

with the bare coupling constant $\hat{a}_s$. These relations apply synonymously to the structure functions $i = F_2, F_3, g_1$.

For those of the structure function F_L one obtains

$$
\begin{aligned}
\hat{C}_{F_L,q}^{\mathrm{NS}} &= \hat{a}_s\Biggl\{c_{F_L,q}^{\mathrm{NS},(1,0)}+\varepsilon c_{F_L,q}^{\mathrm{NS},(1,1)}+\varepsilon^2 c_{F_L,q}^{\mathrm{NS},(1,2)}+O(\varepsilon^3)\Biggr\}\\
&+\hat{a}_s^2\Biggl\{\frac{1}{\varepsilon}c_{F_L,q}^{\mathrm{NS},(2,-1)}+c_{F_L,q}^{\mathrm{NS},(2,0)}+\varepsilon c_{F_L,q}^{\mathrm{NS},(2,1)}+O(\varepsilon^2)\Biggr\}\\
&+\hat{a}_s^3\Biggl\{\frac{1}{\varepsilon^2}c_{F_L,q}^{\mathrm{NS},(3,-2)}+\frac{1}{\varepsilon}c_{F_L,q}^{\mathrm{NS},(3,-1)}+c_{F_L,q}^{\mathrm{NS},(3,0)}+O(\varepsilon)\Biggr\}+O\left(\hat{a}_s^4\right)\ , \qquad (42)
\end{aligned}
$$

$$
\begin{aligned}
\hat{C}_{F_L,q}^{\mathrm{PS}} &= \hat{a}_s^2\Biggl\{\frac{1}{\varepsilon}c_{F_L,q}^{\mathrm{PS},(2,-1)}+c_{F_L,q}^{\mathrm{PS},(2,0)}+\varepsilon c_{F_L,q}^{\mathrm{PS},(2,1)}+O(\varepsilon^2)\Biggr\}\\
&+\hat{a}_s^3\Biggl\{\frac{1}{\varepsilon^2}c_{F_L,q}^{\mathrm{PS},(3,-2)}+\frac{1}{\varepsilon}c_{F_L,q}^{\mathrm{PS},(3,-1)}+c_{F_L,q}^{\mathrm{PS},(3,0)}+O(\varepsilon)\Biggr\}+O\left(\hat{a}_s^4\right)\ , \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
\hat{C}_{F_L,g} &= \hat{a}_s\Biggl\{c_{F_L,g}^{(1,0)}+\varepsilon c_{F_L,g}^{(1,1)}+\varepsilon^2 c_{F_L,g}^{(1,2)}+O(\varepsilon^3)\Biggr\}\\
&+\hat{a}_s.^2\Biggl\{\frac{1}{\varepsilon}c_{F_L,g}^{(2,-1)}+c_{F_L,g}^{(2,0)}+\varepsilon c_{F_L,g}^{(2,1)}+O(\varepsilon^2)\Biggr\}\\
&+\hat{a}_s^3\Biggl\{\frac{1}{\varepsilon^2}c_{F_L,g}^{(3,-2)}+\frac{1}{\varepsilon}c_{F_L,g}^{(3,-1)}+c_{F_L,g}^{(3,0)}+O(\varepsilon)\Biggr\}+O\left(\hat{a}_s^4\right)\ . \qquad (44)
\end{aligned}
$$

The renormalization proceeds in the following way. The coupling constant is renormalized in the $\overline{\mathrm{MS}}$ scheme

$$
\hat{a}_s = a_s\left(1+\frac{2\beta_0}{\varepsilon}a_s+\left[\frac{4\beta_0^2}{\varepsilon^2}+\frac{\beta_1}{\varepsilon}\right]a_s^2+O(a_s^3)\right)\ . \qquad (45)
$$

Here β_i denote the expansion coefficients of the QCD β–function, [45].

$$\beta_0(N_F) = \frac{11}{3}C_A - \frac{4}{3}T_F N_F \ , \tag{46}$$
$$\beta_1(N_F) = \frac{34}{3}C_A^2 - 4\left(\frac{5}{3}C_A + C_F\right) T_F N_F \ . \tag{47}$$

The color factors are $C_F = (N_C^2-1)/(2N_C), C_A = N_C, T_F = 1/2$ for $SU(N_C)$ and $N_C = 3$ for QCD; N_F denotes the number of massless quark flavors. Later we will also need the color factor[3]

$$d_{abc}d^{abc} = \frac{(N_C^2-1)(N_C^2-4)}{16 N_C} \tag{48}$$

and the dimension of the adjoint representation $N_A = N_C^2 - 1$. We are carrying out a partial renormalization of the Wilson coefficient of the structure function xF_3 for the axial vector coupling, which is performed by multiplying the unrenormalized Wilson coefficient by Z_A, [17], see also [25, 46],

$$\begin{aligned} Z_A = {} & 1 + \frac{\hat{a}_s^2}{\varepsilon}\left[\frac{22}{3}C_A C_F - \frac{8}{3}C_F T_F N_F\right] + \hat{a}_s^3\left[\frac{1}{\varepsilon^2}\left(-\frac{484}{27}C_A^2 C_F + \frac{352}{27}C_A C_F T_F N_F\right.\right. \\ & \left. -\frac{64}{27}C_F T_F^2 N_F^2\right) + \frac{1}{\varepsilon}\left(\frac{3578}{81}C_A^2 C_F - \frac{308}{9}C_A C_F^2 - \frac{1664}{81}C_A C_F T_F N_F + \frac{64}{9}C_F^2 T_F N_F\right. \\ & \left.\left. +\frac{32}{81}C_F T_F^2 N_F^2\right)\right] \ . \end{aligned} \tag{49}$$

The non–singlet Wilson coefficients are renormalized via

$$C_{i,q}^{\rm NS} = Z_{qq}^{\rm NS}\hat{C}_{i,q}^{\rm NS} \tag{50}$$

and the singlet Wilson coefficients via

$$\begin{pmatrix} C_{i,q}^{\rm S} \\ C_{i,g} \end{pmatrix} = Z^{{\rm S}\,T} \cdot \begin{pmatrix} \hat{C}_{i,q}^{\rm S} \\ \hat{C}_{i,g} \end{pmatrix} \ , \tag{51}$$

with

$$\begin{aligned} Z_{qq}^{\rm NS} = {} & 1 + a_s\frac{\gamma_{qq}^{(0),\rm NS}}{\varepsilon} + a_s^2\left[\frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{qq}^{(0),\rm NS^2} + \beta_0\gamma_{qq}^{(0),\rm NS}\right) + \frac{1}{2\varepsilon}\gamma_{qq}^{(0),\rm NS}\right] + a_s^3\left[\frac{1}{\varepsilon^3}\left(\frac{1}{6}\gamma_{qq}^{(0),\rm NS^3}\right.\right. \\ & \left. +\beta_0\gamma_{qq}^{(0),\rm NS^2} + \frac{4}{3}\beta_0^2\gamma_{qq}^{(0),\rm NS}\right) + \frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{qq}^{(0),\rm NS}\gamma_{qq}^{(1),\rm NS} + \frac{2}{3}\beta_0\gamma_{qq}^{(1),\rm NS} + \frac{2}{3}\beta_1\gamma_{qq}^{(0),\rm NS}\right) \\ & \left. +\frac{1}{3\varepsilon}\gamma_{qq}^{(2),\rm NS}\right] + O(a_s^4) \ , \end{aligned} \tag{52}$$

$$Z_{ij}^{\rm S} = \delta_{ij} + a_s\frac{\gamma_{ij}^{(0)}}{\varepsilon} + a_s^2\left[\frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{il}^{(0)}\gamma_{lj}^{(0)} + \beta_0\gamma_{ij}^{(0)}\right) + \frac{1}{2\varepsilon}\gamma_{ij}^{(1)}\right]$$

[3]We follow the notation of COLOR, see Ref. [56].

$$+a_s^3\left[\frac{1}{\varepsilon^3}\left(\frac{1}{6}\gamma_{il}^{(0)}\gamma_{lk}^{(0)}\gamma_{kj}^{(0)}+\beta_0\gamma_{il}^{(0)}\gamma_{lj}^{(0)}+\frac{4}{3}\beta_0^2\gamma_{ij}^{(0)}\right)\right.$$

$$\left.+\frac{1}{\varepsilon^2}\left(\frac{1}{6}\gamma_{il}^{(1)}\gamma_{lj}^{(0)}+\frac{1}{3}\gamma_{il}^{(0)}\gamma_{lj}^{(1)}+\frac{2}{3}\beta_0\gamma_{ij}^{(1)}+\frac{2}{3}\beta_1\gamma_{ij}^{(0)}\right)+\frac{1}{3\varepsilon}\gamma_{ij}^{(2)}\right]+O(a_s^4)\ . \tag{53}$$

The pole terms of the Wilson coefficients are predicted by the anomalous dimensions and the expansion coefficients of the unrenormalized Wilson coefficients in lower orders, where in the polarized case the anomalous dimensions γ_{ij} have to be replaced by $\Delta\gamma_{ij}$.

$$c_{i,q}^{\rm NS,(1,-1)} = -\gamma_{qq}^{\rm NS,(0)}\ , \tag{54}$$

$$c_{i,q}^{\rm NS,(2,-2)} = \gamma_{qq}^{\rm NS,(0)}\left(\beta_0+\frac{1}{2}\gamma_{qq}^{\rm NS,(0)}\right)\ , \tag{55}$$

$$c_{i,q}^{\rm NS,(2,-1)} = -c_{i,q}^{\rm NS,(1,0)}\left(\gamma_{qq}^{\rm NS,(0)}+2\beta_0\right)-\frac{1}{2}\gamma_{qq}^{\rm NS,(1)}\ , \tag{56}$$

$$c_{i,q}^{\rm NS,(3,-3)} = -\gamma_{qq}^{\rm NS,(0)}\left(\frac{4}{3}\beta_0^2+\beta_0\gamma_{qq}^{\rm NS,(0)}+\frac{1}{6}\,\gamma_{qq}^{\rm NS,(0)\,2}\right)\ , \tag{57}$$

$$\begin{aligned}c_{i,q}^{\rm NS,(3,-2)} &= 4\beta_0^2c_{i,q}^{\rm NS,(1,0)}+\frac{1}{3}\beta_1\gamma_{qq}^{\rm NS,(0)}\\ &\quad+3\beta_0c_{i,q}^{\rm NS,(1,0)}\gamma_{qq}^{\rm NS,(0)}+\frac{1}{2}c_{i,q}^{\rm NS,(1,0)}\,\gamma_{qq}^{\rm NS,(0)\,2}+\frac{4}{3}\beta_0\gamma_{qq}^{\rm NS,(1)}+\frac{1}{2}\gamma_{qq}^{\rm NS,(0)}\gamma_{qq}^{\rm NS,(1)}\ ,\end{aligned} \tag{58}$$

$$\begin{aligned}c_{i,q}^{\rm NS,(3,-1)} &= -\beta_1c_{i,q}^{\rm NS,(1,0)}-4\beta_0^2c_{i,q}^{\rm NS,(1,1)}-4\beta_0c_{i,q}^{\rm NS,(2,0)}-3\beta_0c_{i,q}^{\rm NS,(1,1)}\gamma_{qq}^{\rm NS,(0)}-c_{i,q}^{\rm NS,(2,0)}\gamma_{qq}^{\rm NS,(0)}\\ &\quad-\frac{1}{2}c_{i,q}^{\rm NS,(1,1)}\,\gamma_{qq}^{\rm NS,(0)\,2}-\frac{1}{2}c_{i,q}^{\rm NS,(1,0)}\gamma_{qq}^{\rm NS,(1)}-\frac{1}{3}\gamma_{qq}^{\rm NS,(2)}\ ,\end{aligned} \tag{59}$$

$$c_{i,q}^{\rm PS,(2,-2)} = \frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\ , \tag{60}$$

$$c_{i,q}^{\rm PS,(2,-1)} = -\gamma_{gq}^{(0)}c_{i,g}^{(1,0)}-\frac{1}{2}\gamma_{qq}^{\rm PS,(1)}\ , \tag{61}$$

$$c_{i,q}^{\rm PS,(3,-3)} = -\beta_0\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}-\frac{1}{6}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}-\frac{1}{3}\gamma_{qq}^{\rm NS,(0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\ , \tag{62}$$

$$\begin{aligned}c_{i,q}^{\rm PS,(3,-2)} &= 3\beta_0\gamma_{gq}^{(0)}c_{i,g}^{(1,0)}+\frac{1}{2}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}c_{i,g}^{(1,0)}+\frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}c_{i,q}^{\rm NS,(1,0)}+\frac{1}{6}\gamma_{gq}^{(1)}\gamma_{qg}^{(0)}+\frac{1}{3}\gamma_{gq}^{(0)}\gamma_{qg}^{(1)}\\ &\quad+\frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qq}^{\rm NS,(0)}c_{i,g}^{(1,0)}+\frac{4}{3}\beta_0\gamma_{qq}^{\rm PS,(1)}+\frac{1}{2}\gamma_{qq}^{\rm NS,(0)}\gamma_{qq}^{\rm PS,(1)}\ ,\end{aligned} \tag{63}$$

$$\begin{aligned}c_{i,q}^{\rm PS,(3,-1)} &= -4\beta_0c_{i,q}^{\rm PS,(2,0)}-3\beta_0\gamma_{gq}^{(0)}c_{i,g}^{(1,1)}-\gamma_{gq}^{(0)}c_{i,g}^{(2,0)}-\frac{1}{2}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}c_{i,g}^{(1,1)}-\frac{1}{2}\gamma_{gq}^{(1)}c_{i,g}^{(1,0)}\\ &\quad-\frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}c_{i,q}^{\rm NS,(1,1)}-\gamma_{qq}^{\rm NS,(0)}c_{i,q}^{\rm PS,(2,0)}-\frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qq}^{\rm NS,(0)}c_{i,g}^{(1,1)}-\frac{1}{2}\gamma_{qq}^{\rm PS,(1)}c_{i,q}^{\rm NS,(1,0)}\\ &\quad-\frac{1}{3}\gamma_{qq}^{\rm PS,(2)}\ ,\end{aligned} \tag{64}$$

and

$$c_{i,g}^{(1,-1)} = -\gamma_{qg}^{(0)}\ , \tag{65}$$

$$c_{i,g}^{(2,-2)} = \beta_0\gamma_{qg}^{(0)}+\frac{1}{2}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}+\frac{1}{2}\gamma_{qg}^{(0)}\gamma_{qq}^{\rm NS,(0)}\ , \tag{66}$$

$$c_{i,g}^{(2,-1)} = -2\beta_0c_{i,g}^{(1,0)}-\gamma_{gg}^{(0)}c_{i,g}^{(1,0)}-\gamma_{qg}^{(0)}c_{i,q}^{\rm NS,(1,0)}-\frac{1}{2}\gamma_{qg}^{(1)}\ , \tag{67}$$

$$
\begin{aligned}
c_{i,g}^{(3,-3)} &= -\frac{4}{3}\beta_0^2\gamma_{qg}^{(0)} - \beta_0\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} - \frac{1}{6}\,\gamma_{gg}^{(0)\,2}\,\gamma_{qg}^{(0)} - \frac{1}{6}\gamma_{gq}^{(0)}\,\gamma_{qg}^{(0)\,2} - \beta_0\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} \\
&\quad -\frac{1}{6}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} - \frac{1}{6}\gamma_{qg}^{(0)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}\ , \qquad (68)\\
c_{i,g}^{(3,-2)} &= 4\beta_0^2 c_{i,g}^{(1,0)} + 3\beta_0 c_{i,g}^{(1,0)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{i,g}^{(1,0)}\,\gamma_{gg}^{(0)\,2} + \frac{1}{3}\beta_1\gamma_{qg}^{(0)} + 3\beta_0 c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)} \\
&\quad +\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{6}\gamma_{gg}^{(1)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{4}{3}\beta_0\gamma_{qg}^{(1)} + \frac{1}{3}\gamma_{gg}^{(0)}\gamma_{qg}^{(1)} \\
&\quad +\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{6}\gamma_{qg}^{(1)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{3}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(1)} + \frac{1}{3}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{PS},(1)}\ , \qquad (69)\\
c_{i,g}^{(3,-1)} &= -\beta_1 c_{i,g}^{(1,0)} - 4\beta_0^2 c_{i,g}^{(1,1)} - 4\beta_0 c_{i,g}^{(2,0)} - 3\beta_0 c_{i,g}^{(1,1)}\gamma_{gg}^{(0)} - c_{i,g}^{(2,0)}\gamma_{gg}^{(0)} \\
&\quad -\frac{1}{2}c_{i,g}^{(1,1)}\,\gamma_{gg}^{(0)\,2} - \frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gg}^{(1)} - 3\beta_0 c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)} - c_{i,q}^{\mathrm{NS},(2,0)}\gamma_{qg}^{(0)} - c_{i,q}^{\mathrm{PS},(2,0)}\gamma_{qg}^{(0)} \\
&\quad -\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} - \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} - \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(1)} - \frac{1}{3}\gamma_{qg}^{(2)} \\
&\quad -\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\ . \qquad (70)
\end{aligned}
$$

For the longitudinal structure function F_L the poles are predicted by

$$
\begin{aligned}
c_{F_L,q}^{\mathrm{NS},(2,-1)} &= -c_{F_L,q}^{\mathrm{NS},(1,0)}\left(\gamma_{qq}^{\mathrm{NS},(0)} + 2\beta_0\right)\ , \qquad (71)\\
c_{F_L,q}^{\mathrm{NS},(3,-2)} &= 4\beta_0^2 c_{F_L,q}^{\mathrm{NS},(1,0)} + 3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}\ , \qquad (72)\\
c_{F_L,q}^{\mathrm{NS},(3,-1)} &= -\beta_1 c_{F_L,q}^{\mathrm{NS},(1,0)} - 4\beta_0^2 c_{F_L,q}^{\mathrm{NS},(1,1)} - 4\beta_0 c_{F_L,q}^{\mathrm{NS},(2,0)} - 3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)} - c_{F_L,q}^{\mathrm{NS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)} \\
&\quad -\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2} - \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(1)}\ , \qquad (73)
\end{aligned}
$$

$$
\begin{aligned}
c_{F_L,q}^{\mathrm{PS},(2,-1)} &= -\gamma_{gq}^{(0)}c_{F_L,g}^{(1,0)}\ , \qquad (74)\\
c_{F_L,q}^{\mathrm{PS},(3,-2)} &= 3\beta_0\gamma_{gq}^{(0)}c_{F_L,g}^{(1,0)} + \frac{1}{2}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}c_{F_L,g}^{(1,0)} + \frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}c_{F_L,q}^{\mathrm{NS},(1,0)} + \frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}c_{F_L,g}^{(1,0)}\ , \qquad (75)\\
c_{F_L,q}^{\mathrm{PS},(3,-1)} &= -4\beta_0 c_{F_L,q}^{\mathrm{PS},(2,0)} - 3\beta_0\gamma_{gq}^{(0)}c_{F_L,g}^{(1,1)} - \gamma_{gq}^{(0)}c_{F_L,g}^{(2,0)} - \frac{1}{2}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}c_{F_L,g}^{(1,1)} - \frac{1}{2}\gamma_{gq}^{(1)}c_{F_L,g}^{(1,0)} \\
&\quad -\frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}c_{F_L,q}^{\mathrm{NS},(1,1)} - \gamma_{qq}^{\mathrm{NS},(0)}c_{F_L,q}^{\mathrm{PS},(2,0)} - \frac{1}{2}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}c_{F_L,g}^{(1,1)} - \frac{1}{2}\gamma_{qq}^{\mathrm{PS},(1)}c_{F_L,q}^{\mathrm{NS},(1,0)}\ , \qquad (76)
\end{aligned}
$$

$$
\begin{aligned}
c_{F_L,g}^{(2,-1)} &= -2\beta_0 c_{F_L,g}^{(1,0)} - \gamma_{gg}^{(0)}c_{F_L,g}^{(1,0)} - \gamma_{qg}^{(0)}c_{F_L,q}^{\mathrm{NS},(1,0)}\ , \qquad (77)\\
c_{F_L,g}^{(3,-2)} &= 4\beta_0^2 c_{F_L,g}^{(1,0)} + 3\beta_0 c_{F_L,g}^{(1,0)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{F_L,g}^{(1,0)}\,\gamma_{gg}^{(0)\,2} + 3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} \\
&\quad +\frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\ , \qquad (78)\\
c_{F_L,g}^{(3,-1)} &= -\beta_1 c_{F_L,g}^{(1,0)} - 4\beta_0^2 c_{F_L,g}^{(1,1)} - 4\beta_0 c_{F_L,g}^{(2,0)} - 3\beta_0 c_{F_L,g}^{(1,1)}\gamma_{gg}^{(0)} - c_{F_L,g}^{(2,0)}\gamma_{gg}^{(0)} - \frac{1}{2}c_{F_L,g}^{(1,1)}\,\gamma_{gg}^{(0)\,2} \\
&\quad -\frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gg}^{(1)} - 3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)} - c_{F_L,q}^{\mathrm{NS},(2,0)}\gamma_{qg}^{(0)} - c_{F_L,q}^{\mathrm{PS},(2,0)}\gamma_{qg}^{(0)} - \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} \\
&\quad -\frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} - \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(1)} - \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\ . \qquad (79)
\end{aligned}
$$

The renormalized Wilson coefficients for the structure functions F_2, F_3 and g_1 have the following structure

$$
C_{i,q}^{\mathrm{NS}} = 1 + a_s\Big\{\ln\left(\frac{Q^2}{\mu^2}\right)\,[-\frac{1}{2}\gamma_{qq}^{\mathrm{NS},(0)}] + c_{i,q}^{\mathrm{NS},(1,0)}\Big\} + a_s^2\Big\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\,[\frac{1}{4}\beta_0\gamma_{qq}^{\mathrm{NS},(0)}
$$

$$
\begin{aligned}
& +\frac{1}{8}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}\big] - \ln\left(\frac{Q^2}{\mu^2}\right)\big[\beta_0 c_{i,q}^{\mathrm{NS},(1,0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}\gamma_{qq}^{\mathrm{NS},(1)}\big] + 2\beta_0 c_{i,q}^{\mathrm{NS},(1,1)} \\
& +c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)} + c_{i,q}^{\mathrm{NS},(2,0)}\Big\} + a_s^3\Big\{-\ln^3\left(\frac{Q^2}{\mu^2}\right)\big[\frac{1}{6}\beta_0^2\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{8}\beta_0\,\gamma_{qq}^{\mathrm{NS},(0)\,2} \\
& +\frac{1}{48}\,\gamma_{qq}^{\mathrm{NS},(0)\,3}\big] + \ln^2\left(\frac{Q^2}{\mu^2}\right)\big[\beta_0^2 c_{i,q}^{\mathrm{NS},(1,0)} + \frac{1}{4}\beta_1\gamma_{qq}^{\mathrm{NS},(0)} + \frac{3}{4}\beta_0 c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)} \\
& +\frac{1}{8}c_{i,q}^{\mathrm{NS},(1,0)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2} + \frac{1}{2}\beta_0\gamma_{qq}^{\mathrm{NS},(1)} + \frac{1}{4}\gamma_{qq}^{\mathrm{NS},(0)}\gamma_{qq}^{\mathrm{NS},(1)}\big] - \ln\left(\frac{Q^2}{\mu^2}\right)\big[\beta_1 c_{i,q}^{\mathrm{NS},(1,0)} \\
& +4\beta_0^2 c_{i,q}^{\mathrm{NS},(1,1)} + 2\beta_0 c_{i,q}^{\mathrm{NS},(2,0)} + 3\beta_0 c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2} \\
& +\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(1)} + \frac{1}{2}\gamma_{qq}^{\mathrm{NS},(2)}\big] + \beta_1 c_{i,q}^{\mathrm{NS},(1,1)} + 4\beta_0^2 c_{i,q}^{\mathrm{NS},(1,2)} + 4\beta_0 c_{i,q}^{\mathrm{NS},(2,1)} \\
& +3\beta_0 c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{qq}^{\mathrm{NS},(0)} + c_{i,q}^{\mathrm{NS},(2,1)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,2)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(1)} + c_{i,q}^{\mathrm{NS},(3,0)}\Big\} \\
& +O(a_s^4)\ , \qquad (80)
\end{aligned}
$$

$$
\begin{aligned}
C_{i,q}^{\mathrm{PS}} =\ & a_s^2\Big\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\big[\frac{1}{8}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\big] - \ln\left(\frac{Q^2}{\mu^2}\right)\big[\frac{1}{2}(c_{i,g}^{(1,0)}\gamma_{gq}^{(0)}) + \frac{1}{2}\gamma_{qq}^{\mathrm{PS},(1)}\big] + c_{i,g}^{(1,1)}\gamma_{gq}^{(0)} \\
& +c_{i,q}^{\mathrm{PS},(2,0)}\Big\} + a_s^3\Big\{-\ln^3\left(\frac{Q^2}{\mu^2}\right)\big[\frac{1}{8}(\beta_0\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}) + \frac{1}{48}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{24}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\big] \\
& +\ln^2\left(\frac{Q^2}{\mu^2}\right)\big[\frac{3}{4}\beta_0 c_{i,g}^{(1,0)}\gamma_{gq}^{(0)} + \frac{1}{8}c_{i,g}^{(1,0)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)} + \frac{1}{8}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{8}\gamma_{gq}^{(1)}\gamma_{qg}^{(0)} \\
& +\frac{1}{8}\gamma_{gq}^{(0)}\gamma_{qg}^{(1)} + \frac{1}{8}c_{i,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}\beta_0\gamma_{qq}^{\mathrm{PS},(1)} + \frac{1}{4}\gamma_{qq}^{\mathrm{NS},(0)}\gamma_{qq}^{\mathrm{PS},(1)}\big] - \ln\left(\frac{Q^2}{\mu^2}\right) \\
& \times\big[2\beta_0 c_{i,q}^{\mathrm{PS},(2,0)} + 3\beta_0 c_{i,g}^{(1,1)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(2,0)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gq}^{(1)} \\
& +\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{i,q}^{\mathrm{PS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{PS},(1)} + \frac{1}{2}\gamma_{qq}^{\mathrm{PS},(2)}\big] \\
& +4\beta_0 c_{i,q}^{\mathrm{PS},(2,1)} + 3\beta_0 c_{i,g}^{(1,2)}\gamma_{gq}^{(0)} + c_{i,g}^{(2,1)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(1,2)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)} + \frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gq}^{(1)} \\
& +\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + c_{i,q}^{\mathrm{PS},(2,1)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,g}^{(1,2)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{PS},(1)} + c_{i,q}^{\mathrm{PS},(3,0)}\Big\} \\
& +O(a_s^4)\ , \qquad (81)
\end{aligned}
$$

$$
\begin{aligned}
C_{i,g} =\ & a_s\Big\{-\ln\left(\frac{Q^2}{\mu^2}\right)\big[\frac{1}{2}\gamma_{qg}^{(0)}\big] + c_{i,g}^{(1,0)}\Big\} + a_s^2\Big\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\big[\frac{1}{4}\beta_0\gamma_{qg}^{(0)} + \frac{1}{8}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} \\
& +\frac{1}{8}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\big] - \ln\left(\frac{Q^2}{\mu^2}\right)\big[\beta_0 c_{i,g}^{(1,0)} + \frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)} + \frac{1}{2}\gamma_{qg}^{(1)}\big] \\
& +2\beta_0 c_{i,g}^{(1,1)} + c_{i,g}^{(1,1)}\gamma_{gg}^{(0)} + c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)} + c_{i,g}^{(2,0)}\Big\} + a_s^3\Big\{-\ln^3\left(\frac{Q^2}{\mu^2}\right)\big[\frac{1}{6}\beta_0^2\gamma_{qg}^{(0)} \\
& +\frac{1}{8}\beta_0\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{48}\,\gamma_{gg}^{(0)\,2}\,\gamma_{qg}^{(0)} + \frac{1}{48}\gamma_{gq}^{(0)}\,\gamma_{qg}^{(0)\,2} + \frac{1}{8}\beta_0\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + \frac{1}{48}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} \\
& +\frac{1}{48}\gamma_{qg}^{(0)}\,\gamma_{qq}^{\mathrm{NS},(0)\,2}\big] + \ln^2\left(\frac{Q^2}{\mu^2}\right)\big[\beta_0^2 c_{i,g}^{(1,0)} + \frac{3}{4}\beta_0 c_{i,g}^{(1,0)}\gamma_{gg}^{(0)} + \frac{1}{8}c_{i,g}^{(1,0)}\,\gamma_{gg}^{(0)\,2} + \frac{1}{4}\beta_1\gamma_{qg}^{(0)} \\
& +\frac{3}{4}\beta_0 c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)} + \frac{1}{8}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{8}\gamma_{gg}^{(1)}\gamma_{qg}^{(0)} + \frac{1}{8}c_{i,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}\beta_0\gamma_{qg}^{(1)} + \frac{1}{8}\gamma_{gg}^{(0)}\gamma_{qg}^{(1)}
\end{aligned}
$$

$$+\frac{1}{8}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{8}\gamma_{qg}^{(1)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{8}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(1)}+\frac{1}{8}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{PS},(1)}\big]$$
$$-\ln\left(\frac{Q^2}{\mu^2}\right)\big[(\beta_1 c_{i,g}^{(1,0)})+4\beta_0^2 c_{i,g}^{(1,1)}+2\beta_0 c_{i,g}^{(2,0)}+3\beta_0 c_{i,g}^{(1,1)}\gamma_{gg}^{(0)}+\frac{1}{2}c_{i,g}^{(2,0)}\gamma_{gg}^{(0)}$$
$$+\frac{1}{2}c_{i,g}^{(1,1)}\ {\gamma_{gg}^{(0)}}^2+\frac{1}{2}c_{i,g}^{(1,0)}\gamma_{gg}^{(1)}+3\beta_0 c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(2,0)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{i,q}^{\mathrm{PS},(2,0)}\gamma_{qg}^{(0)}$$
$$+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(1)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}\gamma_{qg}^{(2)}\big]$$
$$+\beta_1 c_{i,g}^{(1,1)}+4\beta_0^2 c_{i,g}^{(1,2)}+4\beta_0 c_{i,g}^{(2,1)}+3\beta_0 c_{i,g}^{(1,2)}\gamma_{gg}^{(0)}+c_{i,g}^{(2,1)}\gamma_{gg}^{(0)}+\frac{1}{2}c_{i,g}^{(1,2)}\ {\gamma_{gg}^{(0)}}^2$$
$$+\frac{1}{2}c_{i,g}^{(1,1)}\gamma_{gg}^{(1)}+3\beta_0 c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{qg}^{(0)}+c_{i,q}^{\mathrm{NS},(2,1)}\gamma_{qg}^{(0)}+c_{i,q}^{\mathrm{PS},(2,1)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}$$
$$+\frac{1}{2}c_{i,g}^{(1,2)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(1)}+\frac{1}{2}c_{i,q}^{\mathrm{NS},(1,2)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}+c_{i,g}^{(3,0)}\Big\}+O(a_s^4)\ . \quad (82)$$

Accordingly, one obtains for the structure function F_L,

$$C_{F_L,q}^{\mathrm{NS}} = a_s\Big\{c_{F_L,q}^{\mathrm{NS},(1,0)}\Big\}+a_s^2\Big\{-\ln\left(\frac{Q^2}{\mu^2}\right)\big[\beta_0 c_{F_L,q}^{\mathrm{NS},(1,0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)}\big]+2\beta_0 c_{F_L,q}^{\mathrm{NS},(1,1)}$$
$$+c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)}+c_{F_L,q}^{\mathrm{NS},(2,0)}\Big\}+a_s^3\Big\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\big[\beta_0^2 c_{F_L,q}^{\mathrm{NS},(1,0)}+\frac{3}{4}\beta_0 c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(0)}$$
$$+\frac{1}{8}c_{F_L,q}^{\mathrm{NS},(1,0)}\ {\gamma_{qq}^{\mathrm{NS},(0)}}^2\big]-\ln\left(\frac{Q^2}{\mu^2}\right)\big[\beta_1 c_{F_L,q}^{\mathrm{NS},(1,0)}+4\beta_0^2 c_{F_L,q}^{\mathrm{NS},(1,1)}+2\beta_0 c_{F_L,q}^{\mathrm{NS},(2,0)}$$
$$+3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\ {\gamma_{qq}^{\mathrm{NS},(0)}}^2+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{NS},(1)}\big]$$
$$+\beta_1 c_{F_L,q}^{\mathrm{NS},(1,1)}+4\beta_0^2 c_{F_L,q}^{\mathrm{NS},(1,2)}+4\beta_0 c_{F_L,q}^{\mathrm{NS},(2,1)}+3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,2)}\gamma_{qq}^{\mathrm{NS},(0)}+c_{F_L,q}^{\mathrm{NS},(2,1)}\gamma_{qq}^{\mathrm{NS},(0)}$$
$$+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,2)}\ {\gamma_{qq}^{\mathrm{NS},(0)}}^2+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{NS},(1)}+c_{F_L,q}^{\mathrm{NS},(3,0)}\Big\}+O(a_s^4)\ , \quad (83)$$

$$C_{F_L,q}^{\mathrm{PS}} = a_s^2\Big\{-\ln\left(\frac{Q^2}{\mu^2}\right)\big[\frac{1}{2}(c_{F_L,g}^{(1,0)}\gamma_{gq}^{(0)})\big]+c_{F_L,g}^{(1,1)}\gamma_{gq}^{(0)}+c_{F_L,q}^{\mathrm{PS},(2,0)}\Big\}$$
$$+a_s^3\Big\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\big[\frac{3}{4}\beta_0 c_{F_L,g}^{(1,0)}\gamma_{gq}^{(0)}+\frac{1}{8}c_{F_L,g}^{(1,0)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}+\frac{1}{8}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}$$
$$+\frac{1}{8}c_{F_L,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\big]-\ln\left(\frac{Q^2}{\mu^2}\right)\big[2\beta_0 c_{F_L,q}^{\mathrm{PS},(2,0)}+3\beta_0 c_{F_L,g}^{(1,1)}\gamma_{gq}^{(0)}+\frac{1}{2}c_{F_L,g}^{(2,0)}\gamma_{gq}^{(0)}$$
$$+\frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}+\frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gq}^{(1)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{PS},(2,0)}\gamma_{qq}^{\mathrm{NS},(0)}$$
$$+\frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qq}^{\mathrm{PS},(1)}\big]+4\beta_0 c_{F_L,q}^{\mathrm{PS},(2,1)}+3\beta_0 c_{F_L,g}^{(1,2)}\gamma_{gq}^{(0)}$$
$$+c_{F_L,g}^{(2,1)}\gamma_{gq}^{(0)}+\frac{1}{2}c_{F_L,g}^{(1,2)}\gamma_{gg}^{(0)}\gamma_{gq}^{(0)}+\frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gq}^{(1)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,2)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)}$$
$$+c_{F_L,q}^{\mathrm{PS},(2,1)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{F_L,g}^{(1,2)}\gamma_{gq}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qq}^{\mathrm{PS},(1)}+c_{q,(3,0)}^{\mathrm{PS},(L)}\Big\}+O(a_s^4)\ , \quad (84)$$

$$C_{F_L,g} = a_s\Big\{c_{F_L,g}^{(1,0)}\Big\}+a_s^2\Big\{-\ln\left(\frac{Q^2}{\mu^2}\right)\big[\beta_0 c_{F_L,g}^{(1,0)}+\frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gg}^{(0)}+\frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}\big]$$
$$+2\beta_0 c_{F_L,g}^{(1,1)}+c_{F_L,g}^{(1,1)}\gamma_{gg}^{(0)}+c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}+c_{F_L,g}^{(2,0)}\Big\}+a_s^3\Big\{\ln^2\left(\frac{Q^2}{\mu^2}\right)\big[\beta_0^2 c_{F_L,g}^{(1,0)}$$

$$+\frac{3}{4}\beta_0 c_{F_L,g}^{(1,0)}\gamma_{gg}^{(0)} + \frac{1}{8}c_{F_L,g}^{(1,0)}\gamma_{gg}^{(0)\,2} + \frac{3}{4}\beta_0 c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)} + \frac{1}{8}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}$$
$$+\frac{1}{8}c_{F_L,g}^{(1,0)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{8}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\Big] - \ln\left(\frac{Q^2}{\mu^2}\right)\Big[\beta_1 c_{F_L,g}^{(1,0)} + 4\beta_0^2 c_{F_L,g}^{(1,1)}$$
$$+2\beta_0 c_{F_L,g}^{(2,0)} + 3\beta_0 c_{F_L,g}^{(1,1)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{F_L,g}^{(2,0)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gg}^{(0)\,2} + \frac{1}{2}c_{F_L,g}^{(1,0)}\gamma_{gg}^{(1)}$$
$$+3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(2,0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\mathrm{PS},(2,0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}$$
$$+\frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,0)}\gamma_{qg}^{(1)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)}\Big] + \beta_1 c_{F_L,g}^{(1,1)} + 4\beta_0^2 c_{F_L,g}^{(1,2)}$$
$$+4\beta_0 c_{F_L,g}^{(2,1)} + 3\beta_0 c_{F_L,g}^{(1,2)}\gamma_{gg}^{(0)} + c_{F_L,g}^{(2,1)}\gamma_{gg}^{(0)} + \frac{1}{2}c_{F_L,g}^{(1,2)}\gamma_{gg}^{(0)\,2} + \frac{1}{2}c_{F_L,g}^{(1,1)}\gamma_{gg}^{(1)}$$
$$+3\beta_0 c_{F_L,q}^{\mathrm{NS},(1,2)}\gamma_{qg}^{(0)} + c_{F_L,q}^{\mathrm{NS},(2,1)}\gamma_{qg}^{(0)} + c_{F_L,q}^{\mathrm{PS},(2,1)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,2)}\gamma_{gg}^{(0)}\gamma_{qg}^{(0)}$$
$$+\frac{1}{2}c_{F_L,g}^{(1,2)}\gamma_{gq}^{(0)}\gamma_{qg}^{(0)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,1)}\gamma_{qg}^{(1)} + \frac{1}{2}c_{F_L,q}^{\mathrm{NS},(1,2)}\gamma_{qg}^{(0)}\gamma_{qq}^{\mathrm{NS},(0)} + c_{F_L,g}^{(3,0)}\Big\} + O(a_s^4)\,. \tag{85}$$

The Wilson coefficients for the different deep–inelastic structure functions are split w.r.t. the parton contents they are mapping to, the observables and the specific couplings of the gauge bosons exchanged, see Ref. [47].

For pure photon exchange there are two contributions in the flavor non–singlet case. One corresponds to a through-flowing external fermion line to which the photons are coupling to, which is of relative weight $\mathsf{w}_1 = 1$. The second case corresponds to a photon coupling to one through-flowing line and another to a closed internal fermion line, with relative weight

$$\mathsf{w}_2 = \frac{\mathrm{tr}(\hat{Q}_f\lambda_\alpha)}{\mathrm{tr}(\hat{Q}_f^2\lambda_\alpha)}\frac{1}{N_F}\sum_{f=1}^{N_F} e_f, \qquad \frac{\mathrm{tr}(\hat{Q}_f\lambda_\alpha)}{\mathrm{tr}(\hat{Q}_f^2\lambda_\alpha)} = 3\,, \tag{86}$$

for $SU(N_F)$ and λ_a the generalized Pauli–Gell–Mann matrices. e_f denotes the fermion charge and $\hat{Q}_f = \mathrm{diag}(2/3, -1/3, -1/3, 2/3, -1/3)$ the quark charge matrix of the electromagnetic current.

The flavor non–singlet structure functions for photon exchange read [14]

$$F_i^{\mathrm{NS},+}(x,Q^2) = \sum_{f=1}^{N_F} e_f^2\left[C_{i,q}^{\mathrm{NS}}\left(x,\frac{Q^2}{\mu^2}\right) + \mathsf{w}_2 C_{i,q}^{d_{abc}}\left(x,\frac{Q^2}{\mu^2}\right)\right]\otimes f_{q,f}^{\mathrm{NS},+}\left(x,\mu^2\right)\,, \tag{87}$$

with

$$\mathsf{w}_2 = \frac{\mathrm{tr}(\hat{Q}_f)}{N_F}\frac{\mathrm{tr}(\hat{Q}_f\lambda_\alpha)}{\mathrm{tr}(\hat{Q}_f^2\lambda_\alpha)}\,,\quad \mathsf{w}_2(N_F=3)=0\,,\quad \mathsf{w}_2(N_F=4)=\frac{1}{2}\,,\quad \mathsf{w}_2(N_F=5)=\frac{1}{5}\,. \tag{88}$$

The non–singlet distribution function is given by

$$f_{q,f}^{\mathrm{NS},+}\left(x,\mu^2\right) = f_{q,f}(x,\mu^2) + \bar{f}_{q,f}(x,\mu^2) - \frac{1}{N_F}\Sigma(x,\mu^2)\,, \tag{89}$$

with the quark singlet distribution

$$\Sigma(x,\mu^2) = \sum_{f=1}^{N_F} f_{q,f}(x,\mu^2) + \bar{f}_{q,f}(x,\mu^2)\,. \tag{90}$$

In the flavor singlet case the weight factor w_3 appears

$$\mathsf{w}_3 = \frac{1}{N_F}\frac{\left(\sum_{f=1}^{N_F} e_f\right)^2}{\sum_{f=1}^{N_F} e_f^2} \ , \quad \mathsf{w}_3(N_F=3) = 0 \ , \quad \mathsf{w}_3(N_F=4) = \frac{1}{10} \ , \quad \mathsf{w}_3(N_F=5) = \frac{1}{55} \ , \tag{91}$$

because of the overall normalization to the sum of the quark charge squares for all diagrams in which the electromagnetic current couples to two different fermion lines.

The flavor singlet structure functions for photon exchange are given by [14]

$$F_i^{\mathrm{S},+}(x,Q^2) = \left(\frac{1}{N_F}\sum_{f=1}^{N_F} e_f^2\right)\left[C_{i,q}^{\mathrm{S}}\left(x,\frac{Q^2}{\mu^2}\right)\otimes\Sigma(x,\mu^2) + C_{i,g}\left(x,\frac{Q^2}{\mu^2}\right)\otimes G(x,\mu^2)\right] \ , \tag{92}$$

with

$$C_{i,q}^{\mathrm{S}} = C_{i,q}^{\mathrm{NS}} + \mathsf{w}_3 C_{i,q}^{d_{abc}} + C_{i,q}^{\mathrm{PS}} \ , \tag{93}$$

$$C_{i,g} = C_{i,g}^{a} + \mathsf{w}_3 C_{i,g}^{d_{abc}} \ , \tag{94}$$

and $G(x,\mu^2)$ denotes the gluon distribution and $C_{i,q}^{\mathrm{S},1}$ the non–weighted contributions and $C_{i,q}^{\mathrm{S},2}$ the one corresponding to the charge weight factor w_3. Synonymous relations apply to the structure function $g_1(x,Q^2)$ by replacing the unpolarized quantities by the corresponding polarized ones.

Usually the deep–inelastic structure functions are represented referring to $N_F = 3$ massless Wilson coefficients and massive Wilson coefficients [36, 37, 44, 48–53] due to charm and bottom quark corrections. The scaling violations of the heavy flavor Wilson coefficients are very different compared to those in the massless case.

The renormalization of the forward Compton amplitude is performed by renormalizing the strong coupling constant and removing the collinear singularities into the running of the parton distribution functions. In the case of the non–singlet structure function $xF_3(x,Q^2)$ the original calculation is performed in the Larin scheme and we finally switch to the $\overline{\mathsf{MS}}$ scheme by a finite renormalization. In the case of the structure function $g_1(x,Q^2)$ this last step requires the knowledge of the polarized four–loop anomalous dimensions in the singlet case. Because of this we present the corresponding Wilson coefficients in the Larin scheme and switch to the $\overline{\mathsf{MS}}$ in addition only for the non–singlet Wilson coefficient, since here the respective Ward identity is known in explicit form. Thereby, we also derive the Z-factor $Z_5^{\mathrm{NS}}(N,a_s)$ from the ratio of off–shell massless non–singlet operator matrix elements to three–loop order.

3 Details of the calculation

The Feynman diagrams for the different massless Wilson coefficients are generated by `QGRAF` [54] using the forward Compton amplitude.

We maintain all contributions up to the first order in the R_ξ gauge parameter ξ, which is canceling already for the unrenormalized result. For the Compton amplitude the corresponding crossing relations have to be observed [3,31]. We decompose the Wilson coefficients into their flavor non–singlet $\mathbb{C}_{i,q}^{\mathrm{NS}}$, pure singlet $\mathbb{C}_{i,q}^{\mathrm{PS}}$ and gluonic $\mathbb{C}_{i,g}$ parts. At one–loop order the contributions up to $O(\varepsilon^2)$ and two–loop order up to $O(\varepsilon)$ are required to extract the three–loop Wilson coefficients. We perform the calculation keeping the gauge parameter ξ to first order as a test for gauge invariance and show that the corresponding terms disappear. The Dirac and Lorentz algebra is performed by FORM [55] and the color algebra is performed by using Color [56]. The crossing relations [3, 31] imply that for the Wilson coefficients contributing to the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ only the even moments contribute and to $xF_3(x, Q^2)$ and $g_1(x, Q^2)$ only the odd moments. The corresponding generating variable $\omega = 1/x$ appears therefore dominantly quadratically, except of a factor ω in the case of $xF_3(x, Q^2)$ and $g_1(x, Q^2)$.

The irreducible three–loop diagrams are reduced to 293 master integrals using the code Crusher [57] by applying the integration–by–parts relations [58, 59]. The master integrals are mapped to ω^2 dependent structures and a sufficient number of their Mellin moments is calculated and inserted into the forward Compton amplitude. The required numbers of moments are summarized in Table 1. For the gluonic and non-singlet structures we do not map to a manifestly ω^2 dependent structure a priori.

The number of moments in Table 1 in the lower loop orders includes also the terms needed to renormalize the Wilson coefficients at three–loop order. As has been observed before in Ref. [27] there is a small number of master–integral relations which are difficult to prove for vanishing powers in ω. We have verified them explicitly for their moments up to a much higher number than requested and to the respective contributing power in ε, cf. Table 1. We determine recurrence relations for the corresponding color and ζ–value projections [60] of the Wilson coefficients using the method of arbitrary large moments [61] implemented within the package SolveCoupledSystem [62].

This is done by using the method of guessing [63, 64] and its implementation in Sage [65, 66]. For the calculation of the necessary initial values for the difference equations we use the results given in [59, 67]. In the three–loop case the calculation is based on 5000 even or odd moments. We have explicitly checked, that the other moments vanish, which is implied by the amplitude crossing relations. The difference equations are solved by using methods from difference field theory [68] implemented in the package Sigma [69, 70] utilizing functions from HarmonicSums [71–79], to obtain the three–loop Wilson coefficients. The largest difference equation for the individual color, gauge parameter, and ζ_k–factors contributing in the present case has order $\mathtt{o}$ $= 25$ and degree $\mathtt{d} = 778$ and needed 4300 moments. In parallel, we calculated the Wilson coefficients by using the differential equations for the master integrals directly in the variable ω, using the method presented in Ref. [80]. The depth of the initial conditions in the dimensional parameter ε is the same as in the method described previously. The corresponding systems were decoupled using the formalisms of Ref. [81] implemented in the package ORESYS [82] and we further proceeded to find the N-space solution by using algorithms contained in the package HarmonicSums.

Comparing to the reconstruction of the anomalous dimensions and unpolarized Wilson coefficients out of their moments performed in Ref. [64] in 2008 the largest difference equation had order $\mathtt{o} = 35$ and degree $\mathtt{d} = 938$ requiring 5114 moments. Here, however, even and odd moments had been used. The overall computation time using the automated chain of codes described amounted to about one year on Intel(R) Xeon(R) CPU E5-2643 v4 processors, using also parallelizations.

Working in the variable ω^2 rather than ω we obtain the results of the recurrences first for only the even or odd moments expressed in terms of cyclotomic harmonic sums at argument $2N$. For a systematic study of this class of nested sums see [75]. These objects can be algorithmically reduced to simple harmonic sums at argument N using HarmonicSums.

Therefore, all Wilson coefficients can be expressed by harmonic sums [71, 72]

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_{\emptyset} = 1, \quad b, a_i \in \mathbb{Z}\backslash\{0\}, N \in \mathbb{N}\backslash\{0\}. \tag{95}$$

If it is clear from the context we will write $S_{\vec{a}}$ instead of $S_{\vec{a}}(N)$.

Their Mellin inversion to momentum fraction z–space

$$C(N) = \int_0^1 dz z^{N-1} \bar{C}(z) \tag{96}$$

can be performed using routines of the packages `HarmonicSums` and is expressed in terms of harmonic polylogarithms [73] given by

$$\mathrm{H}_{b,\vec{a}}(z) = \int_0^z dx f_b(x) \mathrm{H}_{\vec{a}}(x), \quad \mathrm{H}_\emptyset = 1, \quad b, a_i \in \{-1, 0, 1\}, \tag{97}$$

with the alphabet of letters

$$\mathfrak{A}_\mathrm{H} = \left\{ f_0(z) = \frac{1}{z}, \quad f_{-1}(z) = \frac{1}{1+z}, \quad f_1(z) = \frac{1}{1-z} \right\}. \tag{98}$$

In z–space one distinguishes three contributions to the individual Wilson coefficients because of their different treatment in Mellin convolutions,

$$\bar{C}(z) = \bar{C}^\delta(z) + \bar{C}^\mathrm{plu}(z) + \bar{C}^\mathrm{reg}(z), \tag{99}$$

where $\bar{C}^\delta(z) = c_0 \delta(1-z)$, $\bar{C}^\mathrm{reg}(z)$ is a regular function in $z \in [0, 1]$ and $\bar{C}^\mathrm{plu}(z)$ denotes the remaining genuine +-distribution, the Mellin transformation of which is given by

$$C^\mathrm{plu}(N) = \int_0^1 dz (z^{N-1} - 1) \bar{C}^\mathrm{plu}(z). \tag{100}$$

We will use this representation later on.

Wilson coefficient	1 loop	2 loop	3 loop
F_1^NS	126	1219	4300
F_1^PS	0	374	1708
F_1^g	104	960	3534
F_L^NS	48	560	2387
F_L^PS	0	175	774
F_L^g	54	434	2046
xF_3^NS	126	1219	4171
g_1^NS	126	1219	4171
g_1^PS	0	175	1458
g_1^g	84	1166	2998

Table 1: The necessary maximal number of non–vanishing even (resp. odd) Mellin moments for $F_1, F_L, (xF_3, g_1)$ to determine the Wilson coefficients.

In the Mellin N space representation it can technically occur that there are factors of up to $1/(N-2)^2$ contributing and structures of $1/(N-1)$, in the cases that they are physically not allowed. However, these are all tractable poles, which can be shown by expanding at $N = 2$ and $N = 1$ using `HarmonicSums`. No Kronecker symbols have to be introduced for this case and the usual analytic continuation, described in Ref. [78], can be applied to the N–space expressions directly.

The Wilson coefficients can be represented in N space by harmonic sums weighted by rational functions in N. Here the degree of the numerator needs not to be smaller than that of the denominator, which needs special care in performing the inverse Mellin transform to z–space. After applying the algebraic relations between harmonic sums, c.f. e.g. [79], one obtains the following set of 60 harmonic sums

$$\{S_1; S_2, S_{-2}; S_3, S_{-3}, S_{2,1}, S_{-2,1}; S_4, S_{-4}, S_{-2,2}, S_{3,1}, S_{-3,1}, S_{2,1,1}, S_{-2,1,1}; S_5, S_{-5}, S_{-2,3}, S_{2,3}, S_{2,-3}, S_{-2,-3}, S_{2,2,1}, S_{-2,1,-2}, S_{-2,2,1} S_{4,1}, S_{-4,1}, S_{2,1,-2}, S_{3,1,1}, S_{-3,1,1}, S_{2,1,1,1}, S_{-2,1,1,1}; S_6, S_{-6}, S_{-3,3}, S_{4,2}, S_{4,-2}, S_{-4,2}, S_{-4,-2}, S_{5,1}, S_{-5,1}, S_{-2,2,-2}, S_{-2,2,2}, S_{2,-3,1}, S_{-2,3,1}, S_{-3,1,-2}, S_{-3,-2,1}, S_{-3,2,1}, S_{-4,1,1}, S_{2,3,1}, S_{3,1,-2}, S_{3,2,1}, S_{4,1,1}, S_{-2,-2,1,1}, S_{-2,1,1,2}, S_{-2,2,1,1}, S_{2,-2,1,1}, S_{2,2,1,1}, S_{3,1,1,1}, S_{-3,1,1,1}, S_{2,1,1,1,1}, S_{-2,1,1,1,1}\}. \tag{101}$$

One may, furthermore, apply also the structural relations [78, 83] and obtains the following 31 harmonic sums.

$$\{S_1, S_{-1,1}, S_{-2,1}, S_{-3,1}, S_{-4,1}, S_{-5,1}, S_{2,1}, S_{4,1}, S_{-1,1,1}, S_{2,1,1}, S_{1,2,-1}, S_{2,1,-1}, S_{-2,1,-2}, S_{2,1,-2}, S_{-3,1,1}, S_{3,1,1}, S_{-2,2,-2}, S_{-3,-2,1}, S_{3,1,-2}, S_{-4,1,1}, S_{4,1,1}, S_{-2,1,1,1}, S_{2,1,1,1}, S_{-3,1,1,1}, S_{-2,-2,1,1}, S_{-2,1,1,2}, S_{-2,2,1,1}, S_{2,-2,1,1}, S_{2,2,1,1}, S_{-2,1,1,1,1}, S_{2,1,1,1,1}\} \tag{102}$$

spanning all quantities. In z space, the number of harmonic polylogarithms is, usually, higher if compared to the objects needed in Mellin N space. Here 68 harmonic polylogarithms of up to weight $\mathsf{w} = 5$, weighted by one more letter, contribute

$$\Big\{\mathrm{H}_{-1}, \mathrm{H}_0, \mathrm{H}_1, \mathrm{H}_{0,-1}, \mathrm{H}_{0,1}, \mathrm{H}_{0,-1,-1}, \mathrm{H}_{0,-1,1}, \mathrm{H}_{0,0,-1}, \mathrm{H}_{0,0,1}, \mathrm{H}_{0,1,-1}, \mathrm{H}_{0,1,1}, \mathrm{H}_{0,-1,-1,-1}, \mathrm{H}_{0,-1,-1,1}, \mathrm{H}_{0,-1,0,1}, \mathrm{H}_{0,-1,1,-1}, \mathrm{H}_{0,-1,1,1}, \mathrm{H}_{0,0,-1,-1}, \mathrm{H}_{0,0,-1,1}, \mathrm{H}_{0,0,0,-1}, \mathrm{H}_{0,0,0,1}, \mathrm{H}_{0,0,1,-1}, \mathrm{H}_{0,0,1,1}, \mathrm{H}_{0,1,-1,-1}, \mathrm{H}_{0,1,-1,1}, \mathrm{H}_{0,1,1,-1}, \mathrm{H}_{0,1,1,1}, \mathrm{H}_{0,-1,-1,-1,-1}, \mathrm{H}_{0,-1,-1,-1,1}, \mathrm{H}_{0,-1,-1,0,1}, \mathrm{H}_{0,-1,-1,1,-1}, \mathrm{H}_{0,-1,-1,1,1}, \mathrm{H}_{0,-1,0,-1,-1}, \mathrm{H}_{0,-1,0,-1,1}, \mathrm{H}_{0,-1,0,1,-1}, \mathrm{H}_{0,-1,0,1,1}, \mathrm{H}_{0,-1,1,-1,-1}, \mathrm{H}_{0,-1,1,-1,1}, \mathrm{H}_{0,-1,1,0,1}, \mathrm{H}_{0,-1,1,1,-1}, \mathrm{H}_{0,-1,1,1,1}, \mathrm{H}_{0,0,-1,-1,-1}, \mathrm{H}_{0,0,-1,-1,1}, \mathrm{H}_{0,0,-1,0,-1}, \mathrm{H}_{0,0,-1,0,1}, \mathrm{H}_{0,0,-1,1,-1}, \mathrm{H}_{0,0,-1,1,1}, \mathrm{H}_{0,0,0,-1,-1}, \mathrm{H}_{0,0,0,-1,1}, \mathrm{H}_{0,0,0,0,-1}, \mathrm{H}_{0,0,0,0,1}, \mathrm{H}_{0,0,0,1,-1}, \mathrm{H}_{0,0,0,1,1}, \mathrm{H}_{0,0,1,-1,-1}, \mathrm{H}_{0,0,1,-1,1}, \mathrm{H}_{0,0,1,0,-1}, \mathrm{H}_{0,0,1,0,1}, \mathrm{H}_{0,0,1,1,-1}, \mathrm{H}_{0,0,1,1,1}, \mathrm{H}_{0,1,-1,-1,-1}, \mathrm{H}_{0,1,-1,-1,1}, \mathrm{H}_{0,1,-1,1,-1}, \mathrm{H}_{0,1,-1,1,1}, \mathrm{H}_{0,1,0,1,-1}, \mathrm{H}_{0,1,0,1,1}, \mathrm{H}_{0,1,1,-1,-1}, \mathrm{H}_{0,1,1,-1,1}, \mathrm{H}_{0,1,1,1,-1}, \mathrm{H}_{0,1,1,1,1}\Big\} \tag{103}$$

after algebraic reduction.

4 The one– and two–loop Wilson coefficients

In the following we present the results for the massless Wilson coefficients contributing to the structure functions F_2, F_L, xF_3 and g_1 at one- and two–loop order, together with the higher expansion coefficients in ε of the corresponding forward Compton amplitudes in the unrenormalized case, see Section 2. These contributions are needed for the calculation of the three–loop Wilson coefficients. We will work in Mellin N space and also present the renormalized one- and two–loop Wilson coefficients.

4.1 One-Loop Order

At one loop–order one obtains the following expansion coefficients, cf. (39–41),

$$c_{F_2,q}^{(1,0)} = C_F\left\{\frac{2+5N-2N^2-9N^3}{N^2(1+N)} + \frac{-2+3N+3N^2}{N(1+N)}S_1 + 2S_1^2 - 2S_2\right\}, \tag{104}$$

$$\begin{aligned}c_{F_2,q}^{(1,1)} = C_F\Biggl\{&\frac{P_3}{2N^3(1+N)} + \left(\frac{-2-5N-14N^2-7N^3}{2N^2(1+N)} + S_2 + \zeta_2\right)S_1 - \frac{1}{3}S_1^3\\ &+\frac{\left(2-3N-3N^2\right)}{4N(1+N)}S_1^2 + \frac{\left(-2+3N+3N^2\right)}{4N(1+N)}S_2 + \frac{\left(-2-3N-3N^2\right)}{4N(1+N)}\zeta_2\\ &-\frac{2}{3}S_3\Biggr\},\end{aligned} \tag{105}$$

$$\begin{aligned}c_{F_2,q}^{(1,2)} = C_F\Biggl\{&\frac{P_5}{4N^4(1+N)} + \left(\frac{P_2}{4N^3(1+N)} + \frac{\left(2-3N-3N^2\right)}{8N(1+N)}[S_2+\zeta_2] + \frac{1}{3}S_3 - \frac{7}{3}\zeta_3\right)S_1\\ &+\left(\frac{2+5N+14N^2+7N^3}{8N^2(1+N)} - \frac{1}{4}S_2 - \frac{1}{4}\zeta_2\right)S_1^2 + \frac{\left(-2+3N+3N^2\right)}{24N(1+N)}S_1^3 + \frac{1}{24}S_1^4\\ &+\left(\frac{-2-5N-14N^2-7N^3}{8N^2(1+N)} + \frac{1}{4}\zeta_2\right)S_2 + \frac{1}{8}S_2^2 + \frac{\left(-2+3N+3N^2\right)}{12N(1+N)}S_3 - \frac{1}{4}S_4\\ &+\frac{\left(-2-5N+2N^2+9N^3\right)}{8N^2(1+N)}\zeta_2 + \frac{7\left(2+3N+3N^2\right)}{12N(1+N)}\zeta_3\Biggr\},\end{aligned} \tag{106}$$

$$c_{F_2,g}^{(1,0)} = T_FN_F\left\{-\frac{4\left(-2-N-4N^2+N^3\right)}{N^2(1+N)(2+N)} - \frac{4\left(2+N+N^2\right)}{N(1+N)(2+N)}S_1\right\}, \tag{107}$$

$$\begin{aligned}c_{F_2,g}^{(1,1)} = T_FN_F\Biggl\{&-\frac{2P_1}{N^3(1+N)(2+N)} + \frac{2\left(-2-N-4N^2+N^3\right)}{N^2(1+N)(2+N)}S_1\\ &+\frac{\left(2+N+N^2\right)}{N(1+N)(2+N)}[S_1^2-S_2-\zeta_2]\Biggr\},\end{aligned} \tag{108}$$

$$\begin{aligned}c_{F_2,g}^{(1,2)} = T_FN_F\Biggl\{&\frac{P_6}{N^4(1+N)(2+N)} + \Biggl(\frac{P_1}{N^3(1+N)(2+N)} + \frac{\left(2+N+N^2\right)}{2N(1+N)(2+N)}\\ &\times[S_2+\zeta_2]\Biggr)S_1 + \frac{\left(2+N+4N^2-N^3\right)}{2N^2(1+N)(2+N)}[S_1^2-S_2-\zeta_2] + \frac{\left(-2-N-N^2\right)}{6N(1+N)(2+N)}S_1^3\\ &+\frac{\left(-2-N-N^2\right)}{3N(1+N)(2+N)}S_3 + \frac{7\left(2+N+N^2\right)}{3N(1+N)(2+N)}\zeta_3\Biggr\},\end{aligned} \tag{109}$$

$$c_{F_L,q}^{(1,0)} = C_F\frac{4}{1+N}, \tag{110}$$

$$c_{F_L,q}^{(1,1)} = C_F\left\{-\frac{2}{1+N}[1+S_1]\right\}, \tag{111}$$

$$c_{F_L,q}^{(1,2)} = C_F\left\{\frac{2}{1+N} + \frac{1}{1+N}S_1 + \frac{1}{2(1+N)}[S_1^2-S_2-\zeta_2]\right\}, \tag{112}$$

$$c_{F_L,g}^{(1,0)} = T_F N_F \frac{16}{(1+N)(2+N)} , \tag{113}$$

$$c_{F_L,g}^{(1,1)} = T_F N_F \Biggl\{ -\frac{16}{(1+N)(2+N)} - \frac{8}{(1+N)(2+N)} S_1 \Biggr\} , \tag{114}$$

$$c_{F_L,g}^{(1,2)} = T_F N_F \Biggl\{ \frac{16}{(1+N)(2+N)} + \frac{8}{(1+N)(2+N)} S_1 + \frac{2}{(1+N)(2+N)} [S_1^2 - S_2 - \zeta_2] \Biggr\} , \tag{115}$$

$$c_{F_3,q}^{(1,0),\mathrm{L}} = C_F \Biggl\{ \frac{2+3N-2N^2-5N^3}{N^2(1+N)} + \frac{\bigl(-2+3N+3N^2\bigr)}{N(1+N)} S_1 + 2S_1^2 - 2S_2 \Biggr\} , \tag{116}$$

$$c_{F_3,q}^{(1,1),\mathrm{L}} = C_F \Biggl\{ \frac{2+3N+3N^3+8N^4}{2N^3(1+N)} + \Biggl(\frac{-2-3N-10N^2-7N^3}{2N^2(1+N)} + S_2 \Biggr) S_1 + \frac{\bigl(2-3N-3N^2\bigr)}{4N(1+N)} S_1^2 + \frac{\bigl(-2+3N+3N^2\bigr)}{4N(1+N)} S_2 - \frac{1}{3} S_1^3 - \frac{2}{3} S_3 + \Biggl(\frac{-2-3N-3N^2}{4N(1+N)} + S_1 \Biggr) \zeta_2 \Biggr\} , \tag{117}$$

$$c_{F_3,q}^{(1,2),\mathrm{L}} = C_F \Biggl\{ \frac{P_4}{4N^4(1+N)} + \Biggl(\frac{2+3N+10N^2+7N^3}{8N^2(1+N)} - \frac{1}{4} S_2 \Biggr) S_1^2 + \Biggl(\frac{-2-3N+19N^3+14N^4}{4N^3(1+N)} + \frac{\bigl(2-3N-3N^2\bigr)}{8N(1+N)} S_2 + \frac{1}{3} S_3 \Biggr) S_1 + \frac{\bigl(-2+3N+3N^2\bigr)}{24N(1+N)} S_1^3 + \frac{1}{24} S_1^4 + \frac{\bigl(-2-3N-10N^2-7N^3\bigr)}{8N^2(1+N)} S_2 + \frac{1}{8} S_2^2 + \frac{\bigl(-2+3N+3N^2\bigr)}{12N(1+N)} S_3 - \frac{1}{4} S_4 + \Biggl(\frac{-2-3N+2N^2+5N^3}{8N^2(1+N)} + \frac{\bigl(2-3N-3N^2\bigr) S_1}{8N(1+N)} - \frac{1}{4} S_1^2 + \frac{1}{4} S_2 \Biggr) \zeta_2 + \Biggl(\frac{7\bigl(2+3N+3N^2\bigr)}{12N(1+N)} - \frac{7}{3} S_1 \Biggr) \zeta_3 \Biggr\} , \tag{118}$$

$$c_{g_1,q}^{(1,0),\mathrm{NS,L}} = C_F \Biggl\{ \frac{2-5N-6N^2-9N^3}{N^2(1+N)} + \frac{\bigl(-2+3N+3N^2\bigr)}{N(1+N)} S_1 + 2S_1^2 - 2S_2 \Biggr\} , \tag{119}$$

$$c_{g_1,q}^{(1,1),\mathrm{NS,L}} = C_F \Biggl\{ \frac{P_8}{2N^3(1+N)} + \Biggl(\frac{-2+5N-10N^2-7N^3}{2N^2(1+N)} + S_2 \Biggr) S_1 + \frac{\bigl(2-3N-3N^2\bigr)}{4N(1+N)} S_1^2 - \frac{1}{3} S_1^3 + \frac{\bigl(-2+3N+3N^2\bigr)}{4N(1+N)} S_2 - \frac{2}{3} S_3 + \Biggl(\frac{-2-3N-3N^2}{4N(1+N)} + S_1 \Biggr) \zeta_2 \Biggr\} , \tag{120}$$

$$c_{g_1,q}^{(1,2),\mathrm{NS,L}} = C_F \Biggl\{ \frac{P_9}{4N^4(1+N)} + \Biggl(\frac{P_7}{4N^3(1+N)} + \frac{\bigl(2-3N-3N^2\bigr)}{8N(1+N)} S_2 + \frac{1}{3} S_3 \Biggr) S_1 + \Biggl(\frac{2-5N+10N^2+7N^3}{8N^2(1+N)} - \frac{1}{4} S_2 \Biggr) S_1^2 + \frac{\bigl(-2+3N+3N^2\bigr)}{24N(1+N)} S_1^3 + \frac{1}{24} S_1^4$$

$$+\frac{\big(-2+5N-10N^2-7N^3\big)S_2}{8N^2(1+N)}+\frac{1}{8}S_2^2+\frac{\big(-2+3N+3N^2\big)}{12N(1+N)}S_3-\frac{1}{4}S_4$$
$$+\left(\frac{-2+5N+6N^2+9N^3}{8N^2(1+N)}+\frac{\big(2-3N-3N^2\big)}{8N(1+N)}S_1-\frac{1}{4}S_1^2+\frac{1}{4}S_2\right)\zeta_2$$
$$+\left(\frac{7\big(2+3N+3N^2\big)}{12N(1+N)}-\frac{7}{3}S_1\right)\zeta_3\Bigg\}\,, \tag{121}$$

$$c_{g_1,g}^{(1,0)} = T_F N_F (N-1)\left\{-\frac{4(N-1)}{N^2(1+N)}-\frac{4}{N(1+N)}S_1\right\}\,, \tag{122}$$

$$c_{g_1,g}^{(1,1)} = T_F N_F (N-1)\Bigg\{\frac{2\big(1-N+2N^2\big)}{N^3(1+N)}+\frac{2(N-1)}{N^2(1+N)}S_1+\frac{1}{N(1+N)}S_1^2$$
$$-\frac{1}{N(1+N)}S_2-\frac{1}{N(1+N)}\zeta_2\Bigg\}\,, \tag{123}$$

$$c_{g_1,g}^{(1,2)} = T_F N_F (N-1)\Bigg\{\frac{\big(1-N+2N^2-4N^3\big)}{N^4(1+N)}+\Bigg(-\frac{\big(1-N+2N^2\big)}{N^3(1+N)}$$
$$+\frac{1}{2N(1+N)}S_2\Bigg)S_1-\frac{(N-1)}{2N^2(1+N)}S_1^2-\frac{1}{6N(1+N)}S_1^3+\frac{(N-1)}{2N^2(1+N)}S_2$$
$$-\frac{1}{3N(1+N)}S_3+\left(\frac{(N-1)}{2N^2(1+N)}+\frac{1}{2N(1+N)}S_1\right)\zeta_2+\frac{7}{3N(1+N)}\zeta_3\Bigg\}\,, \tag{124}$$

with

$$P_1 = -3N^4+6N^3-4N^2-N-2, \tag{125}$$
$$P_2 = 14N^4+27N^3+2N^2-5N-2, \tag{126}$$
$$P_3 = 18N^4+5N^3-2N^2+5N+2, \tag{127}$$
$$P_4 = -14N^5-6N^4-N^3+3N+2, \tag{128}$$
$$P_5 = -36N^5-10N^4+5N^3-2N^2+5N+2, \tag{129}$$
$$P_6 = -7N^5+12N^4-6N^3+4N^2+N+2, \tag{130}$$
$$P_7 = 14N^4+23N^3-8N^2+5N-2, \tag{131}$$
$$P_8 = 18N^4+9N^3+8N^2-5N+2, \tag{132}$$
$$P_9 = -36N^5-18N^4-15N^3+8N^2-5N+2. \tag{133}$$

The one–loop Wilson coefficients at $\mu^2 = Q^2$ are given by the expansion coefficients $c_{F_k,q(g)}^{(1,0)}$, $k = 2, L$. There are finite renormalizations for the Wilson coefficients of the structure functions xF_3 and g_1. The respective Wilson coefficients are given by

$$C_{F_3,q}^{(1),\rm NS} = C_F\left[-\frac{(2+3N)\big(-1+3N^2\big)}{N^2(1+N)}+\frac{\big(-2+3N+3N^2\big)S_1}{N(1+N)}+2S_1^2-2S_2\right]\,, \tag{134}$$
$$C_{g_1,q}^{(1),\rm NS,L} = C_F\left[\frac{2-5N-6N^2-9N^3}{N^2(1+N)}+\frac{\big(-2+3N+3N^2\big)S_1}{N(1+N)}+2S_1^2-2S_2\right]\,, \tag{135}$$

$$C_{g_1,q}^{(1),\rm NS} = C_{g_1,q}^{(1),\rm NS,L} - z_{qq}^{(1)} = C_{g_1,q}^{(1),\rm NS,L} + \frac{8C_F}{N(N+1)} = C_{F_3,q}^{(1),\rm NS}\,, \tag{136}$$

where the superscript L denotes the Wilson coefficient obtained in the Larin scheme.

4.2 Two-Loop Order

The corresponding expansion coefficients at two–loop order read, cf. (39–41),

$$
\begin{aligned}
c_{F_2,q}^{\rm NS,(2,0)} = {} & C_F\Biggl\{N_F T_F\Biggl[\frac{P_{78}}{54N^3(1+N)^3} + \Biggl(-\frac{2P_{32}}{27N^2(1+N)^2} + \frac{16}{3}S_2 + \frac{8}{3}\zeta_2\Biggr)S_1 \\
& -\frac{4\bigl(-6+19N+19N^2\bigr)S_1^2}{9N(1+N)} - \frac{16}{9}S_1^3 + \frac{4\bigl(-6+47N+47N^2\bigr)S_2}{9N(1+N)} - \frac{104}{9}S_3 \\
& +\frac{16}{3}S_{2,1} - \frac{2\bigl(2+3N+3N^2\bigr)\zeta_2}{3N(1+N)}\Biggr] + C_A\Biggl[\frac{P_{89}}{216(N-2)N^3(1+N)^3(3+N)} \\
& +\Biggl(\frac{P_{47}}{54N^2(1+N)^3} - \frac{4\bigl(3+11N+11N^2\bigr)S_2}{3N(1+N)} + 24S_3 - 16S_{2,1} - 16S_{-2,1} - \frac{22}{3}\zeta_2 \\
& -48\zeta_3\Biggr)S_1 + \Biggl(\frac{-66+233N+233N^2}{9N(1+N)} + 4S_2\Biggr)S_1^2 + \frac{44}{9}S_1^3 - 4S_2^2 - 8S_4 \\
& +\frac{\bigl(66-481N-1166N^2-583N^3\bigr)S_2}{9N(1+N)^2} + \frac{2\bigl(-72+215N+143N^2\bigr)S_3}{9N(1+N)} \\
& +\Biggl(\frac{4P_{61}}{(N-2)N^2(1+N)^2(3+N)} + \frac{8(-3+4N)S_1}{N(1+N)} + 16S_1^2 - 8S_2\Biggr)S_{-2} - 12S_{-2}^2 \\
& +\Biggl(\frac{16}{1+N} - 8S_1\Biggr)S_{-3} - 20S_{-4} - \frac{4\bigl(-6+11N+11N^2\bigr)S_{2,1}}{3N(1+N)} - \frac{16(-1+2N)S_{-2,1}}{N(1+N)} \\
& -24S_{3,1} + 8S_{-2,2} + 16S_{-3,1} + 24S_{2,1,1} + \frac{11\bigl(2+3N+3N^2\bigr)\zeta_2}{6N(1+N)} \\
& +\frac{6\bigl(6+N+9N^2\bigr)\zeta_3}{N(1+N)}\Biggr]\Biggr\} + C_F^2\Biggl[\frac{2S_2P_{29}}{N^2(1+N)^2} + \frac{P_{106}}{8(N-2)N^4(1+N)^4(3+N)} \\
& +\Biggl(\frac{P_{49}}{2N^3(1+N)^3} - \frac{2\bigl(-14+9N+9N^2\bigr)S_2}{N(1+N)} - \frac{56}{3}S_3 + 16S_{2,1} + 32S_{-2,1} \\
& +\frac{4\bigl(2+3N+3N^2\bigr)\zeta_2}{N(1+N)} + 48\zeta_3\Biggr)S_1 + \Biggl(\frac{2P_{23}}{N^2(1+N)^2} - 28S_2 - 8\zeta_2\Biggr)S_1^2 \\
& +\frac{2\bigl(-14+15N+15N^2\bigr)S_1^3}{3N(1+N)} + \frac{14}{3}S_1^4 + 6S_2^2 - \frac{2\bigl(-26+81N+33N^2\bigr)S_3}{3N(1+N)} + 12S_4 \\
& +\Biggl(-\frac{8P_{61}}{(N-2)N^2(1+N)^2(3+N)} - \frac{16(-3+4N)S_1}{N(1+N)} - 32S_1^2 + 16S_2\Biggr)S_{-2} + 24S_{-2}^2 \\
& +\Biggl(-\frac{32}{1+N} + 16S_1\Biggr)S_{-3} + 40S_{-4} + \frac{4\bigl(-2+3N+3N^2\bigr)S_{2,1}}{N(1+N)} + 40S_{3,1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{32(-1+2N)S_{-2,1}}{N(1+N)} - 16S_{-2,2} - 32S_{-3,1} - 24S_{2,1,1} - \frac{\big(2+3N+3N^2\big)^2\zeta_2}{2N^2(1+N)^2} \\
&-\frac{24\big(2-N+3N^2\big)\zeta_3}{N(1+N)}\Bigg], \qquad (137)
\end{aligned}
$$

$$
\begin{aligned}
c_{F_2,q}^{\mathrm{NS},(2,1)} = {}& C_F\Bigg\{C_A\Bigg[\frac{\zeta_2 P_{10}}{72N^2(1+N)^2} + \frac{2S_{2,1}P_{45}}{9N^2(1+N)^2(3+N)} - \frac{2\zeta_3 P_{77}}{9(N-2)N^2(1+N)^2(3+N)} \\
&+\frac{P_{111}}{2592(N-2)^2N^4(1+N)^4(3+N)^2} + \Bigg(\frac{S_2 P_{44}}{9N^2(1+N)^2(3+N)} + 8S_2^2 + 32S_4 \\
&+\frac{P_{93}}{648(N-2)N^3(1+N)^4(3+N)} - \frac{8\big(-27+56N+11N^2\big)S_3}{9N(1+N)} + \frac{16(N-1)S_{2,1}}{N(1+N)} \\
&-4S_{3,1} + \frac{16(-1+2N)S_{-2,1}}{N(1+N)} - 8S_{-2,2} + \frac{\big(-66+233N+233N^2\big)\zeta_2}{18N(1+N)} \\
&-16S_{-3,1} - 20S_{2,1,1} + \frac{72}{5}\zeta_2^2 + \frac{2\big(-216+689N+257N^2\big)\zeta_3}{9N(1+N)}\Bigg)S_1 \\
&+\Bigg(\frac{P_{34}}{108N^2(1+N)^3} + \frac{\big(15+10N+22N^2\big)S_2}{3N(1+N)} - 14S_3 + 12S_{2,1} + 8S_{-2,1} + \frac{11}{3}\zeta_2 \\
&+24\zeta_3\Bigg)S_1^2 + \Bigg(\frac{66-233N-233N^2}{27N(1+N)} - \frac{8}{3}S_2\Bigg)S_1^3 + \Bigg(\frac{P_{80}}{108N^2(1+N)^3(3+N)} \\
&-\frac{22}{3}S_3 - 8S_{2,1} - 8S_{-2,1} - \frac{22}{3}\zeta_2 - 48\zeta_3\Bigg)S_2 - \frac{11}{9}S_1^4 + \frac{\big(-6-9N-17N^2\big)S_2^2}{N(1+N)} \\
&+\Bigg(\frac{P_{48}}{27(N-2)N^2(1+N)^2(3+N)} + 2\zeta_2\Bigg)S_3 + \frac{\big(-63+119N+59N^2\big)S_4}{3N(1+N)} - 22S_5 \\
&+\Bigg(-\frac{4P_{98}}{(N-2)^2N^3(1+N)^3(3+N)^2} + \Bigg(-\frac{4P_{62}}{(N-2)N^2(1+N)^2(3+N)} + 16S_2 \\
&+4\zeta_2\Bigg)S_1 - \frac{4(-5+8N)S_1^2}{N(1+N)} - 8S_1^3 + \frac{4(-3+4N)S_2}{N(1+N)} - 32S_3 - 8S_{-2,1} - \frac{2\zeta_2}{N(1+N)} \\
&-72\zeta_3\Bigg)S_{-2} + \Bigg(\frac{8(-2+3N)}{N(1+N)} + 20S_1\Bigg)S_{-2}^2 + \Bigg(\frac{4P_{60}}{(N-2)N^2(1+N)^2(3+N)} \\
&+\frac{16(N-1)S_1}{N(1+N)} + 12S_1^2 + 4S_2 - 16S_{-2} + 2\zeta_2\Bigg)S_{-3} + \Bigg(\frac{8(-2+5N)}{N(1+N)} + 12S_1\Bigg)S_{-4} \\
&-38S_{-5} + 20S_{2,3} - 8S_{2,-3} - \frac{2(N-1)(-2+5N)S_{3,1}}{N(1+N)} - 44S_{4,1} + \Bigg(\frac{16}{N^2} - 4\zeta_2\Bigg) \\
&\times S_{-2,1} - \frac{8(-1+2N)S_{-2,2}}{N(1+N)} + 44S_{-2,3} - \frac{16(-1+2N)S_{-3,1}}{N(1+N)} + 8S_{-4,1} \\
&+\frac{4\big(12-7N+11N^2\big)S_{2,1,1}}{3N(1+N)} + 4S_{2,2,1} + 40S_{3,1,1} - 8S_{-2,1,-2} + 4S_{2,1,1,1}
\end{aligned}
$$

$$
-\frac{9(6+N+9N^2)\zeta_2^2}{5N(1+N)}\Bigg] + N_F T_F\Bigg[\frac{\zeta_2 P_{30}}{18N^2(1+N)^2} + \frac{P_{88}}{648N^4(1+N)^4}
+\Bigg(\frac{P_{79}}{162N^3(1+N)^3} - \frac{4(-6+19N+19N^2)S_2}{9N(1+N)} - \frac{2(-6+19N+19N^2)\zeta_2}{9N(1+N)}
+\frac{32}{9}S_3 - \frac{128}{9}\zeta_3\Bigg)S_1 + \Bigg(\frac{P_{32}}{27N^2(1+N)^2} - \frac{8}{3}S_2 - \frac{4}{3}\zeta_2\Bigg)S_1^2 + \frac{4}{9}S_1^4 + 4S_2^2 - \frac{28}{3}S_4
+\frac{4(-6+19N+19N^2)S_1^3}{27N(1+N)} + \Bigg(\frac{P_{11}}{27N^2(1+N)^2} + \frac{8\zeta_2}{3}\Bigg)S_2 - \frac{112}{9}S_{2,1} + 8S_{3,1}
-\frac{16}{3}S_{2,1,1} + \frac{16(-3+41N+41N^2)S_3}{27N(1+N)} + \frac{32(2+3N+3N^2)\zeta_3}{9N(1+N)}\Bigg]\Bigg\}
$$

$$
+C_F^2\Bigg[-\frac{2S_{2,1}P_{73}}{N^2(1+N)^2(2+N)(3+N)} + \frac{4\zeta_3 P_{87}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}
+\frac{\zeta_2 P_{76}}{8N^3(1+N)^3} + \frac{P_{112}}{32(N-2)^2N^5(1+N)^5(3+N)^2}
+\Bigg(\frac{P_{108}}{8(N-2)N^4(1+N)^4(2+N)(3+N)} + \Bigg(\frac{P_{75}}{2N^2(1+N)^2(2+N)(3+N)}
-8\zeta_2\Bigg)S_2 - 23S_2^2 - \frac{4(8-15N+5N^2)S_3}{N(1+N)} - 38S_4 + \frac{4(6-5N+3N^2)S_{2,1}}{N(1+N)} - 8S_{3,1}
-\frac{32(-1+2N)S_{-2,1}}{N(1+N)} + 32S_{-3,1} + 32S_{2,1,1} + \frac{(16+20N-17N^2-45N^3)\zeta_2}{2N^2(1+N)}
+16S_{-2,2} - \frac{72}{5}\zeta_2^2 - \frac{4(-40+165N+21N^2)\zeta_3}{3N(1+N)}\Bigg)S_1 + \Bigg(\frac{(-42+43N+27N^2)S_2}{2N(1+N)}
+\frac{P_{50}}{4N^3(1+N)^3} + 12S_3 - 16S_{2,1} - 16S_{-2,1} + \frac{3(N-1)(2+N)\zeta_2}{N(1+N)} - \frac{16}{3}\zeta_3\Bigg)S_1^2
+\Bigg(\frac{P_{12}}{6N^2(1+N)^2} + \frac{38}{3}S_2 + 4\zeta_2\Bigg)S_1^3 + \frac{(10-13N-13N^2)S_1^4}{4N(1+N)} - S_1^5
+\Bigg(\frac{P_{91}}{4N^3(1+N)^3(2+N)(3+N)} + \frac{28}{3}S_3 + 16S_{2,1} + 16S_{-2,1} + \frac{2(2+3N+3N^2)\zeta_2}{N(1+N)}
+48\zeta_3\Bigg)S_2 + \frac{(62+17N+81N^2)S_2^2}{4N(1+N)} + \Bigg(\frac{2P_{86}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}
-4\zeta_2\Bigg)S_3 + \frac{(58-83N-3N^2)S_4}{2N(1+N)} + 36S_5 + \Bigg(\frac{8P_{98}}{(N-2)^2N^3(1+N)^3(3+N)^2}
+\Bigg(\frac{8P_{62}}{(N-2)N^2(1+N)^2(3+N)} - 32S_2 - 8\zeta_2\Bigg)S_1 + \frac{8(-5+8N)S_1^2}{N(1+N)} + 16S_1^3
-\frac{8(-3+4N)S_2}{N(1+N)} + 64S_3 + 16S_{-2,1} + \frac{4\zeta_2}{N(1+N)} + 144\zeta_3\Bigg)S_{-2} + \Bigg(-\frac{16(-2+3N)}{N(1+N)}
$$

$$-40S_1\Bigg)S_{-2}^2+\Bigg(-\frac{8P_{60}}{(N-2)N^2(1+N)^2(3+N)}-\frac{32(N-1)S_1}{N(1+N)}-24S_1^2-8S_2$$
$$+32S_{-2}-4\zeta_2\Bigg)S_{-3}+\Bigg(-\frac{16(-2+5N)}{N(1+N)}-24S_1\Bigg)S_{-4}+76S_{-5}-32S_{2,3}+16S_{2,-3}$$
$$-\frac{16(-1+3N)S_{3,1}}{N(1+N)}+76S_{4,1}+\Bigg(-\frac{32}{N^2}+8\zeta_2\Bigg)S_{-2,1}+\frac{16(-1+2N)S_{-2,2}}{N(1+N)}-88S_{-2,3}$$
$$+\frac{32(-1+2N)S_{-3,1}}{N(1+N)}-16S_{-4,1}-\frac{2\big(14-9N+15N^2\big)S_{2,1,1}}{N(1+N)}-48S_{3,1,1}+16S_{-2,1,-2}$$
$$-20S_{2,1,1,1}+\frac{36\big(2-N+3N^2\big)\zeta_2^2}{5N(1+N)}\Bigg], \qquad (138)$$

$$c_{F_2,q}^{\mathrm{PS},(2,0)} = C_FN_FT_F\Bigg[\frac{8S_1P_{83}}{(N-1)N^3(1+N)^3(2+N)^2}+\frac{4P_{100}}{(N-1)N^4(1+N)^4(2+N)^3}$$
$$+\frac{8\big(2+N+N^2\big)^2S_1^2}{(N-1)N^2(1+N)^2(2+N)}-\frac{8\big(2+N+N^2\big)^2S_2}{(N-1)N^2(1+N)^2(2+N)}$$
$$+\frac{64S_{-2}}{(N-1)N(1+N)(2+N)}-\frac{4\big(2+N+N^2\big)^2\zeta_2}{(N-1)N^2(1+N)^2(2+N)}\Bigg], \qquad (139)$$

$$c_{F_2,q}^{\mathrm{PS},(2,1)} = C_FN_FT_F\Bigg[-\frac{16S_3P_{16}}{3(N-1)N^2(1+N)^2(2+N)}+\frac{64\zeta_3P_{17}}{3(N-1)N^2(1+N)^2(2+N)}$$
$$-\frac{4S_1^2P_{83}}{(N-1)N^3(1+N)^3(2+N)^2}+\frac{4S_2P_{83}}{(N-1)N^3(1+N)^3(2+N)^2}$$
$$+\frac{2\zeta_2P_{83}}{(N-1)N^3(1+N)^3(2+N)^2}-\frac{2P_{114}}{(N-2)(N-1)N^5(1+N)^5(2+N)^4(3+N)}$$
$$+\Bigg(-\frac{4P_{100}}{(N-1)N^4(1+N)^4(2+N)^3}+\frac{8\big(2+N+N^2\big)^2S_2}{(N-1)N^2(1+N)^2(2+N)}$$
$$+\frac{4\big(2+N+N^2\big)^2\zeta_2}{(N-1)N^2(1+N)^2(2+N)}\Bigg)S_1-\frac{8\big(2+N+N^2\big)^2S_1^3}{3(N-1)N^2(1+N)^2(2+N)}$$
$$+\Bigg(\frac{8P_{82}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}-\frac{64S_1}{(N-1)N(1+N)(2+N)}\Bigg)$$
$$\times S_{-2}+\frac{64S_{-3}}{(N-1)N(1+N)(2+N)}\Bigg], \qquad (140)$$

$$c_{F_2,g}^{(2,0)} = C_AN_FT_F\Bigg[\frac{8S_2P_{51}}{(N-1)N^2(1+N)^2(2+N)^2}-\frac{8S_1^2P_{63}}{(N-1)N^2(1+N)^2(2+N)^2}$$
$$-\frac{4P_{109}}{(N-1)N^4(1+N)^4(2+N)^4}+\Bigg(-\frac{4P_{96}}{(N-1)N^3(1+N)^3(2+N)^3}$$
$$+\frac{28\big(2+N+N^2\big)S_2}{N(1+N)(2+N)}+\frac{8\big(2+N+N^2\big)\zeta_2}{N(1+N)(2+N)}\Bigg)S_1-\frac{28\big(2+N+N^2\big)S_1^3}{3N(1+N)(2+N)}$$

$$
\begin{aligned}
&+\frac{8\big(-2+5N+5N^2\big)S_3}{3N(1+N)(2+N)}+\left(\frac{16P_{38}}{(N-1)N(1+N)^2(2+N)^2}+\frac{16(N-1)S_1}{N(1+N)}\right)S_{-2}\\
&-\frac{16\big(4+N+N^2\big)S_{-3}}{N(1+N)(2+N)}-\frac{16\big(2+N+N^2\big)S_{2,1}}{N(1+N)(2+N)}+\frac{64S_{-2,1}}{N(1+N)(2+N)}\\
&-\frac{16\big(1+N+N^2\big)\big(2+N+N^2\big)\zeta_2}{(N-1)N^2(1+N)^2(2+N)^2}-\frac{24(N-1)\zeta_3}{N(1+N)}\Bigg]\\
&+C_FN_FT_F\Bigg[\frac{4S_2P_{24}}{N^2(1+N)^2(2+N)}+\frac{2P_{103}}{(N-2)N^4(1+N)^4(2+N)(3+N)}\\
&-\frac{4S_1^2P_{25}}{N^2(1+N)^2(2+N)}+\left(-\frac{4P_{64}}{N^3(1+N)^3(2+N)}+\frac{20\big(2+N+N^2\big)S_2}{N(1+N)(2+N)}\right.\\
&\left.+\frac{8\big(2+N+N^2\big)\zeta_2}{N(1+N)(2+N)}\right)S_1-\frac{28\big(2+N+N^2\big)S_1^3}{3N(1+N)(2+N)}-\frac{16\big(-2+5N+5N^2\big)S_3}{3N(1+N)(2+N)}\\
&+\left(\frac{16P_{59}}{(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{128S_1}{N(1+N)(2+N)}\right)S_{-2}\\
&+\frac{64S_{-3}}{N(1+N)(2+N)}+\frac{16\big(2+N+N^2\big)S_{2,1}}{N(1+N)(2+N)}-\frac{128S_{-2,1}}{N(1+N)(2+N)}\\
&-\frac{2\big(2+N+N^2\big)\big(2+3N+3N^2\big)\zeta_2}{N^2(1+N)^2(2+N)}+\frac{48(N-1)\zeta_3}{N(1+N)}\Bigg], \qquad (141)
\end{aligned}
$$

$$
\begin{aligned}
c_{F_2,g}^{(2,1)} = {}& C_FN_FT_F\Bigg[-\frac{8S_{2,1}P_{14}}{N(1+N)^2(2+N)(3+N)}+\frac{2S_1^3P_{27}}{3N^2(1+N)^2(2+N)}\\
&-\frac{16\zeta_3P_{67}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{4S_3P_{72}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}\\
&+\frac{\zeta_2P_{68}}{N^3(1+N)^3(2+N)}+\frac{P_{113}}{(N-2)^2N^5(1+N)^5(2+N)(3+N)^2}\\
&+\left(-\frac{2\zeta_2P_{20}}{N^2(1+N)^2(2+N)}+\frac{2P_{102}}{(N-2)N^4(1+N)^4(2+N)(3+N)}-\frac{16(N-1)S_{2,1}}{N(1+N)}\right.\\
&-\frac{2S_2P_{42}}{N^2(1+N)^2(2+N)(3+N)}+\frac{20(N-2)(3+N)S_3}{N(1+N)(2+N)}+\frac{128S_{-2,1}}{N(1+N)(2+N)}\\
&\left.-\frac{128\big(-7+N+N^2\big)\zeta_3}{3N(1+N)(2+N)}\right)S_1+\left(\frac{P_{74}}{N^3(1+N)^3(2+N)}+\frac{\big(-34-9N-9N^2\big)S_2}{N(1+N)(2+N)}\right.\\
&\left.-\frac{6\big(2+N+N^2\big)\zeta_2}{N(1+N)(2+N)}\right)S_1^2+\frac{5\big(2+N+N^2\big)S_1^4}{2N(1+N)(2+N)}+\left(\frac{P_{81}}{N^3(1+N)^3(2+N)(3+N)}\right.\\
&\left.+\frac{6\big(2+N+N^2\big)\zeta_2}{N(1+N)(2+N)}\right)S_2+\frac{\big(78+7N+7N^2\big)S_2^2}{2N(1+N)(2+N)}-\frac{5\big(-6+5N+5N^2\big)S_4}{N(1+N)(2+N)}\\
&+\left(\frac{16S_1P_{54}}{(N-2)N^2(1+N)^2(2+N)(3+N)}-\frac{8P_{99}}{(N-2)^2N^3(1+N)^3(2+N)(3+N)^2}\right.
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{128S_1^2}{N(1+N)(2+N)}+\frac{64S_2}{N(1+N)(2+N)}\Biggr)S_{-2}+\frac{96S_{-2}^2}{N(1+N)(2+N)} \\
& +\Biggl(\frac{32(N-1)P_{18}}{(N-2)N^2(1+N)^2(3+N)}+\frac{64S_1}{N(1+N)(2+N)}\Biggr)S_{-3}+\frac{160S_{-4}}{N(1+N)(2+N)} \\
& +\frac{16\big(8+N+N^2\big)S_{3,1}}{N(1+N)(2+N)}-\frac{32\big(4+4N-N^2+N^3\big)S_{-2,1}}{N^2(1+N)(2+N)}-\frac{64S_{-2,2}}{N(1+N)(2+N)} \\
& -\frac{128S_{-3,1}}{N(1+N)(2+N)}+\frac{8\big(-10+N+N^2\big)S_{2,1,1}}{N(1+N)(2+N)}-\frac{72(N-1)\zeta_2^2}{5N(1+N)}\Biggr] \\
& +{\color{blue}C_A N_F T_F}\Biggl[\frac{16S_{2,1}P_{35}}{N^2(1+N)^2(2+N)^2}-\frac{8S_3P_{40}}{3(N-1)N(1+N)^2(2+N)^2} \\
& +\frac{4S_1^3P_{65}}{3(N-1)N^2(1+N)^2(2+N)^2}+\frac{8\zeta_3P_{70}}{3(N-1)N^2(1+N)^2(2+N)^2} \\
& -\frac{2\zeta_2P_{95}}{(N-1)N^3(1+N)^3(2+N)^3}+\frac{2P_{115}}{(N-2)(N-1)N^5(1+N)^5(2+N)^5(3+N)} \\
& +\Biggl(-\frac{8\zeta_2P_{37}}{(N-1)N(1+N)^2(2+N)^2}-\frac{4S_2P_{66}}{(N-1)N^2(1+N)^2(2+N)^2} \\
& +\frac{4\big(22+N+N^2\big)S_3}{N(1+N)(2+N)}+\frac{16S_{2,1}}{2+N}-\frac{64S_{-2,1}}{N(1+N)(2+N)}-\frac{64\big(13+2N+2N^2\big)\zeta_3}{3N(1+N)(2+N)}\Biggr)S_1 \\
& +\Biggl(\frac{P_{92}}{(N-1)N^2(1+N)^3(2+N)^3}-\frac{\big(30+19N+19N^2\big)S_2}{N(1+N)(2+N)}-\frac{6\big(2+N+N^2\big)\zeta_2}{N(1+N)(2+N)}\Biggr) \\
& \times S_1^2+\frac{5\big(2+N+N^2\big)S_1^4}{2N(1+N)(2+N)}+\Biggl(\frac{P_{94}}{(N-1)N^3(1+N)^3(2+N)^3}+\frac{2\big(2+N+N^2\big)\zeta_2}{N(1+N)(2+N)}\Biggr) \\
& \times S_2+\frac{\big(-2+15N+15N^2\big)S_2^2}{2N(1+N)(2+N)}+\frac{\big(2+21N+21N^2\big)S_4}{N(1+N)(2+N)}+\Biggl(\frac{8(N-1)S_2}{N(1+N)} \\
& -\frac{32S_1P_{52}}{(N-1)N^2(1+N)^2(2+N)^2}-\frac{8P_{107}}{(N-2)(N-1)N^3(1+N)^3(2+N)^3(3+N)} \\
& -\frac{8(N-2)(3+N)S_1^2}{N(1+N)(2+N)}+\frac{4\big(2+N+N^2\big)\zeta_2}{N(1+N)(2+N)}\Biggr)S_{-2}+\frac{8\big(-4+N+N^2\big)S_{-2}^2}{N(1+N)(2+N)} \\
& +\Biggl(\frac{8P_{55}}{(N-1)N^2(1+N)^2(2+N)^2}+\frac{16S_1}{2+N}\Biggr)S_{-3}-\frac{8\big(12+N+N^2\big)S_{-4}}{N(1+N)(2+N)} \\
& -\frac{16\big(5+N+N^2\big)S_{3,1}}{N(1+N)(2+N)}+\frac{16\big(4+4N-N^2+N^3\big)S_{-2,1}}{N^2(1+N)(2+N)}+\frac{32S_{-2,2}}{N(1+N)(2+N)} \\
& +\frac{64S_{-3,1}}{N(1+N)(2+N)}-\frac{8\big(-4+N+N^2\big)S_{2,1,1}}{N(1+N)(2+N)}+\frac{36(N-1)\zeta_2^2}{5N(1+N)}\Biggr], \qquad (142)
\end{aligned}
$$

$$
c_{F_L,q}^{\mathrm{NS},(2,0)} = {\color{blue}C_F}\Biggl\{{\color{blue}N_F T_F}\Biggl[-\frac{8\big(-6+13N+25N^2\big)}{9N(1+N)^2}-\frac{32S_1}{3(1+N)}\Biggr]+{\color{blue}C_A}\Biggl[\frac{136S_1}{3(1+N)}
$$

$$+\frac{2P_{31}}{9(N-2)N(1+N)^2(3+N)}+\frac{16S_3}{1+N}+\Biggl(-\frac{32P_{15}}{(N-2)N(1+N)^2(3+N)}$$

$$+\frac{32S_1}{1+N}\Biggr)S_{-2}+\frac{16S_{-3}}{1+N}-\frac{32S_{-2,1}}{1+N}-\frac{48\zeta_3}{1+N}\Biggr]\Biggr\}+C_F^2\Biggl[-\frac{16(1+2N)S_1}{N(1+N)}+\frac{24S_1^2}{1+N}$$

$$-\frac{2P_{71}}{(N-2)N^2(1+N)^3(3+N)}-\frac{8S_2}{1+N}-\frac{32S_3}{1+N}+\Biggl(\frac{64P_{15}}{(N-2)N(1+N)^2(3+N)}$$

$$-\frac{64S_1}{1+N}\Biggr)S_{-2}-\frac{32S_{-3}}{1+N}+\frac{64S_{-2,1}}{1+N}+\frac{96\zeta_3}{1+N}\Biggr], \tag{143}$$

$$c_{F_L,q}^{\mathrm{NS},(2,1)} = C_F\Biggl\{N_FT_F\Biggl[\frac{2P_{33}}{27N^2(1+N)^3}+\frac{8\bigl(-6+13N+25N^2\bigr)S_1}{9N(1+N)^2}+\frac{16S_1^2}{3(1+N)}-\frac{16S_2}{3(1+N)}$$

$$-\frac{8\zeta_2}{3(1+N)}\Biggr]+C_A\Biggl[\frac{24\zeta_3P_{13}}{(N-2)N(1+N)^2(3+N)}+\frac{8S_3P_{21}}{(N-2)N(1+N)^2(3+N)}$$

$$+\frac{P_{90}}{54(N-2)^2N^2(1+N)^3(3+N)^2}+\Biggl(-\frac{2P_{46}}{9(N-2)N(1+N)^3(3+N)}-\frac{40S_3}{1+N}$$

$$+\frac{8(N-1)(3+2N)S_2}{(1+N)^2(3+N)}+\frac{16S_{2,1}}{1+N}+\frac{32S_{-2,1}}{1+N}+\frac{96\zeta_3}{1+N}\Biggr)S_1-\Biggl(\frac{80}{3(1+N)}+\frac{4S_2}{1+N}\Biggr)$$

$$\times S_1^2-\frac{16\bigl(-6-8N+N^2\bigr)S_2}{3(1+N)^2(3+N)}+\frac{8S_2^2}{1+N}+\frac{20S_4}{1+N}+\Biggl(\frac{32S_1P_{15}}{(N-2)N(1+N)^2(3+N)}$$

$$-\frac{16P_{69}}{(N-2)^2N^2(1+N)^3(3+N)^2}-\frac{32S_1^2}{1+N}+\frac{16S_2}{1+N}\Biggr)S_{-2}+\frac{24S_{-2}^2}{1+N}+\frac{40S_{-4}}{1+N}$$

$$-\Biggl(\frac{32P_{15}}{(N-2)N(1+N)^2(3+N)}-\frac{16S_1}{1+N}\Biggr)S_{-3}-\frac{16(N-1)(3+2N)S_{2,1}}{(1+N)^2(3+N)}$$

$$+\frac{24S_{3,1}}{1+N}-\frac{16S_{-2,2}}{1+N}-\frac{32S_{-3,1}}{1+N}-\frac{24S_{2,1,1}}{1+N}+\frac{22\zeta_2}{3(1+N)}+\frac{72\zeta_2^2}{5(1+N)}\Biggr]\Biggr\}$$

$$+C_F^2\Biggl[\frac{2S_2P_{28}}{N(1+N)^2(2+N)(3+N)}-\frac{48\zeta_3P_{36}}{(N-2)N(1+N)^2(2+N)(3+N)}$$

$$-\frac{16S_3P_{43}}{3(N-2)N(1+N)^2(2+N)(3+N)}+\frac{P_{105}}{2(N-2)^2N^3(1+N)^4(3+N)^2}$$

$$+\Biggl(\frac{2P_{85}}{(N-2)N^2(1+N)^3(2+N)(3+N)}+\frac{4\bigl(46+41N-2N^2-5N^3\bigr)S_2}{(1+N)^2(2+N)(3+N)}$$

$$+\frac{80S_3}{1+N}-\frac{32S_{2,1}}{1+N}-\frac{64S_{-2,1}}{1+N}+\frac{8\zeta_2}{1+N}-\frac{192\zeta_3}{1+N}\Biggr)S_1+\Biggl(\frac{2(1+3N)(4+3N)}{N(1+N)^2}$$

$$+\frac{8S_2}{1+N}\Biggr)S_1^2-\frac{28S_1^3}{3(1+N)}-\frac{16S_2^2}{1+N}-\frac{40S_4}{1+N}+\Biggl(-\frac{64S_1P_{15}}{(N-2)N(1+N)^2(3+N)}$$

$$+\frac{64S_1^2}{1+N}-\frac{32S_2}{1+N}+\frac{32P_{69}}{(N-2)^2N^2(1+N)^3(3+N)^2}\Biggr)S_{-2}-\frac{48S_{-2}^2}{1+N}+\Biggl(-\frac{32S_1}{1+N}$$

$$+\frac{64P_{15}}{(N-2)N(1+N)^2(3+N)}\Biggr)S_{-3}-\frac{80S_{-4}}{1+N}+\frac{16\bigl(-8+7N+16N^2+5N^3\bigr)S_{2,1}}{(1+N)^2(2+N)(3+N)}$$

$$-\frac{48S_{3,1}}{1+N}+\frac{32S_{-2,2}}{1+N}+\frac{64S_{-3,1}}{1+N}+\frac{48S_{2,1,1}}{1+N}-\frac{2\bigl(2+3N+3N^2\bigr)\zeta_2}{N(1+N)^2}-\frac{144\zeta_2^2}{5(1+N)}\Biggr], \quad (144)$$

$$c_{F_L,q}^{\mathrm{PS},(2,0)} = C_FN_FT_F\Biggl[-\frac{32P_{39}}{(N-1)N^2(1+N)^3(2+N)^2}-\frac{64\bigl(2+N+N^2\bigr)S_1}{(N-1)N(1+N)^2(2+N)}\Biggr], \quad (145)$$

$$c_{F_L,q}^{\mathrm{PS},(2,1)} = C_FN_FT_F\Biggl[\frac{16P_{101}}{(N-2)(N-1)N^3(1+N)^4(2+N)^3(3+N)}$$

$$+\frac{32\bigl(2+N+N^2\bigr)S_1^2}{(N-1)N(1+N)^2(2+N)}+\frac{32S_1P_{39}}{(N-1)N^2(1+N)^3(2+N)^2}$$

$$-\frac{32\bigl(2+N+N^2\bigr)S_2}{(N-1)N(1+N)^2(2+N)}-\frac{32\bigl(6+N+N^2\bigr)S_{-2}}{(N-2)(N-1)(1+N)(2+N)(3+N)}$$

$$-\frac{16\bigl(2+N+N^2\bigr)\zeta_2}{(N-1)N(1+N)^2(2+N)}\Biggr], \quad (146)$$

$$c_{F_L,g}^{(2,0)} = C_AN_FT_F\Biggl[\frac{64S_1P_{22}}{(N-1)N(1+N)^2(2+N)^2}-\frac{32P_{41}}{(N-1)N^2(1+N)^3(2+N)^3}$$

$$+\frac{96S_1^2}{(1+N)(2+N)}-\frac{32S_2}{(1+N)(2+N)}-\frac{64S_{-2}}{(1+N)(2+N)}\Biggr]$$

$$+C_FN_FT_F\Biggl[-\frac{16P_{56}}{(N-2)N^2(1+N)^3(2+N)(3+N)}-\frac{64\bigl(1+N+N^2\bigr)S_1}{N(1+N)^2(2+N)}$$

$$+\frac{64(N-1)S_{-2}}{(N-2)(1+N)(3+N)}\Biggr], \quad (147)$$

$$c_{F_L,g}^{(2,1)} = C_AN_FT_F\Biggl[\frac{16S_2P_{19}}{(N-1)N(1+N)^2(2+N)^2}-\frac{16S_1^2P_{26}}{(N-1)N(1+N)^2(2+N)^2}$$

$$+\frac{16P_{104}}{(N-2)(N-1)N^3(1+N)^4(2+N)^4(3+N)}-\frac{112S_1^3}{3(1+N)(2+N)}$$

$$+\Biggl(-\frac{32P_{84}}{(N-1)N^2(1+N)^3(2+N)^3}+\frac{112S_2}{(1+N)(2+N)}+\frac{32\zeta_2}{(1+N)(2+N)}\Biggr)S_1$$

$$+\Biggl(-\frac{16P_{58}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}+\frac{96S_1}{(1+N)(2+N)}\Biggr)S_{-2}$$

$$-\frac{48S_{-3}}{(1+N)(2+N)}-\frac{64S_{2,1}}{(1+N)(2+N)}-\frac{64\bigl(1+N+N^2\bigr)\zeta_2}{(N-1)N(1+N)^2(2+N)^2}$$

$$+\frac{208S_3}{3(1+N)(2+N)}-\frac{32S_{-2,1}}{(1+N)(2+N)}-\frac{144\zeta_3}{(1+N)(2+N)}\Biggr]$$

$$+C_FN_FT_F\Biggl[\frac{8P_{97}}{(N-2)^2N^3(1+N)^4(2+N)(3+N)^2}+\Biggl(-\frac{16(5+3N)S_2}{(1+N)(2+N)(3+N)}$$

$$
\left.\frac{16P_{53}}{(N-2)N^2(1+N)^3(2+N)(3+N)}\right)S_1+\frac{8(N-1)\big(12+13N+5N^2\big)S_2}{N(1+N)^2(2+N)(3+N)}
+\frac{8\big(4+5N+5N^2\big)S_1^2}{N(1+N)^2(2+N)}-\frac{16\big(-26+3N+7N^2\big)S_3}{(N-2)(1+N)(2+N)(3+N)}
+\left(\frac{128P_{57}}{(N-2)^2N(1+N)^2(2+N)(3+N)^2}-\frac{128\big(-4+N+N^2\big)S_1}{(N-2)(1+N)(2+N)(3+N)}\right)S_{-2}
+\frac{32\big(2+N+N^2\big)S_{-3}}{(N-2)(1+N)(2+N)(3+N)}+\frac{32(5+3N)S_{2,1}}{(1+N)(2+N)(3+N)}+\frac{64S_{-2,1}}{(1+N)(2+N)}
-\left.\frac{8\big(2+N+N^2\big)\zeta_2}{N(1+N)^2(2+N)}+\frac{192\big(-6+3N+N^2\big)\zeta_3}{(N-2)(1+N)(2+N)(3+N)}\right], \tag{148}
$$

with the polynomials

$$P_{10} = -1239N^4-1554N^3-259N^2+584N+396 , \tag{149}$$
$$P_{11} = -1079N^4-2410N^3-1379N^2-192N-108 , \tag{150}$$
$$P_{12} = -75N^4-248N^3-225N^2-108N-50 , \tag{151}$$
$$P_{13} = N^4-12N^3-11N^2+14N+24 , \tag{152}$$
$$P_{14} = N^4-7N^3-7N^2+27N+18 , \tag{153}$$
$$P_{15} = N^4+2N^3-N^2-2N-6 , \tag{154}$$
$$P_{16} = N^4+2N^3+11N^2+10N+4 , \tag{155}$$
$$P_{17} = N^4+2N^3+14N^2+13N+4 , \tag{156}$$
$$P_{18} = N^4+3N^3-6N^2-2N-6 , \tag{157}$$
$$P_{19} = 2N^4+6N^3-9N^2-15N-8 , \tag{158}$$
$$P_{20} = 3N^4+4N^3-N^2-6N-8 , \tag{159}$$
$$P_{21} = 4N^4+N^3-7N^2+2N-12 , \tag{160}$$
$$P_{22} = 4N^4+6N^3-9N^2-9N-4 , \tag{161}$$
$$P_{23} = 5N^4+31N^3+38N^2+24N+11 , \tag{162}$$
$$P_{24} = 6N^4+7N^3+8N^2+5N-6 , \tag{163}$$
$$P_{25} = 6N^4+9N^3+4N^2-5N-10 , \tag{164}$$
$$P_{26} = 12N^4+18N^3-23N^2-23N-8 , \tag{165}$$
$$P_{27} = 15N^4+23N^3+13N^2-9N-22 , \tag{166}$$
$$P_{28} = 15N^4+28N^3-21N^2-38N-24 , \tag{167}$$
$$P_{29} = 26N^4+45N^3+11N^2-8N-5 , \tag{168}$$
$$P_{30} = 111N^4+138N^3+11N^2-64N-36 , \tag{169}$$
$$P_{31} = 281N^4+430N^3-1927N^2-1284N+396 , \tag{170}$$
$$P_{32} = 373N^4+998N^3+673N^2+192N+108 , \tag{171}$$
$$P_{33} = 541N^4+782N^3+199N^2-42N+36 , \tag{172}$$
$$P_{34} = -4541N^5-17529N^4-20631N^3-8099N^2-1968N-1188 , \tag{173}$$
$$P_{35} = N^5-5N^3-6N^2-8N-4 , \tag{174}$$
$$P_{36} = N^5-10N^4-37N^3-6N^2+56N+48 , \tag{175}$$
$$P_{37} = N^5-2N^4-10N^3-5N^2-6N-2 , \tag{176}$$

$$P_{38} = N^5 - N^4 - 5N^3 + 3N^2 + 14N + 12 \ , \tag{177}$$
$$P_{39} = 3N^5 + 10N^4 + 16N^3 + 11N^2 - 4N - 4 \ , \tag{178}$$
$$P_{40} = 8N^5 - 7N^4 - 11N^3 + 68N^2 + 78N + 32 \ , \tag{179}$$
$$P_{41} = 13N^5 + 40N^4 + 45N^3 + 18N^2 - 12N - 8 \ , \tag{180}$$
$$P_{42} = 13N^5 + 80N^4 + 94N^3 - 2N^2 - 51N - 54 \ , \tag{181}$$
$$P_{43} = 14N^5 + 35N^4 - 20N^3 - 65N^2 - 42N - 72 \ , \tag{182}$$
$$P_{44} = 179N^5 + 1219N^4 + 1691N^3 + 111N^2 - 396N - 108 \ , \tag{183}$$
$$P_{45} = 229N^5 + 821N^4 + 1081N^3 + 795N^2 + 198N + 108 \ , \tag{184}$$
$$P_{46} = 461N^5 + 999N^4 - 2037N^3 - 4507N^2 - 1320N + 396 \ , \tag{185}$$
$$P_{47} = 4541N^5 + 17151N^4 + 20199N^3 + 8477N^2 + 2292N + 1188 \ , \tag{186}$$
$$P_{48} = -2311N^6 - 5529N^5 + 6417N^4 + 19097N^3 + 15042N^2 + 4284N + 4536 \ , \tag{187}$$
$$P_{49} = -237N^6 - 699N^5 - 683N^4 - 321N^3 - 316N^2 - 272N - 72 \ , \tag{188}$$
$$P_{50} = -37N^6 - 171N^5 - 231N^4 + 7N^3 + 388N^2 + 352N + 96 \ , \tag{189}$$
$$P_{51} = N^6 - 12N^4 - 13N^3 - 12N^2 - 8N - 4 \ , \tag{190}$$
$$P_{52} = N^6 - 2N^4 + 7N^3 + 10N^2 - 4 \ , \tag{191}$$
$$P_{53} = N^6 - 6N^5 - 22N^4 - 2N^3 + 5N^2 + 16N + 12 \ , \tag{192}$$
$$P_{54} = N^6 - 5N^5 + N^4 + 61N^3 - 26N^2 - 128N - 96 \ , \tag{193}$$
$$P_{55} = N^6 - 3N^5 - 11N^4 - 5N^3 + 22N^2 + 36N + 8 \ , \tag{194}$$
$$P_{56} = N^6 + N^5 - 18N^4 - 27N^3 - 17N^2 + 16N + 12 \ , \tag{195}$$
$$P_{57} = N^6 + 3N^5 - 2N^4 - 7N^3 + 3N^2 + 6N + 24 \ , \tag{196}$$
$$P_{58} = N^6 + 3N^5 + 13N^4 + 21N^3 - 2N^2 - 12N + 72 \ , \tag{197}$$
$$P_{59} = N^6 + 7N^5 - 7N^4 - 39N^3 + 14N^2 + 40N + 48 \ , \tag{198}$$
$$P_{60} = 2N^6 - 2N^5 - 5N^4 + 20N^3 - 39N^2 - 12N - 36 \ , \tag{199}$$
$$P_{61} = 2N^6 - 2N^5 - 3N^4 + 26N^3 - 45N^2 - 34N - 48 \ , \tag{200}$$
$$P_{62} = 2N^6 - 2N^5 + N^4 + 38N^3 - 57N^2 - 78N - 72 \ , \tag{201}$$
$$P_{63} = 3N^6 - 6N^5 - 28N^4 - 11N^3 - 10N^2 + 4 \ , \tag{202}$$
$$P_{64} = 6N^6 + 35N^5 + 57N^4 + 73N^3 + 77N^2 + 56N + 20 \ , \tag{203}$$
$$P_{65} = 7N^6 - 14N^5 - 64N^4 - 23N^3 - 18N^2 + 4N + 12 \ , \tag{204}$$
$$P_{66} = 7N^6 - 10N^5 - 58N^4 - 27N^3 - 24N^2 + 4N + 12 \ , \tag{205}$$
$$P_{67} = 12N^6 - 9N^5 - 133N^4 + 295N^3 + 99N^2 - 524N - 492 \ , \tag{206}$$
$$P_{68} = 13N^6 + 25N^5 + 16N^4 - 27N^3 - 55N^2 - 40N - 12 \ , \tag{207}$$
$$P_{69} = 15N^6 + 53N^5 - 7N^4 - 137N^3 + 40N^2 + 204N + 144 \ , \tag{208}$$
$$P_{70} = 18N^6 - 18N^5 - 31N^4 + 244N^3 + 479N^2 + 168N - 44 \ , \tag{209}$$
$$P_{71} = 23N^6 + 57N^5 - 171N^4 - 357N^3 - 120N^2 + 80N + 24 \ , \tag{210}$$
$$P_{72} = 24N^6 - 25N^5 - 21N^4 + 433N^3 - 403N^2 - 1012N - 516 \ , \tag{211}$$
$$P_{73} = 27N^6 + 107N^5 + 139N^4 + 169N^3 + 238N^2 + 136N + 48 \ , \tag{212}$$
$$P_{74} = 31N^6 + 130N^5 + 187N^4 + 192N^3 + 176N^2 + 128N + 48 \ , \tag{213}$$
$$P_{75} = 67N^6 + 375N^5 + 963N^4 + 1625N^3 + 1656N^2 + 918N + 324 \ , \tag{214}$$
$$P_{76} = 105N^6 + 231N^5 + 159N^4 - 59N^3 - 172N^2 - 112N - 24 \ , \tag{215}$$
$$P_{77} = 834N^6 + 1773N^5 - 2738N^4 - 7729N^3 - 2150N^2 + 2550N + 2916 \ , \tag{216}$$

$$P_{78} = 2667N^6 + 5469N^5 + 3269N^4 + 463N^3 + 860N^2 + 1296N + 504 \; , \tag{217}$$
$$P_{79} = 7081N^6 + 28839N^5 + 35439N^4 + 13693N^3 - 2580N^2 - 3888N - 1512 \; , \tag{218}$$
$$P_{80} = 14311N^6 + 85344N^5 + 179550N^4 + 159592N^3 + 57195N^2 + 9036N + 3564 \; , \tag{219}$$
$$P_{81} = -37N^7 - 157N^6 - 335N^5 - 639N^4 - 924N^3 - 708N^2 - 320N - 96 \; , \tag{220}$$
$$P_{82} = N^7 + N^6 - N^5 + 3N^4 + 8N^3 - 60N^2 - 208N - 192 \; , \tag{221}$$
$$P_{83} = 2N^7 - 25N^5 - 84N^4 - 135N^3 - 114N^2 - 68N - 24 \; , \tag{222}$$
$$P_{84} = 13N^7 + 56N^6 + 52N^5 - 60N^4 - 121N^3 - 56N^2 + 12N + 8 \; , \tag{223}$$
$$P_{85} = 55N^7 + 219N^6 - 41N^5 - 1119N^4 - 1418N^3 - 320N^2 + 184N + 48 \; , \tag{224}$$
$$P_{86} = 108N^7 + 362N^6 - 6N^5 - 908N^4 - 1447N^3 - 2073N^2 - 1604N - 924 \; , \tag{225}$$
$$P_{87} = 198N^7 + 765N^6 + 114N^5 - 2865N^4 - 3904N^3 - 324N^2 + 1984N + 1632 \; , \tag{226}$$
$$\begin{aligned} P_{88} = {} & -56163N^8 - 166680N^7 - 175082N^6 - 69468N^5 - 1323N^4 + 15676N^3 + 27792N^2 \\ & +22608N + 6480 \; , \end{aligned} \tag{227}$$
$$\begin{aligned} P_{89} = {} & -30651N^8 - 94248N^7 + 83888N^6 + 331530N^5 + 210723N^4 + 5534N^3 + 17376N^2 \\ & +67032N + 33264 \; , \end{aligned} \tag{228}$$
$$\begin{aligned} P_{90} = {} & -5885N^8 - 20168N^7 + 51756N^6 + 169186N^5 - 154415N^4 - 399114N^3 \\ & -107208N^2 + 34344N - 14256 \; , \end{aligned} \tag{229}$$
$$\begin{aligned} P_{91} = {} & -397N^8 - 2944N^7 - 8680N^6 - 12850N^5 - 10191N^4 - 5186N^3 - 2760N^2 \\ & -1392N - 288 \; , \end{aligned} \tag{230}$$
$$\begin{aligned} P_{92} = {} & 55N^8 + 111N^7 - 227N^6 - 263N^5 + 1108N^4 + 2112N^3 \\ & +1632N^2 + 688N + 160 \; , \end{aligned} \tag{231}$$
$$\begin{aligned} P_{93} = {} & -86393N^9 - 545545N^8 - 734144N^7 + 1327702N^6 + 4117751N^5 + 3541963N^4 \\ & +1033098N^3 - 171576N^2 - 254232N - 99792 \; , \end{aligned} \tag{232}$$
$$\begin{aligned} P_{94} = {} & -17N^9 - 85N^8 - 63N^7 - 3N^6 - 552N^5 - 1424N^4 - 1584N^3 - 1040N^2 \\ & -480N - 128 \; , \end{aligned} \tag{233}$$
$$P_{95} = N^9 + 6N^8 + 7N^7 + 25N^6 + 124N^5 + 285N^4 + 384N^3 + 304N^2 + 160N + 48 \; , \tag{234}$$
$$\begin{aligned} P_{96} = {} & 19N^9 + 41N^8 - 71N^7 - 71N^6 + 452N^5 + 894N^4 + 800N^3 + 432N^2 \\ & +160N + 32 \; , \end{aligned} \tag{235}$$
$$\begin{aligned} P_{97} = {} & 2N^{10} + 8N^9 - 2N^8 + 8N^7 + 271N^6 + 571N^5 + 491N^4 + 117N^3 + 98N^2 \\ & +156N + 72 \; , \end{aligned} \tag{236}$$
$$\begin{aligned} P_{98} = {} & 3N^{10} + 15N^9 - 11N^8 - 93N^7 - 24N^6 + 52N^5 + 304N^4 + 998N^3 - 356N^2 \\ & -1272N - 720 \; , \end{aligned} \tag{237}$$
$$\begin{aligned} P_{99} = {} & 7N^{10} + 19N^9 - 60N^8 - 186N^7 + 139N^6 + 571N^5 - 542N^4 - 1940N^3 \\ & +1224N^2 + 3456N + 1728 \; , \end{aligned} \tag{238}$$
$$\begin{aligned} P_{100} = {} & 9N^{10} + 44N^9 + 83N^8 + 107N^7 + 291N^6 + 879N^5 + 1579N^4 + 1672N^3 \\ & +1128N^2 + 496N + 112 \; , \end{aligned} \tag{239}$$
$$\begin{aligned} P_{101} = {} & 9N^{10} + 66N^9 + 147N^8 - 41N^7 - 787N^6 - 1627N^5 - 1655N^4 - 904N^3 \\ & -296N^2 - 112N - 48 \; , \end{aligned} \tag{240}$$
$$\begin{aligned} P_{102} = {} & 18N^{10} + 114N^9 + 109N^8 - 331N^7 - 718N^6 - 904N^5 - 623N^4 + 545N^3 \\ & +1386N^2 + 988N + 264 \; , \end{aligned} \tag{241}$$
$$P_{103} = 34N^{10} + 136N^9 + 10N^8 - 372N^7 - 447N^6 - 99N^5 - 215N^4 - 921N^3$$

$$
\begin{aligned}
& -1210N^2 - 716N - 168\ , && (242)\\
P_{104} &= 36N^{10} + 248N^9 + 451N^8 - 523N^7 - 3189N^6 - 5209N^5 - 4382N^4 \\
& -2104N^3 - 704N^2 - 272N - 96\ , && (243)\\
P_{105} &= 133N^{10} + 573N^9 - 1052N^8 - 5646N^7 + 2901N^6 + 21505N^5 + 18130N^4 \\
& +2320N^3 - 160N^2 + 1200N + 288\ , && (244)\\
P_{106} &= 763N^{10} + 3027N^9 - 128N^8 - 9930N^7 - 11309N^6 - 2945N^5 - 350N^4 \\
& -5224N^3 - 7296N^2 - 3824N - 672\ , && (245)\\
P_{107} &= 2N^{11} + 17N^{10} + 34N^9 - 34N^8 - 190N^7 - 255N^6 - 102N^5 + 648N^4 \\
& +1592N^3 + 1136N^2 + 32N - 192\ , && (246)\\
P_{108} &= 1203N^{11} + 9097N^{10} + 19614N^9 - 6786N^8 - 83057N^7 - 122227N^6 \\
& -74000N^5 - 9540N^4 + 26544N^3 + 34608N^2 + 19072N + 3648\ , && (247)\\
P_{109} &= 4N^{12} + 34N^{11} + 76N^{10} - 4N^9 - 257N^8 - 653N^7 - 1597N^6 \\
& -3349N^5 - 4822N^4 - 4520N^3 - 2784N^2 - 1104N - 224\ , && (248)\\
P_{110} &= 68N^{12} + 360N^{11} + 367N^{10} - 1033N^9 - 2581N^8 - 2317N^7 \\
& -3258N^6 - 7354N^5 - 9852N^4 - 7648N^3 - 3776N^2 - 1184N - 192\ , && (249)\\
P_{111} &= 648699N^{12} + 3241902N^{11} - 1257119N^{10} - 24261820N^9 - 20205047N^8 \\
& +35553886N^7 + 60343151N^6 + 28400368N^5 + 5406396N^4 + 1441440N^3 \\
& -4831920N^2 - 7008768N - 2566080\ , && (250)\\
P_{112} &= -4521N^{14} - 26467N^{13} - 9301N^{12} + 178761N^{11} + 272257N^{10} - 131725N^9 \\
& -554287N^8 - 465045N^7 - 251812N^6 - 157892N^5 + 36656N^4 + 261568N^3 \\
& +265824N^2 + 114048N + 17280\ , && (251)\\
P_{113} &= -92N^{14} - 560N^{13} - 355N^{12} + 3542N^{11} + 7046N^{10} + 469N^9 - 11572N^8 \\
& -16160N^7 - 14914N^6 - 9595N^5 + 5423N^4 + 21784N^3 + 22728N^2 + 11088N \\
& +2160\ , && (252)\\
P_{114} &= 28N^{15} + 246N^{14} + 715N^{13} + 259N^{12} - 3044N^{11} - 7282N^{10} - 6158N^9 + 2882N^8 \\
& +18427N^7 + 44383N^6 + 79244N^5 + 99004N^4 + 83536N^3 + 47056N^2 + 16704N \\
& +2880\ , && (253)\\
P_{115} &= 14N^{17} + 174N^{16} + 731N^{15} + 818N^{14} - 2448N^{13} - 7521N^{12} - 3298N^{11} + 11946N^{10} \\
& +15938N^9 - 10959N^8 - 71053N^7 - 162458N^6 - 255636N^5 - 282952N^4 \\
& -216304N^3 - 111584N^2 - 36288N - 5760. && (254)
\end{aligned}
$$

The expansion coefficients for the Compton amplitudes of $xF_3(x,Q^2)$ and $g_1(x,Q^2)$ are

$$
\begin{aligned}
c_{F_3,q}^{(2,0),\mathrm{L}} &= C_F\Biggl\{T_F N_F\Biggl[\frac{P_{136}}{54N^3(1+N)^3} - \frac{4\bigl(-6+19N+19N^2\bigr)}{9N(1+N)}S_1^2 + \Biggl(-\frac{2P_{122}}{27N^2(1+N)^2} \\
& +\frac{16}{3}S_2 + \frac{8}{3}\zeta_2\Biggr)S_1 - \frac{16}{9}S_1^3 + \frac{4\bigl(-6+47N+47N^2\bigr)S_2}{9N(1+N)} - \frac{104}{9}S_3 + \frac{16}{3}S_{2,1} \\
& -\frac{2\bigl(2+3N+3N^2\bigr)\zeta_2}{3N(1+N)}\Biggr] + C_A\Biggl[\frac{P_{144}}{216(N-1)N^4(1+N)^4(2+N)} + \Biggl(\frac{P_{137}}{54N^3(1+N)^3}
\end{aligned}
$$

$$
-\frac{4\big(3+11N+11N^2\big)S_2}{3N(1+N)}+24S_3-16S_{2,1}-16S_{-2,1}-\frac{22}{3}\zeta_2-48\zeta_3\Bigg)S_1
$$
$$
+\Bigg(\frac{-66+233N+233N^2}{9N(1+N)}+4S_2\Bigg)S_1^2+\frac{\big(66-481N-1166N^2-583N^3\big)}{9N(1+N)^2}S_2
$$
$$
+\frac{44}{9}S_1^3-4S_2^2+\frac{2\big(-36+143N+143N^2\big)}{9N(1+N)}S_3+\Bigg(\frac{4P_{123}}{(N-1)N(1+N)^2(2+N)}
$$
$$
-\frac{8S_1}{N(1+N)}+16S_1^2-8S_2\Bigg)S_{-2}-8S_4-12S_{-2}^2+\Bigg(\frac{8}{N(1+N)}-8S_1\Bigg)S_{-3}
$$
$$
-20S_{-4}-\frac{4\big(-6+11N+11N^2\big)}{3N(1+N)}S_{2,1}-24S_{3,1}+8S_{-2,2}+16S_{-3,1}+24S_{2,1,1}
$$
$$
+\frac{11\big(2+3N+3N^2\big)}{6N(1+N)}\zeta_2+\frac{6(1+3N)(2+3N)}{N(1+N)}\zeta_3\Bigg]\Bigg\}+C_F^2\Bigg[\frac{2S_2P_{119}}{N^2(1+N)^2}
$$
$$
+\frac{P_{145}}{8(N-1)N^4(1+N)^4(2+N)}+\Bigg(\frac{P_{130}}{2N^3(1+N)^3}-\frac{2\big(-14+9N+9N^2\big)}{N(1+N)}S_2
$$
$$
-\frac{56}{3}S_3+16S_{2,1}+32S_{-2,1}+\frac{4\big(2+3N+3N^2\big)}{N(1+N)}\zeta_2+48\zeta_3\Bigg)S_1+\Bigg(\frac{2P_{118}}{N^2(1+N)^2}
$$
$$
-28S_2-8\zeta_2\Bigg)S_1^2+\frac{2\big(-14+15N+15N^2\big)}{3N(1+N)}S_1^3-\frac{2\big(-2+33N+33N^2\big)}{3N(1+N)}S_3
$$
$$
+\frac{14}{3}S_1^4+6S_2^2+12S_4+\Bigg(-\frac{8P_{123}}{(N-1)N(1+N)^2(2+N)}+\frac{16S_1}{N(1+N)}-32S_1^2
$$
$$
+16S_2\Bigg)S_{-2}+\Bigg(-\frac{16}{N(1+N)}+16S_1\Bigg)S_{-3}+\frac{4\big(-2+3N+3N^2\big)}{N(1+N)}S_{2,1}
$$
$$
+40S_{-4}+24S_{-2}^2+40S_{3,1}-16S_{-2,2}-32S_{-3,1}-24S_{2,1,1}-\frac{\big(2+3N+3N^2\big)^2}{2N^2(1+N)^2}\zeta_2
$$
$$
-72\zeta_3\Bigg], \tag{255}
$$
$$
c_{F_3,q}^{(2,1),L} = C_F\Bigg\{C_A\Bigg[\frac{2S_{2,1}P_{121}}{9N^2(1+N)(2+N)}+\frac{P_{129}}{72N^3(1+N)^3}\zeta_2-\frac{2P_{135}}{9(N-1)N^2(1+N)^2(2+N)}\zeta_3
$$
$$
+\frac{P_{148}}{2592(N-1)^2N^5(1+N)^5(2+N)^2}+\Bigg(\frac{P_{143}}{648(N-1)N^4(1+N)^4(2+N)}
$$
$$
+\frac{S_2P_{126}}{9N^2(1+N)^2(2+N)}+8S_2^2-\frac{4\big(-9+22N+22N^2\big)S_3}{9N(1+N)}+32S_4-\frac{8S_{2,1}}{N(1+N)}
$$
$$
-4S_{3,1}-8S_{-2,2}-16S_{-3,1}-20S_{2,1,1}+\frac{\big(-66+233N+233N^2\big)\zeta_2}{18N(1+N)}+\frac{72}{5}\zeta_2^2
$$
$$
+\frac{514}{9}\zeta_3\Bigg)S_1+\Bigg(\frac{P_{127}}{108N^3(1+N)^3}+\frac{\big(9+22N+22N^2\big)S_2}{3N(1+N)}-14S_3+12S_{2,1}
$$

$$
+8S_{-2,1}+\frac{11}{3}\zeta_2+24\zeta_3\Biggr)S_1^2+\Biggl(\frac{66-233N-233N^2}{27N(1+N)}-\frac{8}{3}S_2\Biggr)S_1^3-\frac{11}{9}S_1^4
$$

$$
+\Biggl(\frac{P_{140}}{108N^3(1+N)^3(2+N)}-\frac{22}{3}S_3-8S_{2,1}-8S_{-2,1}-\frac{22}{3}\zeta_2-48\zeta_3\Biggr)S_2
$$

$$
+\frac{\bigl(-2-17N-17N^2\bigr)S_2^2}{N(1+N)}+\Biggl(\frac{P_{128}}{27(N-1)N^2(1+N)^2(2+N)}+2\zeta_2\Biggr)S_3
$$

$$
+\frac{\bigl(-33+59N+59N^2\bigr)S_4}{3N(1+N)}-22S_5+\Biggl(-\frac{4P_{142}}{(N-1)^2N(1+N)^3(2+N)^2}
$$

$$
+\Biggl(-\frac{4P_{123}}{(N-1)N(1+N)^2(2+N)}+16S_2+4\zeta_2\Biggr)S_1+\frac{4S_1^2}{N(1+N)}-8S_1^3-\frac{4S_2}{N(1+N)}
$$

$$
-32S_3-8S_{-2,1}-\frac{2\zeta_2}{N(1+N)}-72\zeta_3\Biggr)S_{-2}+\Biggl(-\frac{4}{N(1+N)}+20S_1\Biggr)S_{-2}^2
$$

$$
+\Biggl(\frac{4P_{123}}{(N-1)N(1+N)^2(2+N)}-\frac{8S_1}{N(1+N)}+12S_1^2+4S_2-16S_{-2}+2\zeta_2\Biggr)S_{-3}
$$

$$
+\Biggl(\frac{4}{N(1+N)}+12S_1\Biggr)S_{-4}-38S_{-5}+20S_{2,3}-\frac{2\bigl(-4+5N+5N^2\bigr)}{N(1+N)}S_{3,1}
$$

$$
-8S_{2,-3}-44S_{4,1}+44S_{-2,3}+8S_{-4,1}+\frac{4\bigl(3+11N+11N^2\bigr)}{3N(1+N)}S_{2,1,1}
$$

$$
+4S_{2,2,1}+40S_{3,1,1}-8S_{-2,1,-2}+4S_{2,1,1,1}-4S_{-2,1}\zeta_2-\frac{9(1+3N)(2+3N)\zeta_2^2}{5N(1+N)}\Biggr]
$$

$$
+T_FN_F\Biggl[\frac{P_{120}}{18N^2(1+N)^2}\zeta_2+\frac{P_{141}}{648N^4(1+N)^4}+\Biggl(\frac{P_{138}}{162N^3(1+N)^3}
$$

$$
-\frac{4\bigl(-6+19N+19N^2\bigr)S_2}{9N(1+N)}+\frac{32}{9}S_3-\frac{2\bigl(-6+19N+19N^2\bigr)\zeta_2}{9N(1+N)}-\frac{128}{9}\zeta_3\Biggr)S_1
$$

$$
+\Biggl(\frac{P_{122}}{27N^2(1+N)^2}-\frac{8}{3}S_2-\frac{4}{3}\zeta_2\Biggr)S_1^2+\frac{4\bigl(-6+19N+19N^2\bigr)S_1^3}{27N(1+N)}+\frac{4}{9}S_1^4
$$

$$
+\Biggl(\frac{P_{116}}{27N^2(1+N)^2}+\frac{8\zeta_2}{3}\Biggr)S_2+4S_2^2+\frac{16\bigl(-3+41N+41N^2\bigr)S_3}{27N(1+N)}-\frac{28}{3}S_4
$$

$$
-\frac{112}{9}S_{2,1}+8S_{3,1}-\frac{16}{3}S_{2,1,1}+\frac{32\bigl(2+3N+3N^2\bigr)\zeta_3}{9N(1+N)}\Biggr]\Biggr\}
$$

$$
+C_F^2\Biggl[-\frac{2S_{2,1}P_{124}}{N^2(1+N)^2(2+N)}+\frac{P_{132}}{8N^3(1+N)^3}\zeta_2+\frac{4P_{134}}{3(N-1)N^2(1+N)^2(2+N)}\zeta_3
$$

$$
+\frac{P_{147}}{32(N-1)^2N^5(1+N)^5(2+N)^2}+\Biggl(\frac{P_{146}}{8(N-1)N^4(1+N)^4(2+N)}
$$

$$
\begin{aligned}
&+\Biggl(\frac{P_{125}}{2N^2(1+N)^2(2+N)}-8\zeta_2\Biggr)S_2-23S_2^2-\frac{4\bigl(-2+5N+5N^2\bigr)S_3}{N(1+N)}-38S_4\\
&+\frac{4\bigl(2+3N+3N^2\bigr)S_{2,1}}{N(1+N)}-8S_{3,1}+16S_{-2,2}+32S_{-3,1}+32S_{2,1,1}\\
&+\frac{\bigl(16+12N-17N^2-29N^3\bigr)}{2N^2(1+N)}\zeta_2-\frac{72}{5}\zeta_2^2-\frac{4\bigl(32+21N+21N^2\bigr)}{3N(1+N)}\zeta_3\Biggr)S_1\\
&+\Biggl(\frac{P_{131}}{4N^3(1+N)^3}+\frac{\bigl(-34+27N+27N^2\bigr)S_2}{2N(1+N)}+12S_3-16S_{2,1}-16S_{-2,1}\\
&+\frac{3(N-1)(2+N)}{N(1+N)}\zeta_2-\frac{16}{3}\zeta_3\Biggr)S_1^2+\Biggl(\frac{P_{117}}{6N^2(1+N)^2}+\frac{38}{3}S_2+4\zeta_2\Biggr)S_1^3\\
&+\frac{\bigl(10-13N-13N^2\bigr)}{4N(1+N)}S_1^4-S_1^5+\Biggl(\frac{P_{139}}{4N^3(1+N)^3(2+N)}+\frac{28}{3}S_3+16S_{2,1}\\
&+16S_{-2,1}+\frac{2\bigl(2+3N+3N^2\bigr)}{N(1+N)}\zeta_2+48\zeta_3\Biggr)S_2+\frac{3\bigl(10+27N+27N^2\bigr)}{4N(1+N)}S_2^2\\
&+\Biggl(\frac{2P_{133}}{3(N-1)N^2(1+N)^2(2+N)}-4\zeta_2\Biggr)S_3+\frac{3(2-N)(3+N)S_4}{2N(1+N)}+36S_5\\
&+\Biggl(\frac{8P_{142}}{(N-1)^2N(1+N)^3(2+N)^2}+\Biggl(\frac{8P_{123}}{(N-1)N(1+N)^2(2+N)}-32S_2\\
&-8\zeta_2\Biggr)S_1-\frac{8}{N(1+N)}S_1^2+16S_1^3+\frac{8}{N(1+N)}S_2+64S_3+16S_{-2,1}\\
&+\frac{4}{N(1+N)}\zeta_2+144\zeta_3\Biggr)S_{-2}+\Biggl(-40S_1+\frac{8}{N(1+N)}\Biggr)S_{-2}^2+\Biggl(\frac{16}{N(1+N)}S_1\\
&-\frac{8P_{123}}{(N-1)N(1+N)^2(2+N)}-24S_1^2-8S_2+32S_{-2}-4\zeta_2\Biggr)S_{-3}-\Biggl(24S_1\\
&+\frac{8}{N(1+N)}\Biggr)S_{-4}+76S_{-5}-32S_{2,3}+16S_{2,-3}-\frac{8}{N(1+N)}S_{3,1}+76S_{4,1}\\
&-88S_{-2,3}-16S_{-4,1}-\frac{2\bigl(2+15N+15N^2\bigr)}{N(1+N)}S_{2,1,1}-48S_{3,1,1}+16S_{-2,1,-2}\\
&-20S_{2,1,1,1}+8S_{-2,1}\zeta_2+\frac{108}{5}\zeta_2^2\Biggr]\;, \qquad (256)
\end{aligned}
$$

with

$$
\begin{aligned}
P_{116} &= -1007N^4-2122N^3-1091N^2-120N-108\;, &(257)\\
P_{117} &= -83N^4-208N^3-149N^2-80N-50\;, &(258)\\
P_{118} &= 9N^4+27N^3+24N^2+18N+11\;, &(259)\\
P_{119} &= 22N^4+41N^3+13N^2-6N-5\;, &(260)\\
P_{120} &= 87N^4+138N^3+59N^2-40N-36\;, &(261)
\end{aligned}
$$

$$P_{121} = 229N^4 + 507N^3 + 188N^2 + 72 \ , \qquad (262)$$
$$P_{122} = 373N^4 + 854N^3 + 457N^2 + 120N + 108 \ , \qquad (263)$$
$$P_{123} = 2N^5 + 6N^4 + 5N^3 + 2N^2 + 7N + 2 \ , \qquad (264)$$
$$P_{124} = 27N^5 + 66N^4 + 33N^3 - 2N^2 + 24N + 16 \ , \qquad (265)$$
$$P_{125} = 75N^5 + 246N^4 + 285N^3 + 202N^2 + 198N + 108 \ , \qquad (266)$$
$$P_{126} = 179N^5 + 896N^4 + 1261N^3 + 394N^2 - 204N - 72 \ , \qquad (267)$$
$$P_{127} = -4541N^6 - 14649N^5 - 16023N^4 - 7523N^3 - 2256N^2 - 324N + 432 \ , \qquad (268)$$
$$P_{128} = -2311N^6 - 6393N^5 - 1423N^4 + 6333N^3 + 2654N^2 - 372N + 216 \ , \qquad (269)$$
$$P_{129} = -975N^6 - 2529N^5 - 2341N^4 - 467N^3 + 428N^2 + 108N - 144 \ , \qquad (270)$$
$$P_{130} = -133N^6 - 355N^5 - 379N^4 - 281N^3 - 228N^2 - 160N - 40 \ , \qquad (271)$$
$$P_{131} = -89N^6 - 351N^5 - 331N^4 + 83N^3 + 308N^2 + 224N + 64 \ , \qquad (272)$$
$$P_{132} = 57N^6 + 135N^5 + 103N^4 - 43N^3 - 68N^2 - 32N + 8 \ , \qquad (273)$$
$$P_{133} = 104N^6 + 246N^5 + 14N^4 - 208N^3 - 3N^2 + N - 10 \ , \qquad (274)$$
$$P_{134} = 180N^6 + 351N^5 - 120N^4 - 447N^3 - 232N^2 - 76N - 88 \ , \qquad (275)$$
$$P_{135} = 726N^6 + 1557N^5 - 320N^4 - 1921N^3 - 1108N^2 - 122N - 108 \ , \qquad (276)$$
$$P_{136} = 1923N^6 + 4725N^5 + 3893N^4 + 679N^3 + 20N^2 + 864N + 504 \ , \qquad (277)$$
$$P_{137} = 4541N^6 + 14703N^5 + 15807N^4 + 7037N^3 + 1932N^2 + 324N - 432 \ , \qquad (278)$$
$$P_{138} = 7081N^6 + 24375N^5 + 26871N^4 + 10813N^3 - 60N^2 - 2592N - 1512 \ , \qquad (279)$$
$$P_{139} = -345N^7 - 1629N^6 - 2849N^5 - 2367N^4 - 1142N^3 - 456N^2 - 208N - 32 \ , \qquad (280)$$
$$P_{140} = 13519N^7 + 67649N^6 + 122235N^5 + 96259N^4 + 31370N^3 + 3540N^2 + 216N - 864 \ , \qquad (281)$$
$$P_{141} = -38715N^8 - 131784N^7 - 168338N^6 - 89028N^5 - 11355N^4 + 4564N^3 + 11808N^2 + 16560N + 6480 \ , \qquad (282)$$
$$P_{142} = 3N^8 + 15N^7 + 13N^6 - 41N^5 - 94N^4 - 78N^3 - 50N^2 - 36N - 20 \ , \qquad (283)$$
$$P_{143} = -86393N^{10} - 469657N^9 - 848380N^8 - 376714N^7 + 589247N^6 + 803483N^5 + 303838N^4 - 78072N^3 - 102168N^2 + 432N + 15552 \ , \qquad (284)$$
$$P_{144} = -20163N^{10} - 89115N^9 - 115804N^8 + 11018N^7 + 141341N^6 + 95849N^5 + 15002N^4 + 21272N^3 + 28872N^2 - 144N - 5184 \ , \qquad (285)$$
$$P_{145} = 187N^{10} + 707N^9 + 468N^8 - 1258N^7 - 2037N^6 - 289N^5 - 202N^4 - 1848N^3 - 1856N^2 - 176N + 160 \ , \qquad (286)$$
$$P_{146} = 619N^{10} + 3003N^9 + 4988N^8 + 2654N^7 - 2525N^6 - 5553N^5 - 3274N^4 + 1480N^3 + 3136N^2 + 1392N + 224 \ , \qquad (287)$$
$$P_{147} = -41N^{14} + 461N^{13} + 4075N^{12} + 10649N^{11} + 5585N^{10} - 28621N^9 - 66255N^8 - 64133N^7 - 59540N^6 - 74900N^5 - 47952N^4 + 10560N^3 + 16480N^2 + 384N - 1664 \ , \qquad (288)$$
$$P_{148} = 370659N^{14} + 2352657N^{13} + 4903159N^{12} + 1576653N^{11} - 7789003N^{10} - 9288369N^9 + 1194501N^8 + 8087991N^7 + 5995676N^6 + 3609228N^5 + 2013520N^4 - 526896N^3 - 789120N^2 + 88128N + 145152. \qquad (289)$$

and

$$
\begin{aligned}
c_{g_1,q}^{\mathrm{NS},(2,0),\mathrm{L}} = {} & C_F \Biggl\{ T_F N_F \Biggl[\frac{P_{190}}{54N^3(1+N)^3} + \Biggl(-\frac{2P_{159}}{27N^2(1+N)^2} + \frac{16}{3} S_2 + \frac{8}{3}\zeta_2 \Biggr) S_1 - \frac{16}{9} S_1^3 \\
& - \frac{4\bigl(-6+19N+19N^2\bigr)}{9N(1+N)} S_1^2 + \frac{4\bigl(-6+47N+47N^2\bigr)}{9N(1+N)} S_2 - \frac{104}{9} S_3 + \frac{16}{3} S_{2,1} \\
& - \frac{2\bigl(2+3N+3N^2\bigr)}{3N(1+N)} \zeta_2 \Biggr] + C_A \Biggl[\frac{P_{202}}{216(N-1)N^4(1+N)^4(2+N)} + \Biggl(24 S_3 - 16 S_{2,1} \\
& + \frac{P_{191}}{54N^3(1+N)^3} - \frac{4\bigl(3+11N+11N^2\bigr)}{3N(1+N)} S_2 - 16 S_{-2,1} - \frac{22}{3}\zeta_2 - 48\zeta_3 \Biggr) S_1 \\
& + \Biggl(\frac{-66+233N+233N^2}{9N(1+N)} + 4S_2 \Biggr) S_1^2 + \frac{\bigl(66-481N-1166N^2-583N^3\bigr)}{9N(1+N)^2} S_2 \\
& + \frac{44}{9} S_1^3 - 4 S_2^2 + \frac{2\bigl(-36+143N+143N^2\bigr)}{9N(1+N)} S_3 - 8 S_4 - 12 S_{-2}^2 + \Biggl(\frac{8}{N(1+N)} \\
& - 8 S_1 \Biggr) S_{-3} + \Biggl(\frac{4P_{165}}{(N-1)N(1+N)^2(2+N)} - \frac{8}{N(1+N)} S_1 + 16 S_1^2 - 8 S_2 \Biggr) S_{-2} \\
& - 20 S_{-4} - \frac{4\bigl(-6+11N+11N^2\bigr)}{3N(1+N)} S_{2,1} - 24 S_{3,1} + 8 S_{-2,2} + 16 S_{-3,1} + 24 S_{2,1,1} \\
& + \frac{11\bigl(2+3N+3N^2\bigr)}{6N(1+N)} \zeta_2 + \frac{6(1+3N)(2+3N)}{N(1+N)} \zeta_3 \Biggr] \Biggr\} + C_F^2 \Biggl[\frac{2P_{156}}{N^2(1+N)^2} S_2 \\
& + \frac{P_{207}}{8(N-1)N^4(1+N)^4(2+N)} + \Biggl(\frac{P_{174}}{2N^3(1+N)^3} - \frac{2\bigl(-14+9N+9N^2\bigr)}{N(1+N)} S_2 \\
& - \frac{56}{3} S_3 + 16 S_{2,1} + 32 S_{-2,1} + \frac{4\bigl(2+3N+3N^2\bigr)\zeta_2}{N(1+N)} + 48\zeta_3 \Biggr) S_1 + \Biggl(\frac{2P_{154}}{N^2(1+N)^2} \\
& - 28 S_2 - 8\zeta_2 \Biggr) S_1^2 + \frac{2\bigl(-14+15N+15N^2\bigr)}{3N(1+N)} S_1^3 - \frac{2\bigl(-2+33N+33N^2\bigr)}{3N(1+N)} S_3 \\
& + \frac{14}{3} S_1^4 + 6 S_2^2 + 12 S_4 + \Biggl(-\frac{8P_{165}}{(N-1)N(1+N)^2(2+N)} + \frac{16}{N(1+N)} S_1 \\
& - 32 S_1^2 + 16 S_2 \Biggr) S_{-2} + 24 S_{-2}^2 + \Biggl(-\frac{16}{N(1+N)} + 16 S_1 \Biggr) S_{-3} + 40 S_{-4} \\
& + \frac{4\bigl(-2+3N+3N^2\bigr)}{N(1+N)} S_{2,1} + 40 S_{3,1} - 16 S_{-2,2} - 32 S_{-3,1} - 24 S_{2,1,1} \\
& - \frac{\bigl(2+3N+3N^2\bigr)^2}{2N^2(1+N)^2} \zeta_2 - 72\zeta_3 \Biggr], \qquad (290)
\end{aligned}
$$

$$
c_{g_1,q}^{\mathrm{NS},(2,1),\mathrm{L}} = C_F \Biggl\{ C_A \Biggl[\frac{2P_{158}}{9N^2(1+N)(2+N)} S_{2,1} - \frac{2P_{189}}{9(N-1)N^2(1+N)^2(2+N)} \zeta_3
$$

$$
+\frac{P_{173}}{72N^3(1+N)^3}\zeta_2+\frac{P_{213}}{2592(N-1)^2N^5(1+N)^5(2+N)^2}+\Biggl(8S_2^2+32S_4
-\frac{8}{N(1+N)}S_{2,1}+\frac{P_{170}}{9N^2(1+N)^2(2+N)}S_2+\frac{P_{201}}{648(N-1)N^4(1+N)^4(2+N)}
+\frac{4\bigl(9-22N-22N^2\bigr)}{9N(1+N)}S_3-4S_{3,1}-8S_{-2,2}-\frac{\bigl(66-233N-233N^2\bigr)}{18N(1+N)}\zeta_2-16S_{-3,1}
-20S_{2,1,1}+\frac{514}{9}\zeta_3+\frac{72}{5}\zeta_2^2\Biggr)S_1+\Biggl(\frac{\bigl(9+22N+22N^2\bigr)}{3N(1+N)}S_2-14S_3+12S_{2,1}
+\frac{P_{171}}{108N^3(1+N)^3}+8S_{-2,1}+\frac{11}{3}\zeta_2+24\zeta_3\Biggr)S_1^2+\Biggl(\frac{66-233N-233N^2}{27N(1+N)}-\frac{8}{3}S_2\Biggr)
\times S_1^3-\frac{11}{9}S_1^4+\Biggl(\frac{P_{196}}{108N^3(1+N)^3(2+N)}-\frac{22}{3}S_3-8S_{2,1}-8S_{-2,1}-\frac{22}{3}\zeta_2
-48\zeta_3\Biggr)S_2+\frac{\bigl(-2-17N-17N^2\bigr)}{N(1+N)}S_2^2+\Biggl(\frac{P_{172}}{27(N-1)N^2(1+N)^2(2+N)}+2\zeta_2\Biggr)
\times S_3+\frac{\bigl(-33+59N+59N^2\bigr)S_4}{3N(1+N)}-22S_5+\Biggl(-\frac{4P_{199}}{(N-1)^2N(1+N)^3(2+N)^2}
+\Biggl(-\frac{4P_{165}}{(N-1)N(1+N)^2(2+N)}+16S_2+4\zeta_2\Biggr)S_1+\frac{4}{N(1+N)}S_1^2-8S_1^3
-\frac{4}{N(1+N)}S_2-32S_3-8S_{-2,1}-\frac{2}{N(1+N)}\zeta_2-72\zeta_3\Biggr)S_{-2}+\Biggl(-\frac{4}{N(1+N)}
+20S_1\Biggr)S_{-2}^2+\Biggl(\frac{4P_{165}}{(N-1)N(1+N)^2(2+N)}-\frac{8}{N(1+N)}S_1+12S_1^2+4S_2
-16S_{-2}+2\zeta_2\Biggr)S_{-3}+\Biggl(\frac{4}{N(1+N)}+12S_1\Biggr)S_{-4}-38S_{-5}+20S_{2,3}-8S_{2,-3}
-\frac{2\bigl(-4+5N+5N^2\bigr)}{N(1+N)}S_{3,1}-44S_{4,1}+44S_{-2,3}+8S_{-4,1}+\frac{4\bigl(3+11N+11N^2\bigr)}{3N(1+N)}
\times S_{2,1,1}+4S_{2,2,1}+40S_{3,1,1}-8S_{-2,1,-2}+4S_{2,1,1,1}-4S_{-2,1}\zeta_2
-\frac{9(1+3N)(2+3N)}{5N(1+N)}\zeta_2^2\Biggr]+{\color{blue}T_FN_F}\Biggl[\frac{P_{157}}{18N^2(1+N)^2}\zeta_2+\frac{P_{197}}{648N^4(1+N)^4}
+\Biggl(\frac{P_{192}}{162N^3(1+N)^3}-\frac{4\bigl(-6+19N+19N^2\bigr)}{9N(1+N)}S_2-\frac{2\bigl(-6+19N+19N^2\bigr)}{9N(1+N)}\zeta_2
+\frac{32}{9}S_3-\frac{128}{9}\zeta_3\Biggr)S_1+\frac{4\bigl(-6+19N+19N^2\bigr)}{27N(1+N)}S_1^3+\Biggl(\frac{P_{159}}{27N^2(1+N)^2}-\frac{8}{3}S_2
-\frac{4}{3}\zeta_2\Biggr)S_1^2+\frac{4}{9}S_1^4+\Biggl(\frac{P_{149}}{27N^2(1+N)^2}+\frac{8}{3}\zeta_2\Biggr)S_2+\frac{16\bigl(-3+41N+41N^2\bigr)}{27N(1+N)}S_3
$$

$$
+4S_2^2-\frac{28}{3}S_4-\frac{112}{9}S_{2,1}+8S_{3,1}-\frac{16}{3}S_{2,1,1}+\frac{32\big(2+3N+3N^2\big)}{9N(1+N)}\zeta_3\Bigg]\Bigg\}
$$

$$
+C_F^2\Bigg[-\frac{2P_{168}}{N^2(1+N)^2(2+N)}S_{2,1}+\frac{4P_{188}}{3(N-1)N^2(1+N)^2(2+N)}\zeta_3
$$

$$
+\frac{P_{186}}{8N^3(1+N)^3}\zeta_2+\frac{P_{210}}{32(N-1)^2N^5(1+N)^5(2+N)^2}
$$

$$
+\Bigg(\Bigg(\frac{P_{169}}{2N^2(1+N)^2(2+N)}-8\zeta_2\Bigg)S_2+\frac{\big(16-20N-33N^2-45N^3\big)}{2N^2(1+N)}\zeta_2
$$

$$
+\frac{P_{208}}{8(N-1)N^4(1+N)^4(2+N)}-23S_2^2-\frac{4\big(-2+5N+5N^2\big)}{N(1+N)}S_3-38S_4
$$

$$
+\frac{4\big(2+3N+3N^2\big)}{N(1+N)}S_{2,1}-8S_{3,1}+16S_{-2,2}+32S_{-3,1}+32S_{2,1,1}-\frac{72}{5}\zeta_2^2
$$

$$
-\frac{4\big(32+21N+21N^2\big)}{3N(1+N)}\zeta_3\Bigg)S_1+\Bigg(\frac{P_{150}}{6N^2(1+N)^2}+\frac{38}{3}S_2+4\zeta_2\Bigg)S_1^3
$$

$$
+\Bigg(\frac{P_{175}}{4N^3(1+N)^3}+\frac{\big(-34+27N+27N^2\big)}{2N(1+N)}S_2+12S_3-16S_{2,1}
$$

$$
-16S_{-2,1}+\frac{3(N-1)(2+N)\zeta_2}{N(1+N)}-\frac{16}{3}\zeta_3\Bigg)S_1^2
$$

$$
+\frac{\big(10-13N-13N^2\big)}{4N(1+N)}S_1^4-S_1^5+\Bigg(\frac{P_{193}}{4N^3(1+N)^3(2+N)}+\frac{28}{3}S_3+16S_{2,1}
$$

$$
+16S_{-2,1}+\frac{2\big(2+3N+3N^2\big)}{N(1+N)}\zeta_2+48\zeta_3\Bigg)S_2+\frac{3\big(10+27N+27N^2\big)}{4N(1+N)}S_2^2
$$

$$
+\Bigg(\frac{2P_{187}}{3(N-1)N^2(1+N)^2(2+N)}-4\zeta_2\Bigg)S_3+\frac{3(2-N)(3+N)}{2N(1+N)}S_4+36S_5
$$

$$
+\Bigg(\frac{8P_{199}}{(N-1)^2N(1+N)^3(2+N)^2}+(\frac{8P_{165}}{(N-1)N(1+N)^2(2+N)}-32S_2
$$

$$
-8\zeta_2\Bigg)S_1-\frac{8}{N(1+N)}S_1^2+16S_1^3+\frac{8}{N(1+N)}S_2+64S_3+16S_{-2,1}
$$

$$
+\frac{4}{N(1+N)}\zeta_2+144\zeta_3\Bigg)S_{-2}+\Bigg(\frac{8}{N(1+N)}-40S_1\Bigg)S_{-2}^2+\Bigg(\frac{16}{N(1+N)}S_1
$$

$$
-\frac{8P_{165}}{(N-1)N(1+N)^2(2+N)}-24S_1^2-8S_2+32S_{-2}-4\zeta_2\Bigg)S_{-3}-\Bigg(24S_1
$$

$$
+\frac{8}{N(1+N)}\Bigg)S_{-4}+76S_{-5}-32S_{2,3}+16S_{2,-3}-\frac{8}{N(1+N)}S_{3,1}+76S_{4,1}
$$

$$
-88S_{-2,3}-16S_{-4,1}-\frac{2\big(2+15N+15N^2\big)}{N(1+N)}S_{2,1,1}-48S_{3,1,1}+16S_{-2,1,-2}
$$

$$-20S_{2,1,1,1}+8S_{-2,1}\zeta_2+\frac{108}{5}\zeta_2^2\Biggr], \tag{291}$$

$$c_{g_1,q}^{\mathrm{PS},(2,0),L} = C_FT_FN_F\Biggl[\frac{4P_{200}}{(N-1)N^4(1+N)^4(2+N)}+\frac{8(2+N)\bigl(3-2N^2+3N^3\bigr)}{N^3(1+N)^3}S_1$$
$$+\frac{8(N-1)(2+N)}{N^2(1+N)^2}S_1^2-\frac{8(N-1)(2+N)}{N^2(1+N)^2}S_2-\frac{64}{(N-1)N(1+N)(2+N)}S_{-2}$$
$$-\frac{4(N-1)(2+N)}{N^2(1+N)^2}\zeta_2\Biggr], \tag{292}$$

$$c_{g_1,q}^{\mathrm{PS},(2,1),L} = C_FT_FN_F\Biggl[\frac{64P_{151}}{3(N-1)N^2(1+N)^2(2+N)}\zeta_3-\frac{16P_{152}}{3(N-1)N^2(1+N)^2(2+N)}S_3$$
$$-\frac{2P_{209}}{(N-1)^2N^5(1+N)^5(2+N)^2}+\Biggl(-\frac{4P_{200}}{(N-1)N^4(1+N)^4(2+N)}$$
$$+\frac{8(N-1)(2+N)}{N^2(1+N)^2}S_2+\frac{4(N-1)(2+N)}{N^2(1+N)^2}\zeta_2\Biggr)S_1-\frac{4(2+N)\bigl(3-2N^2+3N^3\bigr)}{N^3(1+N)^3}S_1^2$$
$$-\frac{64}{(N-1)N(1+N)(2+N)}S_{-3}+\frac{4(2+N)\bigl(3-2N^2+3N^3\bigr)}{N^3(1+N)^3}S_2$$
$$+\frac{2(2+N)\bigl(3-2N^2+3N^3\bigr)}{N^3(1+N)^3}\zeta_2-\frac{8(N-1)(2+N)}{3N^2(1+N)^2}S_1^3$$
$$+\Biggl(\frac{64}{(N-1)N(1+N)(2+N)}S_1-\frac{32\bigl(4+13N+6N^2+N^3\bigr)}{(N-1)^2N(1+N)^2(2+N)^2}\Biggr)S_{-2}\Biggr], \tag{293}$$

$$c_{g_1,g}^{(2,0)} = C_AT_FN_F\Biggl[-\frac{4P_{203}}{(N-1)N^4(1+N)^4(2+N)^2}+\Biggl(-\frac{4P_{181}}{N^3(1+N)^3(2+N)}$$
$$+\frac{28(N-1)}{N(1+N)}S_2+\frac{8(N-1)\zeta_2}{N(1+N)}\Biggr)S_1-\frac{8\bigl(7-4N-3N^2+3N^3\bigr)}{N^2(1+N)^2}S_1^2$$
$$-\frac{28(N-1)}{3N(1+N)}S_1^3+\frac{8\bigl(5-4N+N^2+N^3\bigr)}{N^2(1+N)^2}S_2+\frac{8\bigl(2+5N+5N^2\bigr)}{3N(1+N)(2+N)}S_3$$
$$+\Biggl(\frac{16P_{163}}{(N-1)N(1+N)^2(2+N)^2}+\frac{16\bigl(2+N+N^2\bigr)}{N(1+N)(2+N)}S_1\Biggr)S_{-2}-\frac{16(N-1)}{N(1+N)}S_{2,1}$$
$$-\frac{16\bigl(-4+N+N^2\bigr)}{N(1+N)(2+N)}S_{-3}-\frac{64}{N(1+N)(2+N)}S_{-2,1}-\frac{16(N-1)}{N^2(1+N)^2}\zeta_2$$
$$-\frac{24\bigl(2+N+N^2\bigr)}{N(1+N)(2+N)}\zeta_3\Biggr]+C_FT_FN_F\Biggl[\frac{2P_{205}}{(N-1)N^4(1+N)^4(2+N)^2}$$
$$+\Biggl(-\frac{4P_{178}}{N^3(1+N)^3(2+N)}+\frac{20(N-1)}{N(1+N)}S_2+\frac{8(N-1)}{N(1+N)}\zeta_2\Biggr)S_1-\frac{28(N-1)}{3N(1+N)}$$
$$\times S_1^3-\frac{4(N-1)\bigl(-5+3N+6N^2\bigr)}{N^2(1+N)^2}S_1^2+\frac{4\bigl(3-12N-5N^2+6N^3\bigr)}{N^2(1+N)^2}S_2$$

$$
\begin{aligned}
&-\frac{16\big(2+5N+5N^2\big)}{3N(1+N)(2+N)}S_3+\left(\frac{16\big(10+N+N^2\big)}{(N-1)(2+N)^2}-\frac{128}{N(1+N)(2+N)}S_1\right)S_{-2}\\
&-\frac{64}{N(1+N)(2+N)}S_{-3}+\frac{16(N-1)}{N(1+N)}S_{2,1}+\frac{128}{N(1+N)(2+N)}S_{-2,1}\\
&-\frac{2(N-1)\big(2+3N+3N^2\big)}{N^2(1+N)^2}\zeta_2+\frac{48\big(2+N+N^2\big)}{N(1+N)(2+N)}\zeta_3\Bigg], \qquad (294)
\end{aligned}
$$

$$
\begin{aligned}
c_{g_1,g}^{(2,1)} = {}& C_F T_F N_F\Bigg[-\frac{8P_{153}}{N(1+N)^2(2+N)^2}S_{2,1}-\frac{16P_{180}}{3(N-1)N^2(1+N)^2(2+N)^2}\zeta_3\\
&+\frac{4P_{183}}{3(N-1)N^2(1+N)^2(2+N)^2}S_3+\frac{P_{211}}{(N-1)^2N^5(1+N)^5(2+N)^3}\\
&+\frac{(N-1)P_{155}}{N^3(1+N)^3}\zeta_2-\Bigg(\frac{2P_{167}}{N^2(1+N)^2(2+N)^2}S_2-\frac{2P_{204}}{(N-1)N^4(1+N)^4(2+N)^2}\\
&-\frac{20\big(6+N+N^2\big)}{N(1+N)(2+N)}S_3+\frac{16\big(2+N+N^2\big)}{N(1+N)(2+N)}S_{2,1}+\frac{128}{N(1+N)(2+N)}S_{-2,1}\\
&+\frac{2(N-1)^2(4+3N)}{N^2(1+N)^2}\zeta_2+\frac{128\big(7+N+N^2\big)}{3N(1+N)(2+N)}\zeta_3\Bigg)S_1+\Bigg(\frac{P_{184}}{N^3(1+N)^3(2+N)}\\
&+\frac{\big(34-9N-9N^2\big)}{N(1+N)(2+N)}S_2-\frac{6(N-1)}{N(1+N)}\zeta_2\Bigg)S_1^2+\frac{2(N-1)\big(-11+8N+15N^2\big)}{3N^2(1+N)^2}\\
&\times S_1^3+\frac{5(N-1)}{2N(1+N)}S_1^4+\Bigg(\frac{P_{194}}{N^3(1+N)^3(2+N)^2}+\frac{6(N-1)}{N(1+N)}\zeta_2\Bigg)S_2\\
&+\frac{\big(-78+7N+7N^2\big)}{2N(1+N)(2+N)}S_2^2-\frac{5\big(6+5N+5N^2\big)}{N(1+N)(2+N)}S_4-\frac{96}{N(1+N)(2+N)}S_{-2}^2\\
&+\Bigg(\frac{8(2-N)\big(60+49N+28N^2+7N^3\big)}{(N-1)^2(2+N)^3}+\frac{16P_{161}}{(N-1)N^2(1+N)(2+N)^2}S_1\\
&+\frac{128}{N(1+N)(2+N)}S_1^2-\frac{64}{N(1+N)(2+N)}S_2\Bigg)S_{-2}+\Bigg(-\frac{64}{N(1+N)(2+N)}\\
&\times S_1\\
&+\frac{32P_{162}}{(N-1)N^2(1+N)(2+N)^2}\Bigg)S_{-3}+\frac{64}{N(1+N)(2+N)}S_{-2,2}\\
&-\frac{160}{N(1+N)(2+N)}S_{-4}-\frac{16\big(8-N-N^2\big)}{N(1+N)(2+N)}S_{3,1}+\frac{32\big(4+2N-N^2-N^3\big)}{N^2(1+N)(2+N)}\\
&\times S_{-2,1}+\frac{128}{N(1+N)(2+N)}S_{-3,1}+\frac{8\big(10+N+N^2\big)}{N(1+N)(2+N)}S_{2,1,1}\\
&-\frac{72\big(2+N+N^2\big)}{5N(1+N)(2+N)}\zeta_2^2\Bigg]+C_A T_F N_F\Bigg[\frac{16P_{164}}{N^2(1+N)^2(2+N)^2}S_{2,1}\\
&-\frac{2P_{160}}{N^3(1+N)^3}\zeta_2-\frac{8P_{179}}{3(N-1)N^2(1+N)^2(2+N)^2}S_3
\end{aligned}
$$

$$
+\frac{8P_{182}}{3(N-1)N^2(1+N)^2(2+N)^2}\zeta_3+\frac{2P_{212}}{(N-1)^2N^5(1+N)^5(2+N)^3}
+\Biggl(-\frac{4P_{166}}{N^2(1+N)^2(2+N)^2}S_2+\frac{2P_{206}}{(N-1)N^4(1+N)^4(2+N)^2}
+\frac{4\bigl(-22+N+N^2\bigr)}{N(1+N)(2+N)}S_3+\frac{16}{2+N}S_{2,1}+\frac{64}{N(1+N)(2+N)}S_{-2,1}
-\frac{8\bigl(3-2N-N^2+N^3\bigr)}{N^2(1+N)^2}\zeta_2-\frac{64\bigl(-13+2N+2N^2\bigr)}{3N(1+N)(2+N)}\zeta_3\Biggr)S_1
+\Biggl(\frac{P_{185}}{N^3(1+N)^3(2+N)}+\frac{\bigl(30-19N-19N^2\bigr)}{N(1+N)(2+N)}S_2-\frac{6(N-1)}{N(1+N)}\zeta_2\Biggr)S_1^2
+\frac{5(N-1)}{2N(1+N)}S_1^4+\frac{4\bigl(15-8N-7N^2+7N^3\bigr)}{3N^2(1+N)^2}S_1^3+\Biggl(\frac{P_{195}}{N^3(1+N)^3(2+N)^2}
+\frac{2(N-1)}{N(1+N)}\zeta_2\Biggr)S_2+\frac{\bigl(2+15N+15N^2\bigr)}{2N(1+N)(2+N)}S_2^2+\frac{\bigl(-2+21N+21N^2\bigr)}{N(1+N)(2+N)}S_4
+\Biggl(-\frac{32P_{177}}{(N-1)N^2(1+N)^2(2+N)^2}S_1-\frac{8P_{198}}{(N-1)^2N(1+N)^3(2+N)^3}
-\frac{8\bigl(6+N+N^2\bigr)}{N(1+N)(2+N)}S_1^2+\frac{8\bigl(2+N+N^2\bigr)}{N(1+N)(2+N)}S_2+\frac{4(N-1)}{N(1+N)}\zeta_2\Biggr)S_{-2}
+\frac{8\bigl(4+N+N^2\bigr)}{N(1+N)(2+N)}S_{-2}^2+\Biggl(\frac{8P_{176}}{(N-1)N^2(1+N)^2(2+N)^2}+\frac{16}{2+N}S_1\Biggr)S_{-3}
-\frac{8(N-3)(4+N)}{N(1+N)(2+N)}S_{-4}-\frac{16\bigl(4+2N-N^2-N^3\bigr)}{N^2(1+N)(2+N)}S_{-2,1}-\frac{32}{N(1+N)(2+N)}
\times S_{-2,2}-\frac{16\bigl(-5+N+N^2\bigr)}{N(1+N)(2+N)}S_{3,1}-\frac{64}{N(1+N)(2+N)}S_{-3,1}
-\frac{8\bigl(4+N+N^2\bigr)}{N(1+N)(2+N)}S_{2,1,1}+\frac{36\bigl(2+N+N^2\bigr)}{5N(1+N)(2+N)}\zeta_2^2\Biggr]\,. \tag{295}
$$

with

$$P_{149} = -1079N^4-2266N^3-1019N^2+24N-108\,, \tag{296}$$
$$P_{150} = -75N^4-192N^3-29N^2+32N-50\,, \tag{297}$$
$$P_{151} = N^4+2N^3-12N^2-13N+4\,, \tag{298}$$
$$P_{152} = N^4+2N^3-9N^2-10N+4\,, \tag{299}$$
$$P_{153} = N^4+4N^3-13N^2-40N-20\,, \tag{300}$$
$$P_{154} = 5N^4+19N^3-4N^2-6N+11\,, \tag{301}$$
$$P_{155} = 13N^4+20N^3+14N^2-N-6\,, \tag{302}$$
$$P_{156} = 26N^4+49N^3+33N^2+10N-5\,, \tag{303}$$
$$P_{157} = 111N^4+186N^3+131N^2+8N-36\,, \tag{304}$$
$$P_{158} = 229N^4+507N^3+116N^2-144N+72\,, \tag{305}$$

$$P_{159} = 373N^4 + 854N^3 + 313N^2 - 24N + 108 \; , \tag{306}$$
$$P_{160} = N^5 + N^4 - 12N^3 + 11N^2 + N - 10 \; , \tag{307}$$
$$P_{161} = N^5 + 2N^4 - 17N^3 - 26N^2 + 16 \; , \tag{308}$$
$$P_{162} = N^5 + 2N^4 + 4N^3 + N^2 + 4 \; , \tag{309}$$
$$P_{163} = N^5 + 3N^4 - 3N^3 - 9N^2 - 8N - 8 \; , \tag{310}$$
$$P_{164} = N^5 + 4N^4 + N^3 - 4N^2 + 4N + 4 \; , \tag{311}$$
$$P_{165} = 2N^5 + 6N^4 + N^3 - 6N^2 + 11N + 10 \; , \tag{312}$$
$$P_{166} = 7N^5 + 25N^4 + 2N^3 - 39N^2 + 28N + 60 \; , \tag{313}$$
$$P_{167} = 13N^5 + 43N^4 + 27N^3 - 39N^2 - 16N + 36 \; , \tag{314}$$
$$P_{168} = 27N^5 + 66N^4 + 25N^3 - 26N^2 + 8N + 16 \; , \tag{315}$$
$$P_{169} = 67N^5 + 214N^4 + 133N^3 - 150N^2 - 26N + 108 \; , \tag{316}$$
$$P_{170} = 179N^5 + 896N^4 + 1333N^3 + 610N^2 - 60N - 72 \; , \tag{317}$$
$$P_{171} = -4541N^6 - 14649N^5 - 14439N^4 - 4355N^3 - 672N^2 - 324N + 432 \; , \tag{318}$$
$$P_{172} = -2311N^6 - 6393N^5 - 991N^4 + 7197N^3 + 2222N^2 - 1236N + 216 \; , \tag{319}$$
$$P_{173} = -1239N^6 - 3321N^5 - 3661N^4 - 1787N^3 - 100N^2 + 108N - 144 \; , \tag{320}$$
$$P_{174} = -237N^6 - 635N^5 - 675N^4 - 321N^3 + 12N^2 - 40 \; , \tag{321}$$
$$P_{175} = -37N^6 - 243N^5 - 135N^4 + 167N^3 + 44N^2 + 16N + 64 \; , \tag{322}$$
$$P_{176} = N^6 + 3N^5 - 5N^4 - 7N^3 - 8N^2 - 24N - 8 \; , \tag{323}$$
$$P_{177} = N^6 + 3N^5 - 2N^4 - 10N^3 - 8N^2 + 4 \; , \tag{324}$$
$$P_{178} = 6N^6 + 19N^5 + 5N^4 - 19N^3 + 9N^2 - 4N - 20 \; , \tag{325}$$
$$P_{179} = 8N^6 + 19N^5 - 28N^4 - 56N^3 - 31N^2 - 32N + 48 \; , \tag{326}$$
$$P_{180} = 12N^6 + 27N^5 - 97N^4 - 281N^3 - 177N^2 + 64N + 20 \; , \tag{327}$$
$$P_{181} = 15N^6 + 36N^5 - 19N^4 - 42N^3 + 30N^2 - 40N - 56 \; , \tag{328}$$
$$P_{182} = 18N^6 + 36N^5 - 31N^4 - 134N^3 - 357N^2 - 200N + 236 \; , \tag{329}$$
$$P_{183} = 24N^6 + 59N^5 - 211N^4 - 487N^3 - 29N^2 + 280N + 76 \; , \tag{330}$$
$$P_{184} = 31N^6 + 90N^5 + 23N^4 - 84N^3 - 8N^2 - 16N - 48 \; , \tag{331}$$
$$P_{185} = 43N^6 + 102N^5 - 37N^4 - 100N^3 + 44N^2 - 104N - 128 \; , \tag{332}$$
$$P_{186} = 105N^6 + 279N^5 + 375N^4 + 261N^3 + 124N^2 + 32N + 8 \; , \tag{333}$$
$$P_{187} = 108N^6 + 258N^5 + 14N^4 - 228N^3 - 7N^2 + 9N - 10 \; , \tag{334}$$
$$P_{188} = 198N^6 + 405N^5 - 102N^4 - 501N^3 - 268N^2 - 76N - 88 \; , \tag{335}$$
$$P_{189} = 834N^6 + 1881N^5 - 104N^4 - 2029N^3 - 1432N^2 - 338N - 108 \; , \tag{336}$$
$$P_{190} = 2667N^6 + 6669N^5 + 6605N^4 + 2095N^3 - 940N^2 + 504 \; , \tag{337}$$
$$P_{191} = 4541N^6 + 14703N^5 + 14223N^4 + 3869N^3 + 348N^2 + 324N - 432 \; , \tag{338}$$
$$P_{192} = 7081N^6 + 25239N^5 + 25431N^4 + 8797N^3 + 2820N^2 - 1512 \; , \tag{339}$$
$$P_{193} = -397N^7 - 1873N^6 - 3293N^5 - 2811N^4 - 1302N^3 - 200N^2 + 16N - 32 \; , \tag{340}$$
$$P_{194} = -37N^7 - 160N^6 - 85N^5 + 426N^4 + 612N^3 + 184N^2 + 32N + 64 \; , \tag{341}$$
$$P_{195} = -13N^7 - 80N^6 - 121N^5 + 30N^4 + 28N^3 - 96N^2 + 208N + 192 \; , \tag{342}$$
$$P_{196} = 14311N^7 + 71609N^6 + 126915N^5 + 92875N^4 + 23306N^3 + 372N^2 + 216N - 864 \; , \tag{343}$$
$$P_{197} = -56163N^8 - 192648N^7 - 268610N^6 - 170340N^5 - 27651N^4 + 4084N^3$$

$$-8928N^2 + 4464N + 6480 \ , \tag{344}$$

$$P_{198} = 2N^8 + 11N^7 + 10N^6 - 26N^5 - 2N^4 + 151N^3 + 262N^2 + 160N + 8 \ , \tag{345}$$

$$P_{199} = 3N^8 + 15N^7 + 11N^6 - 45N^5 - 74N^4 - 30N^3 - 52N^2 - 80N - 36 \ , \tag{346}$$

$$P_{200} = 10N^8 + 34N^7 + 4N^6 - 64N^5 - 51N^4 - 48N^3 - 53N^2 + 12N + 28 \ , \tag{347}$$

$$P_{201} = -86393N^{10} - 479161N^9 - 842908N^8 - 273610N^7 + 665567N^6 \\ +681947N^5 + 206494N^4 - 29400N^3 - 86616N^2 - 20304N + 15552 \ , \tag{348}$$

$$P_{202} = -30651N^{10} - 138387N^9 - 203260N^8 - 49510N^7 + 185861N^6 + 211505N^5 \\ +73610N^4 + 8504N^3 + 23688N^2 + 6768N - 5184 \ , \tag{349}$$

$$P_{203} = 4N^{10} + 18N^9 - 18N^8 - 130N^7 - 3N^6 + 253N^5 + 131N^4 + 115N^3 + 146N^2 \\ -44N - 88 \ , \tag{350}$$

$$P_{204} = 16N^{10} + 96N^9 + 137N^8 - 155N^7 - 584N^6 - 618N^5 - 431N^4 - 67N^3 \\ +178N^2 - 20N - 88 \ , \tag{351}$$

$$P_{205} = 34N^{10} + 144N^9 + 134N^8 - 156N^7 - 147N^6 + 449N^5 + 825N^4 + 355N^3 \\ -130N^2 - 28N + 56 \ , \tag{352}$$

$$P_{206} = 50N^{10} + 214N^9 + 99N^8 - 533N^7 - 355N^6 + 605N^5 + 562N^4 + 338N^3 \\ +244N^2 - 216N - 240 \ , \tag{353}$$

$$P_{207} = 763N^{10} + 3395N^9 + 5268N^8 + 2262N^7 - 3701N^6 - 5729N^5 - 4554N^4 \\ -2872N^3 - 1216N^2 + 80N + 160 \ , \tag{354}$$

$$P_{208} = 1203N^{10} + 5811N^9 + 9468N^8 + 4366N^7 - 5429N^6 - 8825N^5 - 3450N^4 \\ +1256N^3 + 1152N^2 + 368N + 224 \ , \tag{355}$$

$$P_{209} = 32N^{12} + 172N^{11} + 187N^{10} - 377N^9 - 635N^8 + 560N^7 + 1634N^6 \\ +1246N^5 + 488N^4 - 145N^3 - 294N^2 + 84N + 120 \ , \tag{356}$$

$$P_{210} = -4521N^{14} - 28595N^{13} - 64213N^{12} - 42855N^{11} + 63121N^{10} + 130099N^9 \\ +49137N^8 - 79237N^7 - 137428N^6 - 123924N^5 - 65104N^4 - 448N^3 \\ +11360N^2 - 640N - 1664 \ , \tag{357}$$

$$P_{211} = -84N^{14} - 520N^{13} - 1019N^{12} - 186N^{11} + 1538N^{10} + 1433N^9 + 2372N^8 \\ +9376N^7 + 14990N^6 + 9125N^5 + 163N^4 - 868N^3 + 720N^2 + 64N - 240 \ , \tag{358}$$

$$P_{212} = 12N^{14} + 76N^{13} + 33N^{12} - 576N^{11} - 806N^{10} + 1161N^9 + 1790N^8 \\ -2494N^7 - 5322N^6 - 3207N^5 - 715N^4 + 664N^3 + 840N^2 - 304N - 368 \ , \tag{359}$$

$$P_{213} = 648699N^{14} + 4173081N^{13} + 9225583N^{12} + 4949301N^{11} - 11882659N^{10} \\ -20345337N^9 - 5717331N^8 + 11889903N^7 + 12065660N^6 + 4779564N^5 \\ +2272528N^4 + 613584N^3 - 713088N^2 - 160704N + 145152. \tag{360}$$

The evanescent poles at $N = 2$ are tractable, and the rightmost singularities are at $N = 1$ for the singlet contributions and at $N = 0$ for the non–singlet contributions, as generally expected, cf. [84].

4.3 The Two–Loop Wilson Coefficients

The two–loop Wilson coefficients for $\mu^2 = Q^2$ are given by

$$
\begin{aligned}
C_{F_2,q}^{\rm NS,(2)} &= C_F^2\Biggl[\frac{S_2 P_{218}}{2N^2(1+N)^2}+\frac{P_{233}}{8(N-2)N^4(1+N)^4(3+N)}+\Biggl(\frac{P_{225}}{2N^3(1+N)^3} \\
&-\frac{2(-2+3N)(5+3N)S_2}{N(1+N)}-24S_3+16S_{2,1}+32S_{-2,1}+48\zeta_3\Biggr)S_1+\Biggl(\frac{P_{214}}{2N^2(1+N)^2} \\
&-20S_2\Biggr)S_1^2+\frac{2\bigl(-2+3N+3N^2\bigr)S_1^3}{N(1+N)}+2S_1^4+6S_2^2-\frac{2\bigl(-10+25N+9N^2\bigr)S_3}{N(1+N)} \\
&+12S_4+\Biggl(-\frac{8P_{61}}{(N-2)N^2(1+N)^2(3+N)}-\frac{16(-3+4N)S_1}{N(1+N)}-32S_1^2+16S_2\Biggr)S_{-2} \\
&+24S_{-2}^2+\Biggl(-\frac{32}{1+N}+16S_1\Biggr)S_{-3}+40S_{-4}+\frac{4\bigl(-2+3N+3N^2\bigr)S_{2,1}}{N(1+N)}+40S_{3,1} \\
&+\frac{32(-1+2N)S_{-2,1}}{N(1+N)}-16S_{-2,2}-32S_{-3,1}-24S_{2,1,1}-\frac{24\bigl(2-N+3N^2\bigr)\zeta_3}{N(1+N)}\Biggr] \\
&+C_AC_F\Biggl[\frac{P_{229}}{216(N-2)N^3(1+N)^3(3+N)}+\Biggl(-\frac{2\bigl(6+11N+11N^2\bigr)S_2}{3N(1+N)} \\
&+\frac{P_{224}}{54N^2(1+N)^3}+24S_3-16S_{2,1}-16S_{-2,1}-48\zeta_3\Biggr)S_1+\Biggl(\frac{-66+367N+367N^2}{18N(1+N)} \\
&+4S_2\Biggr)S_1^2+\frac{\bigl(66-929N-2134N^2-1067N^3\bigr)S_2}{18N(1+N)^2}+\frac{2\bigl(-72+193N+121N^2\bigr)S_3}{9N(1+N)} \\
&-4S_2^2+\frac{22}{9}S_1^3-8S_4+\Biggl(\frac{4P_{61}}{(N-2)N^2(1+N)^2(3+N)}+\frac{8(-3+4N)S_1}{N(1+N)} \\
&+16S_1^2-8S_2\Biggr)S_{-2}+\Biggl(\frac{16}{1+N}-8S_1\Biggr)S_{-3}-\frac{4\bigl(-6+11N+11N^2\bigr)S_{2,1}}{3N(1+N)}-24S_{3,1} \\
&-20S_{-4}-12S_{-2}^2-\frac{16(-1+2N)S_{-2,1}}{N(1+N)}+8S_{-2,2}+16S_{-3,1}+24S_{2,1,1} \\
&+\frac{6\bigl(6+N+9N^2\bigr)\zeta_3}{N(1+N)}\Biggr]+C_FT_FN_F\Biggl[\frac{P_{235}}{54N^3(1+N)^3}+\Biggl(\frac{8}{3}S_2-\frac{2P_{236}}{27N^2(1+N)^2}\Biggr)S_1 \\
&+\frac{2\bigl(6-29N-29N^2\bigr)}{9N(1+N)}S_1^2-\frac{8}{9}S_1^3-\frac{2\bigl(6-85N-85N^2\bigr)}{9N(1+N)}S_2 \\
&-\frac{88}{9}S_3+\frac{16}{3}S_{2,1}\Biggr], \qquad (361)
\end{aligned}
$$

$$
\begin{aligned}
C_{F_2,q}^{\rm PS,(2)} &= C_FT_FN_F\Biggl[\frac{8S_1P_{228}}{(N-1)N^3(1+N)^3(2+N)^2}+\frac{4P_{232}}{(N-1)N^4(1+N)^4(2+N)^3} \\
&+\frac{4\bigl(2+N+N^2\bigr)^2S_1^2}{(N-1)N^2(1+N)^2(2+N)}-\frac{4\bigl(2+N+N^2\bigr)^2S_2}{(N-1)N^2(1+N)^2(2+N)}
\end{aligned}
$$

$$+\frac{64S_{-2}}{(N-1)N(1+N)(2+N)}\Biggr], \tag{362}$$

$$\begin{aligned}
C_{F_2,g}^{(2)} &= C_A T_F N_F \Biggl[\frac{8S_2 P_{220}}{(N-1)N(1+N)^2(2+N)^2} - \frac{8S_1^2 P_{221}}{(N-1)N(1+N)^2(2+N)^2} \\
&-\frac{4P_{234}}{(N-1)N^4(1+N)^4(2+N)^4} + \Biggl(-\frac{4P_{230}}{(N-1)N^3(1+N)^3(2+N)^3} \\
&+\frac{20\bigl(2+N+N^2\bigr)S_2}{N(1+N)(2+N)}\Biggr)S_1 - \frac{4\bigl(2+N+N^2\bigr)S_1^3}{3N(1+N)(2+N)} + \frac{8\bigl(-2+5N+5N^2\bigr)S_3}{3N(1+N)(2+N)} \\
&+\Biggl(\frac{16P_{38}}{(N-1)N(1+N)^2(2+N)^2} + \frac{16(N-1)S_1}{N(1+N)}\Biggr)S_{-2} - \frac{16\bigl(4+N+N^2\bigr)S_{-3}}{N(1+N)(2+N)} \\
&-\frac{16\bigl(2+N+N^2\bigr)S_{2,1}}{N(1+N)(2+N)} + \frac{64S_{-2,1}}{N(1+N)(2+N)} - \frac{24(N-1)\zeta_3}{N(1+N)}\Biggr] \\
&+C_F T_F N_F \Biggl[\frac{2S_2 P_{216}}{N^2(1+N)^2(2+N)} - \frac{2P_{231}}{(N-2)N^4(1+N)^4(2+N)(3+N)} \\
&-\frac{2S_1^2 P_{217}}{N^2(1+N)^2(2+N)} + \Biggl(\frac{4P_{226}}{N^3(1+N)^3(2+N)} + \frac{12\bigl(2+N+N^2\bigr)S_2}{N(1+N)(2+N)}\Biggr)S_1 \\
&-\frac{20\bigl(2+N+N^2\bigr)S_1^3}{3N(1+N)(2+N)} - \frac{64\bigl(-1+N+N^2\bigr)S_3}{3N(1+N)(2+N)} + \Biggl(\frac{128S_1}{N(1+N)(2+N)} \\
&+\frac{16P_{59}}{(N-2)N^2(1+N)^2(2+N)(3+N)}\Biggr)S_{-2} + \frac{64S_{-3}}{N(1+N)(2+N)} \\
&+\frac{16\bigl(2+N+N^2\bigr)S_{2,1}}{N(1+N)(2+N)} - \frac{128S_{-2,1}}{N(1+N)(2+N)} + \frac{48(N-1)\zeta_3}{N(1+N)}\Biggr],
\end{aligned} \tag{363}$$

$$\begin{aligned}
C_{F_L,q}^{\mathrm{NS},(2)} &= C_F\Biggl\{T_F N_F\Biggl[-\frac{8\bigl(-6+7N+19N^2\bigr)}{9N(1+N)^2} - \frac{16S_1}{3(1+N)}\Biggr] + C_A\Biggl[\frac{92S_1}{3(1+N)} \\
&+\frac{2P_{219}}{9(N-2)N(1+N)^2(3+N)} + \frac{16S_3}{1+N} + \Biggl(-\frac{32P_{15}}{(N-2)N(1+N)^2(3+N)} \\
&+\frac{32S_1}{1+N}\Biggr)S_{-2} + \frac{16S_{-3}}{1+N} - \frac{32S_{-2,1}}{1+N} - \frac{48\zeta_3}{1+N}\Biggr]\Biggr\} + C_F^2\Biggl\{-\frac{4\bigl(2+13N+9N^2\bigr)S_1}{N(1+N)^2} \\
&-\frac{2P_{227}}{(N-2)N^2(1+N)^3(3+N)} + \frac{8S_1^2}{1+N} - \frac{8S_2}{1+N} - \frac{32S_3}{1+N} + \Biggl(-\frac{64S_1}{1+N} \\
&+\frac{64P_{15}}{(N-2)N(1+N)^2(3+N)}\Biggr)S_{-2} - \frac{32S_{-3}}{1+N} + \frac{64S_{-2,1}}{1+N} + \frac{96\zeta_3}{1+N}\Biggr\},
\end{aligned} \tag{364}$$

$$C_{F_L,q}^{\mathrm{PS}(2)} = C_F T_F N_F\Biggl[-\frac{32P_{222}}{(N-1)N^2(1+N)^3(2+N)^2} - \frac{32\bigl(2+N+N^2\bigr)S_1}{(N-1)N(1+N)^2(2+N)}\Biggr], \tag{365}$$

$$
\begin{aligned}
C_{F_L,g}^{(2)} &= C_A T_F N_F \Bigg[-\frac{32 P_{223}}{(N-1)N^2(1+N)^3(2+N)^3}+\frac{64\big(-1-N-2N^2+2N^3\big)S_1}{(N-1)N(1+N)^2(2+N)} \\
&\quad +\frac{32 S_1^2}{(1+N)(2+N)}-\frac{32 S_2}{(1+N)(2+N)}-\frac{64 S_{-2}}{(1+N)(2+N)}\Bigg] \\
&\quad +C_F T_F N_F \Bigg[\frac{16(N-1)P_{215}}{(N-2)N^2(1+N)^3(2+N)(3+N)}-\frac{16\big(2+3N+3N^2\big)S_1}{N(1+N)^2(2+N)} \\
&\quad +\frac{64(N-1)S_{-2}}{(N-2)(1+N)(3+N)}\Bigg] ,
\end{aligned}
\tag{366}
$$

with

$$P_{214} = -27N^4-26N^3+9N^2+40N+24 , \tag{367}$$
$$P_{215} = 2N^4+19N^3+39N^2+40N+12 , \tag{368}$$
$$P_{216} = 9N^4+8N^3+9N^2+6N-8 , \tag{369}$$
$$P_{217} = 9N^4+12N^3+N^2-14N-16 , \tag{370}$$
$$P_{218} = 95N^4+162N^3+35N^2-32N-16 , \tag{371}$$
$$P_{219} = 215N^4+298N^3-1597N^2-888N+396 , \tag{372}$$
$$P_{220} = N^5-10N^3-9N^2-4N-2 , \tag{373}$$
$$P_{221} = N^5-2N^4-6N^3-3N^2-12N-2 , \tag{374}$$
$$P_{222} = N^5+2N^4+2N^3-5N^2-12N-4 , \tag{375}$$
$$P_{223} = 5N^5+8N^4-3N^3-22N^2-28N-8 , \tag{376}$$
$$P_{224} = 3155N^5+11607N^4+12279N^3+3329N^2+510N+792 , \tag{377}$$
$$P_{225} = -51N^6-203N^5-207N^4-33N^3-106N^2-160N-48 , \tag{378}$$
$$P_{226} = N^6-7N^5-3N^4-5N^3-30N^2-40N-16 , \tag{379}$$
$$P_{227} = 17N^6+39N^5-157N^4-299N^3-64N^2+104N+24 , \tag{380}$$
$$P_{228} = N^7-15N^5-58N^4-92N^3-76N^2-48N-16 , \tag{381}$$
$$
\begin{aligned}
P_{229} &= -16395N^8-47520N^7+51416N^6+162042N^5+99843N^4-7930N^3 \\
&\quad -21432N^2+25848N+23760 ,
\end{aligned}
\tag{382}
$$
$$
\begin{aligned}
P_{230} &= 7N^9+5N^8-43N^7+25N^6+296N^5+498N^4+524N^3+336N^2+144N \\
&\quad +32 ,
\end{aligned}
\tag{383}
$$
$$
\begin{aligned}
P_{231} &= 2N^{10}+18N^9+98N^8+98N^7-425N^6-1071N^5-477N^4+651N^3 \\
&\quad +886N^2+484N+120 ,
\end{aligned}
\tag{384}
$$
$$
\begin{aligned}
P_{232} &= 3N^{10}+14N^9+33N^8+79N^7+297N^6+849N^5+1373N^4+1312N^3 \\
&\quad +840N^2+368N+80 ,
\end{aligned}
\tag{385}
$$
$$
\begin{aligned}
P_{233} &= 331N^{10}+1179N^9-848N^8-4754N^7-2157N^6+4247N^5+3474N^4 \\
&\quad -2528N^3-4976N^2-2704N-480 ,
\end{aligned}
\tag{386}
$$
$$
\begin{aligned}
P_{234} &= 4N^{12}+34N^{11}+100N^{10}+116N^9-81N^8-637N^7-1677N^6 \\
&\quad -3093N^5-3998N^4-3472N^3-2064N^2-816N-160 ,
\end{aligned}
\tag{387}
$$
$$P_{235} = 360+648N+140N^2+31N^3+1397N^4+2517N^5+1371N^6 , \tag{388}$$
$$P_{236} = 72+66N+331N^2+620N^3+247N^4. \tag{389}$$

$$
\begin{aligned}
C_{F_3,q}^{(2),\rm NS} &= C_F\Biggl\{T_FN_F\Biggl[\frac{P_{241}}{54N^3(1+N)^3}+\Biggl(-\frac{2P_{239}}{27N^2(1+N)^2}+\frac{8S_2}{3}\Biggr)S_1-\frac{8}{9}S_1^3 \\
&-\frac{2\bigl(-6+29N+29N^2\bigr)S_1^2}{9N(1+N)}+\frac{2\bigl(-6+85N+85N^2\bigr)S_2}{9N(1+N)}-\frac{88}{9}S_3+\frac{16}{3}S_{2,1}\Biggr] \\
&+C_A\Biggl[\frac{P_{243}}{216(N-1)N^4(1+N)^4(2+N)}+\Biggl(\frac{P_{242}}{54N^3(1+N)^3}+24S_3-16S_{2,1} \\
&-16S_{-2,1}-\frac{2\bigl(6+11N+11N^2\bigr)S_2}{3N(1+N)}-48\zeta_3\Biggr)S_1+\Biggl(\frac{-66+367N+367N^2}{18N(1+N)}+4S_2\Biggr) \\
&\times S_1^2+\frac{22}{9}S_1^3+\frac{\bigl(66-929N-2134N^2-1067N^3\bigr)S_2}{18N(1+N)^2}-4S_2^2-8S_4-12S_{-2}^2 \\
&+\frac{2\bigl(-36+121N+121N^2\bigr)S_3}{9N(1+N)}+\Biggl(\frac{4P_{123}}{(N-1)N(1+N)^2(2+N)}-\frac{8S_1}{N(1+N)} \\
&+16S_1^2-8S_2\Biggr)S_{-2}+\Biggl(\frac{8}{N(1+N)}-8S_1\Biggr)S_{-3}-20S_{-4}+\frac{4\bigl(6-11N-11N^2\bigr)S_{2,1}}{3N(1+N)} \\
&-24S_{3,1}+8S_{-2,2}+16S_{-3,1}+24S_{2,1,1}+\frac{6(1+3N)(2+3N)\zeta_3}{N(1+N)}\Biggr]\Biggr\} \\
&+C_F^2\Biggl[\frac{S_2P_{238}}{2N^2(1+N)^2}+\frac{P_{244}}{8(N-1)N^4(1+N)^4(2+N)}+\Biggl(\frac{P_{240}}{2N^3(1+N)^3} \\
&-\frac{2(-2+3N)(5+3N)S_2}{N(1+N)}-24S_3+16S_{2,1}+32S_{-2,1}+48\zeta_3\Biggr)S_1+\Biggl(-20S_2 \\
&+\frac{P_{237}}{2N^2(1+N)^2}\Biggr)S_1^2+\frac{2\bigl(-2+3N+3N^2\bigr)S_1^3}{N(1+N)}+2S_1^4+6S_2^2+12S_4 \\
&-\frac{2\bigl(-2+9N+9N^2\bigr)S_3}{N(1+N)}+\Biggl(-\frac{8P_{123}}{(N-1)N(1+N)^2(2+N)}+\frac{16S_1}{N(1+N)} \\
&-32S_1^2+16S_2\Biggr)S_{-2}+24S_{-2}^2+\Biggl(-\frac{16}{N(1+N)}+16S_1\Biggr)S_{-3}+40S_{-4} \\
&+\frac{4\bigl(-2+3N+3N^2\bigr)S_{2,1}}{N(1+N)}+40S_{3,1}-16S_{-2,2}-32S_{-3,1}-24S_{2,1,1}-72\zeta_3\Biggr], \qquad (390)
\end{aligned}
$$

with the polynomials

$$P_{237} = -27N^4-42N^3-15N^2+32N+24, \qquad (391)$$
$$P_{238} = 95N^4+178N^3+59N^2-24N-16, \qquad (392)$$
$$P_{239} = 247N^4+548N^3+223N^2+30N+72, \qquad (393)$$
$$P_{240} = -51N^6-131N^5-115N^4-49N^3-54N^2-72N-16, \qquad (394)$$
$$P_{241} = 1371N^6+3429N^5+2957N^4+271N^3-556N^2+360N+360, \qquad (395)$$

$$P_{242} = 3155N^6+9951N^5+9867N^4+3473N^3+546N^2-72N-432\,, \quad (396)$$

$$P_{243} = -16395N^{10}-74235N^9-101500N^8+5090N^7+133973N^6+105113N^5 +16970N^4+6224N^3+16200N^2-3312N-5184\,, \quad (397)$$

$$P_{244} = 331N^{10}+1451N^9+1492N^8-1826N^7-4341N^6-1673N^5+246N^4 -928N^3-1232N^2+112N+224 \quad (398)$$

$$C_{g_1,q}^{(2),\mathrm{NS,L}} = C_F\Biggl\{T_FN_F\Biggl[\frac{P_{251}}{54N^3(1+N)^3}+\Biggl(-\frac{2P_{239}}{27N^2(1+N)^2}+\frac{8}{3}S_2\Biggr)S_1-\frac{8}{9}S_1^3 -\frac{2\bigl(-6+29N+29N^2\bigr)}{9N(1+N)}S_1^2+\frac{2\bigl(-6+85N+85N^2\bigr)}{9N(1+N)}S_2-\frac{88}{9}S_3+\frac{16}{3}S_{2,1}\Biggr] +C_A\Biggl[\frac{P_{253}}{216(N-1)N^4(1+N)^4(2+N)}+\Biggl(-\frac{2\bigl(6+11N+11N^2\bigr)}{3N(1+N)}S_2 +\frac{P_{242}}{54N^3(1+N)^3}+24S_3-16S_{2,1}-16S_{-2,1}-48\zeta_3\Biggr)S_1+\frac{22}{9}S_1^3+\Biggl(4S_2 +\frac{-66+367N+367N^2}{18N(1+N)}\Biggr)S_1^2+\frac{\bigl(66-929N-2134N^2-1067N^3\bigr)}{18N(1+N)^2}S_2-4S_2^2 +\frac{2\bigl(-36+121N+121N^2\bigr)}{9N(1+N)}S_3-8S_4+\Biggl(\frac{4P_{165}}{(N-1)N(1+N)^2(2+N)} -\frac{8}{N(1+N)}S_1+16S_1^2-8S_2\Biggr)S_{-2}-12S_{-2}^2+\Biggl(\frac{8}{N(1+N)}-8S_1\Biggr)S_{-3} -20S_{-4}-\frac{4\bigl(-6+11N+11N^2\bigr)}{3N(1+N)}S_{2,1}-24S_{3,1}+8S_{-2,2}+16S_{-3,1}+24S_{2,1,1} +\frac{6(1+3N)(2+3N)}{N(1+N)}\zeta_3\Biggr]\Biggr\}+C_F^2\Biggl[\frac{P_{256}}{8(N-1)N^4(1+N)^4(2+N)} +\frac{P_{246}}{2N^2(1+N)^2}S_2+\Biggl(\frac{P_{248}}{2N^3(1+N)^3}-\frac{2(-2+3N)(5+3N)S_2}{N(1+N)}-24S_3+16S_{2,1} +32S_{-2,1}+48\zeta_3\Biggr)S_1+\Biggl(\frac{P_{245}}{2N^2(1+N)^2}-20S_2\Biggr)S_1^2+\frac{2\bigl(-2+3N+3N^2\bigr)}{N(1+N)}S_1^3 +2S_1^4+6S_2^2-\frac{2\bigl(-2+9N+9N^2\bigr)S_3}{N(1+N)}+12S_4+\Biggl(\frac{16}{N(1+N)}S_1 -\frac{8P_{165}}{(N-1)N(1+N)^2(2+N)}-32S_1^2+16S_2\Biggr)S_{-2}+24S_{-2}^2+16\Biggl(-\frac{1}{N(1+N)} +S_1\Biggr)S_{-3}+40S_{-4}+\frac{4\bigl(-2+3N+3N^2\bigr)}{N(1+N)}S_{2,1}+40S_{3,1}-16S_{-2,2}-32S_{-3,1} -24S_{2,1,1}-72\zeta_3\Biggr]\,, \quad (399)$$

$$
\begin{aligned}
C_{g_1,q}^{(2),\mathrm{NS,M}} &= C_{g_1,q}^{(2),\mathrm{NS,L}} - \Big(C_{g1,q}^{(1),\mathrm{NS,L}} - z_{qq}^{(1)}\Big) z_{qq}^{(1)} - z_{qq}^{(2),\mathrm{NS}} \\
&= C_{g_1,q}^{(2),\mathrm{NS,L}} + C_F \Bigg\{ -T_F N_F \frac{16\big(-3-N+5N^2\big)}{9N^2(1+N)^2} + C_A \Bigg[\frac{4P_{247}}{9N^3(1+N)^3} \\
&+ \frac{16}{N(1+N)} S_{-2} \Bigg] \Bigg\} + C_F^2 \Bigg[-\frac{8\big(11+16N+11N^2\big)}{N(1+N)^3} + \frac{8(-4+3N)}{N^2(1+N)} S_1 \\
&+ \frac{16}{N(1+N)} S_1^2 - \frac{32}{N(1+N)} S_2 - \frac{32}{N(1+N)} S_{-2} \Bigg], \qquad (400) \\
C_{g_1,q}^{(2),\mathrm{PS,L}} &= C_F T_F N_F \Bigg[\frac{4P_{252}}{(N-1)N^4(1+N)^4(2+N)} + \frac{8(2+N)\big(2+N-N^2+2N^3\big)}{N^3(1+N)^3} S_1 \\
&+ \frac{4(N-1)(2+N)}{N^2(1+N)^2} S_1^2 - \frac{4(N-1)(2+N)}{N^2(1+N)^2} S_2 - \frac{64}{(N-1)N(1+N)(2+N)} \\
&\times S_{-2} \Bigg], \qquad (401) \\
C_{g_1,q}^{(2),\mathrm{PS,M}} &= C_{g_1,q}^{(2),\mathrm{NS,L}} - z_{qq}^{(2),\mathrm{PS}} = C_{g_1,q}^{(2),\mathrm{NS,L}} + C_F T_F N_F \frac{8(2+N)\big(-1-N+N^2\big)}{N^3(1+N)^3}, \qquad (402) \\
C_{g_1,g}^{(2)} &= C_A T_F N_F \Bigg[-\frac{4P_{255}}{(N-1)N^4(1+N)^4(2+N)^2} + \Bigg(-\frac{4P_{250}}{N^3(1+N)^3(2+N)} \\
&+ \frac{20(N-1)}{N(1+N)} S_2 \Bigg) S_1 - \frac{8\big(3-N^2+N^3\big)}{N^2(1+N)^2} S_1^2 + \frac{8\big(3-2N+N^2+N^3\big)}{N^2(1+N)^2} S_2 \\
&- \frac{4(N-1)}{3N(1+N)} S_1^3 + \frac{8\big(2+5N+5N^2\big)}{3N(1+N)(2+N)} S_3 + \Bigg(\frac{16P_{163}}{(N-1)N(1+N)^2(2+N)^2} \\
&+ \frac{16\big(2+N+N^2\big)}{N(1+N)(2+N)} S_1 \Bigg) S_{-2} - \frac{16\big(-4+N+N^2\big)}{N(1+N)(2+N)} S_{-3} - \frac{16(N-1)}{N(1+N)} S_{2,1} \\
&- \frac{64}{N(1+N)(2+N)} S_{-2,1} - \frac{24\big(2+N+N^2\big)}{N(1+N)(2+N)} \zeta_3 \Bigg] \\
&+ C_F T_F N_F \Bigg[-\frac{2P_{254}}{(N-1)N^4(1+N)^4(2+N)^2} + \Bigg(\frac{4P_{249}}{N^3(1+N)^3(2+N)} \\
&+ \frac{12(N-1)}{N(1+N)} S_2 \Bigg) S_1 - \frac{2(N-1)\big(-8+3N+9N^2\big)}{N^2(1+N)^2} S_1^2 - \frac{20(N-1)}{3N(1+N)} S_1^3 \\
&+ \frac{2\big(4-19N-10N^2+9N^3\big)}{N^2(1+N)^2} S_2 - \frac{64\big(1+N+N^2\big)}{3N(1+N)(2+N)} S_3 \\
&+ \Bigg(-\frac{128}{N(1+N)(2+N)} S_1 + \frac{16\big(10+N+N^2\big)}{(N-1)(2+N)^2} \Bigg) S_{-2} - \frac{64}{N(1+N)(2+N)} S_{-3} \\
&+ \frac{16(N-1)}{N(1+N)} S_{2,1} + \frac{128}{N(1+N)(2+N)} S_{-2,1} + \frac{48\big(2+N+N^2\big)}{N(1+N)(2+N)} \zeta_3 \Bigg], \qquad (403)
\end{aligned}
$$

with the polynomials

$$
\begin{aligned}
P_{245} &= -27N^4-42N^3-47N^2+24\ , & (404)\\
P_{246} &= 95N^4+178N^3+123N^2+40N-16\ , & (405)\\
P_{247} &= 103N^4+140N^3+58N^2+21N+36\ , & (406)\\
P_{239} &= 247N^4+548N^3+223N^2+30N+72\ , & (407)\\
P_{248} &= -51N^6-131N^5-163N^4-81N^3+26N^2-8N-16\ , & (408)\\
P_{249} &= N^6+5N^5+3N^4-13N^3-20N^2+12N+16\ , & (409)\\
P_{250} &= 7N^6+16N^5-11N^4-18N^3+6N^2-44N-32\ , & (410)\\
P_{251} &= 1371N^6+3429N^5+3437N^4+655N^3-940N^2+72N+360\ , & (411)\\
P_{242} &= 3155N^6+9951N^5+9867N^4+3473N^3+546N^2-72N-432\ , & (412)\\
P_{252} &= 6N^8+20N^7+2N^6-36N^5-43N^4-62N^3-47N^2+12N+20\ , & (413)\\
P_{253} &= -16395N^{10}-74235N^9-111388N^8-28126N^7+111413N^6+125177N^5 & \\
&\quad +41930N^4+12464N^3+23688N^2+3600N-5184\ , & (414)\\
P_{254} &= 2N^{10}+18N^9+26N^8-42N^7-253N^6-563N^5-633N^4-181N^3+126N^2 & \\
&\quad +4N-40\ , & (415)\\
P_{255} &= 4N^{10}+18N^9-2N^8-74N^7+5N^6+141N^5+99N^4+171N^3+122N^2 & \\
&\quad -44N-56\ , & (416)\\
P_{256} &= 331N^{10}+1451N^9+2196N^8+606N^7-2293N^6-2697N^5-2506N^4 & \\
&\quad -2336N^3-1232N^2+112N+224\ , & (417)
\end{aligned}
$$

with the coefficients $z_{ij}^{(l)}$ given in [28,85]. Up to two–loop order, the Wilson coefficients are given in different schemes [21,23,24,52]. In particular the non–singlet Wilson coefficients are also given in the $\overline{\mathsf{MS}}$ scheme [17,53].

5 The three–loop Wilson coefficients for the structure functions $F_2(x,Q^2)$ and $F_L(x,Q^2)$

Before we present the three–loop Wilson coefficients we would like to remark that the unrenormalized three–loop forward Compton amplitudes depend on the complete anomalous dimensions up to two–loop order, cf. e.g. [29], and contain the following three–loop anomalous dimensions in their pole terms of $O(1/\varepsilon)$: $\gamma_{qq}^{(2),\rm NS}, \gamma_{qq}^{(2),\rm PS}, \Delta\gamma_{qq}^{(2),\rm NS}, \Delta\gamma_{qq}^{(2),\rm PS}, \gamma_{qg}^{(2)}$ and $\Delta\gamma_{qg}^{(2)}$. We confirm the previous results given in Refs. [27,28,85–92].

We now turn to the unpolarized three–loop Wilson coefficients for neutral–current deep–inelastic scattering for pure virtual photon exchange, $F_2(x,Q^2)$ and $F_L(x,Q^2)$. In the structure of the following expressions there are some evanescent poles at $N=1,2$, which are, however, all tractable. Therefore, the analytic continuation [78] to $N\in\mathbb{N}$ can be carried out directly.

The non–singlet and gluonic Wilson coefficients are split for convenience as

$$
\begin{aligned}
\mathbb{C}_{i,q}^{(3)} &= C_{i,q}^{\rm NS,(3)}+C_{i,q}^{d_{abc},(3)}+C_{i,q}^{\rm PS,(3)} & (418)\\
\mathbb{C}_{i,g}^{(3)} &= C_{i,g}^{\rm NS,(3)}+C_{i,g}^{d_{abc},(3)}. & (419)
\end{aligned}
$$

The Wilson coefficients are given by

$C_{F_2,q}^{\rm NS,(3)} =$

$$
\begin{aligned}
&C_F^2\Biggl\{C_A\Biggl[\frac{20P_{262}}{3N(1+N)}\zeta_5-\frac{8S_{-2,1,-2}P_{277}}{3N(1+N)}-\frac{16S_{-2,3}P_{283}}{9N(1+N)}+\frac{36P_{284}}{5N^2(1+N)^2}\zeta_2^2+\frac{4S_5P_{285}}{9N(1+N)}\\
&+\frac{S_2^2P_{327}}{27N^2(1+N)^2}-\frac{4S_{-3,1}P_{375}}{9(N-2)N^2(1+N)^2(3+N)}+\frac{8S_{-2,2}P_{380}}{27(N-2)N^2(1+N)^2(3+N)}\\
&-\frac{4S_{2,1,1}P_{316}}{3N^2(1+N)^2}-\frac{8S_{-2,1,1}P_{390}}{27(N-2)N^2(1+N)^2(3+N)}+\frac{4S_4P_{418}}{27(N-2)N^2(1+N)^2(2+N)(3+N)}\\
&+\frac{4S_{3,1}P_{419}}{27(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{\zeta_3P_{435}}{3(N-2)N^3(1+N)^3(2+N)(3+N)}\\
&+\frac{P_{472}}{648(N-2)^2N^5(1+N)^5(3+N)}+\Biggl(\frac{4S_{2,1}P_{323}}{9N^2(1+N)^2}+\frac{8S_{-2,1}P_{389}}{27(N-2)N^2(1+N)^2(3+N)}\\
&-\frac{2S_3P_{409}}{27(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{2\zeta_3P_{411}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}\\
&+\frac{P_{474}}{648(N-2)^2N^5(1+N)^5(2+N)(3+N)}+\Biggl(\frac{P_{422}}{162N^3(1+N)^3(2+N)(3+N)}\\
&-\frac{2960}{3}S_3+\frac{160}{3}S_{2,1}-\frac{15104}{3}S_{-2,1}+480\zeta_3\Biggr)S_2+\frac{8\bigl(15+236N+236N^2\bigr)S_2^2}{9N(1+N)}\\
&+\frac{8\bigl(297+806N+734N^2\bigr)S_4}{9N(1+N)}+256S_5+\frac{2080}{3}S_{2,3}-\frac{8\bigl(-636+1445N+869N^2\bigr)S_{3,1}}{9N(1+N)}\\
&+3920S_{2,-3}+240S_{4,1}-\frac{8\bigl(3750-4115N+61N^2\bigr)S_{-2,2}}{9N(1+N)}-3840S_{-2,3}+64S_{-2,-3}\\
&-\frac{32\bigl(287-310N+50N^2\bigr)S_{-3,1}}{3N(1+N)}-\frac{4000}{3}S_{-4,1}-\frac{40\bigl(6+13N+13N^2\bigr)S_{2,1,1}}{3N(1+N)}-\frac{11552}{3}S_{2,1,-2}\\
&-\frac{32}{3}S_{2,2,1}+1984S_{3,1,1}-\frac{32\bigl(-2028+2255N+239N^2\bigr)S_{-2,1,1}}{9N(1+N)}-\frac{11744}{3}S_{-2,1,-2}\\
&-\frac{12224}{3}S_{-2,2,1}-\frac{6784}{3}S_{-3,1,1}-128S_{2,1,1,1}+11776S_{-2,1,1,1}+560\zeta_5\Biggr)S_1+\Biggl(-\frac{4S_2P_{326}}{9N^2(1+N)^2}\\
&+\frac{P_{451}}{36(N-2)N^4(1+N)^4(3+N)}-\frac{40}{3}S_2^2+\frac{4\bigl(-492+1093N+733N^2\bigr)S_3}{9N(1+N)}-88S_4\\
&+\frac{16\bigl(30+29N+29N^2\bigr)S_{2,1}}{9N(1+N)}-\frac{2464}{3}S_{3,1}+\frac{16\bigl(-1188+1219N+247N^2\bigr)S_{-2,1}}{9N(1+N)}\\
&+\frac{7216}{3}S_{-2,2}+\frac{6304}{3}S_{-3,1}+80S_{2,1,1}-\frac{16256}{3}S_{-2,1,1}+\frac{4\bigl(-22+75N+35N^2\bigr)\zeta_3}{N(1+N)}\Biggr)S_1^2\\
&+\Biggl(\frac{P_{340}}{54N^2(1+N)^3}-\frac{4\bigl(108+491N+491N^2\bigr)S_2}{27N(1+N)}+\frac{1648}{9}S_3-\frac{320}{9}S_{2,1}+1120S_{-2,1}-96\zeta_3\Biggr)\\
&\times S_1^3+\Biggl(\frac{-110+433N+433N^2}{9N(1+N)}+8S_2\Biggr)S_1^4+\Biggl(\frac{P_{467}}{324(N-2)N^4(1+N)^4(2+N)(3+N)}\\
&-\frac{16\bigl(-522+1861N+1159N^2\bigr)S_3}{27N(1+N)}-\frac{616}{3}S_4-\frac{20\bigl(4+9N+9N^2\bigr)S_{2,1}}{3N(1+N)}+\frac{3136}{3}S_{3,1}
\end{aligned}
$$

$$
-\frac{8\big(-1442+2071N+79N^2\big)S_{-2,1}}{3N(1+N)}-\frac{3760}{3}S_{-2,2}+880S_{-3,1}-\frac{544}{3}S_{2,1,1}+\frac{12736}{3}S_{-2,1,1}
$$

$$
-\frac{12\big(22+69N+77N^2\big)\zeta_3}{N(1+N)}\Bigg)S_2+\frac{44}{9}S_1^5+\frac{64}{3}S_2^3+\Bigg(\frac{P_{432}}{81(N-2)N^3(1+N)^3(2+N)(3+N)}
$$

$$
-\frac{1072}{9}S_{2,1}+2656S_{-2,1}+256\zeta_3\Bigg)S_3+\frac{3392}{9}S_3^2+224S_6+\Bigg(\frac{16\zeta_3P_{263}}{3N(1+N)}+\frac{8S_{-2,1}P_{276}}{3N(1+N)}
$$

$$
+\frac{8S_3P_{296}}{27N(1+N)}+\frac{4S_2P_{387}}{27(N-2)N^2(1+N)^2(3+N)}-\frac{2P_{470}}{81(N-2)^2N^4(1+N)^4(2+N)(3+N)}
$$

$$
+\Bigg(-\frac{4P_{457}}{81(N-2)^2N^3(1+N)^3(2+N)(3+N)}+\frac{8\big(60+113N+185N^2\big)S_2}{3N(1+N)}+608S_3
$$

$$
+\frac{11264}{3}S_{2,1}+\frac{1312}{3}S_{-2,1}+1248\zeta_3\Bigg)S_1+\Bigg(-\frac{4P_{382}}{9(N-2)N^2(1+N)^2(3+N)}-112S_2\Bigg)S_1^2
$$

$$
-3136S_{3,1}-\frac{16\big(135+31N+139N^2\big)S_1^3}{27N(1+N)}+32S_1^4+\frac{128}{3}S_2^2+2224S_4
$$

$$
-\frac{32\big(906-1441N+71N^2\big)S_{2,1}}{9N(1+N)}-1728S_{-2,2}-2048S_{-3,1}+4480S_{-2,1,1}+\frac{288}{5}\zeta_2^2\Bigg)S_{-2}
$$

$$
+\Bigg(\frac{2P_{376}}{3(N-2)N^2(1+N)^2(3+N)}+\frac{4\big(-366+449N+161N^2\big)S_1}{3N(1+N)}+392S_1^2-168S_2\Bigg)S_{-2}^2
$$

$$
-\frac{32}{3}S_{-2}^3+\Bigg(\frac{4P_{449}}{81(N-2)N^3(1+N)^3(2+N)(3+N)}+\Bigg(\frac{4P_{381}}{27(N-2)N^2(1+N)^2(3+N)}
$$

$$
+\frac{2176}{3}S_2\Bigg)S_1+\frac{8\big(1674-925N+335N^2\big)S_1^2}{9N(1+N)}-\frac{2416}{3}S_1^3-\frac{4\big(2310-5203N+773N^2\big)S_2}{9N(1+N)}
$$

$$
-2720S_3+\Bigg(\frac{8\big(-296+169N+N^2\big)}{3N(1+N)}+\frac{4064}{3}S_1\Bigg)S_{-2}-80S_{2,1}+3584S_{-2,1}-64\zeta_3\Big)S_{-3}
$$

$$
-1184S_{-3}^2+\Bigg(\frac{2P_{391}}{27(N-2)N^2(1+N)^2(3+N)}-\frac{4\big(-1926+3631N+31N^2\big)S_1}{9N(1+N)}-456S_1^2
$$

$$
+\frac{1400}{3}S_2+960S_{-2}\Bigg)S_{-4}+\Bigg(\frac{4P_{286}}{9N(1+N)}+\frac{3920S_1}{3}\Bigg)S_{-5}-\frac{544}{3}S_{-6}
$$

$$
+\Bigg(\frac{4P_{428}}{9N^3(1+N)^3(2+N)(3+N)}-288\zeta_3\Bigg)S_{2,1}+40S_{2,1}^2+\frac{16\big(-87+647N+431N^2\big)S_{2,3}}{9N(1+N)}
$$

$$
-\frac{4\big(5922-5561N+487N^2\big)S_{2,-3}}{9N(1+N)}-\frac{4\big(702+2671N+1807N^2\big)S_{4,1}}{9N(1+N)}+552S_{4,2}
$$

$$
+\Bigg(\frac{8P_{444}}{81(N-2)N^3(1+N)^3(2+N)(3+N)}+96S_{2,1}-640\zeta_3\Bigg)S_{-2,1}-2080S_{4,-2}+\frac{5888}{3}S_{5,1}
$$

$$
-2688S_{-2,1}^2+\frac{16(N-1)(2+N)S_{-2,-3}}{N(1+N)}+3456S_{-3,3}-\frac{8\big(-16-747N+189N^2\big)S_{-4,1}}{3N(1+N)}
$$

$$
-2384S_{-4,2}-1216S_{-4,-2}+288S_{-5,1}+\frac{8\big(3678-5755N+293N^2\big)S_{2,1,-2}}{9N(1+N)}
-\frac{4\big(-4+65N+65N^2\big)S_{2,2,1}}{3N(1+N)}-\frac{1760}{3}S_{2,3,1}+\frac{32\big(-171+577N+361N^2\big)S_{3,1,1}}{9N(1+N)}
-2160S_{2,-3,1}+2784S_{3,1,-2}+48S_{3,2,1}+\frac{320}{3}S_{4,1,1}+704S_{-2,2,2}+64S_{-2,2,-2}
+\frac{16\big(1902-2891N+133N^2\big)S_{-2,2,1}}{9N(1+N)}+2816S_{-2,3,1}+\frac{16\big(464-881N+127N^2\big)S_{-3,1,1}}{3N(1+N)}
+960S_{-3,1,-2}-112S_{-3,2,1}-704S_{-3,-2,1}+5504S_{-4,1,1}+\frac{32\big(6+13N+13N^2\big)S_{2,1,1,1}}{3N(1+N)}
+\frac{208}{3}S_{2,2,1,1}+\frac{32\big(-2412+3139N+115N^2\big)S_{-2,1,1,1}}{9N(1+N)}-1792S_{2,-2,1,1}-\frac{6368}{3}S_{3,1,1,1}
-288S_{-2,1,1,2}+192S_{-2,2,1,1}-1408S_{-2,-2,1,1}-896S_{-3,1,1,1}+\frac{640}{3}S_{2,1,1,1,1}-9728S_{-2,1,1,1,1}\Bigg]
$$

$$
+T_FN_F\Bigg[-\frac{16S_{2,1}P_{364}}{9N^2(1+N)^2(2+N)(3+N)}+\frac{4S_3P_{392}}{81N^2(1+N)^2(2+N)(3+N)}
+\frac{16\zeta_3P_{406}}{9(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{P_{471}}{162(N-2)^2N^5(1+N)^5(3+N)}
-\frac{128S_{-2,1}P_{332}}{81N^2(1+N)^2(2+N)}+\Bigg(\frac{P_{463}}{162(N-2)N^4(1+N)^4(2+N)(3+N)}
+\frac{2S_2P_{394}}{81N^2(1+N)^2(2+N)(3+N)}-\frac{752}{9}S_2^2-\frac{8\big(-138-83N+61N^2\big)S_3}{27N(1+N)}-\frac{1664}{9}S_4
-\frac{16\big(-34+61N+13N^2\big)S_{2,1}}{9N(1+N)}+\frac{1088}{9}S_{3,1}-\frac{1024\big(-6+14N+5N^2\big)S_{-2,1}}{27N(1+N)}+\frac{256}{9}S_{-2,2}
+\frac{512}{3}S_{-3,1}+\frac{320}{3}S_{2,1,1}+\frac{2048}{9}S_{-2,1,1}+\frac{64}{5}\zeta_2^2-\frac{16\big(-78+313N+73N^2\big)\zeta_3}{9N(1+N)}\Bigg)S_1
+\Bigg(\frac{P_{361}}{9N^3(1+N)^3}+\frac{16\big(-29+94N+82N^2\big)S_2}{9N(1+N)}-\frac{464}{9}S_3-\frac{352}{9}S_{2,1}-\frac{1280}{9}S_{-2,1}-32\zeta_3\Bigg)S_1^2
+\Bigg(-\frac{2P_{324}}{27N^2(1+N)^2}+\frac{832S_2}{27}\Bigg)S_1^3-\frac{20\big(-2+7N+7N^2\big)S_1^4}{9N(1+N)}-\frac{16}{9}S_1^5+\Bigg(\frac{4976}{27}S_3
+\frac{P_{423}}{81N^3(1+N)^3(2+N)(3+N)}+32S_{2,1}+\frac{256}{3}S_{-2,1}+\frac{352}{3}\zeta_3\Bigg)S_2
-\frac{4\big(-354+455N+167N^2\big)S_2^2}{27N(1+N)}-\frac{16\big(-309+782N+170N^2\big)S_4}{27N(1+N)}
+\Bigg(\frac{32P_{454}}{81(N-2)^2N^3(1+N)^3(2+N)(3+N)}+\Bigg(\frac{128P_{400}}{81(N-2)N^2(1+N)^2(2+N)(3+N)}
-\frac{512}{3}S_2\Bigg)S_1+\frac{256\big(-4+11N+5N^2\big)S_1^2}{9N(1+N)}+\frac{1280}{27}S_1^3-\frac{128\big(-27+56N+20N^2\big)S_2}{27N(1+N)}
$$

$$+\frac{3712}{27}S_3+\frac{512}{9}S_{2,1}+\frac{896}{3}\zeta_3\Big)S_{-2}+\left(-\frac{32(-6+21N+13N^2)}{3N(1+N)}-\frac{256}{3}S_1\right)S_{-2}^2$$

$$+\Big(-\frac{64P_{397}}{81(N-2)N^2(1+N)^2(2+N)(3+N)}-\frac{256(-15+28N+10N^2)S_1}{27N(1+N)}-\frac{896}{9}S_1^2$$

$$+\frac{640}{9}S_2+\frac{512}{3}S_{-2}\Big)S_{-3}+\left(-\frac{32(-66+413N+197N^2)}{27N(1+N)}-\frac{256}{9}S_1\right)S_{-4}+\frac{3008}{9}S_{-5}$$

$$-\frac{16(156+215N+431N^2)S_{3,1}}{27N(1+N)}+\frac{128(-3+10N+10N^2)S_{-2,2}}{27N(1+N)}-\frac{1984}{9}S_{2,3}+\frac{512}{9}S_{2,-3}$$

$$+\frac{3008}{9}S_{4,1}-\frac{896}{9}S_{-2,3}+\frac{128(-9+22N+10N^2)S_{-3,1}}{9N(1+N)}+\frac{16(-16+29N+5N^2)S_{2,1,1}}{3N(1+N)}$$

$$-\frac{512}{9}S_{2,1,-2}+\frac{64}{3}S_{2,2,1}-\frac{2944}{9}S_{3,1,1}+\frac{256(-21+46N+10N^2)S_{-2,1,1}}{27N(1+N)}+\frac{512}{3}S_{-2,1,-2}$$

$$-\frac{512}{9}S_{-2,2,1}-\frac{512}{3}S_{-3,1,1}-\frac{256}{3}S_{2,1,1,1}-\frac{1024}{9}S_{-2,1,1,1}-\frac{16(2+3N+3N^2)\zeta_2^2}{5N(1+N)}+\frac{2368}{9}S_5\Big]\Big\}$$

$$+C_F\Big\{C_A^2\Big[-\frac{5\zeta_5P_{269}}{3N(1+N)}+\frac{4S_{-2,1,-2}P_{271}}{3N(1+N)}-\frac{2S_5P_{280}}{9N(1+N)}+\frac{4S_{-2,3}P_{282}}{9N(1+N)}-\frac{12\zeta_2^2P_{284}}{5N^2(1+N)^2}$$

$$+\frac{4S_{2,1,1}P_{322}}{9N^2(1+N)^2}-\frac{8S_{-2,2}P_{370}}{27(N-2)N^2(1+N)^2(3+N)}+\frac{4S_{-3,1}P_{372}}{9(N-2)N^2(1+N)^2(3+N)}$$

$$+\frac{S_2^2P_{319}}{9N^2(1+N)^2}+\frac{8S_{-2,1,1}P_{383}}{27(N-2)N^2(1+N)^2(3+N)}-\frac{4S_{3,1}P_{404}}{27(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$-\frac{2S_4P_{414}}{9(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{2\zeta_3P_{429}}{27(N-2)N^3(1+N)^2(2+N)(3+N)}$$

$$+\frac{P_{458}}{5832(N-2)^2N^4(1+N)^4(3+N)}+\Big[-\frac{4S_{2,1}P_{320}}{9N^2(1+N)^2}-\frac{8S_{-2,1}P_{384}}{27(N-2)N^2(1+N)^2(3+N)}$$

$$+\frac{4S_3P_{412}}{27(N-2)N^2(1+N)^2(2+N)(3+N)}-\frac{8\zeta_3P_{416}}{27(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$+\frac{P_{473}}{729(N-2)N^5(1+N)^5(2+N)(3+N)}+\Big(-32S_{2,1}+\frac{P_{395}}{27N^3(1+N)^3(3+N)}+\frac{1184}{3}S_3$$

$$+1376S_{-2,1}-64\zeta_3\Big)S_2-\frac{4(-6+145N+145N^2)S_2^2}{9N(1+N)}-\frac{4(-162+1325N+893N^2)S_4}{9N(1+N)}$$

$$+\frac{448}{3}S_5-\frac{1120}{3}S_{2,3}-1040S_{2,-3}+\frac{16(-73+120N+48N^2)S_{3,1}}{3N(1+N)}-\frac{176}{3}S_{4,1}$$

$$+\frac{32(249-277N+11N^2)S_{-2,2}}{9N(1+N)}+\frac{2912}{3}S_{-2,3}-\frac{64}{3}S_{-2,-3}+\frac{64(36-37N+11N^2)S_{-3,1}}{3N(1+N)}$$

$$+\frac{1184}{3}S_{-4,1}+\frac{8(12+55N+55N^2)S_{2,1,1}}{3N(1+N)}+992S_{2,1,-2}+64S_{2,2,1}-\frac{2816}{3}S_{3,1,1}$$

$$+\frac{128(-141+166N+22N^2)S_{-2,1,1}}{9N(1+N)}+\frac{3424}{3}S_{-2,1,-2}+\frac{3200}{3}S_{-2,2,1}+\frac{1600}{3}S_{-3,1,1}$$

$$
+128 S_{2,1,1,1} - 3200 S_{-2,1,1,1} + 240\zeta_5 \Big] S_1 + \Big[\frac{2 S_2 P_{312}}{9N^2(1+N)^2} + \frac{P_{431}}{162 N^4 (1+N)^4} - \frac{8}{3} S_2^2
+ \frac{4\big(240 - 205N + 11N^2\big) S_3}{9N(1+N)} - 72 S_4 - \frac{4\big(12 + 209N + 209N^2\big) S_{2,1}}{9N(1+N)} + \frac{1168}{3} S_{3,1}
- \frac{32\big(-171 + 199N + 55N^2\big) S_{-2,1}}{9N(1+N)} - \frac{1888}{3} S_{-2,2} - 512 S_{-3,1} - 32 S_{2,1,1} + \frac{4480}{3} S_{-2,1,1}
- \frac{16\big(-4 + 15N + 11N^2\big)\zeta_3}{N(1+N)} \Big] S_1^2 + \Big[\frac{-726 + 4649N + 4649N^2}{81N(1+N)} + \frac{616}{27} S_2 - \frac{640}{9} S_3 + \frac{32}{9} S_{2,1}
-320 S_{-2,1} \Big] S_1^3 + \frac{121}{27} S_1^4 + \Big[\frac{P_{433}}{54N^4(1+N)^4(3+N)} + \frac{8\big(-666 + 1453N + 481N^2\big) S_3}{27N(1+N)}
+ \frac{424}{3} S_4 + \frac{16\big(1 + 3N + 3N^2\big) S_{2,1}}{N(1+N)} - \frac{1168}{3} S_{3,1} + \frac{16\big(-201 + 310N + 22N^2\big) S_{-2,1}}{3N(1+N)}
+352 S_{-2,2} - \frac{1040}{3} S_{-3,1} + \frac{272}{3} S_{2,1,1} - 1088 S_{-2,1,1} + \frac{32\big(1 + 18N + 18N^2\big)\zeta_3}{N(1+N)} \Big] S_2 - \frac{56}{9} S_2^3
+ \Big[\frac{2 P_{450}}{81(N-2)N^3(1+N)^3(2+N)(3+N)} + \frac{880}{9} S_{2,1} - \frac{2080}{3} S_{-2,1} - 128\zeta_3 \Big] S_3 - \frac{1280}{9} S_3^2
+ \frac{80}{9} S_6 + \Big[-\frac{4\zeta_3 P_{260}}{3N(1+N)} - \frac{4 S_{-2,1} P_{264}}{3N(1+N)} + \frac{2 P_{469}}{81(N-2)^2 N^4 (1+N)^4 (2+N)(3+N)}
- \frac{4 S_3 P_{287}}{27N(1+N)} + \Big(\frac{4 P_{448}}{81(N-2)N^3(1+N)^3(2+N)(3+N)} - \frac{32\big(-3 + 22N + 22N^2\big) S_2}{3N(1+N)}
- \frac{320}{3} S_3 - 960 S_{2,1} - 160 S_{-2,1} - 320\zeta_3 \Big) S_1 + \Big(\frac{4 P_{321}}{9N^2(1+N)^2} - 32 S_2 \Big) S_1^2 + \frac{1760}{27} S_1^3
- \frac{4\big(-72 - 1206N + 2737N^2 + 1108N^3\big) S_2}{27N^2(1+N)} - \frac{16}{3} S_2^2 + \frac{32\big(243 - 410N + 22N^2\big) S_{2,1}}{9N(1+N)}
- \frac{1648}{3} S_4 + 768 S_{3,1} + \frac{1280}{3} S_{-2,2} + \frac{1600}{3} S_{-3,1} - \frac{3584}{3} S_{-2,1,1} - \frac{96}{5}\zeta_2^2 \Big] S_{-2}
+ \Big[-\frac{8 P_{353}}{3(N-2)N^2(1+N)^2(3+N)} - \frac{8\big(-62 + 95N + 47N^2\big) S_1}{3N(1+N)} - \frac{400}{3} S_1^2 + \frac{208}{3} S_2 \Big] S_{-2}^2
+ \frac{32}{9} S_{-2}^3 + \Big[-\frac{2 P_{447}}{81(N-2)N^3(1+N)^3(2+N)(3+N)} + \Big(-\frac{4 P_{379}}{27(N-2)N^2(1+N)^2(3+N)}
-144 S_2 \Big) S_1 - \frac{16\big(189 - 67N + 77N^2\big) S_1^2}{9N(1+N)} + \frac{544}{3} S_1^3 + \frac{8\big(297 - 754N + 110N^2\big) S_2}{9N(1+N)} + \frac{2080}{3} S_3
+ \Big(\frac{8\big(64 - 35N + 13N^2\big)}{3N(1+N)} - \frac{832}{3} S_1 \Big) S_{-2} + \frac{80}{3} S_{2,1} - \frac{3136}{3} S_{-2,1} + 32\zeta_3 \Big] S_{-3} + \frac{1072}{3} S_{-3}^2
+ \Big(-\frac{4 P_{385}}{27(N-2)N^2(1+N)^2(3+N)} - \frac{8\big(228 - 379N + 53N^2\big) S_1}{9N(1+N)} + \frac{320}{3} S_1^2 - \frac{320}{3} S_2
$$

$$
-\frac{736}{3}S_{-2}\Big)S_{-4}+\Big[-\frac{2P_{281}}{9N(1+N)}-208S_1\Big]S_{-5}+\frac{544}{9}S_{-6}+\Big[-\frac{4P_{417}}{27N^3(1+N)^3(3+N)}
+96\zeta_3\Big]S_{2,1}-\frac{56}{3}S_{2,1}^2-\frac{16\big(-35+106N+52N^2\big)S_{2,3}}{3N(1+N)}
$$

$$
+\frac{8\big(801-758N+106N^2\big)S_{2,-3}}{9N(1+N)}+\frac{8\big(33+691N+367N^2\big)S_{4,1}}{9N(1+N)}-\frac{952}{3}S_{4,2}+\frac{1600}{3}S_{4,-2}
$$

$$
-\frac{2144}{3}S_{5,1}+\Big[-\frac{4P_{445}}{81(N-2)N^3(1+N)^3(2+N)(3+N)}-32S_{2,1}+192\zeta_3\Big]S_{-2,1}+\frac{2240}{3}S_{-2,1}^2
$$

$$
+\frac{32S_{-2,-3}}{3N(1+N)}-\frac{2848}{3}S_{-3,3}+\frac{8\big(-8-207N+57N^2\big)S_{-4,1}}{3N(1+N)}+\frac{2096}{3}S_{-4,2}+\frac{1216}{3}S_{-4,-2}
$$

$$
-224S_{-5,1}-\frac{16\big(495-820N+44N^2\big)S_{2,1,-2}}{9N(1+N)}+\frac{4\big(-24+29N+29N^2\big)S_{2,2,1}}{3N(1+N)}+\frac{640}{3}S_{2,3,1}
$$

$$
+\frac{2096}{3}S_{2,-3,1}-\frac{16\big(-264+611N+287N^2\big)S_{3,1,1}}{9N(1+N)}-\frac{1952}{3}S_{3,1,-2}-\frac{208}{3}S_{3,2,1}+\frac{320}{3}S_{4,1,1}
$$

$$
-\frac{64\big(129-205N+11N^2\big)S_{-2,2,1}}{9N(1+N)}-\frac{832}{3}S_{-2,2,2}-\frac{64}{3}S_{-2,2,-2}-\frac{1984}{3}S_{-2,3,1}
$$

$$
-\frac{32\big(61-122N+22N^2\big)S_{-3,1,1}}{3N(1+N)}-320S_{-3,1,-2}+\frac{112}{3}S_{-3,2,1}+\frac{704}{3}S_{-3,-2,1}-\frac{4480}{3}S_{-4,1,1}
$$

$$
-\frac{16\big(36+11N+11N^2\big)S_{2,1,1,1}}{9N(1+N)}-\frac{272}{3}S_{2,2,1,1}+384S_{2,-2,1,1}+\frac{2720}{3}S_{3,1,1,1}+384S_{-3,1,1,1}
$$

$$
-\frac{64\big(-333+454N+22N^2\big)S_{-2,1,1,1}}{9N(1+N)}+96S_{-2,1,1,2}-64S_{-2,2,1,1}+\frac{1408}{3}S_{-2,-2,1,1}
$$

$$
-\frac{640}{3}S_{2,1,1,1,1}+2560S_{-2,1,1,1,1}\Big]
$$

$$
+C_AT_FN_F\Big[\frac{64S_{-2,1}P_{332}}{81N^2(1+N)^2(2+N)}-\frac{8S_{3,1}P_{334}}{27(N-2)N(1+N)(2+N)(3+N)}
$$

$$
+\frac{8S_{2,1}P_{337}}{27N^2(1+N)^2(3+N)}+\frac{4S_4P_{338}}{27(N-2)N(1+N)(2+N)(3+N)}
$$

$$
-\frac{4\zeta_3P_{339}}{27(N-2)N^2(1+N)(3+N)}-\frac{8S_3P_{413}}{27(N-2)N^2(1+N)^2(2+N)(3+N)}
$$

$$
+\frac{P_{466}}{729(N-2)^2N^4(1+N)^4(3+N)}+\Big[-\frac{32S_3P_{333}}{27(N-2)N(1+N)(2+N)(3+N)}
$$

$$
+\frac{16\zeta_3P_{336}}{27(N-2)N(1+N)(2+N)(3+N)}-\frac{4P_{465}}{729(N-2)N^4(1+N)^4(2+N)(3+N)}
$$

$$
+\frac{8S_2P_{335}}{27N^2(1+N)^2(3+N)}+\frac{224}{9}S_2^2+\frac{1312}{9}S_4+\frac{16\big(-25+64N+40N^2\big)S_{2,1}}{9N(1+N)}-96S_{3,1}
$$

$$
+\frac{512\big(-6+14N+5N^2\big)S_{-2,1}}{27N(1+N)}-\frac{128}{9}S_{-2,2}-\frac{256}{3}S_{-3,1}-\frac{160}{3}S_{2,1,1}-\frac{1024}{9}S_{-2,1,1}-\frac{64}{5}\zeta_2^2\Big]S_1
$$

$$
+\Biggl[-\frac{2P_{393}}{81N^3(1+N)^3}+\frac{8\bigl(17-10N+2N^2\bigr)S_2}{9N(1+N)}-\frac{16}{9}S_3+\frac{304}{9}S_{2,1}+\frac{640}{9}S_{-2,1}+32\zeta_3\Biggr]S_1^2
+\Biggl[-\frac{16\bigl(-33+194N+194N^2\bigr)}{81N(1+N)}-\frac{224}{27}S_2\Biggr]S_1^3-\frac{88}{27}S_1^4+\Biggl[\frac{2P_{421}}{81N^3(1+N)^3(3+N)}-\frac{1360}{27}S_3
-16S_{2,1}-\frac{128}{3}S_{-2,1}-\frac{352}{3}\zeta_3\Biggr]S_2-\frac{4\bigl(40+61N+109N^2\bigr)S_2^2}{9N(1+N)}-\frac{1312}{9}S_5
+\Biggl[-\frac{16P_{454}}{81(N-2)^2N^3(1+N)^3(2+N)(3+N)}+\Biggl(-\frac{64P_{400}}{81(N-2)N^2(1+N)^2(2+N)(3+N)}
+\frac{256}{3}S_2\Biggr)S_1-\frac{128\bigl(-4+11N+5N^2\bigr)S_1^2}{9N(1+N)}-\frac{640}{27}S_1^3+\frac{64\bigl(-27+56N+20N^2\bigr)S_2}{27N(1+N)}-\frac{1856}{27}S_3
-\frac{256}{9}S_{2,1}-\frac{448}{3}\zeta_3\Biggr]S_{-2}+\Biggl[\frac{128}{3}S_1+\frac{16\bigl(-6+21N+13N^2\bigr)}{3N(1+N)}\Biggr]S_{-2}^2
+\Biggl[-\frac{128\bigl(15-28N-10N^2\bigr)S_1}{27N(1+N)}+\frac{32P_{397}}{81(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{448}{9}S_1^2
-\frac{320}{9}S_2-\frac{256}{3}S_{-2}\Biggr]S_{-3}+\Biggl[\frac{16\bigl(-66+413N+197N^2\bigr)}{27N(1+N)}+\frac{128}{9}S_1\Biggr]S_{-4}-\frac{1504}{9}S_{-5}+\frac{320}{3}S_{2,3}
-\frac{64\bigl(-3+10N+10N^2\bigr)S_{-2,2}}{27N(1+N)}+\frac{448}{9}S_{-2,3}-\frac{64\bigl(-9+22N+10N^2\bigr)S_{-3,1}}{9N(1+N)}+\frac{1696}{9}S_{3,1,1}
-\frac{256}{9}S_{2,-3}-\frac{1120}{9}S_{4,1}-\frac{64\bigl(-6+13N+4N^2\bigr)S_{2,1,1}}{9N(1+N)}+\frac{256}{9}S_{2,1,-2}-\frac{64}{3}S_{2,2,1}-\frac{256}{3}S_{-2,1,-2}
-\frac{128\bigl(-21+46N+10N^2\bigr)S_{-2,1,1}}{27N(1+N)}+\frac{256}{9}S_{-2,2,1}+\frac{256}{3}S_{-3,1,1}+\frac{64}{9}S_{2,1,1,1}
+\frac{512}{9}S_{-2,1,1,1}+\frac{16\bigl(2+3N+3N^2\bigr)\zeta_2^2}{5N(1+N)}\Biggr]
+T_F^2N_F^2\Biggl[-\frac{8S_2P_{325}}{81N^2(1+N)^2}-\frac{2P_{430}}{729N^4(1+N)^4}+\Biggl[-\frac{16\bigl(-6+29N+29N^2\bigr)S_2}{27N(1+N)}
+\frac{8P_{388}}{729N^3(1+N)^3}+\frac{128}{27}S_3+\frac{128}{27}\zeta_3\Biggr]S_1+\Biggl[\frac{8P_{314}}{81N^2(1+N)^2}-\frac{32S_2}{9}\Biggr]S_1^2-\frac{992}{27}S_4-\frac{1216}{27}S_{2,1}
+\frac{16\bigl(-6+29N+29N^2\bigr)}{81N(1+N)}S_1^3+\frac{16}{27}S_1^4+\frac{80}{9}S_2^2+\frac{32\bigl(-6+247N+247N^2\bigr)S_3}{81N(1+N)}+\frac{256}{9}S_{3,1}
-\frac{128}{9}S_{2,1,1}-\frac{32\bigl(2+3N+3N^2\bigr)\zeta_3}{27N(1+N)}\Biggr]\Biggr\}+C_F^3\Biggl\{\frac{S_2^2P_{257}}{3N^2(1+N)^2}-\frac{20\zeta_5P_{259}}{3N(1+N)}-\frac{8S_5P_{261}}{3N(1+N)}
+\frac{16S_{-2,3}P_{268}}{3N(1+N)}+\frac{16S_{-2,1,-2}P_{270}}{3N(1+N)}-\frac{24\zeta_2^2P_{284}}{5N^2(1+N)^2}-\frac{8S_{-3,1}P_{350}}{3(N-2)N^2(1+N)^2(3+N)}
-\frac{16S_{-2,2}P_{356}}{3(N-2)N^2(1+N)^2(3+N)}+\frac{16S_{-2,1,1}P_{371}}{3(N-2)N^2(1+N)^2(3+N)}
$$

$$
-\frac{8S_{3,1}P_{402}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}-\frac{2S_4P_{405}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}
-\frac{4\zeta_3P_{440}}{3(N-2)N^3(1+N)^3(2+N)(3+N)}+\frac{P_{476}}{24(N-2)^2N^6(1+N)^6(3+N)}
+\Biggl[-\frac{4S_{2,1}P_{308}}{3N^2(1+N)^2}-\frac{16S_{-2,1}P_{369}}{3(N-2)N^2(1+N)^2(3+N)}-\frac{8\zeta_3P_{398}}{(N-2)N^2(1+N)^2(2+N)(3+N)}
+\frac{2S_3P_{399}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{P_{475}}{24(N-2)^2N^5(1+N)^5(2+N)(3+N)}
+\biggl(\frac{P_{386}}{6N^3(1+N)^3}+\frac{976}{3}S_3-\frac{64}{3}S_{2,1}+\frac{13696}{3}S_{-2,1}-416\zeta_3\biggr)S_2+\frac{2\bigl(-54+25N+25N^2\bigr)S_2^2}{N(1+N)}
-\frac{4\bigl(174-75N+85N^2\bigr)S_4}{N(1+N)}-\frac{2560}{3}S_5-64S_{2,3}-3680S_{2,-3}+\frac{8\bigl(22+45N+109N^2\bigr)S_{3,1}}{N(1+N)}
-\frac{1696}{3}S_{4,1}-\frac{16\bigl(-586+633N+9N^2\bigr)S_{-2,2}}{3N(1+N)}+\frac{11392}{3}S_{-2,3}-\frac{128}{3}S_{-2,-3}+\frac{11200}{3}S_{2,1,-2}
+\frac{64\bigl(143-162N+6N^2\bigr)S_{-3,1}}{3N(1+N)}+1088S_{-4,1}-\frac{24\bigl(-2+3N+3N^2\bigr)S_{2,1,1}}{N(1+N)}-\frac{160}{3}S_{2,2,1}
-\frac{1984}{3}S_{3,1,1}+\frac{64\bigl(-100+103N+7N^2\bigr)S_{-2,1,1}}{N(1+N)}+3264S_{-2,1,-2}+\frac{11648}{3}S_{-2,2,1}+\frac{7168}{3}S_{-3,1,1}
-10752S_{-2,1,1,1}-1120\zeta_5\Biggr]S_1+\Biggl[\frac{2S_2P_{317}}{3N^2(1+N)^2}+\frac{P_{455}}{12(N-2)N^4(1+N)^4(3+N)}+108S_2^2
-\frac{4\bigl(10+59N+75N^2\bigr)S_3}{N(1+N)}+\frac{24\bigl(-2+3N+3N^2\bigr)S_{2,1}}{N(1+N)}-\frac{32\bigl(-56+47N+3N^2\bigr)S_{-2,1}}{N(1+N)}
+336S_{3,1}+344S_4-\frac{6880}{3}S_{-2,2}-\frac{6464}{3}S_{-3,1}-48S_{2,1,1}+4864S_{-2,1,1}-\frac{64(N-1)\zeta_3}{N(1+N)}\Biggr]S_1^2
+\Biggl[\frac{P_{341}}{6N^3(1+N)^3}-\frac{12\bigl(-6+7N+7N^2\bigr)S_2}{N(1+N)}-48S_3+32S_{2,1}-960S_{-2,1}+96\zeta_3\Biggr]S_1^3
+\Biggl[\frac{P_{258}}{N^2(1+N)^2}-36S_2\Biggr]S_1^4+\frac{2\bigl(-2+3N+3N^2\bigr)S_1^5}{N(1+N)}+\frac{4}{3}S_1^6+\Biggl[40S_4-432S_{3,1}+\frac{3296}{3}S_{-2,2}
+\frac{P_{452}}{12(N-2)N^4(1+N)^4(3+N)}+\frac{4\bigl(154+273N+513N^2\bigr)S_3}{3N(1+N)}-\frac{8\bigl(-4+15N+15N^2\bigr)S_{2,1}}{3N(1+N)}
-\frac{16\bigl(638-831N+9N^2\bigr)S_{-2,1}}{3N(1+N)}-\frac{1120}{3}S_{-3,1}+\frac{272}{3}S_{2,1,1}+\frac{64\bigl(4+3N+6N^2\bigr)\zeta_3}{N(1+N)}
-\frac{12416}{3}S_{-2,1,1}\Biggr]S_2-\frac{148}{9}S_2^3+\Biggl[\frac{P_{443}}{3(N-2)N^3(1+N)^3(2+N)(3+N)}+\frac{64}{3}S_{2,1}-\frac{7616}{3}S_{-2,1}
-\frac{320}{3}\zeta_3\Biggr]S_3-\frac{512}{3}S_3^2-\frac{1664}{9}S_6+\Biggl[-\frac{16S_{-2,1}P_{266}}{3N(1+N)}-\frac{8S_2P_{358}}{3(N-2)N^2(1+N)^2(3+N)}
-\frac{16\zeta_3P_{267}}{3N(1+N)}+\frac{4P_{468}}{3(N-2)^2N^4(1+N)^4(2+N)(3+N)}+\Biggl[-\frac{11008}{3}S_{2,1}-\frac{704}{3}S_{-2,1}
$$

$$
-\frac{8P_{453}}{3(N-2)^2N^3(1+N)^3(2+N)(3+N)}-\frac{16\big(28-21N+3N^2\big)S_2}{N(1+N)}-\frac{2368}{3}S_3-1216\zeta_3\Bigg]S_1
$$

$$
+\Bigg[\frac{8P_{362}}{3(N-2)N^2(1+N)^2(3+N)}+352S_2\Bigg]S_1^2-\frac{32\big(-5+7N+3N^2\big)S_1^3}{N(1+N)}-64S_1^4-64S_2^2
$$

$$
-\frac{16\big(-80+N-46N^2-22N^3+N^4\big)S_3}{3N(1+N)}-\frac{6752}{3}S_4+\frac{64\big(140-207N+9N^2\big)S_{2,1}}{3N(1+N)}
$$

$$
+3200S_{3,1}+\frac{5248}{3}S_{-2,2}+\frac{5888}{3}S_{-3,1}-\frac{12544}{3}S_{-2,1,1}-\frac{192}{5}\zeta_2^2\Bigg]S_{-2}+\Bigg[-\frac{752}{3}S_1^2+\frac{176}{3}S_2
$$

$$
-\frac{4P_{351}}{3(N-2)N^2(1+N)^2(3+N)}+\frac{8\big(118-69N+27N^2\big)S_1}{3N(1+N)}\Bigg]S_{-2}^2+\frac{64}{9}S_{-2}^3
$$

$$
+\Bigg[-\frac{8P_{439}}{3(N-2)N^3(1+N)^3(2+N)(3+N)}+\Bigg(\frac{8P_{348}}{3(N-2)N^2(1+N)^2(3+N)}-\frac{2624}{3}S_2\Bigg)S_1
$$

$$
-\frac{16\big(102-73N+3N^2\big)S_1^2}{N(1+N)}+\frac{8\big(374-729N+111N^2\big)S_2}{3N(1+N)}+\frac{2656}{3}S_1^3+\frac{8000}{3}S_3
$$

$$
+\Bigg(-\frac{16\big(-56+33N+9N^2\big)}{N(1+N)}-1600S_1\Bigg)S_{-2}+\frac{160}{3}S_{2,1}-\frac{8960}{3}S_{-2,1}\Bigg]S_{-3}+\frac{2816}{3}S_{-3}^2
$$

$$
+\Bigg[-\frac{4P_{347}}{3(N-2)N^2(1+N)^2(3+N)}+\frac{8\big(-338+705N+81N^2\big)S_1}{3N(1+N)}+\frac{1456}{3}S_1^2-\frac{1520}{3}S_2
$$

$$
-\frac{2816}{3}S_{-2}\Bigg]S_{-4}+\Bigg[-\frac{8P_{265}}{3N(1+N)}-\frac{5344S_1}{3}\Bigg]S_{-5}+\frac{1088}{9}S_{-6}+\Bigg[-\frac{4P_{363}}{3N^3(1+N)^3}+192\zeta_3\Bigg]
$$

$$
\times S_{2,1}-\frac{64}{3}S_{2,1}^2-\frac{32\big(11+5N+17N^2\big)S_{2,3}}{N(1+N)}+\frac{8\big(302-281N+7N^2\big)S_{2,-3}}{N(1+N)}
$$

$$
+\frac{8\big(250-9N+135N^2\big)S_{4,1}}{3N(1+N)}-\frac{1088}{3}S_{4,2}+\frac{6080}{3}S_{4,-2}-1120S_{5,1}+\Bigg[-64S_{2,1}+512\zeta_3
$$

$$
+\frac{16P_{438}}{3(N-2)N^3(1+N)^3(2+N)(3+N)}\Bigg]S_{-2,1}+\frac{7168}{3}S_{-2,1}^2-\frac{32\big(-2+3N+3N^2\big)S_{-2,-3}}{3N(1+N)}
$$

$$
-\frac{9344}{3}S_{-3,3}+\frac{16(-111+25N)S_{-4,1}}{1+N}+\frac{5920}{3}S_{-4,2}+\frac{2432}{3}S_{-4,-2}+320S_{-5,1}+\frac{80S_{2,2,1}}{3N(1+N)}
$$

$$
+\frac{8\big(-14-20N+10N^2+29N^3\big)S_{2,1,1}}{N^2(1+N)}-\frac{16\big(566-825N+39N^2\big)S_{2,1,-2}}{3N(1+N)}+\frac{352}{3}S_{2,3,1}
$$

$$
+\frac{4576}{3}S_{2,-3,1}-\frac{16\big(82+51N+195N^2\big)S_{3,1,1}}{3N(1+N)}-\frac{8896}{3}S_{3,1,-2}+\frac{64}{3}S_{3,2,1}+\frac{128}{3}S_{4,1,1}
$$

$$
-\frac{32\big(290-417N+15N^2\big)S_{-2,2,1}}{3N(1+N)}-\frac{896}{3}S_{-2,2,2}-\frac{128}{3}S_{-2,2,-2}-\frac{8960}{3}S_{-2,3,1}
$$

$$
-\frac{32\big(220-393N+39N^2\big)S_{-3,1,1}}{3N(1+N)}-640S_{-3,1,-2}+\frac{224}{3}S_{-3,2,1}+\frac{1408}{3}S_{-3,-2,1}-\frac{15104}{3}S_{-4,1,1}
$$

$$
+\frac{64}{3}S_{2,2,1,1}+2048S_{2,-2,1,1}+704S_{3,1,1,1}-\frac{192\big(-40+49N+N^2\big)S_{-2,1,1,1}}{N(1+N)}+192S_{-2,1,1,2}
$$

$$-128S_{-2,2,1,1}+\frac{2816}{3}S_{-2,-2,1,1}+256S_{-3,1,1,1}+9216S_{-2,1,1,1,1}\Bigg\}, \tag{420}$$

$$C_{F_L,q}^{\mathrm{NS},(3)} =$$
$$C_F^2\Bigg\{T_FN_F\Bigg[\frac{8S_2P_{303}}{3N(1+N)^2(2+N)(3+N)}-\frac{64\zeta_3P_{331}}{3(N-2)(N-1)N(1+N)(2+N)(3+N)}$$
$$+\frac{2P_{446}}{27(N-2)^2N^3(1+N)^4(3+N)}+\Bigg[\frac{32P_{425}}{27(N-2)(N-1)N^2(1+N)^3(2+N)(3+N)}$$
$$-\frac{64(4+3N)\big(-5+N^2\big)S_2}{3(1+N)^2(2+N)(3+N)}+\frac{128S_3}{3(1+N)}-\frac{256S_{2,1}}{3(1+N)}-\frac{1024S_{-2,1}}{3(1+N)}-\frac{1280\zeta_3}{3(1+N)}\Bigg]S_1$$
$$+\Bigg[\frac{8\big(30+83N+23N^2\big)}{9N(1+N)^2}+\frac{64S_2}{3(1+N)}\Bigg]S_1^2+\frac{32\big(294+553N+270N^2+35N^3\big)S_3}{9(1+N)^2(2+N)(3+N)}$$
$$-\frac{128S_1^3}{9(1+N)}-\frac{128S_2^2}{3(1+N)}-\frac{1088S_4}{3(1+N)}+\Bigg[\frac{512S_1P_{329}}{9(N-2)(N-1)N(1+N)(2+N)(3+N)}$$
$$-\frac{256P_{436}}{9(N-2)^2(N-1)N^2(1+N)^3(2+N)(3+N)}+\frac{512S_1^2}{3(1+N)}-\frac{512S_2}{3(1+N)}\Bigg]S_{-2}-\frac{256S_{-2}^2}{3(1+N)}$$
$$+\Bigg[\frac{256P_{343}}{9(N-2)(N-1)N(1+N)^2(2+N)(3+N)}-\frac{512S_1}{3(1+N)}\Bigg]S_{-3}-\frac{256S_{-4}}{1+N}$$
$$+\frac{64\big(-10+25N+38N^2+11N^3\big)S_{2,1}}{3(1+N)^2(2+N)(3+N)}+\frac{128S_{3,1}}{1+N}-\frac{512\big(-16-23N+N^2+2N^3\big)S_{-2,1}}{9(N-1)(1+N)^2(2+N)}$$
$$+\frac{512S_{-3,1}}{3(1+N)}+\frac{128S_{2,1,1}}{1+N}+\frac{1024S_{-2,1,1}}{3(1+N)}\Bigg]+C_A\Bigg[-\frac{64S_{-2,2}P_{300}}{(N-2)N(1+N)^2(3+N)}$$
$$+\frac{128S_{-2,1,1}P_{306}}{3(N-2)N(1+N)^2(3+N)}-\frac{64S_{-3,1}P_{310}}{3(N-2)N(1+N)^2(3+N)}-\frac{16S_{2,1}P_{311}}{3N(1+N)^2(2+N)(3+N)}$$
$$-\frac{64S_{3,1}P_{344}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}+\frac{32S_4P_{367}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$
$$+\frac{16\zeta_3P_{407}}{3(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}+\frac{P_{434}}{54(N-2)^2N^3(1+N)^4(3+N)}$$
$$+\frac{32S_{-2,1}P_{408}}{9(N-2)(N-1)N(1+N)^3(2+N)(3+N)}$$
$$-\frac{8S_3P_{410}}{9(N-2)(N-1)N(1+N)^3(2+N)(3+N)}+\Bigg(\frac{16S_2P_{294}}{3N(1+N)^2(2+N)(3+N)}$$
$$-\frac{16\zeta_3P_{349}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}+\frac{64S_4}{1+N}+\frac{512S_{2,1}}{3(1+N)}-\frac{512S_{3,1}}{1+N}$$
$$+\frac{16S_3P_{360}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}-\frac{32S_{-2,1}P_{309}}{3(N-2)N(1+N)^2(3+N)}$$
$$+\frac{3712S_{-2,2}}{1+N}+\frac{3840S_{-3,1}}{1+N}-\frac{7168S_{-2,1,1}}{1+N}$$

$$
-\frac{4P_{464}}{27(N-2)^2(N-1)N^3(1+N)^4(2+N)(3+N)}\Biggr)S_1+\Biggl(-\frac{128S_2}{3(1+N)}+\frac{160S_3}{1+N}+\frac{1728S_{-2,1}}{1+N}
+\frac{160\zeta_3}{1+N}-\frac{2P_{315}}{9(N-2)N(1+N)^2(3+N)}\Biggr)S_1^2+\frac{640S_1^3}{9(1+N)}+\Biggl(-\frac{416S_3}{1+N}-\frac{5312S_{-2,1}}{1+N}+\frac{96\zeta_3}{1+N}
-\frac{2P_{377}}{3(N-2)N(1+N)^3(2+N)(3+N)}\Biggr)S_2+\frac{304S_2^2}{3(1+N)}-\frac{64(N-1)(2+N)S_5}{1+N}
+\Biggl(\frac{64S_2P_{299}}{3(N-2)N(1+N)^2(3+N)}+\frac{16P_{461}}{9(N-2)^2(N-1)N^3(1+N)^4(2+N)(3+N)}
+\Biggl(-\frac{32P_{441}}{9(N-2)^2(N-1)N^2(1+N)^3(2+N)(3+N)}-\frac{192S_2}{1+N}\Biggr)S_1+\frac{64S_1^3}{1+N}
-\frac{64\bigl(-5+2N+2N^2\bigr)S_3}{1+N}-\frac{32\bigl(-72-108N+35N^2+23N^3\bigr)S_1^2}{3(N-2)N(1+N)(3+N)}+\frac{5376S_{2,1}}{1+N}
-\frac{128\bigl(-3+N+N^2\bigr)S_{-2,1}}{1+N}-\frac{64\bigl(-21+2N+2N^2\bigr)\zeta_3}{1+N}\Biggr)S_{-2}
+\Biggl(\frac{16P_{307}}{3(N-2)N(1+N)^2(3+N)}+\frac{384S_1}{1+N}\Biggr)S_{-2}^2+\Biggl(\frac{16S_1P_{318}}{3(N-2)N(1+N)^2(3+N)}
-\frac{16P_{427}}{9(N-2)(N-1)N^2(1+N)^3(2+N)(3+N)}-\frac{1120S_1^2}{1+N}+\frac{2656S_2}{1+N}+\frac{448S_{-2}}{1+N}\Biggr)S_{-3}
+\Biggl(\frac{32P_{305}}{(N-2)N(1+N)^2(3+N)}-\frac{1600S_1}{1+N}\Biggr)S_{-4}-\frac{64\bigl(-34+N+N^2\bigr)S_{-5}}{1+N}+\frac{384S_{2,3}}{1+N}
+\frac{2688S_{2,-3}}{1+N}-\frac{384S_{4,1}}{1+N}+\frac{64\bigl(-65+2N+2N^2\bigr)S_{-2,3}}{1+N}+\frac{2496S_{-4,1}}{1+N}-\frac{256S_{2,1,1}}{1+N}
-\frac{5376S_{2,1,-2}}{1+N}+\frac{768S_{3,1,1}}{1+N}+\frac{128\bigl(-10+N+N^2\bigr)S_{-2,1,-2}}{1+N}-\frac{5376S_{-2,2,1}}{1+N}-\frac{5376S_{-3,1,1}}{1+N}
+\frac{10752S_{-2,1,1,1}}{1+N}-\frac{160\bigl(-21+N+N^2\bigr)\zeta_5}{1+N}\Biggr]\Biggr\}
$$

$$
+C_F\Biggl\{T_F^2N_F^2\Biggl[\frac{32P_{313}}{81N^2(1+N)^3}+\frac{64\bigl(-6+7N+19N^2\bigr)S_1}{27N(1+N)^2}+\frac{64S_1^2}{9(1+N)}-\frac{64S_2}{9(1+N)}\Biggr]
+C_AT_FN_F\Biggl[\frac{64S_{3,1}P_{274}}{(N-2)(N-1)(1+N)(2+N)(3+N)}
+\frac{32S_4P_{291}}{3(N-2)(N-1)(1+N)(2+N)(3+N)}+\frac{64S_3P_{328}}{9(N-2)(N-1)(1+N)^2(2+N)(3+N)}
-\frac{16P_{415}}{81(N-2)^2N^2(1+N)^3(3+N)}+\Biggl(\frac{1024\zeta_3P_{272}}{3(N-2)(N-1)(1+N)(2+N)(3+N)}
-\frac{256S_3P_{273}}{3(N-2)(N-1)(1+N)(2+N)(3+N)}-\frac{16P_{426}}{27(N-2)(N-1)N^2(1+N)^3(2+N)(3+N)}
$$

$$
+\frac{64(N-1)(3+2N)S_2}{3(1+N)^2(3+N)}+\frac{128S_{2,1}}{3(1+N)}+\frac{512S_{-2,1}}{3(1+N)}\Bigg)S_1+\Bigg(-\frac{640}{9(1+N)}-\frac{32S_2}{3(1+N)}\Bigg)S_1^2
$$

$$
+\frac{64\big(21+28N+N^2\big)S_2}{9(1+N)^2(3+N)}+\frac{64S_2^2}{3(1+N)}+\Bigg(-\frac{256S_1P_{329}}{9(N-2)(N-1)N(1+N)(2+N)(3+N)}
$$

$$
+\frac{128P_{436}}{9(N-2)^2(N-1)N^2(1+N)^3(2+N)(3+N)}-\frac{256S_1^2}{3(1+N)}+\frac{256S_2}{3(1+N)}\Bigg)S_{-2}+\frac{128S_{-2}^2}{3(1+N)}
$$

$$
+\Bigg(-\frac{128P_{343}}{9(N-2)(N-1)N(1+N)^2(2+N)(3+N)}+\frac{256S_1}{3(1+N)}\Bigg)S_{-3}+\frac{128S_{-4}}{1+N}
$$

$$
-\frac{128(N-1)(3+2N)S_{2,1}}{3(1+N)^2(3+N)}+\frac{256\big(-16-23N+N^2+2N^3\big)S_{-2,1}}{9(N-1)(1+N)^2(2+N)}-\frac{256S_{-3,1}}{3(1+N)}-\frac{64S_{2,1,1}}{1+N}
$$

$$
-\frac{512S_{-2,1,1}}{3(1+N)}+\frac{128\big(36-45N-22N^2+2N^3\big)\zeta_3}{3(N-2)N(1+N)(3+N)}\Bigg]
$$

$$
+C_A^2\Bigg[-\frac{1024S_{-2,1,1}P_{279}}{3(N-2)N(1+N)^2(3+N)}+\frac{64S_{-2,2}P_{288}}{(N-2)N(1+N)^2(3+N)}
$$

$$
+\frac{128S_{-3,1}P_{297}}{3(N-2)N(1+N)^2(3+N)}-\frac{16S_{3,1}P_{342}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}
$$

$$
-\frac{8S_4P_{365}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}+\frac{16S_3P_{368}}{9(N-2)(N-1)N(1+N)^2(2+N)(3+N)}
$$

$$
-\frac{32S_{-2,1}P_{373}}{9(N-2)(N-1)N(1+N)^2(2+N)(3+N)}-\frac{8\zeta_3P_{374}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}
$$

$$
+\frac{4P_{420}}{81(N-2)^2N^2(1+N)^3(3+N)}+\Bigg(\frac{512S_{-2,1}P_{275}}{3(N-2)N(1+N)^2(3+N)}
$$

$$
+\frac{16S_3P_{352}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}-\frac{32\zeta_3P_{355}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}
$$

$$
+\frac{4P_{456}}{27(N-2)(N-1)N^3(1+N)^4(2+N)(3+N)}-\frac{176(N-1)(3+2N)S_2}{3(1+N)^2(3+N)}-\frac{192S_4}{1+N}
$$

$$
-\frac{352S_{2,1}}{3(1+N)}+\frac{384S_{3,1}}{1+N}-\frac{1024S_{-2,2}}{1+N}-\frac{1024S_{-3,1}}{1+N}+\frac{2048S_{-2,1,1}}{1+N}\Bigg)S_1+\Bigg(\frac{1276}{9(1+N)}
$$

$$
+\frac{88S_2}{3(1+N)}-\frac{96S_3}{1+N}-\frac{512S_{-2,1}}{1+N}-\frac{64\zeta_3}{1+N}\Bigg)S_1^2+\Bigg(\frac{44\big(-51-68N+7N^2\big)}{9(1+N)^2(3+N)}+\frac{288S_3}{1+N}
$$

$$
+\frac{1536S_{-2,1}}{1+N}\Bigg)S_2-\frac{176S_2^2}{3(1+N)}+\frac{16\big(4+N+N^2\big)S_5}{1+N}+\Bigg(\frac{512S_1^2}{3(1+N)}-\frac{704S_2}{3(1+N)}
$$

$$
+\frac{32S_1P_{366}}{9(N-2)(N-1)N(1+N)^2(2+N)(3+N)}+\frac{32\big(-4+N+N^2\big)S_3}{1+N}
$$

$$
-\frac{16P_{442}}{9(N-2)^2(N-1)N^2(1+N)^3(2+N)(3+N)}-\frac{1536S_{2,1}}{1+N}+\frac{32\big(-4+N+N^2\big)S_{-2,1}}{1+N}
$$

$$
+\frac{32(N-3)(4+N)\zeta_3}{1+N}\Bigg)S_{-2}+\Bigg(-\frac{16P_{301}}{3(N-2)N(1+N)^2(3+N)}-\frac{128S_1}{1+N}\Bigg)S_{-2}^2
$$

$$+\left(-\frac{128S_1P_{293}}{3(N-2)N(1+N)^2(3+N)}+\frac{16P_{378}}{9(N-2)(N-1)N(1+N)^2(2+N)(3+N)}\right.$$
$$\left.+\frac{256S_1^2}{1+N}-\frac{768S_2}{1+N}-\frac{128S_{-2}}{1+N}\right)S_{-3}+\left(-\frac{16P_{302}}{(N-2)N(1+N)^2(3+N)}+\frac{384S_1}{1+N}\right)S_{-4}$$
$$+\frac{16\big(-40+N+N^2\big)S_{-5}}{1+N}+\frac{352(N-1)(3+2N)S_{2,1}}{3(1+N)^2(3+N)}-\frac{288S_{2,3}}{1+N}-\frac{768S_{2,-3}}{1+N}+\frac{288S_{4,1}}{1+N}$$
$$-\frac{32\big(-38+N+N^2\big)S_{-2,3}}{1+N}-\frac{704S_{-4,1}}{1+N}+\frac{176S_{2,1,1}}{1+N}+\frac{1536S_{2,1,-2}}{1+N}-\frac{576S_{3,1,1}}{1+N}$$
$$-\frac{32(N-3)(4+N)S_{-2,1,-2}}{1+N}+\frac{1536S_{-2,2,1}}{1+N}+\frac{1536S_{-3,1,1}}{1+N}-\frac{3072S_{-2,1,1,1}}{1+N}$$
$$\left.\left.+\frac{40\big(-16+N+N^2\big)\zeta_5}{1+N}\right]\right\}$$
$$+C_F^3\left\{-\frac{256S_{-2,1,1}P_{289}}{(N-2)N(1+N)^2(3+N)}+\frac{128S_{-2,2}P_{290}}{(N-2)N(1+N)^2(3+N)}\right.$$
$$+\frac{128S_{-3,1}P_{292}}{(N-2)N(1+N)^2(3+N)}+\frac{32S_{3,1}P_{357}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$
$$-\frac{16S_4P_{359}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}-\frac{64S_{-2,1}P_{396}}{(N-2)(N-1)N(1+N)^3(2+N)(3+N)}$$
$$-\frac{16\zeta_3P_{401}}{(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}+\frac{8S_3P_{403}}{(N-2)(N-1)N(1+N)^3(2+N)(3+N)}$$
$$+\frac{P_{462}}{6(N-2)^2N^4(1+N)^5(3+N)}+\left(-\frac{64S_3P_{345}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}\right.$$
$$+\frac{64S_{-2,1}P_{298}}{(N-2)N(1+N)^2(3+N)}+\frac{32\zeta_3P_{354}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$
$$+\frac{8P_{460}}{(N-2)^2(N-1)N^3(1+N)^4(2+N)(3+N)}+\frac{8(1+3N)(10+9N)S_2}{N(1+N)^2}+\frac{640S_4}{1+N}+\frac{64S_{2,1}}{1+N}$$
$$\left.-\frac{512S_{3,1}}{1+N}-\frac{3328S_{-2,2}}{1+N}-\frac{3584S_{-3,1}}{1+N}+\frac{6144S_{-2,1,1}}{1+N}\right)S_1+\left(-\frac{2P_{330}}{(N-2)N^2(1+N)^2(3+N)}\right.$$
$$\left.-\frac{80S_2}{1+N}+\frac{64S_3}{1+N}-\frac{1408S_{-2,1}}{1+N}-\frac{64\zeta_3}{1+N}\right)S_1^2-\frac{8\big(2+13N+9N^2\big)S_1^3}{N(1+N)^2}+\frac{8S_1^4}{1+N}$$
$$+\left(\frac{2P_{346}}{(N-2)N^2(1+N)^3(3+N)}-\frac{320S_3}{1+N}+\frac{4480S_{-2,1}}{1+N}-\frac{192\zeta_3}{1+N}\right)S_2+\frac{24S_2^2}{1+N}$$
$$+\frac{64\big(-8+N+N^2\big)S_5}{1+N}+\left(-\frac{32P_{459}}{(N-2)^2(N-1)N^3(1+N)^4(2+N)(3+N)}\right.$$
$$-\frac{128S_2P_{278}}{(N-2)N(1+N)^2(3+N)}+\left(\frac{64P_{437}}{(N-2)^2(N-1)N^2(1+N)^3(2+N)(3+N)}+\frac{384S_2}{1+N}\right)S_1$$
$$-\frac{64\big(24-28N-N^2+3N^3\big)S_1^2}{(N-2)N(1+N)(3+N)}-\frac{128S_1^3}{1+N}+\frac{128\big(-1+N+N^2\big)S_3}{1+N}-\frac{4608S_{2,1}}{1+N}$$

$$
+\frac{128(N-1)(2+N)S_{-2,1}}{1+N}+\frac{128\big(-9+N+N^2\big)\zeta_3}{1+N}\Bigg)S_{-2}+\Bigg(-\frac{256S_1}{1+N}
-\frac{32\big(-8+N+N^2\big)\big(-6+5N+5N^2\big)}{(N-2)N(1+N)^2(3+N)}\Bigg)S_{-2}^2+\Bigg(-\frac{32S_1P_{304}}{(N-2)N(1+N)^2(3+N)}
+\frac{32P_{424}}{(N-2)(N-1)N^2(1+N)^3(2+N)(3+N)}+\frac{1216S_1^2}{1+N}-\frac{2240S_2}{1+N}-\frac{384S_{-2}}{1+N}\Bigg)S_{-3}
+\Bigg(-\frac{64P_{295}}{(N-2)N(1+N)^2(3+N)}+\frac{1664S_1}{1+N}\Bigg)S_{-4}+\frac{64\big(-28+N+N^2\big)S_{-5}}{1+N}
+\frac{16\big(-2+3N+3N^2\big)S_{2,1}}{N(1+N)^2}+\frac{384S_{2,3}}{1+N}-\frac{2304S_{2,-3}}{1+N}-\frac{384S_{4,1}}{1+N}-\frac{128\big(-27+N+N^2\big)S_{-2,3}}{1+N}
-\frac{2176S_{-4,1}}{1+N}-\frac{96S_{2,1,1}}{1+N}+\frac{4608S_{2,1,-2}}{1+N}+\frac{768S_{3,1,1}}{1+N}-\frac{128\big(-8+N+N^2\big)S_{-2,1,-2}}{1+N}
+\frac{4608S_{-2,2,1}}{1+N}+\frac{4608S_{-3,1,1}}{1+N}-\frac{9216S_{-2,1,1,1}}{1+N}+\frac{160\big(-26+N+N^2\big)\zeta_5}{1+N}\Bigg\}
\tag{421}
$$

with

$$P_{257} = -617N^4-1108N^3-281N^2+222N+148, \tag{422}$$
$$P_{258} = -9N^4-4N^3+3N^2+26N+20, \tag{423}$$
$$P_{259} = N^4-22N^3-325N^2+322N-312, \tag{424}$$
$$P_{260} = N^4-22N^3-237N^2+74N-192, \tag{425}$$
$$P_{261} = N^4-22N^3-226N^2-11N-280, \tag{426}$$
$$P_{262} = N^4-22N^3-217N^2+310N-240, \tag{427}$$
$$P_{263} = N^4-22N^3-137N^2+138N-180, \tag{428}$$
$$P_{264} = N^4-22N^3-133N^2-14N-84, \tag{429}$$
$$P_{265} = N^4-22N^3-112N^2+583N-502, \tag{430}$$
$$P_{266} = N^4-22N^3-64N^2+7N-34, \tag{431}$$
$$P_{267} = N^4-22N^3-37N^2+202N-168, \tag{432}$$
$$P_{268} = N^4-22N^3-37N^2+634N-518, \tag{433}$$
$$P_{269} = N^4-22N^3-N^2+406N-96, \tag{434}$$
$$P_{270} = N^4-22N^3+98N^2+313N-354, \tag{435}$$
$$P_{271} = N^4-22N^3+271N^2+582N-500, \tag{436}$$
$$P_{272} = N^4+2N^3-4N^2-5N+9, \tag{437}$$
$$P_{273} = N^4+2N^3-N^2-2N+6, \tag{438}$$
$$P_{274} = N^4+2N^3+9N^2+8N-4, \tag{439}$$
$$P_{275} = N^4+2N^3+16N^2+15N-36, \tag{440}$$
$$P_{276} = 2N^4-44N^3-197N^2-7N-118, \tag{441}$$
$$P_{277} = 2N^4-44N^3+369N^2+895N-854, \tag{442}$$
$$P_{278} = 2N^4+4N^3-5N^2-7N-12, \tag{443}$$

$$P_{279} = 2N^4 + 4N^3 + 11N^2 + 9N - 36, \tag{444}$$
$$P_{280} = 3N^4 - 66N^3 - 1501N^2 - 1720N + 192, \tag{445}$$
$$P_{281} = 3N^4 - 66N^3 - 1387N^2 + 1562N - 1188, \tag{446}$$
$$P_{282} = 3N^4 - 66N^3 - 221N^2 + 2584N - 1776, \tag{447}$$
$$P_{283} = 3N^4 - 66N^3 - 166N^2 + 2243N - 1665, \tag{448}$$
$$P_{284} = 5N^4 + 10N^3 + N^2 - 4N - 4, \tag{449}$$
$$P_{285} = 6N^4 - 132N^3 - 2003N^2 - 1577N - 792, \tag{450}$$
$$P_{286} = 6N^4 - 132N^3 - 1723N^2 + 3311N - 2694, \tag{451}$$
$$P_{287} = 9N^4 - 198N^3 - 1123N^2 - 52N - 576, \tag{452}$$
$$P_{288} = 9N^4 + 18N^3 + 11N^2 + 2N - 96, \tag{453}$$
$$P_{289} = 9N^4 + 18N^3 + 43N^2 + 34N - 144, \tag{454}$$
$$P_{290} = 10N^4 + 20N^3 + 39N^2 + 29N - 150, \tag{455}$$
$$P_{291} = 11N^4 + 22N^3 - 125N^2 - 136N + 180, \tag{456}$$
$$P_{292} = 11N^4 + 22N^3 + 35N^2 + 24N - 156, \tag{457}$$
$$P_{293} = 13N^4 + 26N^3 - 23N^2 - 36N - 72, \tag{458}$$
$$P_{294} = 15N^4 - 64N^3 - 366N^2 - 343N - 18, \tag{459}$$
$$P_{295} = 15N^4 + 30N^3 + 8N^2 - 7N - 138, \tag{460}$$
$$P_{296} = 18N^4 - 396N^3 - 1537N^2 - 43N - 1296, \tag{461}$$
$$P_{297} = 19N^4 + 38N^3 - 11N^2 - 30N - 144, \tag{462}$$
$$P_{298} = 23N^4 + 42N^3 + 55N^2 + 60N - 276, \tag{463}$$
$$P_{299} = 28N^4 + 56N^3 - 125N^2 - 153N - 36, \tag{464}$$
$$P_{300} = 28N^4 + 56N^3 + 61N^2 + 33N - 342, \tag{465}$$
$$P_{301} = 31N^4 + 62N^3 - 203N^2 - 234N + 144, \tag{466}$$
$$P_{302} = 43N^4 + 86N^3 - 119N^2 - 162N - 144, \tag{467}$$
$$P_{303} = 43N^4 + 118N^3 + 19N^2 - 104N - 60, \tag{468}$$
$$P_{304} = 47N^4 + 90N^3 + 7N^2 - 12N - 420, \tag{469}$$
$$P_{305} = 58N^4 + 116N^3 - 111N^2 - 169N - 282, \tag{470}$$
$$P_{306} = 59N^4 + 118N^3 + 305N^2 + 246N - 1008, \tag{471}$$
$$P_{307} = 77N^4 + 154N^3 - 529N^2 - 606N + 432, \tag{472}$$
$$P_{308} = 85N^4 + 86N^3 - 47N^2 - 136N - 84, \tag{473}$$
$$P_{309} = 101N^4 + 190N^3 + 677N^2 + 660N - 1980, \tag{474}$$
$$P_{310} = 109N^4 + 218N^3 + 61N^2 - 48N - 1044, \tag{475}$$
$$P_{311} = 121N^4 + 418N^3 + 269N^2 - 140N - 36, \tag{476}$$
$$P_{312} = 147N^4 + 426N^3 + 80N^2 - 211N - 18, \tag{477}$$
$$P_{313} = 203N^4 + 178N^3 - 31N^2 - 6N + 36, \tag{478}$$
$$P_{314} = 235N^4 + 596N^3 + 319N^2 + 66N + 72, \tag{479}$$
$$P_{315} = 265N^4 + 1478N^3 + 745N^2 - 6156N - 2844, \tag{480}$$
$$P_{316} = 277N^4 + 728N^3 + 152N^2 - 383N - 108, \tag{481}$$
$$P_{317} = 291N^4 + 368N^3 + 21N^2 - 380N - 272, \tag{482}$$
$$P_{318} = 349N^4 + 686N^3 - 347N^2 - 612N - 2412, \tag{483}$$

$$P_{319} = 423N^4 + 264N^3 + 293N^2 + 452N + 54, \quad (484)$$
$$P_{320} = 536N^4 + 1351N^3 + 498N^2 - 329N - 42, \quad (485)$$
$$P_{321} = 536N^4 + 1459N^3 + 565N^2 - 358N - 102, \quad (486)$$
$$P_{322} = 562N^4 + 1565N^3 + 649N^2 - 354N - 72, \quad (487)$$
$$P_{323} = 575N^4 + 1468N^3 + 324N^2 - 845N - 294, \quad (488)$$
$$P_{324} = 683N^4 + 1702N^3 + 767N^2 + 164N + 292, \quad (489)$$
$$P_{325} = 1055N^4 + 2236N^3 + 1139N^2 + 66N + 72, \quad (490)$$
$$P_{326} = 1124N^4 + 2320N^3 + 823N^2 - 451N - 93, \quad (491)$$
$$P_{327} = 4513N^4 + 11762N^3 + 2455N^2 - 4794N - 684, \quad (492)$$
$$P_{328} = 2N^5 - 9N^4 + 113N^3 + 273N^2 - 61N - 174, \quad (493)$$
$$P_{329} = 2N^5 + N^4 - 38N^3 - 13N^2 + 120N - 36, \quad (494)$$
$$P_{330} = 17N^5 - 62N^4 - 413N^3 + 218N^2 + 504N + 144, \quad (495)$$
$$P_{331} = 31N^5 - 34N^4 - 301N^3 + 148N^2 + 540N - 288, \quad (496)$$
$$P_{332} = 83N^5 + 404N^4 + 643N^3 - 50N^2 - 57N + 126, \quad (497)$$
$$P_{333} = 97N^5 + 460N^4 + 32N^3 - 1669N^2 - 1518N + 72, \quad (498)$$
$$P_{334} = 131N^5 + 308N^4 - 347N^3 - 3392N^2 - 2004N + 2592, \quad (499)$$
$$P_{335} = 328N^5 + 1985N^4 + 2899N^3 + 927N^2 - 231N - 180, \quad (500)$$
$$P_{336} = 361N^5 + 2020N^4 + 899N^3 - 7756N^2 - 8724N + 1728, \quad (501)$$
$$P_{337} = 1103N^5 + 4825N^4 + 6329N^3 + 2733N^2 + 66N + 360, \quad (502)$$
$$P_{338} = 1687N^5 + 7540N^4 + 539N^3 - 34066N^2 - 30444N + 9576, \quad (503)$$
$$P_{339} = 3153N^5 + 3318N^4 - 13819N^3 - 2896N^2 - 11820N + 9936, \quad (504)$$
$$P_{340} = 8425N^5 + 30483N^4 + 30615N^3 + 7625N^2 + 2712N + 3212, \quad (505)$$
$$P_{341} = -279N^6 - 1137N^5 - 1215N^4 - 63N^3 - 714N^2 - 1344N - 496, \quad (506)$$
$$P_{342} = 13N^6 + 39N^5 - 33N^4 - 131N^3 - 380N^2 - 308N + 48, \quad (507)$$
$$P_{343} = 14N^6 + 39N^5 - 46N^4 - 129N^3 + 50N^2 + 72N + 144, \quad (508)$$
$$P_{344} = 17N^6 + 51N^5 + 11N^4 - 63N^3 + 284N^2 + 324N + 48, \quad (509)$$
$$P_{345} = 18N^6 + 52N^5 - 37N^4 - 142N^3 + 101N^2 + 148N + 60, \quad (510)$$
$$P_{346} = 21N^6 + 15N^5 - 383N^4 - 463N^3 + 378N^2 + 608N + 96, \quad (511)$$
$$P_{347} = 33N^6 + 729N^5 - 997N^4 - 4225N^3 + 5052N^2 + 3240N + 6264, \quad (512)$$
$$P_{348} = 34N^6 - 450N^5 + 657N^4 + 3376N^3 - 4743N^2 - 3234N - 6192, \quad (513)$$
$$P_{349} = 35N^6 + 69N^5 + 1763N^4 + 3747N^3 + 902N^2 - 1548N + 3096, \quad (514)$$
$$P_{350} = 36N^6 - 390N^5 + 1249N^4 + 2912N^3 - 7001N^2 - 3342N - 7632, \quad (515)$$
$$P_{351} = 37N^6 + 169N^5 + 149N^4 - 749N^3 - 1170N^2 + 484N - 1032, \quad (516)$$
$$P_{352} = 41N^6 + 123N^5 - 157N^4 - 519N^3 - 484N^2 - 204N + 72, \quad (517)$$
$$P_{353} = 49N^6 + 210N^5 + 17N^4 - 848N^3 - 807N^2 + 29N - 318, \quad (518)$$
$$P_{354} = 51N^6 + 141N^5 + 15N^4 - 93N^3 + 226N^2 + 100N + 360, \quad (519)$$
$$P_{355} = 55N^6 + 165N^5 - 395N^4 - 1065N^3 + 28N^2 + 588N - 504, \quad (520)$$
$$P_{356} = 61N^6 - 50N^5 + 490N^4 + 842N^3 - 3659N^2 - 1536N - 3780, \quad (521)$$
$$P_{357} = 65N^6 + 195N^5 - 93N^4 - 511N^3 + 284N^2 + 572N + 288, \quad (522)$$
$$P_{358} = 74N^6 + 302N^5 - 307N^4 - 1512N^3 + 137N^2 + 826N + 840, \quad (523)$$

$$P_{359} = 75N^6 + 225N^5 - 131N^4 - 637N^3 + 240N^2 + 596N + 432, \quad (524)$$
$$P_{360} = 77N^6 + 219N^5 + 161N^4 + 69N^3 + 1742N^2 + 1548N + 216, \quad (525)$$
$$P_{361} = 83N^6 - 109N^5 - 77N^4 - 567N^3 - 42N^2 + 1096N + 576, \quad (526)$$
$$P_{362} = 96N^6 + 178N^5 - 595N^4 - 740N^3 - 13N^2 + 1274N + 1536, \quad (527)$$
$$P_{363} = 100N^6 + 313N^5 + 235N^4 + 31N^3 - 79N^2 - 60N - 20, \quad (528)$$
$$P_{364} = 112N^6 + 517N^5 + 773N^4 + 539N^3 + 311N^2 + 184N + 156, \quad (529)$$
$$P_{365} = 115N^6 + 345N^5 - 671N^4 - 1917N^3 + 1756N^2 + 2772N - 144, \quad (530)$$
$$P_{366} = 122N^6 + 300N^5 - 871N^4 - 1626N^3 + 2009N^2 + 1794N + 468, \quad (531)$$
$$P_{367} = 134N^6 + 402N^5 - 373N^4 - 1416N^3 + 1121N^2 + 1896N + 252, \quad (532)$$
$$P_{368} = 146N^6 + 603N^5 - 733N^4 - 2823N^3 + 2234N^2 + 3471N + 486, \quad (533)$$
$$P_{369} = 158N^6 + 178N^5 + 85N^4 + 28N^3 - 4043N^2 - 1126N - 3576, \quad (534)$$
$$P_{370} = 182N^6 - 1416N^5 + 1896N^4 + 9632N^3 - 15849N^2 - 8379N - 19494, \quad (535)$$
$$P_{371} = 208N^6 + 190N^5 + 711N^4 + 456N^3 - 7635N^2 - 2802N - 7488, \quad (536)$$
$$P_{372} = 248N^6 + 2595N^5 - 1572N^4 - 13771N^3 + 6576N^2 + 7356N + 12672, \quad (537)$$
$$P_{373} = 257N^6 + 705N^5 - 913N^4 - 2385N^3 + 3410N^2 + 3642N - 324, \quad (538)$$
$$P_{374} = 385N^6 + 99N^5 - 3089N^4 - 1239N^3 + 9376N^2 + 4644N - 2736, \quad (539)$$
$$P_{375} = 388N^6 + 6360N^5 - 6891N^4 - 36278N^3 + 34155N^2 + 24738N + 48240, \quad (540)$$
$$P_{376} = 429N^6 + 1849N^5 + 285N^4 - 7533N^3 - 7626N^2 + 716N - 3576, \quad (541)$$
$$P_{377} = 449N^6 + 705N^5 - 2395N^4 - 4669N^3 - 198N^2 + 3860N + 1896, \quad (542)$$
$$P_{378} = 521N^6 + 1497N^5 - 1177N^4 - 4233N^3 + 1826N^2 + 3114N + 2844, \quad (543)$$
$$P_{379} = 676N^6 + 5853N^5 - 2718N^4 - 32261N^3 + 8853N^2 + 19179N + 23166, \quad (544)$$
$$P_{380} = 913N^6 - 3282N^5 + 8202N^4 + 26842N^3 - 64629N^2 - 30582N - 73008, \quad (545)$$
$$P_{381} = 1046N^6 + 15756N^5 - 11349N^4 - 94906N^3 + 60393N^2 + 67464N + 102060, \quad (546)$$
$$P_{382} = 1360N^6 + 4524N^5 - 4169N^4 - 19314N^3 - 7739N^2 + 7914N + 5832, \quad (547)$$
$$P_{383} = 1472N^6 + 2121N^5 + 2868N^4 - 2785N^3 - 43668N^2 - 11448N - 39960, \quad (548)$$
$$P_{384} = 1540N^6 + 4053N^5 - 150N^4 - 13805N^3 - 28359N^2 - 3609N - 20682, \quad (549)$$
$$P_{385} = 1540N^6 + 9345N^5 - 792N^4 - 42161N^3 - 13683N^2 + 8649N + 13122, \quad (550)$$
$$P_{386} = 1545N^6 + 5807N^5 + 5289N^4 + 833N^3 + 2254N^2 + 3776N + 1232, \quad (551)$$
$$P_{387} = 2882N^6 + 12624N^5 - 5307N^4 - 59242N^3 - 19839N^2 + 22626N + 8424, \quad (552)$$
$$P_{388} = 4357N^6 + 20253N^5 + 23997N^4 + 10171N^3 + 666N^2 - 2700N - 1944, \quad (553)$$
$$P_{389} = 4502N^6 + 9708N^5 + 465N^4 - 27358N^3 - 93105N^2 - 17352N - 73548, \quad (554)$$
$$P_{390} = 4816N^6 + 5952N^5 + 12135N^4 - 1466N^3 - 156051N^2 - 48114N - 147312, \quad (555)$$
$$P_{391} = 6457N^6 + 43941N^5 - 12141N^4 - 206669N^3 - 9264N^2 + 63756N + 108864, \quad (556)$$
$$P_{392} = 7241N^6 + 50345N^5 + 124855N^4 + 122983N^3 + 29388N^2 - 3516N + 4032, \quad (557)$$
$$P_{393} = 7531N^6 + 26499N^5 + 27861N^4 + 8401N^3 + 336N^2 + 504N - 540, \quad (558)$$
$$P_{394} = 9187N^6 + 58477N^5 + 134975N^4 + 133571N^3 + 52890N^2 + 27036N + 24408, \quad (559)$$
$$P_{395} = -3221N^7 - 26286N^6 - 57930N^5 - 47240N^4 - 11865N^3 + 942N^2 - 72N - 1080, \quad (560)$$
$$P_{396} = 23N^7 + 88N^6 + 42N^5 - 144N^4 + 271N^3 + 824N^2 + 408N - 72, \quad (561)$$
$$P_{397} = 25N^7 - 379N^6 - 1119N^5 + 2215N^4 + 4787N^3 - 4173N^2 - 5436N - 4860, \quad (562)$$

$$P_{398} = 79N^7 + 169N^6 - 407N^5 - 1353N^4 - 912N^3 + 88N^2 + 1536N + 144, \quad (563)$$
$$P_{399} = 81N^7 - 1463N^6 - 5889N^5 - 333N^4 + 16148N^3 + 10448N^2 + 7072N + 3216, \quad (564)$$
$$P_{400} = 83N^7 + 487N^6 + 582N^5 - 1789N^4 - 4016N^3 - 339N^2 - 180N - 1836, \quad (565)$$
$$P_{401} = 115N^7 + 309N^6 + 141N^5 + 79N^4 + 1552N^3 + 1180N^2 - 48N - 288, \quad (566)$$
$$P_{402} = 173N^7 + 32N^6 - 2263N^5 - 2810N^4 + 3582N^3 + 4014N^2 + 2120N - 120, \quad (567)$$
$$P_{403} = 179N^7 + 700N^6 + 352N^5 - 1242N^4 - 407N^3 + 1830N^2 + 1572N + 216, \quad (568)$$
$$P_{404} = 379N^7 + 3227N^6 + 2631N^5 - 20843N^4 - 22294N^3 + 6252N^2 + 21744N + 18576, \quad (569)$$
$$P_{405} = 403N^7 + 3797N^6 + 6527N^5 - 9957N^4 - 29446N^3 - 8996N^2 + 4936N + 6240, \quad (570)$$
$$P_{406} = 417N^7 + 1362N^6 - 523N^5 - 4588N^4 - 4220N^3 - 2720N^2 + 864N + 2880, \quad (571)$$
$$P_{407} = 713N^7 + 1029N^6 - 2740N^5 - 1095N^4 + 11969N^3 + 6480N^2 - 2772N - 432, \quad (572)$$
$$P_{408} = 721N^7 + 2716N^6 - 38N^5 - 7892N^4 + 4489N^3 + 21520N^2 + 10308N - 1296, \quad (573)$$
$$P_{409} = 781N^7 - 2755N^6 - 22167N^5 - 45617N^4 + 12110N^3 + 59256N^2 + 100728N + 67824, \quad (574)$$
$$P_{410} = 1669N^7 + 7264N^6 + 3778N^5 - 16112N^4 - 2843N^3 + 29584N^2 + 22020N + 3024, \quad (575)$$
$$P_{411} = 1873N^7 + 8797N^6 + 5775N^5 - 38361N^4 - 70804N^3 - 27520N^2 + 20832N - 3888, \quad (576)$$
$$P_{412} = 1964N^7 + 11260N^6 + 8946N^5 - 44536N^4 - 72725N^3 - 15489N^2 + 21528N + 12204, \quad (577)$$
$$P_{413} = 2251N^7 + 11006N^6 + 6514N^5 - 39436N^4 - 64685N^3 - 23230N^2 + 1596N - 1656, \quad (578)$$
$$P_{414} = 2284N^7 + 12899N^6 + 12631N^5 - 41611N^4 - 89702N^3 - 32317N^2 + 10896N - 2700, \quad (579)$$
$$P_{415} = 2686N^7 - 128N^6 - 28605N^5 + 15784N^4 + 49639N^3 + 2700N^2 - 7740N + 4752, \quad (580)$$
$$P_{416} = 3098N^7 + 17416N^6 + 16611N^5 - 60352N^4 - 122465N^3 - 52368N^2 + 12312N - 7128, \quad (581)$$
$$P_{417} = 3691N^7 + 18675N^6 + 34038N^5 + 26401N^4 + 6405N^3 - 618N^2 - 36N - 540, \quad (582)$$
$$P_{418} = 4591N^7 + 32765N^6 + 46752N^5 - 90491N^4 - 251779N^3 - 89754N^2 + 43020N + 10584, \quad (583)$$
$$P_{419} = 4645N^7 + 15287N^6 - 15678N^5 - 107039N^4 - 63475N^3 + 26208N^2 + 51948N + 28080, \quad (584)$$
$$P_{420} = 16828N^7 + 802N^6 - 194889N^5 + 102862N^4 + 378583N^3 + 53370N^2 - 56556N + 26136, \quad (585)$$
$$P_{421} = 32287N^7 + 193848N^6 + 395214N^5 + 329536N^4 + 98223N^3 + 5076N^2 + 3996N - 324, \quad (586)$$
$$P_{422} = -116957N^8 - 894616N^7 - 2631828N^6 - 3627014N^5 - 2222155N^4 - 543882N^3 - 286164N^2 - 177768N + 23328, \quad (587)$$
$$P_{423} = -28023N^8 - 203826N^7 - 566428N^6 - 738922N^5 - 430121N^4 - 91484N^3$$

$$-69852N^2 - 97776N - 34560, \quad (588)$$

$$P_{424} = 35N^8 + 136N^7 + 72N^6 - 222N^5 + 117N^4 + 702N^3 + 472N^2 + 32N + 96, \quad (589)$$

$$P_{425} = 296N^8 + 920N^7 - 1687N^6 - 6931N^5 - 679N^4 + 9827N^3 + 5634N^2 - 252 N - 216, \quad (590)$$

$$P_{426} = 805N^8 + 2632N^7 - 2678N^6 - 12656N^5 + 661N^4 + 16144N^3 + 5856N^2 -1044N + 648, \quad (591)$$

$$P_{427} = 1357N^8 + 5260N^7 + 1288N^6 - 12818N^5 - 3761N^4 + 16198N^3 + 16164N^2 +5976N + 864, \quad (592)$$

$$P_{428} = 1511N^8 + 9442N^7 + 22584N^6 + 26891N^5 + 16961N^4 + 5241N^3 + 38N^2 - 804N -648, \quad (593)$$

$$P_{429} = 17529N^8 + 60627N^7 - 8987N^6 - 211343N^5 - 245062N^4 - 225100N^3 + 6936N^2 +155088N + 23328, \quad (594)$$

$$P_{430} = 28551N^8 + 63048N^7 + 57098N^6 + 23004N^5 + 3519N^4 + 3044N^3 - 9864N^2 -21888N - 9936, \quad (595)$$

$$P_{431} = 50689N^8 + 231286N^7 + 374706N^6 + 252688N^5 + 58645N^4 + 3486N^3 +22752N^2 + 19980N + 8100, \quad (596)$$

$$P_{432} = -97249N^9 - 674988N^8 - 1131968N^7 + 1296434N^6 + 5370597N^5 + 4078570N^4 -68076N^3 - 537576N^2 + 352224N + 114048, \quad (597)$$

$$P_{433} = -71503N^9 - 495343N^8 - 1283176N^7 - 1561498N^6 - 898003N^5 - 199315N^4 -12042N^3 - 12852N^2 - 4536N - 1620, \quad (598)$$

$$P_{434} = -47295N^9 - 50958N^8 + 608888N^7 + 213650N^6 - 1767525N^5 - 1823284N^4 +42212N^3 + 300888N^2 - 153216N - 85536, \quad (599)$$

$$P_{435} = -5021N^9 - 13792N^8 + 14010N^7 + 70516N^6 + 68039N^5 + 94644N^4 + 83628N^3 -75096N^2 - 108576N - 25056, \quad (600)$$

$$P_{436} = 10N^9 + 14N^8 - 108N^7 - 168N^6 + 213N^5 + 438N^4 + 29N^3 - 824N^2 - 180N +288, \quad (601)$$

$$P_{437} = 14N^9 + 20N^8 - 133N^7 - 173N^6 + 407N^5 + 397N^4 - 384N^3 - 964N^2 - 192N +288, \quad (602)$$

$$P_{438} = 25N^9 - 121N^8 - 775N^7 - 546N^6 + 1772N^5 - 1113N^4 - 5854N^3 - 2060N^2 +1112N + 384, \quad (603)$$

$$P_{439} = 79N^9 + 61N^8 - 785N^7 - 936N^6 + 1418N^5 - 2005N^4 - 8840N^3 - 7848N^2 -3104N - 1104, \quad (604)$$

$$P_{440} = 307N^9 + 3168N^8 + 6674N^7 - 2144N^6 - 18819N^5 - 6472N^4 + 12218N^3 - 3564N^2 -15128N - 6288, \quad (605)$$

$$P_{441} = 370N^9 + 536N^8 - 4027N^7 - 4267N^6 + 14417N^5 + 9647N^4 - 14144N^3 -16788N^2 - 3600N + 2592, \quad (606)$$

$$P_{442} = 430N^9 + 728N^8 - 3699N^7 - 6279N^6 + 7179N^5 + 13731N^4 - 616N^3 - 21338N^2 -5472N + 6552, \quad (607)$$

$$P_{443} = 575N^9 + 7672N^8 + 19410N^7 - 7588N^6 - 68521N^5 - 45276N^4 + 23872N^3 +21648N^2 - 8624N - 3840, \quad (608)$$

$$P_{444} = 1288N^9 + 33279N^8 + 109499N^7 - 4538N^6 - 391455N^5 - 190195N^4 + 311202N^3 + 104832N^2 - 74520N - 16848, \quad (609)$$

$$P_{445} = 1963N^9 + 30012N^8 + 88574N^7 - 19280N^6 - 343611N^5 - 220246N^4 + 153144N^3 + 49212N^2 - 44496N - 6480, \quad (610)$$

$$P_{446} = 2133N^9 + 810N^8 - 35584N^7 + 4418N^6 + 120351N^5 + 97028N^4 - 21724N^3 - 15336N^2 + 17856N + 7776, \quad (611)$$

$$P_{447} = 3761N^9 - 14568N^8 - 104870N^7 - 23608N^6 + 335763N^5 + 109918N^4 - 430176N^3 - 423972N^2 - 170640N + 9072, \quad (612)$$

$$P_{448} = 4258N^9 + 34062N^8 + 65735N^7 - 85433N^6 - 339093N^5 - 148117N^4 + 85182N^3 - 99882N^2 - 153360N - 48600, \quad (613)$$

$$P_{449} = 5894N^9 - 12921N^8 - 126065N^7 - 48880N^6 + 374049N^5 + 55783N^4 - 668856N^3 - 635868N^2 - 254448N - 20736, \quad (614)$$

$$P_{450} = 43949N^9 + 268194N^8 + 376282N^7 - 637870N^6 - 2123703N^5 - 1675742N^4 - 209052N^3 + 96174N^2 - 61992N - 8424, \quad (615)$$

$$P_{451} = -5563N^{10} - 24261N^9 - 16394N^8 + 58642N^7 + 172417N^6 + 124231N^5 - 76780N^4 - 55652N^3 + 153696N^2 + 131616N + 34560, \quad (616)$$

$$P_{452} = -4253N^{10} - 18719N^9 - 3594N^8 + 69026N^7 + 89183N^6 + 6165N^5 - 37744N^4 + 8424N^3 + 50664N^2 + 35696N + 8160, \quad (617)$$

$$P_{453} = 24N^{10} - 230N^9 - 267N^8 + 2755N^7 - 219N^6 - 13309N^5 + 8190N^4 + 29248N^3 - 144N^2 - 28400N - 17376, \quad (618)$$

$$P_{454} = 261N^{10} + 324N^9 - 1405N^8 + 651N^7 - 1688N^6 - 14839N^5 + 15318N^4 + 41614N^3 + 29772N^2 - 27720N - 28944, \quad (619)$$

$$P_{455} = 561N^{10} + 2115N^9 + 794N^8 - 3146N^7 - 15755N^6 - 9537N^5 + 39368N^4 + 15976N^3 - 84440N^2 - 83120N - 24864, \quad (620)$$

$$P_{456} = 6439N^{10} + 27179N^9 - 8632N^8 - 154834N^7 - 149641N^6 + 93575N^5 + 183334N^4 + 75768N^3 - 2448N^2 - 12636N - 9720, \quad (621)$$

$$P_{457} = 7868N^{10} + 57302N^9 + 2431N^8 - 508191N^7 - 330541N^6 + 1419481N^5 + 541702N^4 - 1330188N^3 + 96696N^2 + 1283040N + 663552, \quad (622)$$

$$P_{458} = -5729259N^{11} - 8768673N^{10} + 46945558N^9 + 46575050N^8 - 44721647N^7 - 98312261N^6 - 77570092N^5 - 6099524N^4 + 7097376N^3 - 18265104N^2 + 8239104N + 14427072, \quad (623)$$

$$P_{459} = 27N^{11} + 69N^{10} - 176N^9 - 554N^8 + 209N^7 + 1231N^6 + 396N^5 - 1730N^4 - 2104N^3 - 792N^2 + 352N + 192, \quad (624)$$

$$P_{460} = 33N^{11} + 90N^{10} - 827N^9 - 2100N^8 + 2393N^7 + 7786N^6 + 2037N^5 - 7676N^4 - 8880N^3 - 936N^2 + 1712N + 1248, \quad (625)$$

$$P_{461} = 1103N^{11} + 2937N^{10} - 7526N^9 - 24942N^8 + 3681N^7 + 52899N^6 + 29794N^5 - 59478N^4 - 72556N^3 - 4968N^2 + 16272N + 1728, \quad (626)$$

$$P_{462} = 1937N^{11} + 4659N^{10} - 17094N^9 - 30326N^8 + 45693N^7 + 123927N^6 + 59528N^5 - 29940N^4 - 13440N^3 + 26832N^2 + 17088N + 2880, \quad (627)$$

$$P_{463} = 6009N^{11} + 89967N^{10} + 289824N^9 - 63490N^8 - 1473715N^7 - 1627513N^6$$

$$
\begin{aligned}
&& +315866N^5+800300N^4+227984N^3+702960N^2+787680N+229824, && (628)\\
P_{464} &=& 8183N^{11}+18327N^{10}-109486N^9-251134N^8+289655N^7+850715N^6 \\
&& +154440N^5-665500N^4-552096N^3-42048N^2+84672N+62208, && (629)\\
P_{465} &=& 80453N^{11}+688424N^{10}+1757044N^9+44376N^8-6544585N^7-10461842N^6 \\
&& -6460412N^5-1603590N^4-41364N^3+220968N^2+31104N-69984, && (630)\\
P_{466} &=& 428649N^{11}+601299N^{10}-3536240N^9-3241402N^8+3843517N^7+7383499N^6 \\
&& +4538354N^5+20668N^4+528168N^3+1514448N^2-1263168N-1311552, && (631)\\
P_{467} &=& 463143N^{11}+3002511N^{10}+5392892N^9-3891846N^8-24947513N^7 \\
&& -31706485N^6-15500034N^5-1653956N^4-1227864N^3-2638368N^2 \\
&& -1159488N-124416, && (632)\\
P_{468} &=& 241N^{12}+521N^{11}-1523N^{10}-3268N^9+659N^8-1815N^7+2291N^6+25482N^5 \\
&& +45116N^4+19656N^3-27296N^2-32512N-9984, && (633)\\
P_{469} &=& 10629N^{12}+22185N^{11}-70738N^{10}-118858N^9+91525N^8-54183N^7-58576N^6 \\
&& +818608N^5+1687828N^4+888516N^3-727992N^2-580176N-28512, && (634)\\
P_{470} &=& 27765N^{12}+58437N^{11}-182597N^{10}-325952N^9+200843N^8-157371N^7 \\
&& -55295N^6+2325230N^5+4593788N^4+2307744N^3-2192976N^2-2038176N \\
&& -326592, && (635)\\
P_{471} &=& -3069N^{13}+31221N^{12}+144399N^{11}-301293N^{10}-795373N^9+250503N^8 \\
&& +1886057N^7+1623213N^6-63486N^5-894316N^4-31760N^3+890256N^2 \\
&& +677088N+157248, && (636)\\
P_{472} &=& 494694N^{13}+1075005N^{12}-4137924N^{11}-6752619N^{10}+5087252N^9 \\
&& +13650627N^8+2421476N^7-9000225N^6-3343434N^5+7931228N^4 \\
&& +4200112N^3-6598896N^2-6774048N-1729728, && (637)\\
P_{473} &=& 599375N^{13}+5709355N^{12}+18035106N^{11}+12196222N^{10}-51833983N^9 \\
&& -129886893N^8-124003084N^7-53668604N^6-7785942N^5+2544156N^4 \\
&& +6838020N^3+7118928N^2+4129056N+1049760, && (638)\\
P_{474} &=& -458487N^{14}-3433746N^{13}-4940481N^{12}+20878856N^{11}+64591263N^{10} \\
&& +10248134N^9-139889479N^8-158068804N^7-33854512N^6+17126664N^5 \\
&& -5950960N^4+25711872N^3+46793088N^2+27948672N+5971968, && (639)\\
P_{475} &=& 3003N^{14}+22594N^{13}+47885N^{12}-148240N^{11}-659235N^{10}-247430N^9 \\
&& +1482531N^8+1874548N^7+339992N^6-373632N^5-131264N^4-1500992N^3 \\
&& -2232704N^2-1344000N-301824, && (640)\\
P_{476} &=& -7255N^{15}-25524N^{14}+35986N^{13}+183186N^{12}+88824N^{11}-295334N^{10} \\
&& -321342N^9+60030N^8+10607N^7-441646N^6-382164N^5+194648N^4 \\
&& +529968N^3+385824N^2+130752N+17280. && (641)
\end{aligned}
$$

Further non–singlet contributions are

$$
C_{F_2,q}^{d_{abc},(3)} = \frac{d_{abc}d^{abc}N_F}{N_C}\Biggl\{\frac{64\bigl(-3-N+N^2\bigr)}{(N-2)(3+N)} - \frac{128(N-1)S_3P_{479}}{(N-2)N(1+N)(2+N)(3+N)}
$$

$$
-\frac{256S_4P_{483}}{(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{512S_{3,1}P_{483}}{(N-2)N^2(1+N)^2(2+N)(3+N)}
-\frac{256S_{-2,1}P_{484}}{(N-2)N(1+N)(2+N)(3+N)}+\Biggl[-\frac{128\bigl(-66+5N-9N^2+N^3\bigr)}{(N-2)N(1+N)(2+N)(3+N)}
-\frac{256S_3P_{483}}{(N-2)N^2(1+N)^2(2+N)(3+N)}-\frac{1024(N-3)S_{-2,1}}{(N-2)(1+N)(3+N)}\Biggr]S_1
-\Biggl[\frac{256S_1P_{480}}{(N-2)N(1+N)^2(2+N)(3+N)}+\frac{64P_{487}}{(N-2)N^2(1+N)^2(2+N)(3+N)}\Biggr]
\times S_{-2}+\Biggl[\frac{128P_{484}}{(N-2)N(1+N)(2+N)(3+N)}+\frac{512(N-3)S_1}{(N-2)(1+N)(3+N)}\Biggr]S_{-3}
+\frac{64(N-1)S_{-2}^2}{1+N}+\frac{64\bigl(-18+N+N^3\bigr)S_{-4}}{(N-2)(1+N)(3+N)}-\frac{1024(N-3)S_{-2,2}}{(N-2)(1+N)(3+N)}
-\frac{512(N-1)S_{-2,3}}{N(1+N)}-\frac{1024(N-3)S_{-3,1}}{(N-2)(1+N)(3+N)}+\frac{512(N-1)S_{-4,1}}{N(1+N)}
+\frac{2048(N-3)S_{-2,1,1}}{(N-2)(1+N)(3+N)}+\Biggl[\frac{32P_{485}}{3(N-2)N(1+N)(2+N)(3+N)}
+\frac{1024\bigl(-12-4N-3N^2+N^3\bigr)S_1}{(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{1024(N-1)S_{-2}}{N(1+N)}\Biggr]\zeta_3
-\frac{1280(N-3)(N-2)}{3N(1+N)}\zeta_5\Biggr\}, \quad (642)
$$

$$
C_{F_L,q}^{d_{abc},(3)} = \frac{d_{abc}d^{abc}N_F}{N_C}\Biggl\{-\frac{128N}{(N-2)(3+N)}-\frac{128S_3P_{478}}{(N-2)(N-1)(1+N)(2+N)(3+N)}
+\frac{512S_{-2,1}P_{482}}{(N-2)(N-1)(1+N)(2+N)(3+N)}+\Biggl[\frac{4096S_{-2,1}}{(N-2)(1+N)(3+N)}
+\frac{1024\bigl(6+N+2N^2+2N^3+N^4\bigr)S_3}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}
+\frac{1536\bigl(8+N+N^2\bigr)}{(N-2)(N-1)(1+N)(2+N)(3+N)}\Biggr]S_1
+\frac{1024\bigl(6+N+2N^2+2N^3+N^4\bigr)S_4}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}
+\Biggl[\frac{1536S_1P_{481}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}
+\frac{128P_{486}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}\Biggr]S_{-2}-\frac{128S_{-2}^2}{1+N}
+\Biggl[-\frac{256P_{482}}{(N-2)(N-1)(1+N)(2+N)(3+N)}-\frac{2048S_1}{(N-2)(1+N)(3+N)}\Biggr]S_{-3}
$$

$$-\frac{128(10+N+N^2)S_{-4}}{(N-2)(1+N)(3+N)}-\frac{2048(6+N+2N^2+2N^3+N^4)S_{3,1}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}$$
$$+\frac{4096S_{-2,2}}{(N-2)(1+N)(3+N)}-\frac{512S_{-2,3}}{1+N}+\frac{4096S_{-3,1}}{(N-2)(1+N)(3+N)}+\frac{512S_{-4,1}}{1+N}$$
$$-\frac{8192S_{-2,1,1}}{(N-2)(1+N)(3+N)}+\Biggl[-\frac{64P_{477}}{(N-2)(N-1)(1+N)(2+N)(3+N)}$$
$$+\frac{1024S_{-2}}{1+N}-\frac{6144(2+N+N^2)S_1}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}\Biggr]\zeta_3+\frac{2560}{1+N}\zeta_5\Biggr\}, \quad (643)$$

with

$$\begin{aligned}
P_{477} &= N^4+2N^3-43N^2-44N+36, && (644)\\
P_{478} &= N^4+2N^3+5N^2+4N+36, && (645)\\
P_{479} &= N^4+6N^3+6N^2-8N-24, && (646)\\
P_{480} &= 2N^4-4N^3-17N^2-23N-6, && (647)\\
P_{481} &= 2N^4+4N^3+3N^2+N+2, && (648)\\
P_{482} &= 4N^4+8N^3-31N^2-35N+18, && (649)\\
P_{483} &= N^5-5N^3-12N^2-8N-24, && (650)\\
P_{484} &= 2N^5-21N^3+21N^2+46N-24, && (651)\\
P_{485} &= 7N^5+22N^4-25N^3+128N^2+60N-432, && (652)\\
P_{486} &= 7N^6+21N^5-26N^4-87N^3-47N^2-12, && (653)\\
P_{487} &= 5N^7+11N^6-31N^5-49N^4+28N^3+68N^2+40N+48. && (654)
\end{aligned}$$

The pure singlet contributions are given by

$$C_{F_2,q}^{\mathrm{PS},(3)} =$$
$$C_F\Biggl\{T_F^2N_F^2\Biggl[-\frac{512S_{-2}P_{495}}{9(N-2)(N-1)N^2(1+N)(2+N)^2(3+N)}$$
$$+\frac{128\zeta_3P_{492}}{9(N-1)N^2(1+N)^2(2+N)}+\frac{16S_1^2P_{516}}{27(N-1)N^3(1+N)^3(2+N)^2}$$
$$-\frac{32P_{547}}{243(N-2)(N-1)N^5(1+N)^5(2+N)^4(3+N)}$$
$$+\frac{16S_2P_{522}}{27(N-1)N^3(1+N)^3(2+N)^2}+\Biggl(-\frac{32P_{537}}{81(N-1)N^4(1+N)^4(2+N)^3}$$
$$-\frac{32(2+N+N^2)^2S_2}{9(N-1)N^2(1+N)^2(2+N)}\Biggr)S_1+\frac{64(2+N+N^2)^2S_1^3}{27(N-1)N^2(1+N)^2(2+N)}$$
$$-\frac{160(2+N+N^2)^2S_3}{27(N-1)N^2(1+N)^2(2+N)}+\frac{1024S_{-3}}{3(N-1)N(1+N)(2+N)}\Biggr]$$
$$+C_AT_FN_F\Biggl[-\frac{48(2+N+N^2)^2}{(N-1)N^2(1+N)^2(2+N)}\zeta_4-\frac{16S_{-2,2}P_{488}}{(N-1)N^2(1+N)^2(2+N)}$$

$$
+\frac{64S_{-3,1}P_{493}}{3(N-1)N^2(1+N)^2(2+N)}+\frac{64S_{-2,1,1}P_{494}}{3(N-1)N^2(1+N)^2(2+N)}
-\frac{64S_{3,1}P_{497}}{(N-2)N^2(1+N)^2(2+N)(3+N)}-\frac{8S_{-4}P_{501}}{3(N-1)N^2(1+N)^2(2+N)}
-\frac{4S_4P_{508}}{3(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}-\frac{8\big(2+N+N^2\big)S_1^3P_{510}}{27(N-1)^2N^3(1+N)^3(2+N)^2}
-\frac{16S_{2,1}P_{519}}{3(N-1)N^3(1+N)^3(2+N)^2}-\frac{16S_{-2,1}P_{520}}{3(N-1)N^3(1+N)^3(2+N)^2}
+\frac{8S_3P_{538}}{27(N-2)(N-1)^2N^3(1+N)^3(2+N)^2(3+N)}-\frac{4S_2P_{542}}{27(N-1)^2N^4(1+N)^4(2+N)^3}
+\frac{8P_{550}}{243(N-2)(N-1)^2N^6(1+N)^6(2+N)^5(3+N)}+\Bigg(-\frac{16S_{-2,1}P_{499}}{3(N-1)N^2(1+N)^2(2+N)}
+\frac{8S_3P_{513}}{9(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}+\frac{8S_2P_{526}}{9(N-1)^2N^3(1+N)^3(2+N)^2}
+\frac{4P_{548}}{81(N-2)(N-1)^2N^5(1+N)^5(2+N)^4(3+N)}\Bigg)S_1
+\Bigg(\frac{4P_{541}}{27(N-1)^2N^4(1+N)^4(2+N)^3}-\frac{80\big(2+N+N^2\big)^2S_2}{3(N-1)N^2(1+N)^2(2+N)}\Bigg)S_1^2
+\frac{28\big(2+N+N^2\big)^2S_1^4}{9(N-1)N^2(1+N)^2(2+N)}+\frac{16\big(2+N+N^2\big)\big(5+2N+2N^2\big)S_2^2}{3(N-1)N^2(1+N)^2(2+N)}
+\Bigg(-\frac{16S_1P_{531}}{3(N-2)(N-1)^2N^3(1+N)^3(2+N)^2(3+N)}
-\frac{8P_{543}}{9(N-2)(N-1)^2N^4(1+N)^4(2+N)^3(3+N)}
-\frac{16\big(-22+N+N^2\big)\big(2+N+N^2\big)S_1^2}{3(N-1)N^2(1+N)^2(2+N)}+\frac{32\big(-8+N+N^2\big)\big(2+N+N^2\big)S_2}{3(N-1)N^2(1+N)^2(2+N)}\Bigg)S_{-2}
-\frac{16\big(2+N+N^2\big)\big(2+3N+3N^2\big)S_{-2}^2}{3(N-1)N^2(1+N)^2(2+N)}+\Bigg(\frac{8S_1P_{503}}{3(N-1)N^2(1+N)^2(2+N)}
+\frac{8P_{527}}{3(N-1)^2N^3(1+N)^3(2+N)^2}\Bigg)S_{-3}+\frac{32\big(2+N+N^2\big)^2S_{2,1,1}}{3(N-1)N^2(1+N)^2(2+N)}
+\Bigg(-\frac{16S_1P_{509}}{3(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}
-\frac{16P_{534}}{9(N-2)(N-1)^2N^3(1+N)^3(2+N)^2(3+N)}\Bigg)\zeta_3\Bigg]\Bigg\}
$$

$$
+C_F^2T_FN_F\Bigg\{\frac{48\big(2+N+N^2\big)^2}{(N-1)N^2(1+N)^2(2+N)}\zeta_4-\frac{256S_{-2,1}P_{496}}{(N-1)N^3(1+N)^3(2+N)^2}
-\frac{64S_{3,1}P_{506}}{(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}
$$

$$
\begin{aligned}
&+\frac{8S_4P_{511}}{3(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)} \\
&+\frac{64S_{2,1}P_{518}}{3(N-1)N^3(1+N)^3(2+N)^2}+\frac{16S_1^3P_{521}}{9(N-1)N^3(1+N)^3(2+N)^2} \\
&-\frac{8S_3P_{530}}{9(N-2)(N-1)N^3(1+N)^3(2+N)^2(3+N)}-\frac{4S_2P_{532}}{3(N-1)N^4(1+N)^4(2+N)^3} \\
&-\frac{4P_{549}}{3(N-2)^2(N-1)N^6(1+N)^6(2+N)^4(3+N)} \\
&+\Biggl(\frac{16S_3P_{515}}{9(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)}-\frac{8S_2P_{523}}{3(N-1)N^3(1+N)^3(2+N)^2} \\
&+\frac{8P_{546}}{3(N-2)(N-1)N^5(1+N)^5(2+N)^4(3+N)}-\frac{256S_{-2,1}}{N^2(1+N)^2}\Biggr)S_1 \\
&+\Biggl(\frac{4P_{535}}{3(N-1)N^4(1+N)^4(2+N)^3}-\frac{112\bigl(2+N+N^2\bigr)^2S_2}{3(N-1)N^2(1+N)^2(2+N)}\Biggr)S_1^2 \\
&+\frac{44\bigl(2+N+N^2\bigr)^2S_1^4}{9(N-1)N^2(1+N)^2(2+N)}+\frac{68\bigl(2+N+N^2\bigr)^2S_2^2}{3(N-1)N^2(1+N)^2(2+N)} \\
&+\Biggl(-\frac{32S_1P_{524}}{(N-2)(N-1)N^3(1+N)^2(2+N)^2(3+N)} \\
&+\frac{32P_{539}}{(N-2)^2(N-1)N^4(1+N)^4(2+N)^2(3+N)}-\frac{128\bigl(4+N+N^2\bigr)S_1^2}{(N-1)N^2(1+N)^2(2+N)} \\
&+\frac{128\bigl(4+N+N^2\bigr)S_2}{(N-1)N^2(1+N)^2(2+N)}\Biggr)S_{-2}+\Biggl(\frac{128\bigl(6+N+N^2\bigr)S_1}{(N-1)N^2(1+N)^2(2+N)} \\
&+\frac{64P_{528}}{(N-2)(N-1)N^3(1+N)^3(2+N)^2(3+N)}\Biggr)S_{-3}+\frac{128\bigl(6+5N+5N^2\bigr)S_{-4}}{(N-1)N^2(1+N)^2(2+N)} \\
&-\frac{256\bigl(2+3N+3N^2\bigr)S_{-2,2}}{(N-1)N^2(1+N)^2(2+N)}-\frac{1024\bigl(1+N+N^2\bigr)S_{-3,1}}{(N-1)N^2(1+N)^2(2+N)} \\
&-\frac{32\bigl(2+N+N^2\bigr)^2S_{2,1,1}}{3(N-1)N^2(1+N)^2(2+N)}+\frac{1024S_{-2,1,1}}{(N-1)N(1+N)(2+N)} \\
&+\Biggl(-\frac{128S_1P_{507}}{3(N-2)(N-1)N^2(1+N)^2(2+N)(3+N)} \\
&+\frac{32P_{529}}{3(N-2)(N-1)N^3(1+N)^3(2+N)^2(3+N)}\Biggr)\zeta_3\Biggr\} \qquad (655)
\end{aligned}
$$

and

$$
\begin{aligned}
&C_{F_L,q}^{\mathrm{PS},(3)}= \\
&C_F\Biggl\{T_F^2N_F^2\Biggl[\frac{128P_{536}}{27(N-2)(N-1)N^3(1+N)^4(2+N)^3(3+N)} \\
&+\frac{128\bigl(16+27N+13N^2+8N^3\bigr)S_1}{9(N-1)N(1+N)^3(2+N)}-\frac{64\bigl(2+N+N^2\bigr)S_1^2}{3(N-1)N(1+N)^2(2+N)}
\end{aligned}
$$

$$
-\frac{64\left(2+N+N^2\right)S_2}{3(N-1)N(1+N)^2(2+N)}-\frac{2048S_{-2}}{3(N-2)(N-1)(1+N)(2+N)(3+N)}\Bigg]
$$

$$
+C_AT_FN_F\Bigg[-\frac{32S_3P_{502}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}-\frac{16S_1^2P_{512}}{3(N-1)^2N^2(1+N)^3(2+N)^2}
$$

$$
+\frac{16S_2P_{514}}{3(N-1)^2N^2(1+N)^3(2+N)^2}-\frac{32P_{545}}{27(N-2)(N-1)^2N^4(1+N)^5(2+N)^4(3+N)}
$$

$$
+\Bigg(-\frac{32P_{540}}{9(N-2)(N-1)^2N^3(1+N)^4(2+N)^3(3+N)}+\frac{160\left(2+N+N^2\right)S_2}{(N-1)N(1+N)^2(2+N)}
$$

$$
+\frac{256\left(2+N+N^2\right)S_3}{(N-2)(N-1)(1+N)(2+N)(3+N)}\Bigg)S_1-\frac{160\left(2+N+N^2\right)S_1^3}{3(N-1)N(1+N)^2(2+N)}
$$

$$
+\frac{256\left(2+N+N^2\right)S_4}{(N-2)(N-1)(1+N)(2+N)(3+N)}+\Bigg(\frac{128S_1P_{490}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}
$$

$$
+\frac{64P_{525}}{3(N-2)(N-1)^2N^2(1+N)^3(2+N)^2(3+N)}\Bigg)S_{-2}-\frac{576\left(2+N+N^2\right)S_{-3}}{(N-1)N(1+N)^2(2+N)}
$$

$$
-\frac{512\left(2+N+N^2\right)S_{3,1}}{(N-2)(N-1)(1+N)(2+N)(3+N)}+\frac{384\left(2+N+N^2\right)S_{-2,1}}{(N-1)N(1+N)^2(2+N)}
$$

$$
+\Bigg(\frac{192\left(18+5N+5N^2\right)}{(N-2)N(1+N)^2(3+N)}-\frac{512\left(2+N+N^2\right)S_1}{(N-2)(N-1)(1+N)(2+N)(3+N)}\Bigg)\zeta_3\Bigg]\Bigg\}
$$

$$
+C_F^2T_FN_F\Bigg\{\frac{256S_{-3}P_{491}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}+\frac{16S_2P_{504}}{(N-1)N^2(1+N)^3(2+N)^2}
$$

$$
+\frac{64S_3P_{500}}{3(N-2)(N-1)N(1+N)^2(2+N)(3+N)}-\frac{16S_1^2P_{505}}{(N-1)N^2(1+N)^3(2+N)^2}
$$

$$
+\frac{32P_{544}}{(N-2)^2(N-1)N^4(1+N)^5(2+N)^3(3+N)}+\Bigg(\frac{96\left(2+N+N^2\right)S_2}{(N-1)N(1+N)^2(2+N)}
$$

$$
-\frac{32P_{533}}{(N-2)(N-1)N^3(1+N)^4(2+N)^3(3+N)}-\frac{512\left(2+N+N^2\right)S_3}{(N-2)(N-1)(1+N)(2+N)(3+N)}\Bigg)
$$

$$
\times S_1-\frac{32\left(2+N+N^2\right)S_1^3}{3(N-1)N(1+N)^2(2+N)}-\frac{512\left(2+N+N^2\right)S_4}{(N-2)(N-1)(1+N)(2+N)(3+N)}
$$

$$
+\Bigg(-\frac{128S_1P_{489}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}
$$

$$
+\frac{256P_{517}}{(N-2)^2(N-1)N^2(1+N)^3(2+N)(3+N)}\Bigg)S_{-2}
$$

$$
+\frac{1024\left(2+N+N^2\right)S_{3,1}}{(N-2)(N-1)(1+N)(2+N)(3+N)}-\frac{1024S_{-2,1}}{(N-1)N(1+N)^2(2+N)}
$$

$$
+\Bigg(-\frac{128P_{498}}{(N-2)(N-1)N(1+N)^2(2+N)(3+N)}
$$

$$+\frac{1024(2+N+N^2)S_1}{(N-2)(N-1)(1+N)(2+N)(3+N)}\Bigg)\zeta_3\Bigg\} \tag{656}$$

with

$$P_{488} = N^4+2N^3-71N^2-72N-20, \tag{657}$$
$$P_{489} = N^4+2N^3-15N^2-16N+44, \tag{658}$$
$$P_{490} = N^4+2N^3-13N^2-14N+16, \tag{659}$$
$$P_{491} = N^4+2N^3+3N^2+2N-16, \tag{660}$$
$$P_{492} = N^4+2N^3+41N^2+40N+4, \tag{661}$$
$$P_{493} = N^4+2N^3+65N^2+64N+28, \tag{662}$$
$$P_{494} = 5N^4+10N^3-83N^2-88N-4, \tag{663}$$
$$P_{495} = 5N^4+15N^3-38N^2-36N+72, \tag{664}$$
$$P_{496} = 7N^4+16N^3+13N^2+12N+4, \tag{665}$$
$$P_{497} = 8N^4+13N^3+15N^2-20N-60, \tag{666}$$
$$P_{498} = 11N^4+22N^3+39N^2+28N-84, \tag{667}$$
$$P_{499} = 25N^4+50N^3-87N^2-112N+60, \tag{668}$$
$$P_{500} = 41N^4+82N^3+117N^2+76N-204, \tag{669}$$
$$P_{501} = 47N^4+94N^3+487N^2+440N+308, \tag{670}$$
$$P_{502} = 49N^4+98N^3+105N^2+56N-372, \tag{671}$$
$$P_{503} = 63N^4+126N^3-N^2-64N+4, \tag{672}$$
$$P_{504} = N^5-6N^4-19N^3-52N^2-76N-32, \tag{673}$$
$$P_{505} = 11N^5+30N^4+47N^3+36N^2-36N-32, \tag{674}$$
$$P_{506} = N^6-13N^5-7N^4-3N^3+34N^2+44N-120, \tag{675}$$
$$P_{507} = 4N^6-12N^5-11N^4-18N^3-43N^2-20N+84, \tag{676}$$
$$P_{508} = 5N^6-177N^5+65N^4+297N^3+190N^2+140N-3720, \tag{677}$$
$$P_{509} = 7N^6+117N^5+67N^4+3N^3-382N^2-428N-600, \tag{678}$$
$$P_{510} = 7N^6+246N^5+558N^4+532N^3+377N^2-340N-228, \tag{679}$$
$$P_{511} = 19N^6-135N^5+67N^4+231N^3-178N^2-188N-3336, \tag{680}$$
$$P_{512} = 19N^6+9N^5-111N^4-329N^3-364N^2+32N+24, \tag{681}$$
$$P_{513} = 37N^6+399N^5+301N^4+129N^3-2326N^2-2516N+264, \tag{682}$$
$$P_{514} = 41N^6+87N^5+3N^4-175N^3-284N^2-80N-24, \tag{683}$$
$$P_{515} = 43N^6-159N^5-173N^4-273N^3+278N^2+580N-1896, \tag{684}$$
$$P_{516} = N^7-37N^6-248N^5-799N^4-1183N^3-970N^2-580N-168, \tag{685}$$
$$P_{517} = N^7+2N^6-10N^5+37N^3-2N^2+4N-40, \tag{686}$$
$$P_{518} = N^7+3N^6+2N^5+33N^4+73N^3+52N^2+36N+16, \tag{687}$$
$$P_{519} = 3N^7+7N^6+7N^5+125N^4+274N^3+248N^2+200N+80, \tag{688}$$
$$P_{520} = 15N^7-26N^6-187N^5-464N^4-734N^3-644N^2-504N-176, \tag{689}$$
$$P_{521} = 23N^7+94N^6+147N^5-28N^4-288N^3-348N^2-384N-176, \tag{690}$$
$$P_{522} = 49N^7+185N^6+340N^5+287N^4+65N^3+62N^2-196N-168, \tag{691}$$
$$P_{523} = 51N^7+175N^6+139N^5-435N^4-1022N^3-1060N^2-952N-384, \tag{692}$$

$$P_{524} = N^8+9N^7-9N^6-165N^5-148N^4+408N^3+464N^2+432N+576, \quad (693)$$
$$P_{525} = 6N^8-6N^7-17N^6+141N^5+61N^4+297N^3+58N^2-900N-216, \quad (694)$$
$$P_{526} = 11N^8+221N^7+754N^6+1826N^5+2301N^4+897N^3-150N^2-1180N -648, \quad (695)$$
$$P_{527} = 37N^8-119N^7-883N^6-2063N^5-986N^4+1994N^3+2252N^2+1576N +496, \quad (696)$$
$$P_{528} = N^9+10N^8+24N^7+16N^6-35N^5-214N^4-246N^3+100N^2+168N +144, \quad (697)$$
$$P_{529} = 9N^9-54N^8-154N^7-140N^6-1159N^5-1550N^4+1192N^3+2896N^2 +3664N+2208, \quad (698)$$
$$P_{530} = 37N^9-744N^8-2698N^7-1652N^6+1309N^5+2820N^4+4416N^3 +560N^2+560N-768, \quad (699)$$
$$P_{531} = 3N^{10}-15N^9-121N^8+106N^7+993N^6+693N^5-1643N^4-2376N^3 -2088N^2-688N+528, \quad (700)$$
$$P_{532} = 6N^{10}+53N^9+192N^8-30N^7+160N^6+4697N^5+12778N^4+17152N^3 +14848N^2+7920N+1888, \quad (701)$$
$$P_{533} = 28N^{10}+205N^9+641N^8+1139N^7+1035N^6+92N^5-748N^4-1656N^3 -2944N^2-2464N-768, \quad (702)$$
$$P_{534} = 67N^{10}-142N^9-27N^8+1572N^7-5543N^6-12210N^5+5535N^4+16444N^3 +20992N^2+3696N-5040, \quad (703)$$
$$P_{535} = 114N^{10}+681N^9+1398N^8-264N^7-4306N^6-2417N^5+7466N^4+14832N^3 +15008N^2+9040N+2400, \quad (704)$$
$$P_{536} = 127N^{10}+842N^9+1753N^8-236N^7-8468N^6-17624N^5-14130N^4 -2296N^3+288N^2-1440N-864, \quad (705)$$
$$P_{537} = 176N^{10}+973N^9+1824N^8-948N^7-10192N^6-19173N^5-20424N^4 -16036N^3-7816N^2-1248N+288, \quad (706)$$
$$P_{538} = 220N^{10}-673N^9-3885N^8-2682N^7+1450N^6+10779N^5+19215N^4 -3632N^3+712N^2-2928N-2448, \quad (707)$$
$$P_{539} = 9N^{11}+23N^{10}-100N^9-182N^8+333N^7+519N^6+446N^5+520N^4- 1040N^3-2288N^2-2784N-1152, \quad (708)$$
$$P_{540} = 64N^{11}+343N^{10}-190N^9-3834N^8-7845N^7-4410N^6+6491N^5+4865N^4 -9500N^3-11760N^2-9216N-3024, \quad (709)$$
$$P_{541} = 82N^{11}-107N^{10}+1405N^9+19278N^8+66823N^7+123752N^6+143766N^5 +112177N^4+57120N^3+5000N^2-7584N-1584, \quad (710)$$
$$P_{542} = 824N^{11}+3749N^{10}+6953N^9+11052N^8+22103N^7+50950N^6+83556N^5 +75245N^4+44472N^3+22840N^2+7584N+720, \quad (711)$$
$$P_{543} = 39N^{13}+66N^{12}-1159N^{11}-6196N^{10}-3405N^9+38262N^8+50307N^7 -84768N^6-167366N^5+10252N^4+198720N^3+200928N^2+92064N +17856, \quad (712)$$
$$P_{544} = 43N^{13}+253N^{12}+286N^{11}-985N^{10}-2923N^9-1277N^8+5318N^7$$

$$+9049N^6+4744N^5+1348N^4+5408N^3+8624N^2+4736N+960, \quad (713)$$

$$P_{545} = 1844N^{14}+16184N^{13}+50096N^{12}+39487N^{11}-152593N^{10}-478032N^9 -483740N^8+106345N^7+667861N^6+593144N^5+330028N^4+334800N^3 +331056N^2+152064N+25920, \quad (714)$$

$$P_{546} = 78N^{15}+542N^{14}+845N^{13}-3182N^{12}-15821N^{11}-29382N^{10}-37143N^9 -60252N^8-74387N^7+58234N^6+375052N^5+647480N^4+655920N^3+ 432480N^2+173632N+31872, \quad (715)$$

$$P_{547} = 1733N^{15}+14466N^{14}+39452N^{13}+4062N^{12}-191762N^{11}-320533N^{10} +130420N^9+775488N^8+348241N^7-756255N^6-802132N^5+107380N^4 +801072N^3+878256N^2+502848N+119232, \quad (716)$$

$$P_{548} = 4373N^{16}+39739N^{15}+116771N^{14}-491N^{13}-811481N^{12}-1974485N^{11} -1526011N^{10}+1448375N^9+6113428N^8+13620182N^7+22903928N^6 +27507368N^5+24009536N^4+15346080N^3+7254144N^2+2401920N +414720, \quad (717)$$

$$P_{549} = 424N^{18}+3206N^{17}+5830N^{16}-13976N^{15}-63725N^{14}-46169N^{13} +105366N^{12}+112942N^{11}-193953N^{10}-59123N^9+1048298N^8+1685392N^7 +242464N^6-2245408N^5-3487424N^4-2893056N^3-1520896N^2- 480000N-69120, \quad (718)$$

$$P_{550} = 24754N^{19}+255605N^{18}+933618N^{17}+783522N^{16}-4107236N^{15} -12029850N^{14}-4276384N^{13}+29789918N^{12}+46452820N^{11}-4027635N^{10} -75545302N^9-98368740N^8-114975366N^7-183461620N^6-250266040N^5 -242010000N^4-166111776N^3-78065856N^2-22156416N-2799360. \quad (719)$$

Finally, the gluonic contributions read

$$C_{F_2,g}^{(3)} =$$

$$C_F\Biggl\{T_F^2N_F^2\Biggl[-\frac{256S_{2,1}P_{553}}{9N(1+N)^2(2+N)(3+N)}+\frac{16S_1^3P_{608}}{81(N-1)N^3(1+N)^3(2+N)^2}$$

$$+\frac{32S_3P_{627}}{81(N-1)N^3(1+N)^3(2+N)^2(3+N)}-\frac{4S_2P_{648}}{27(N-1)N^4(1+N)^4(2+N)^3(3+N)}$$

$$+\frac{P_{663}}{243(N-2)^2(N-1)N^6(1+N)^6(2+N)^5(3+N)}$$

$$-\frac{64\zeta_3P_{630}}{9(N-2)(N-1)N^3(1+N)^3(2+N)^2(3+N)}$$

$$+\Biggl(-\frac{16S_2P_{625}}{27(N-1)N^3(1+N)^3(2+N)^2(3+N)}$$

$$-\frac{8P_{659}}{243(N-2)(N-1)N^5(1+N)^5(2+N)^4(3+N)}+\frac{32\bigl(-74+35N+35N^2\bigr)S_3}{27N(1+N)(2+N)}$$

$$-\frac{128(N-1)S_{2,1}}{3N(1+N)}+\frac{512S_{-2,1}}{3N(1+N)(2+N)}+\frac{64\bigl(46+5N+5N^2\bigr)\zeta_3}{9N(1+N)(2+N)}\Biggr)S_1$$

$$+\Biggl(\frac{4P_{641}}{81(N-1)N^4(1+N)^4(2+N)^3}-\frac{8\bigl(62+7N+7N^2\bigr)S_2}{9N(1+N)(2+N)}\Biggr)S_1^2+\frac{68\bigl(2+N+N^2\bigr)S_1^4}{27N(1+N)(2+N)}$$
$$+\frac{4\bigl(226+17N+17N^2\bigr)S_2^2}{9N(1+N)(2+N)}-\frac{56\bigl(-14+17N+17N^2\bigr)S_4}{9N(1+N)(2+N)}$$
$$+\Biggl(-\frac{64S_1P_{594}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$
$$-\frac{32P_{636}}{9(N-2)^2(N-1)N^3(1+N)^2(2+N)^3(3+N)}-\frac{512S_1^2}{3N(1+N)(2+N)}$$
$$+\frac{512S_2}{3N(1+N)(2+N)}\Biggr)S_{-2}+\frac{256S_{-2}^2}{3N(1+N)(2+N)}+\Biggl(\frac{256S_1}{N(1+N)(2+N)}$$
$$+\frac{128P_{593}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}\Biggr)S_{-3}+\frac{1024S_{-4}}{3N(1+N)(2+N)}$$
$$+\frac{64\bigl(10+3N+3N^2\bigr)S_{3,1}}{3N(1+N)(2+N)}+\frac{512(-3+5N)S_{-2,1}}{9N^2(1+N)(2+N)}-\frac{512S_{-2,2}}{3N(1+N)(2+N)}$$
$$-\frac{1024S_{-3,1}}{3N(1+N)(2+N)}+\frac{64\bigl(-10+N+N^2\bigr)S_{2,1,1}}{3N(1+N)(2+N)}\Biggr]$$
$$+C_AT_FN_F\Biggl[-\frac{96\zeta_2^2P_{563}}{5(N-1)N^2(1+N)^2(2+N)^2}-\frac{16S_{2,1,1}P_{564}}{3(N-1)N^2(1+N)^2(2+N)^2}$$
$$-\frac{2S_1^4P_{580}}{27(N-1)N^2(1+N)^2(2+N)^2}-\frac{2S_2^2P_{582}}{9(N-1)N^2(1+N)^2(2+N)^2}$$
$$-\frac{32S_{-2,1,1}P_{595}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$
$$+\frac{16S_{-2,2}P_{596}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$
$$+\frac{16S_{-3,1}P_{597}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$
$$-\frac{16S_{3,1}P_{601}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$
$$+\frac{4S_4P_{616}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}+\frac{8S_{2,1}P_{632}}{9(N-1)N^3(1+N)^3(2+N)^3(3+N)}$$
$$-\frac{16S_{-2,1}P_{638}}{9(N-2)(N-1)N^3(1+N)^3(2+N)^3(3+N)}$$
$$-\frac{8\zeta_3P_{640}}{9(N-2)(N-1)N^3(1+N)^3(2+N)^3(3+N)}$$
$$-\frac{8S_3P_{642}}{81(N-2)(N-1)N^3(1+N)^3(2+N)^3(3+N)}$$
$$+\frac{P_{664}}{972(N-2)^2(N-1)N^6(1+N)^6(2+N)^5(3+N)}+\Biggl(\frac{8S_{2,1}P_{555}}{3N^2(1+N)^2(2+N)^2}$$
$$-\frac{32S_{-2,1}P_{584}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}-\frac{8S_3P_{612}}{27(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$
-\frac{8\zeta_3 P_{618}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}+\frac{4S_2P_{635}}{27(N-1)N^3(1+N)^3(2+N)^3(3+N)}
+\frac{2P_{661}}{243(N-2)^2(N-1)N^5(1+N)^5(2+N)^5(3+N)}-\frac{80\big(2+N+N^2\big)S_2^2}{N(1+N)(2+N)}
+\frac{8\big(382+47N+47N^2\big)S_4}{3N(1+N)(2+N)}+\frac{64\big(-130+7N+7N^2\big)S_{3,1}}{3N(1+N)(2+N)}+\frac{128\big(54+5N+5N^2\big)S_{-2,2}}{N(1+N)(2+N)}
+\frac{16\big(1378+113N+113N^2\big)S_{-3,1}}{3N(1+N)(2+N)}-\frac{32\big(1214+127N+127N^2\big)S_{-2,1,1}}{3N(1+N)(2+N)}
-\frac{80\big(2+N+N^2\big)S_{2,1,1}}{N(1+N)(2+N)}-\frac{96\big(2+N+N^2\big)\zeta_2^2}{5N(1+N)(2+N)}\Bigg)S_1+\Bigg(\frac{4S_2P_{581}}{9(N-1)N^2(1+N)^2(2+N)^2}
+\frac{P_{654}}{81(N-2)(N-1)N^4(1+N)^4(2+N)^4(3+N)}-\frac{8\big(-310+13N+13N^2\big)S_3}{3N(1+N)(2+N)}
-\frac{32\big(2+N+N^2\big)S_{2,1}}{N(1+N)(2+N)}+\frac{32\big(286+35N+35N^2\big)S_{-2,1}}{3N(1+N)(2+N)}+\frac{32\big(-166+19N+19N^2\big)\zeta_3}{3N(1+N)(2+N)}\Bigg)
\times S_1^2+\Bigg(-\frac{4P_{629}}{81(N-1)N^3(1+N)^3(2+N)^3}+\frac{136\big(2+N+N^2\big)S_2}{3N(1+N)(2+N)}\Bigg)S_1^3
+\Bigg(\frac{P_{656}}{27(N-2)(N-1)N^4(1+N)^4(2+N)^4(3+N)}+\frac{8\big(-898+7N+7N^2\big)S_3}{3N(1+N)(2+N)}
-\frac{128\big(72+7N+7N^2\big)S_{-2,1}}{N(1+N)(2+N)}-\frac{96\big(2+3N+3N^2\big)\zeta_3}{N(1+N)(2+N)}\Bigg)S_2-\frac{256\big(11+N+N^2\big)S_5}{3N(1+N)(2+N)}
+\Bigg(-\frac{8S_2P_{603}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}
+\frac{8S_1^2P_{606}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}
-\frac{8P_{655}}{9(N-2)^2(N-1)N^4(1+N)^4(2+N)^3(3+N)}
+\Bigg(-\frac{16P_{644}}{9(N-2)^2(N-1)N^3(1+N)^3(2+N)^3(3+N)}-\frac{48\big(18+N+N^2\big)S_2}{N(1+N)(2+N)}\Bigg)S_1
+\frac{16\big(178+17N+17N^2\big)S_1^3}{9N(1+N)(2+N)}-\frac{64\big(-2+35N+35N^2\big)S_3}{9N(1+N)(2+N)}+\frac{64\big(146+13N+13N^2\big)S_{2,1}}{N(1+N)(2+N)}
-\frac{128\big(2+N+N^2\big)S_{-2,1}}{3N(1+N)(2+N)}-\frac{1024\big(4-2N+N^2\big)\zeta_3}{N(1+N)(2+N)}\Bigg)S_{-2}+\Bigg(\frac{256\big(11+N+N^2\big)S_1}{3N(1+N)(2+N)}
+\frac{32P_{586}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}\Bigg)S_{-2}^2
+\Bigg(-\frac{16S_1P_{600}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}-\frac{16\big(490+41N+41N^2\big)S_1^2}{3N(1+N)(2+N)}
+\frac{8P_{639}}{9(N-2)(N-1)N^3(1+N)^3(2+N)^3(3+N)}+\frac{128\big(107+10N+10N^2\big)S_2}{3N(1+N)(2+N)}
$$

$$-\frac{32\big(-22+5N+5N^2\big)S_{-2}}{N(1+N)(2+N)}\Bigg)S_{-3}+\Bigg(\frac{8P_{602}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$
$$-\frac{16\big(198+19N+19N^2\big)S_1}{N(1+N)(2+N)}\Bigg)S_{-4}-\frac{16\big(-562+31N+31N^2\big)S_{-5}}{3N(1+N)(2+N)}+\frac{64\big(38+N+N^2\big)S_{2,3}}{N(1+N)(2+N)}$$
$$+\frac{160\big(86+7N+7N^2\big)S_{2,-3}}{3N(1+N)(2+N)}-\frac{64\big(134+13N+13N^2\big)S_{4,1}}{3N(1+N)(2+N)}-\frac{256\big(50+19N+N^2\big)S_{-2,3}}{3N(1+N)(2+N)}$$
$$+\frac{256\big(7+7N+N^2\big)S_{-4,1}}{N(1+N)(2+N)}-\frac{64\big(434+37N+37N^2\big)S_{2,1,-2}}{3N(1+N)(2+N)}-\frac{16\big(2+N+N^2\big)S_{2,2,1}}{3N(1+N)(2+N)}$$
$$-\frac{96\big(-46+N+N^2\big)S_{3,1,1}}{N(1+N)(2+N)}-\frac{64\big(118+11N+11N^2\big)S_{-2,1,-2}}{3N(1+N)(2+N)}$$
$$-\frac{2560\big(11+N+N^2\big)S_{-2,2,1}}{3N(1+N)(2+N)}-\frac{160\big(58+5N+5N^2\big)S_{-3,1,1}}{N(1+N)(2+N)}+\frac{160\big(2+N+N^2\big)S_{2,1,1,1}}{N(1+N)(2+N)}$$
$$+\frac{320\big(178+17N+17N^2\big)S_{-2,1,1,1}}{3N(1+N)(2+N)}-\frac{80\big(118-95N+N^2\big)\zeta_5}{N(1+N)(2+N)}\Bigg]\Bigg\}$$
$$+C_AT_F^2N_F^2\Bigg\{\frac{32S_1^3P_{556}}{81(N-1)N(1+N)^2(2+N)^2}+\frac{128S_{2,1}P_{557}}{27N^2(1+N)^2(2+N)^2}$$
$$-\frac{64S_3P_{559}}{81(N-1)N(1+N)^2(2+N)^2}-\frac{16S_2P_{611}}{81(N-1)N^2(1+N)^3(2+N)^3}$$
$$+\frac{8P_{660}}{243(N-2)(N-1)N^5(1+N)^5(2+N)^5(3+N)}+\frac{32\zeta_3P_{577}}{9(N-1)N^2(1+N)^2(2+N)^2}$$
$$+\Bigg(-\frac{32S_2P_{578}}{27(N-1)N^2(1+N)^2(2+N)^2}+\frac{8P_{650}}{243(N-1)N^4(1+N)^4(2+N)^4}$$
$$+\frac{32\big(110+19N+19N^2\big)S_3}{27N(1+N)(2+N)}+\frac{256\big(-1+N+N^2\big)S_{2,1}}{9N(1+N)(2+N)}-\frac{256S_{-2,1}}{3N(1+N)(2+N)}$$
$$-\frac{32\big(62+13N+13N^2\big)\zeta_3}{9N(1+N)(2+N)}\Bigg)S_1+\Bigg(\frac{16P_{615}}{81(N-1)N^2(1+N)^3(2+N)^3}$$
$$-\frac{8\big(14+19N+19N^2\big)S_2}{9N(1+N)(2+N)}\Bigg)S_1^2+\frac{28\big(2+N+N^2\big)S_1^4}{27N(1+N)(2+N)}+\frac{4\big(-34+31N+31N^2\big)S_2^2}{9N(1+N)(2+N)}$$
$$+\frac{56\big(-10+7N+7N^2\big)S_4}{9N(1+N)(2+N)}+\Bigg(-\frac{64S_1P_{568}}{9(N-1)N^2(1+N)^2(2+N)^2}$$
$$-\frac{32P_{631}}{81(N-2)(N-1)N^3(1+N)^2(2+N)^3(3+N)}-\frac{32(N-2)(3+N)S_1^2}{3N(1+N)(2+N)}$$
$$+\frac{32(N-2)(3+N)S_2}{3N(1+N)(2+N)}\Bigg)S_{-2}-\frac{128S_{-2}^2}{3N(1+N)(2+N)}+\Bigg(\frac{128P_{573}}{27(N-1)N^2(1+N)^2(2+N)^2}$$
$$+\frac{128\big(-1+N+N^2\big)S_1}{3N(1+N)(2+N)}\Bigg)S_{-3}-\frac{64\big(38+7N+7N^2\big)S_{-4}}{9N(1+N)(2+N)}-\frac{128\big(13+5N+5N^2\big)S_{3,1}}{9N(1+N)(2+N)}$$
$$-\frac{256(-3+5N)S_{-2,1}}{9N^2(1+N)(2+N)}+\frac{256S_{-2,2}}{3N(1+N)(2+N)}+\frac{512S_{-3,1}}{3N(1+N)(2+N)}$$

$$
-\frac{64\big(-16+N+N^2\big)S_{2,1,1}}{9N(1+N)(2+N)}\Bigg\}
$$

$$
+C_A^2T_FN_F\Bigg\{\frac{48(N-2)(3+N)\zeta_2^2P_{554}}{5(N-1)N^2(1+N)^2(2+N)^2}+\frac{S_2^2P_{561}}{9(N-1)N^2(1+N)^2(2+N)^2}
$$

$$
+\frac{S_1^4P_{562}}{27(N-1)N^2(1+N)^2(2+N)^2}+\frac{16S_{2,1,1}P_{569}}{9(N-1)N^2(1+N)^2(2+N)^2}
$$

$$
+\frac{8S_{3,1}P_{589}}{9(N-2)N^2(1+N)^2(2+N)^2(3+N)}-\frac{32S_{-2,1,1}P_{604}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}
$$

$$
+\frac{16S_{-2,2}P_{605}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}
$$

$$
+\frac{16S_{-3,1}P_{607}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}
$$

$$
-\frac{2S_4P_{617}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}-\frac{8S_{2,1}P_{628}}{27(N-1)N^3(1+N)^3(2+N)^3}
$$

$$
-\frac{32S_{-2,1}P_{637}}{9(N-2)(N-1)N^3(1+N)^3(2+N)^3(3+N)}
$$

$$
+\frac{4\zeta_3P_{643}}{9(N-2)(N-1)^2N^3(1+N)^3(2+N)^3(3+N)}
$$

$$
+\frac{8S_3P_{649}}{81(N-2)(N-1)^2N^3(1+N)^3(2+N)^3(3+N)}
$$

$$
-\frac{4P_{665}}{243(N-2)(N-1)^2N^6(1+N)^6(2+N)^6(3+N)}+\Bigg(-\frac{16S_{2,1}P_{558}}{9N^2(1+N)^2(2+N)^2}
$$

$$
+\frac{16S_{-2,1}P_{598}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}-\frac{32\big(-70+N+N^2\big)S_{3,1}}{3N(1+N)(2+N)}
$$

$$
+\frac{8\zeta_3P_{613}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}-\frac{48\big(2+N+N^2\big)S_4}{N(1+N)(2+N)}
$$

$$
-\frac{8S_3P_{614}}{27(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}+\frac{8S_2P_{634}}{27(N-1)^2N^3(1+N)^3(2+N)^3}
$$

$$
-\frac{4P_{662}}{243(N-2)(N-1)^2N^5(1+N)^5(2+N)^5(3+N)}-\frac{124\big(2+N+N^2\big)S_2^2}{3N(1+N)(2+N)}
$$

$$
-\frac{32\big(226+5N+5N^2\big)S_{-2,2}}{3N(1+N)(2+N)}-\frac{80\big(106+5N+5N^2\big)S_{-3,1}}{3N(1+N)(2+N)}-\frac{16\big(2+N+N^2\big)S_{2,1,1}}{3N(1+N)(2+N)}
$$

$$
-\frac{32\big(-374+5N+5N^2\big)S_{-2,1,1}}{3N(1+N)(2+N)}+\frac{96\big(2+N+N^2\big)\zeta_2^2}{5N(1+N)(2+N)}\Bigg)S_1
$$

$$
+\Bigg(\frac{2S_2P_{583}}{9(N-1)N^2(1+N)^2(2+N)^2}-\frac{2P_{653}}{81(N-1)^2N^4(1+N)^4(2+N)^4}
$$

$$
-\frac{8\big(154+29N+29N^2\big)S_3}{3N(1+N)(2+N)}+\frac{32\big(2+N+N^2\big)S_{2,1}}{3N(1+N)(2+N)}+\frac{32\big(-22+3N+3N^2\big)S_{-2,1}}{N(1+N)(2+N)}
$$

$$
+\frac{16\big(190+17N+17N^2\big)\zeta_3}{3N(1+N)(2+N)}\Bigg)S_1^2+\Bigg(-\frac{8P_{633}}{81(N-1)^2N^3(1+N)^3(2+N)^3}
$$

$$
+\frac{128\big(2+N+N^2\big)S_2}{9N(1+N)(2+N)}\Bigg)S_1^3-\frac{4\big(2+N+N^2\big)S_1^5}{3N(1+N)(2+N)}+\Bigg(\frac{2P_{652}}{81(N-1)^2N^4(1+N)^4(2+N)^4}
$$

$$
-\frac{8\big(-658+31N+31N^2\big)S_3}{9N(1+N)(2+N)}+\frac{128\big(27+2N+2N^2\big)S_{-2,1}}{N(1+N)(2+N)}+\frac{48(N-1)\zeta_3}{N(1+N)}\Bigg)S_2
$$

$$
+\frac{32\big(26+7N+7N^2\big)S_5}{3N(1+N)(2+N)}+\Bigg[\frac{8S_1^2P_{575}}{3(N-1)N^2(1+N)^2(2+N)^2}-\frac{8S_2P_{576}}{3(N-1)N^2(1+N)^2(2+N)^2}
$$

$$
+\frac{8P_{658}}{81(N-2)(N-1)^2N^4(1+N)^4(2+N)^4(3+N)}
$$

$$
+\Bigg(\frac{16P_{646}}{9(N-2)(N-1)^2N^3(1+N)^3(2+N)^3(3+N)}-\frac{16\big(-50+11N+11N^2\big)S_2}{3N(1+N)(2+N)}\Bigg)S_1
$$

$$
-\frac{16\big(26+N+N^2\big)S_1^3}{3N(1+N)(2+N)}+\frac{128\big(5+4N+4N^2\big)S_3}{3N(1+N)(2+N)}-\frac{1024\big(11+N+N^2\big)S_{2,1}}{3N(1+N)(2+N)}
$$

$$
+\frac{128\big(2+N+N^2\big)S_{-2,1}}{3N(1+N)(2+N)}+\frac{128\big(7-4N+2N^2\big)\zeta_3}{N(1+N)(2+N)}\Bigg]S_{-2}
$$

$$
+\Bigg(-\frac{8P_{599}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}-\frac{16\big(18+N+N^2\big)S_1}{N(1+N)(2+N)}\Bigg)S_{-2}^2
$$

$$
+\Bigg(-\frac{8S_1P_{610}}{3(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}-\frac{16\big(-122+29N+29N^2\big)S_1^2}{3N(1+N)(2+N)}
$$

$$
-\frac{16P_{647}}{27(N-2)(N-1)^2N^3(1+N)^3(2+N)^3(3+N)}-\frac{32\big(64+9N+9N^2\big)S_2}{N(1+N)(2+N)}
$$

$$
+\frac{32\big(-10+13N+13N^2\big)S_{-2}}{3N(1+N)(2+N)}\Bigg)S_{-3}+\Bigg(-\frac{8P_{609}}{9(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}
$$

$$
+\frac{128\big(13+2N+2N^2\big)S_1}{N(1+N)(2+N)}\Bigg)S_{-4}+\frac{16\big(-238+N+N^2\big)S_{-5}}{3N(1+N)(2+N)}-\frac{48\big(14+N+N^2\big)S_{2,3}}{N(1+N)(2+N)}
$$

$$
+\frac{128\big(-34+N+N^2\big)S_{2,-3}}{3N(1+N)(2+N)}+\frac{16\big(146+19N+19N^2\big)S_{4,1}}{3N(1+N)(2+N)}-\frac{128\big(-38-7N+2N^2\big)S_{-2,3}}{3N(1+N)(2+N)}
$$

$$
-\frac{32\big(142+71N+35N^2\big)S_{-4,1}}{3N(1+N)(2+N)}+\frac{1024\big(11+N+N^2\big)S_{2,1,-2}}{3N(1+N)(2+N)}+\frac{176\big(2+N+N^2\big)S_{2,2,1}}{3N(1+N)(2+N)}
$$

$$
-\frac{1152S_{3,1,1}}{N(1+N)(2+N)}-\frac{64\big(-22+7N+7N^2\big)S_{-2,1,-2}}{3N(1+N)(2+N)}+\frac{64\big(178+17N+17N^2\big)S_{-2,2,1}}{3N(1+N)(2+N)}
$$

$$
+\frac{32\big(130+17N+17N^2\big)S_{-3,1,1}}{N(1+N)(2+N)}-\frac{64\big(322+17N+17N^2\big)S_{-2,1,1,1}}{3N(1+N)(2+N)}
$$

$$
+\frac{160\big(14-11N+N^2\big)\zeta_5}{N(1+N)(2+N)}\Bigg\}
$$

$$
+C_F^2T_FN_F\Bigg\{\frac{S_1^4P_{551}}{9N^2(1+N)^2(2+N)}-\frac{64S_{-2,2}P_{566}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}
$$

$$
-\frac{32S_{-3,1}P_{570}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}+\frac{64S_{-2,1,1}P_{571}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}
$$

$$+\frac{S_2^2 P_{552}}{3N^2(1+N)^2(2+N)}+\frac{32S_{3,1}P_{585}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}-\frac{16S_{2,1}P_{587}}{3N^3(1+N)^3(2+N)^2}$$
$$-\frac{2S_4P_{590}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}+\frac{32S_{-2,1}P_{621}}{3(N-2)N^3(1+N)^3(2+N)^2(3+N)}$$
$$+\frac{4S_3P_{626}}{9(N-2)N^3(1+N)^3(2+N)^2(3+N)}+\frac{P_{657}}{12(N-2)^2N^6(1+N)^6(2+N)(3+N)}$$
$$+\frac{16\zeta_3P_{624}}{3(N-2)N^3(1+N)^3(2+N)^2(3+N)}+\Biggl(-\frac{64S_{-2,1}P_{567}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}$$
$$+\frac{16\zeta_3P_{591}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}-\frac{8S_3P_{592}}{9(N-2)N^2(1+N)^2(2+N)^2(3+N)}$$
$$+\frac{2S_2P_{588}}{3N^3(1+N)^3(2+N)^2}-\frac{2P_{651}}{3(N-2)N^5(1+N)^5(2+N)^2(3+N)}-\frac{188\bigl(2+N+N^2\bigr)S_2^2}{3N(1+N)(2+N)}$$
$$+\frac{32\bigl(-20+12N+18N^2+3N^3\bigr)S_{2,1}}{3N(1+N)(2+N)^2}-\frac{32\bigl(-262+13N+13N^2\bigr)S_{3,1}}{3N(1+N)(2+N)}$$
$$-\frac{8\bigl(766+95N+95N^2\bigr)S_4}{3N(1+N)(2+N)}+\frac{128\bigl(-14+N+N^2\bigr)S_{-2,2}}{N(1+N)(2+N)}+\frac{64\bigl(-70+13N+13N^2\bigr)S_{-3,1}}{3N(1+N)(2+N)}$$
$$+\frac{256\bigl(2+N+N^2\bigr)S_{2,1,1}}{3N(1+N)(2+N)}+\frac{128\bigl(98+N+N^2\bigr)S_{-2,1,1}}{3N(1+N)(2+N)}\Biggr)S_1+\Biggl(\frac{2S_2P_{560}}{3N^2(1+N)^2(2+N)^2}$$
$$+\frac{P_{619}}{3N^4(1+N)^4(2+N)^2}-\frac{48\bigl(18+N+N^2\bigr)S_3}{N(1+N)(2+N)}+\frac{64\bigl(2+N+N^2\bigr)S_{2,1}}{3N(1+N)(2+N)}$$
$$-\frac{256\bigl(14+N+N^2\bigr)S_{-2,1}}{3N(1+N)(2+N)}-\frac{128\bigl(-14+5N+5N^2\bigr)\zeta_3}{3N(1+N)(2+N)}\Biggr)S_1^2+\Biggl(-\frac{2P_{579}}{9N^3(1+N)^3(2+N)}$$
$$+\frac{472\bigl(2+N+N^2\bigr)S_2}{9N(1+N)(2+N)}\Biggr)S_1^3-\frac{20\bigl(2+N+N^2\bigr)S_1^5}{3N(1+N)(2+N)}+\Biggl(\frac{P_{623}}{3N^4(1+N)^4(2+N)^2}$$
$$+\frac{16\bigl(1462+83N+83N^2\bigr)S_3}{9N(1+N)(2+N)}+\frac{256\bigl(38+N+N^2\bigr)S_{-2,1}}{3N(1+N)(2+N)}+\frac{192\bigl(2+N+N^2\bigr)\zeta_3}{N(1+N)(2+N)}\Biggr)S_2$$
$$+\frac{64\bigl(82+17N+17N^2\bigr)S_5}{3N(1+N)(2+N)}+\Biggl(\frac{8P_{645}}{3(N-2)^2N^4(1+N)^4(2+N)^2(3+N)}$$
$$+\Biggl(\frac{32P_{620}}{3(N-2)N^3(1+N)^3(2+N)^2(3+N)}-\frac{640\bigl(2+N+N^2\bigr)S_2}{3N(1+N)(2+N)}\Biggr)S_1$$
$$+\frac{64\bigl(-27+30N+22N^2+6N^3\bigr)S_1^2}{3N^2(1+N)^2(2+N)}-\frac{64\bigl(-7+3N-N^2+2N^3\bigr)S_2}{N^2(1+N)^2(2+N)}$$
$$+\frac{640\bigl(2+N+N^2\bigr)S_1^3}{9N(1+N)(2+N)}+\frac{512\bigl(5+7N+7N^2\bigr)S_3}{9N(1+N)(2+N)}-\frac{3072S_{2,1}}{N(1+N)(2+N)}$$
$$-\frac{128\bigl(2+N+N^2\bigr)S_{-2,1}}{3N(1+N)(2+N)}+\frac{512\bigl(9-4N+2N^2\bigr)\zeta_3}{N(1+N)(2+N)}\Biggr)S_{-2}+\Biggl(-\frac{256\bigl(8+N+N^2\bigr)S_1}{3N(1+N)(2+N)}$$
$$-\frac{32P_{574}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}\Biggr)S_{-2}^2+\Biggl(\frac{32S_1P_{565}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}$$

$$
-\frac{16P_{622}}{3(N-2)N^3(1+N)^3(2+N)^2(3+N)}-\frac{64\big(-2+11N+11N^2\big)S_1^2}{3N(1+N)(2+N)}
-\frac{64\big(74+N+N^2\big)S_2}{3N(1+N)(2+N)}+\frac{512\big(1+N+N^2\big)S_{-2}}{N(1+N)(2+N)}\Bigg)S_{-3}+\Bigg(\frac{512S_1}{N(1+N)(2+N)}
-\frac{32P_{572}}{3(N-2)N^2(1+N)^2(2+N)(3+N)}\Bigg)S_{-4}+\frac{32\big(-10+67N+67N^2\big)S_{-5}}{3N(1+N)(2+N)}
-\frac{128\big(20+N+N^2\big)S_{2,3}}{N(1+N)(2+N)}+\frac{192(N-2)(3+N)S_{2,-3}}{N(1+N)(2+N)}+\frac{32\big(98+13N+13N^2\big)S_{4,1}}{N(1+N)(2+N)}
-\frac{64\big(2-47N+25N^2\big)S_{-2,3}}{3N(1+N)(2+N)}-\frac{32\big(20+13N+13N^2+12N^3\big)S_{2,1,1}}{3N^2(1+N)^2(2+N)}
-\frac{64\big(-2+27N+3N^2\big)S_{-4,1}}{N(1+N)(2+N)}-\frac{128\big(-70+N+N^2\big)S_{2,1,-2}}{3N(1+N)(2+N)}-\frac{160\big(2+N+N^2\big)S_{2,2,1}}{3N(1+N)(2+N)}
-\frac{4608S_{3,1,1}}{N(1+N)(2+N)}-\frac{128\big(-22+N+N^2\big)S_{-2,1,-2}}{3N(1+N)(2+N)}-\frac{64\big(-142+N+N^2\big)S_{-3,1,1}}{3N(1+N)(2+N)}
+\frac{3072S_{-2,2,1}}{N(1+N)(2+N)}-\frac{160\big(2+N+N^2\big)S_{2,1,1,1}}{N(1+N)(2+N)}-\frac{128\big(146+N+N^2\big)S_{-2,1,1,1}}{3N(1+N)(2+N)}
+\frac{48(N-1)\big(-2+3N+3N^2\big)\zeta_2^2}{5N^2(1+N)^2}-\frac{160\big(-66+49N+N^2\big)\zeta_5}{N(1+N)(2+N)}\Bigg\}. \qquad (720)
$$

with

$$
\begin{aligned}
P_{551} &= -249N^4-346N^3-65N^2+336N+452, && (721)\\
P_{552} &= -117N^4+14N^3-109N^2-256N+244, && (722)\\
P_{553} &= 2N^4+N^3+6N^2+34N+21, && (723)\\
P_{554} &= 3N^4+6N^3+7N^2+4N+4, && (724)\\
P_{555} &= 17N^5+14N^4-45N^3-74N^2-68N-24, && (725)\\
P_{556} &= 19N^5+12N^4-48N^3-29N^2-112N-58, && (726)\\
P_{557} &= 19N^5+40N^4+37N^3+44N^2-32N-36, && (727)\\
P_{558} &= 44N^5+149N^4+221N^3+160N^2-10N-36, && (728)\\
P_{559} &= 53N^5+33N^4+12N^3+74N^2-68N+112, && (729)\\
P_{560} &= 177N^5+500N^4+317N^3-134N^2-692N-712, && (730)\\
P_{561} &= -305N^6-531N^5+1305N^4+2359N^3+2600N^2+1676N+1248, && (731)\\
P_{562} &= -293N^6+201N^5+1473N^4+595N^3+1628N^2+380N-240, && (732)\\
P_{563} &= 3N^6+9N^5-5N^4-25N^3-14N^2-16, && (733)\\
P_{564} &= 5N^6-21N^5-147N^4-181N^3+100N^2+220N+120, && (734)\\
P_{565} &= 6N^6+24N^5-37N^4-22N^3-93N^2+210N+144, && (735)\\
P_{566} &= 6N^6+36N^5-65N^4-146N^3+81N^2+228N+468, && (736)\\
P_{567} &= 6N^6+48N^5-31N^4-154N^3-51N^2-66N+720, && (737)\\
P_{568} &= 8N^6+15N^5-14N^4-17N^3+60N^2+44N-24, && (738)\\
P_{569} &= 11N^6-3N^5-132N^4-193N^3+187N^2+298N+120, && (739)\\
P_{570} &= 12N^6+60N^5-139N^4-238N^3+159N^2+618N+720, && (740)
\end{aligned}
$$

$$P_{571} = 12N^6 + 84N^5 - 121N^4 - 346N^3 + 165N^2 + 294N + 1152, \quad (741)$$
$$P_{572} = 15N^6 + 83N^5 + 81N^4 - 315N^3 - 482N^2 - 370N - 12, \quad (742)$$
$$P_{573} = 19N^6 + 27N^5 + 12N^4 + 52N^3 + 41N^2 + 47N + 18, \quad (743)$$
$$P_{574} = 21N^6 + 105N^5 - 9N^4 - 425N^3 - 368N^2 + 84N + 432, \quad (744)$$
$$P_{575} = 23N^6 + 13N^5 - 3N^4 + 299N^3 + 484N^2 - 72N + 24, \quad (745)$$
$$P_{576} = 35N^6 - 3N^5 - 177N^4 - 65N^3 + 298N^2 + 328N + 160, \quad (746)$$
$$P_{577} = 42N^6 + 81N^5 - 61N^4 - 53N^3 + 383N^2 + 228N - 92, \quad (747)$$
$$P_{578} = 67N^6 + 102N^5 - 12N^4 - 35N^3 - 304N^2 - 106N + 72, \quad (748)$$
$$P_{579} = 127N^6 + 375N^5 + 165N^4 + 737N^3 + 1668N^2 + 1960N + 872, \quad (749)$$
$$P_{580} = 206N^6 + 522N^5 + 777N^4 - 556N^3 - 3197N^2 - 884N - 180, \quad (750)$$
$$P_{581} = 349N^6 + 1029N^5 + 588N^4 - 1499N^3 - 3295N^2 - 616N + 420, \quad (751)$$
$$P_{582} = 368N^6 + 1404N^5 + 1419N^4 - 190N^3 - 3353N^2 - 2636N - 324, \quad (752)$$
$$P_{583} = 497N^6 - 9N^5 - 1449N^4 - 1027N^3 - 4304N^2 - 1388N - 96, \quad (753)$$
$$P_{584} = 3N^7 - 260N^6 - 461N^5 + 1640N^4 + 2358N^3 - 1832N^2 - 5040N - 7056, \quad (754)$$
$$P_{585} = 12N^7 + 74N^6 - 315N^5 - 1024N^4 + 829N^3 + 4244N^2 + 2996N + 5568, \quad (755)$$
$$P_{586} = 15N^7 + 155N^6 + 292N^5 - 489N^4 - 1527N^3 - 594N^2 + 916N + 1128, \quad (756)$$
$$P_{587} = 16N^7 + 36N^6 - 89N^5 - 342N^4 - 395N^3 - 246N^2 - 124N - 24, \quad (757)$$
$$P_{588} = 23N^7 - 83N^6 - 933N^5 - 1405N^4 + 670N^3 + 2664N^2 + 2616N + 1072, \quad (758)$$
$$P_{589} = 85N^7 + 701N^6 + 305N^5 - 1809N^4 - 1686N^3 + 412N^2 + 8520N + 27360, \quad (759)$$
$$P_{590} = 93N^7 + 801N^6 - 2057N^5 - 9261N^4 + 4880N^3 + 31656N^2 + 23760N + 57456, \quad (760)$$
$$P_{591} = 147N^7 + 483N^6 - 1091N^5 - 3551N^4 + 844N^3 + 9776N^2 + 7552N + 6384, \quad (761)$$
$$P_{592} = 213N^7 + 985N^6 - 2533N^5 - 10357N^4 + 2972N^3 + 33504N^2 + 27296N + 33936, \quad (762)$$
$$P_{593} = 3N^8 + 24N^7 - 16N^6 - 204N^5 + 71N^4 + 588N^3 - 58N^2 - 336N - 648, \quad (763)$$
$$P_{594} = 3N^8 + 24N^7 + 34N^6 - 84N^5 - 269N^4 - 12N^3 + 592N^2 + 144N - 1008, \quad (764)$$
$$P_{595} = 9N^8 + 512N^7 + 186N^6 - 4406N^5 - 1547N^4 + 9986N^3 + 9116N^2 + 1936N - 13488, \quad (765)$$
$$P_{596} = 9N^8 + 512N^7 + 244N^6 - 4234N^5 - 2009N^4 + 8834N^3 + 9964N^2 + 3096N - 11808, \quad (766)$$
$$P_{597} = 9N^8 + 512N^7 + 302N^6 - 4062N^5 - 2471N^4 + 7682N^3 + 10812N^2 + 4256N - 10128, \quad (767)$$
$$P_{598} = 20N^8 - 34N^7 - 295N^6 + 167N^5 + 885N^4 + 243N^3 - 726N^2 - 1452N + 936, \quad (768)$$
$$P_{599} = 27N^8 + 174N^7 + 76N^6 - 902N^5 - 931N^4 + 1008N^3 + 1380N^2 + 32N - 1632, \quad (769)$$
$$P_{600} = 30N^8 + 493N^7 + 213N^6 - 3539N^5 - 1915N^4 + 5886N^3 + 6880N^2 + 3280N - 6720, \quad (770)$$
$$P_{601} = 33N^8 + 194N^7 - 666N^6 - 1212N^5 + 2993N^4 + 5026N^3 - 440N^2 + 6888N - 15120, \quad (771)$$
$$P_{602} = 39N^8 + 4N^7 + 188N^6 + 886N^5 - 895N^4 - 2378N^3 - 6316N^2 - 5352N + 4608, \quad (772)$$

$$P_{603} = 39N^8 + 200N^7 - 1006N^5 - 1299N^4 + 166N^3 + 3348N^2 + 1816N + 1344, \quad (773)$$
$$P_{604} = 60N^8 - 96N^7 - 793N^6 + 835N^5 + 2809N^4 - 603N^3 - 4488N^2 - 4252N + 2688, \quad (774)$$
$$P_{605} = 66N^8 - 106N^7 - 877N^6 + 933N^5 + 3225N^4 - 427N^3 - 5330N^2 - 5284N + 2040, \quad (775)$$
$$P_{606} = 69N^8 + 176N^7 - 576N^6 - 1906N^5 - 521N^4 + 6258N^3 + 7340N^2 - 4888N - 1344, \quad (776)$$
$$P_{607} = 72N^8 - 116N^7 - 961N^6 + 1031N^5 + 3641N^4 - 251N^3 - 6172N^2 - 6316N + 1392, \quad (777)$$
$$P_{608} = 124N^8 + 481N^7 + 609N^6 + 181N^5 - 759N^4 - 1226N^3 - 730N^2 - 192N - 216, \quad (778)$$
$$P_{609} = 143N^8 - 454N^7 - 2349N^6 + 3611N^5 + 12288N^4 + 4331N^3 - 27470N^2 - 32364N - 7272, \quad (779)$$
$$P_{610} = 176N^8 - 10N^7 - 2127N^6 - 153N^5 + 5809N^4 + 3419N^3 - 2982N^2 - 8524N + 1320, \quad (780)$$
$$P_{611} = 200N^8 + 1179N^7 + 952N^6 - 2962N^5 - 4454N^4 - 1081N^3 + 530N^2 + 236N + 216, \quad (781)$$
$$P_{612} = 205N^8 + 646N^7 + 7902N^6 + 6508N^5 - 42471N^4 - 77522N^3 + 3764N^2 - 16824N + 137376, \quad (782)$$
$$P_{613} = 233N^8 + 392N^7 - 3288N^6 - 4054N^5 + 3891N^4 + 9254N^3 + 28540N^2 + 22920N - 10656, \quad (783)$$
$$P_{614} = 263N^8 + 602N^7 - 4467N^6 - 6067N^5 + 9090N^4 + 17633N^3 + 33166N^2 + 27756N - 35352, \quad (784)$$
$$P_{615} = 274N^8 + 654N^7 - 544N^6 - 1793N^5 - 1288N^4 - 1853N^3 - 782N^2 - 68N + 216, \quad (785)$$
$$P_{616} = 434N^8 + 1964N^7 - 5853N^6 - 14263N^5 + 25449N^4 + 54611N^3 - 15746N^2 + 19140N - 129672, \quad (786)$$
$$P_{617} = 485N^8 + 1172N^7 - 6414N^6 - 8680N^5 + 23145N^4 + 38828N^3 - 5048N^2 - 6192N - 96048, \quad (787)$$
$$P_{618} = 641N^8 + 1412N^7 - 8862N^6 - 11728N^5 + 24861N^4 + 49412N^3 + 9640N^2 + 8784N - 51120, \quad (788)$$
$$P_{619} = -205N^9 - 474N^8 + 744N^7 + 3298N^6 + 5881N^5 + 10528N^4 + 17388N^3 + 20760N^2 + 13184N + 3232, \quad (789)$$
$$P_{620} = 13N^9 + 88N^8 + 136N^7 - 122N^6 - 177N^5 - 738N^4 - 2500N^3 - 2060N^2 - 1984N - 1776, \quad (790)$$
$$P_{621} = 35N^9 - 40N^8 - 760N^7 - 28N^6 + 3417N^5 + 2604N^4 - 1580N^3 - 2032N^2 + 928N + 768, \quad (791)$$
$$P_{622} = 71N^9 + 84N^8 - 660N^7 + 36N^6 + 3113N^5 + 1232N^4 - 4772N^3 - 8240N^2 - 3968N - 2112, \quad (792)$$
$$P_{623} = 93N^9 + 1002N^8 + 5144N^7 + 12790N^6 + 13999N^5 + 2456N^4 - 9140N^3$$

$$-11528N^2 - 6880N - 1632, \quad (793)$$

$$P_{624} = 114N^9 + 1125N^8 + 2886N^7 - 2796N^6 - 16498N^5 - 15241N^4 + 830N^3 +6052N^2 + 6248N + 4992, \quad (794)$$

$$P_{625} = 140N^9 + 1139N^8 + 2790N^7 + 1892N^6 - 1836N^5 - 4153N^4 - 3614N^3 -1902N^2 - 720N - 648, \quad (795)$$

$$P_{626} = 293N^9 + 2208N^8 + 754N^7 - 7064N^6 + 1245N^5 + 38896N^4 + 54988N^3 +40360N^2 + 512N - 6816, \quad (796)$$

$$P_{627} = 388N^9 + 2233N^8 + 4500N^7 + 4324N^6 + 2124N^5 - 3167N^4 - 8164N^3 -6414N^2 - 2088N - 648, \quad (797)$$

$$P_{628} = 1067N^9 + 4674N^8 + 6232N^7 + 2504N^6 - 2267N^5 - 3250N^4 + 6992N^3 +17600N^2 + 13680N + 3744, \quad (798)$$

$$P_{629} = 1721N^9 + 10404N^8 + 25600N^7 + 22820N^6 - 13643N^5 - 11908N^4 +45842N^3 + 52772N^2 + 23496N + 4464, \quad (799)$$

$$P_{630} = 30N^{10} + 177N^9 - 26N^8 - 932N^7 - 534N^6 + 589N^5 - 282N^4 - 314N^3 +2852N^2 + 2520N + 144, \quad (800)$$

$$P_{631} = 191N^{10} + 663N^9 - 914N^8 - 3481N^7 + 3013N^6 + 7856N^5 - 9352N^4 -14488N^3 + 5280N^2 + 6048N - 5184, \quad (801)$$

$$P_{632} = 383N^{10} + 2370N^9 + 5056N^8 + 4994N^7 + 5029N^6 + 5516N^5 + 2420N^4 +5360N^3 + 13992N^2 + 13344N + 3744, \quad (802)$$

$$P_{633} = 764N^{10} + 436N^9 - 4817N^8 + 1183N^7 + 16949N^6 + 12811N^5 +9684N^4 + 4650N^3 + 2836N^2 - 2160N - 864, \quad (803)$$

$$P_{634} = 1427N^{10} + 2932N^9 - 2816N^8 + 1429N^7 + 18422N^6 + 15883N^5 +15795N^4 + 6756N^3 - 6620N^2 - 12456N - 4464, \quad (804)$$

$$P_{635} = 1471N^{10} + 17463N^9 + 74204N^8 + 138364N^7 + 86603N^6 - 47561N^5 -29066N^4 + 91534N^3 + 81060N^2 + 9720N - 5616, \quad (805)$$

$$P_{636} = 19N^{11} + 134N^{10} - 120N^9 - 1514N^8 + 237N^7 + 5580N^6 - 1912N^5 -7208N^4 + 4848N^3 + 6464N^2 - 1920N - 17280, \quad (806)$$

$$P_{637} = 42N^{11} - 21N^{10} - 902N^9 - 1743N^8 - 123N^7 + 3687N^6 + 5601N^5 +3531N^4 + 6320N^3 + 14040N^2 + 13488N + 5040, \quad (807)$$

$$P_{638} = 90N^{11} + 399N^{10} - 448N^9 + 2160N^8 + 17304N^7 + 9873N^6 - 29826N^5 -27432N^4 - 1208N^3 - 7968N^2 - 36096N - 16704, \quad (808)$$

$$P_{639} = 240N^{11} + 633N^{10} - 1246N^9 + 5868N^8 + 31038N^7 - 5913N^6 - 104064N^5 -90228N^4 + 28888N^3 + 118368N^2 + 64800N + 27648, \quad (809)$$

$$P_{640} = 339N^{11} + 4164N^{10} + 17783N^9 - 2526N^8 - 133535N^7 - 158392N^6 +173869N^5 + 322418N^4 - 27256N^3 - 251632N^2 - 232368N - 113760, \quad (810)$$

$$P_{641} = 1565N^{11} + 11465N^{10} + 30742N^9 + 35334N^8 + 10185N^7 - 2331N^6 +34756N^5 + 92380N^4 + 111968N^3 + 73104N^2 + 41472N + 15552, \quad (811)$$

$$P_{642} = 4724N^{11} + 32993N^{10} + 49126N^9 - 60426N^8 - 161406N^7 - 29073N^6 +68536N^5 + 87850N^4 + 173588N^3 - 14280N^2 + 92016N + 95040, \quad (812)$$

$$P_{643} = 39N^{12} + 1656N^{11} + 7771N^{10} - 9028N^9 - 78031N^8 - 36088N^7 + 227761N^6$$

$$+194300N^5-257892N^4-310104N^3-149632N^2+33696N+71424, \quad (813)$$

$$P_{644} = 63N^{12}-381N^{11}-2219N^{10}+1918N^9+14631N^8-6291N^7-35967N^6 +62346N^5+83132N^4-143680N^3-160656N^2-10464N-6336, \quad (814)$$

$$P_{645} = 153N^{12}+49N^{11}-1696N^{10}+126N^9+5857N^8-3075N^7-10674N^6 +19412N^5+39320N^4-14912N^3-43712N^2-41344N-19200, \quad (815)$$

$$P_{646} = 226N^{12}+579N^{11}-1701N^{10}-2617N^9+8109N^8+1923N^7 -33361N^6-11565N^5+57183N^4+42280N^3-7200N^2-6768N-5616, \quad (816)$$

$$P_{647} = 430N^{12}+2883N^{11}+2713N^{10}-8201N^9-3396N^8+3522N^7-44182N^6 -83352N^5-43465N^4+53360N^3+113496N^2+99072N+31536, \quad (817)$$

$$P_{648} = 651N^{12}+4796N^{11}+12675N^{10}+13520N^9+6837N^8+19740N^7+57489N^6 +94024N^5+124188N^4+127120N^3+85008N^2+46656N +15552, \quad (818)$$

$$P_{649} = 1973N^{12}+8607N^{11}-2581N^{10}-29530N^9-4464N^8-5301N^7-29117N^6 +129132N^5+193261N^4-30332N^3+31872N^2-54864N-63504, \quad (819)$$

$$P_{650} = 5399N^{12}+21717N^{11}-8182N^{10}-141466N^9-174693N^8+104127N^7 +435784N^6+484686N^5+327700N^4+206368N^3+101088N^2+31968N +5184, \quad (820)$$

$$P_{651} = 254N^{13}+807N^{12}-2683N^{11}-10266N^{10}-3135N^9+52217N^8+158159N^7 +189724N^6+56845N^5-102970N^4-172072N^3-143952N^2-66544N -12576, \quad (821)$$

$$P_{652} = 5459N^{13}+43255N^{12}+82772N^{11}-53316N^{10}-306641N^9-409197N^8 -673052N^7-1415068N^6-1869746N^5-1266354N^4-356752N^3+99792N^2 +139104N+38880, \quad (822)$$

$$P_{653} = 13789N^{13}+50057N^{12}-25496N^{11}-233268N^{10}-154315N^9-125559N^8 -1154968N^7-2581772N^6-2738026N^5-1636170N^4-717392N^3 -119664N^2+52704N+7776, \quad (823)$$

$$P_{654} = -12043N^{14}-178942N^{13}-923869N^{12}-1714868N^{11}+1262555N^{10} +9673170N^9+13425517N^8+4916752N^7-5024960N^6-9988720N^5 -15826400N^4-20182656N^3-16420608N^2-7236864N-1306368, \quad (824)$$

$$P_{655} = 205N^{14}-1070N^{13}-6997N^{12}+9444N^{11}+57515N^{10}-20678N^9 -188539N^8+10016N^7+350968N^6-25824N^5-496784N^4+19520N^3 +470400N^2+418560N+156672, \quad (825)$$

$$P_{656} = 9429N^{14}+70894N^{13}+159471N^{12}-47508N^{11}-712173N^{10}-1057954N^9 -675639N^8-551176N^7-224640N^6+1703952N^5+3636768N^4 +4242688N^3+3770112N^2+1951488N+380160, \quad (826)$$

$$P_{657} = 617N^{15}+1213N^{14}+3647N^{13}-5135N^{12}-62711N^{11}-59557N^{10} +245733N^9+472003N^8-3502N^7-620780N^6-330504N^5+457568N^4 +783264N^3+552896N^2+210048N+34560, \quad (827)$$

$$P_{658} = 3805N^{15}+25467N^{14}+31782N^{13}-102029N^{12}-229612N^{11}-7783N^{10} -46240N^9-603459N^8+610299N^7+3183968N^6+1748734N^5-3156692N^4$$

$$
\begin{aligned}
& -5185536N^3-3830112N^2-1561248N-212544, && (828)\\
P_{659} &= 1538N^{16}+6349N^{15}-4604N^{14}-4419N^{13}+247590N^{12}+595697N^{11}\\
&-155918N^{10}-1529965N^9-660880N^8+802290N^7-3242454N^6\\
&-12852560N^5-19050600N^4-16010208N^3-9510048N^2-4292352N\\
&-1026432, && (829)\\
P_{660} &= 4920N^{17}+59829N^{16}+268187N^{15}+460995N^{14}-275823N^{13}-2426947N^{12}\\
&-3950033N^{11}-2689715N^{10}-806789N^9-436062N^8+2361354N^7\\
&+9732268N^6+15162440N^5+13427504N^4+7410144N^3+2426688N^2\\
&+328320N-20736, && (830)\\
P_{661} &= 32968N^{18}+200687N^{17}-76724N^{16}-2085743N^{15}+226312N^{14}\\
&+17762935N^{13}+21073412N^{12}-46005765N^{11}-129344416N^{10}-84661090N^9\\
&+89668640N^8+206886704N^7+116843424N^6-124546688N^5-362045568N^4\\
&-451275264N^3-326343168N^2-127858176N-20404224, && (831)\\
P_{662} &= 34763N^{18}+253932N^{17}+235627N^{16}-2269948N^{15}-5497382N^{14}\\
&+4841882N^{13}+30639918N^{12}+32530946N^{11}-9999541N^{10}-49961642N^9\\
&-76271245N^8-149423714N^7-245858156N^6-261281936N^5\\
&-186926496N^4-92761632N^3-31078080N^2-7713792N-1306368, && (832)\\
P_{663} &= -10701N^{20}-117180N^{19}-383698N^{18}+362182N^{17}+4985840N^{16}\\
&+7991006N^{15}-14095594N^{14}-52618222N^{13}-4462979N^{12}+154482966N^{11}\\
&+143027116N^{10}-208752656N^9-456973712N^8-131024384N^7+630587904N^6\\
&+1328877824N^5+1500652800N^4+1142401536N^3+629462016N^2\\
&+238215168N+44789760, && (833)\\
P_{664} &= 295317N^{20}+3132540N^{19}+8925794N^{18}-15503126N^{17}-128158720N^{16}\\
&-159520558N^{15}+378838970N^{14}+1195612190N^{13}+526578043N^{12}\\
&-1946889990N^{11}-3106593740N^{10}-1641770960N^9-862068320N^8\\
&-1429854848N^7+580862976N^6+5174659328N^5+7525373184N^4\\
&+5839105536N^3+2748681216N^2+747823104N+89579520, && (834)\\
P_{665} &= 23379N^{21}+334890N^{20}+1845809N^{19}+4416076N^{18}+910717N^{17}\\
&-19493138N^{16}-43527868N^{15}-31459862N^{14}+7207567N^{13}-2971600N^{12}\\
&-47214885N^{11}+27754946N^{10}+294517297N^9+665343056N^8+1059128256N^7\\
&+1391632944N^6+1471240544N^5+1188614208N^4+711979776N^3\\
&+304328448N^2+83960064N+11197440 && (835)
\end{aligned}
$$

and

$$
\begin{aligned}
C_{F_L,g}^{(3)} &= C_F\Biggl\{T_F^2N_F^2\Biggl[\frac{64S_1^2P_{685}}{3(N-1)N^2(1+N)^3(2+N)^2}+\frac{64S_2P_{698}}{3(N-1)N^2(1+N)^3(2+N)^2(3+N)}\\
&-\frac{32P_{721}}{9(N-2)^2(N-1)N^4(1+N)^5(2+N)^4(3+N)}+\Biggl(-\frac{128(5+3N)S_2}{3(1+N)(2+N)(3+N)}
\end{aligned}
$$

$$+\frac{32P_{716}}{9(N-2)(N-1)N^3(1+N)^4(2+N)^3(3+N)}\Bigg)S_1$$

$$-\frac{128(5+3N)S_3}{3(1+N)(2+N)(3+N)}+\Bigg(-\frac{128P_{699}}{9(N-2)^2(N-1)N(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{256(N-1)S_1}{3(N-2)(1+N)(3+N)}\Bigg)S_{-2}+\frac{512(N-1)S_{-3}}{3(N-2)(1+N)(3+N)}$$

$$+\frac{256(5+3N)S_{2,1}}{3(1+N)(2+N)(3+N)}-\frac{256\big(-2-3N+N^2\big)\zeta_3}{(N-2)(1+N)(2+N)(3+N)}\Bigg]+C_AT_FN_F\Bigg[$$

$$-\frac{128S_{-4}P_{667}}{(N-2)N(1+N)^2(2+N)(3+N)}+\frac{512S_{-3,1}P_{669}}{(N-2)N(1+N)^2(2+N)(3+N)}$$

$$+\frac{128S_{-2,2}P_{672}}{(N-2)N(1+N)^2(2+N)(3+N)}-\frac{256S_{3,1}P_{683}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{64S_4P_{688}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{256S_{-2,1}P_{697}}{(N-2)(N-1)N(1+N)^3(2+N)^2(3+N)}$$

$$-\frac{32S_2P_{703}}{3(N-2)(N-1)N(1+N)^3(2+N)^2(3+N)}$$

$$-\frac{32S_3P_{704}}{3(N-2)(N-1)N(1+N)^3(2+N)^2(3+N)}$$

$$-\frac{64\zeta_3P_{705}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{32S_1^2P_{706}}{3(N-2)(N-1)N(1+N)^3(2+N)^2(3+N)}$$

$$-\frac{8P_{722}}{9(N-2)^2(N-1)N^4(1+N)^5(2+N)^4(3+N)}$$

$$+\Bigg(-\frac{64S_3P_{678}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{64\zeta_3P_{692}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}$$

$$-\frac{8P_{719}}{9(N-2)^2(N-1)N^3(1+N)^4(2+N)^3(3+N)}$$

$$+\frac{128S_{-2,1}P_{673}}{(N-2)N(1+N)^2(2+N)(3+N)}+\frac{16\big(90+293N+380N^2+117N^3\big)S_2}{3N(1+N)^2(2+N)(3+N)}\Bigg)S_1$$

$$-\frac{16\big(2+13N+13N^2\big)S_1^3}{3N(1+N)^2(2+N)}+\Bigg(-\frac{256S_1^2P_{666}}{(N-2)N(1+N)^2(2+N)(3+N)}$$

$$+\frac{256S_1P_{707}}{3(N-2)^2(N-1)N^2(1+N)^2(2+N)^2(3+N)}$$

$$+\frac{32P_{717}}{9(N-2)^2(N-1)N^3(1+N)^4(2+N)^2(3+N)}+\frac{3072\zeta_3}{(1+N)(2+N)}$$

$$
-\frac{256\big(6+N+N^2\big)S_2}{(N-2)N(1+N)^2(2+N)(3+N)}\Bigg)S_{-2}+\frac{64\big(-54+13N+13N^2\big)S_{-2}^2}{(N-2)(1+N)(2+N)(3+N)}
$$

$$
+\Bigg(-\frac{64P_{709}}{3(N-2)(N-1)N^2(1+N)^3(2+N)^2(3+N)}
$$

$$
-\frac{64S_1P_{674}}{(N-2)N(1+N)^2(2+N)(3+N)}\Bigg)S_{-3}-\frac{128\big(9+35N+50N^2+18N^3\big)S_{2,1}}{3N(1+N)^2(2+N)(3+N)}
$$

$$
-\frac{1536S_{-2,3}}{(1+N)(2+N)}+\frac{1536S_{-4,1}}{(1+N)(2+N)}-\frac{4608(N-1)S_{-2,1,1}}{(N-2)(1+N)(3+N)}
$$

$$
+\frac{7680\zeta_5}{(1+N)(2+N)}\Bigg]\Bigg\}
$$

$$
+C_AT_F^2N_F^2\Bigg\{\frac{128S_{-2}P_{691}}{9(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}
$$

$$
-\frac{32P_{718}}{27(N-2)(N-1)N^3(1+N)^4(2+N)^4(3+N)}-\frac{64\zeta_3}{(N+1)(N+2)}
$$

$$
+\Bigg(-\frac{64P_{677}}{27(N-1)N(1+N)^3(2+N)^2}+\frac{128S_2}{(1+N)(2+N)}+\frac{256S_{-2}}{3(1+N)(2+N)}\Bigg)S_1
$$

$$
-\frac{128\big(-3-8N-6N^2+11N^3\big)S_1^2}{9(N-1)N(1+N)^2(2+N)}+\frac{128\big(-6-19N+11N^2+5N^3\big)S_2}{9(N-1)N(1+N)(2+N)^2}
$$

$$
-\frac{128S_1^3}{9(1+N)(2+N)}+\frac{512S_3}{9(1+N)(2+N)}-\frac{512S_{-3}}{3(1+N)(2+N)}-\frac{512S_{2,1}}{3(1+N)(2+N)}\Bigg\}
$$

$$
+C_F^2T_FN_F\Bigg\{\frac{4096S_{3,1}P_{668}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}+\frac{8S_1^2P_{676}}{N^2(1+N)^3(2+N)}
$$

$$
-\frac{128S_{-4}P_{670}}{(N-2)N(1+N)^2(2+N)(3+N)}-\frac{128S_4P_{687}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}
$$

$$
-\frac{8S_2P_{671}}{N^2(1+N)^3(2+N)}+\frac{64S_3P_{700}}{3(N-2)(N-1)N(1+N)^3(2+N)^2(3+N)}
$$

$$
-\frac{128S_{-2,1}P_{701}}{(N-2)(N-1)N(1+N)^3(2+N)^2(3+N)}
$$

$$
+\frac{128\zeta_3P_{702}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}
$$

$$
-\frac{8P_{714}}{(N-2)^2N^4(1+N)^5(2+N)(3+N)}
$$

$$
+\Bigg(\frac{8P_{712}}{(N-2)(N-1)N^3(1+N)^4(2+N)^2(3+N)}
$$

$$
+\frac{128S_3P_{679}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}
$$

$$
-\frac{128\zeta_3P_{690}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}+\frac{48\big(2+N+N^2\big)S_2}{N(1+N)^2(2+N)}
$$

$$
-\frac{768(N-1)\big(2+N+N^2\big)S_{-2,1}}{(N-2)N(1+N)^2(3+N)}\Bigg)S_1-\frac{80\big(2+N+N^2\big)S_1^3}{3N(1+N)^2(2+N)}
$$

$$
+\Bigg(\frac{128S_1P_{696}}{(N-2)(N-1)N^2(1+N)^2(2+N)^2(3+N)}+\frac{256\big(2+N+N^2\big)S_1^2}{N(1+N)^2(2+N)}
$$

$$
-\frac{32P_{715}}{(N-2)^2(N-1)N^3(1+N)^4(2+N)^2(3+N)}-\frac{256\big(2+N+N^2\big)S_2}{N(1+N)^2(2+N)}
$$

$$
-\frac{3072\zeta_3}{(1+N)(2+N)}\Bigg)S_{-2}-\frac{512\big(-4+N+N^2\big)S_{-2}^2}{(N-2)(1+N)(2+N)(3+N)}
$$

$$
+\Bigg(\frac{64P_{708}}{(N-2)(N-1)N^2(1+N)^3(2+N)^2(3+N)}
$$

$$
-\frac{128\big(-18+N+N^2\big)\big(2+N+N^2\big)S_1}{(N-2)N(1+N)^2(2+N)(3+N)}\Bigg)S_{-3}+\frac{64\big(2+N+N^2\big)S_{2,1}}{N(1+N)^2(2+N)}
$$

$$
-\frac{256\big(2+N+N^2\big)\big(6+N+N^2\big)S_{-2,2}}{(N-2)N(1+N)^2(2+N)(3+N)}-\frac{3072\big(2+N+N^2\big)S_{-3,1}}{(N-2)N(1+N)^2(2+N)(3+N)}
$$

$$
+\frac{1536S_{-2,3}}{(1+N)(2+N)}-\frac{1536S_{-4,1}}{(1+N)(2+N)}+\frac{1024\big(2+N+N^2\big)S_{-2,1,1}}{(N-2)(1+N)(2+N)(3+N)}
$$

$$
-\frac{7680\zeta_5}{(1+N)(2+N)}\Bigg\}
$$

$$
+C_A^2T_FN_F\Bigg\{\frac{64S_4P_{680}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}
$$

$$
+\frac{256S_{3,1}P_{682}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}
$$

$$
+\frac{128S_{-2,1}P_{689}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}
$$

$$
+\frac{16\zeta_3P_{693}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}
$$

$$
-\frac{32S_3P_{694}}{9(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}-\frac{32S_2P_{710}}{9(N-1)^2N^2(1+N)^3(2+N)^3}
$$

$$
+\frac{16P_{723}}{27(N-2)(N-1)^2N^4(1+N)^5(2+N)^5(3+N)}
$$

$$
+\Bigg[-\frac{32S_2P_{675}}{(N-1)N(1+N)^2(2+N)^2}+\frac{128S_3P_{681}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}
$$

$$
-\frac{128\zeta_3P_{686}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}-\frac{512\big(-14+3N+3N^2\big)S_{-2,1}}{(N-2)(1+N)(2+N)(3+N)}
$$

$$
+\frac{32P_{720}}{27(N-2)(N-1)^2N^3(1+N)^4(2+N)^4(3+N)}\Bigg]S_1+\Bigg[-\frac{256S_2}{(1+N)(2+N)}
$$

$$
+\frac{32P_{711}}{9(N-1)^2N^2(1+N)^3(2+N)^3}\Bigg]S_1^2+\frac{32\big(-36-47N-72N^2+83N^3\big)S_1^3}{9(N-1)N(1+N)^2(2+N)}
$$

$$
\begin{aligned}
&+\frac{32S_1^4}{(1+N)(2+N)}+\frac{32S_2^2}{(1+N)(2+N)}+\Biggl[-\frac{128S_1^2}{(1+N)(2+N)}\\
&-\frac{64S_1P_{695}}{3(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}+\frac{256S_2}{(1+N)(2+N)}\\
&-\frac{32P_{713}}{9(N-2)(N-1)^2N^2(1+N)^3(2+N)^3(3+N)}-\frac{768\zeta_3}{(1+N)(2+N)}\Biggr]S_{-2}\\
&-\frac{32\bigl(-14+5N+5N^2\bigr)S_{-2}^2}{(N-2)(1+N)(2+N)(3+N)}+\Biggl[\frac{512\bigl(-16+3N+3N^2\bigr)S_1}{(N-2)(1+N)(2+N)(3+N)}\\
&-\frac{64P_{684}}{3(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}\Biggr]S_{-3}+\frac{1408S_{2,1}}{3(1+N)(2+N)}\\
&+\frac{32\bigl(-122+23N+23N^2\bigr)S_{-4}}{(N-2)(1+N)(2+N)(3+N)}-\frac{128\bigl(-62+13N+13N^2\bigr)S_{-2,2}}{(N-2)(1+N)(2+N)(3+N)}\\
&+\frac{384S_{-2,3}}{(1+N)(2+N)}-\frac{256\bigl(-34+7N+7N^2\bigr)S_{-3,1}}{(N-2)(1+N)(2+N)(3+N)}-\frac{384S_{-4,1}}{(1+N)(2+N)}\\
&+\frac{1024\bigl(-14+3N+3N^2\bigr)S_{-2,1,1}}{(N-2)(1+N)(2+N)(3+N)}-\frac{1920\zeta_5}{(1+N)(2+N)}\Biggr\}. \qquad (836)
\end{aligned}
$$

with

$$
\begin{aligned}
P_{666} &= N^4+2N^3-6N^2-7N-6, && (837)\\
P_{667} &= N^4+2N^3+16N^2+15N+18, && (838)\\
P_{668} &= 2N^4+4N^3+N^2-N+12, && (839)\\
P_{669} &= 4N^4+8N^3-3N^2-7N+6, && (840)\\
P_{670} &= 5N^4+10N^3-27N^2-32N-36, && (841)\\
P_{671} &= 15N^4+48N^3+71N^2+58N+8, && (842)\\
P_{672} &= 17N^4+34N^3-15N^2-32N+12, && (843)\\
P_{673} &= 21N^4+42N^3-31N^2-52N-12, && (844)\\
P_{674} &= 25N^4+50N^3-27N^2-52N+36, && (845)\\
P_{675} &= 35N^4+46N^3-79N^2-66N-32, && (846)\\
P_{676} &= 35N^4+84N^3+99N^2+78N+16, && (847)\\
P_{677} &= 302N^5+597N^4-286N^3-1017N^2-844N-120, && (848)\\
P_{678} &= N^6+3N^5-87N^4-179N^3-46N^2+44N-312, && (849)\\
P_{679} &= N^6+3N^5-35N^4-75N^3-26N^2+12N-168, && (850)\\
P_{680} &= N^6+3N^5-33N^4-71N^3-4N^2+32N-72, && (851)\\
P_{681} &= N^6+3N^5-19N^4-43N^3+22N-36, && (852)\\
P_{682} &= N^6+3N^5+9N^4+13N^3+8N^2+2N+36, && (853)\\
P_{683} &= N^6+3N^5+37N^4+69N^3+22N^2-12N+168, && (854)\\
P_{684} &= 2N^6-6N^5+143N^4+396N^3-403N^2-720N-612, && (855)\\
P_{685} &= 2N^6+6N^5+7N^4+4N^3+9N^2+8N+12, && (856)\\
P_{686} &= 3N^6+9N^5-35N^4-85N^3-4N^2+40N-72, && (857)
\end{aligned}
$$

$$P_{687} = 3N^6+9N^5+23N^4+31N^3-14N^2-28N+264, \quad (858)$$
$$P_{688} = 5N^6+15N^5+65N^4+105N^3+14N^2-36N+408, \quad (859)$$
$$P_{689} = 8N^6+20N^5-13N^4-26N^3-9N^2-32N-60, \quad (860)$$
$$P_{690} = 11N^6+33N^5-85N^4-225N^3-130N^2-12N-168, \quad (861)$$
$$P_{691} = 16N^6+48N^5-83N^4-246N^3+13N^2+144N-36, \quad (862)$$
$$P_{692} = 17N^6+51N^5-211N^4-507N^3-154N^2+108N-456, \quad (863)$$
$$P_{693} = 33N^6+99N^5-421N^4-1007N^3+68N^2+588N-1152, \quad (864)$$
$$P_{694} = 35N^6+33N^5-976N^4-1407N^3+131N^2+132N-3132, \quad (865)$$
$$P_{695} = 59N^6+129N^5-427N^4-669N^3+794N^2+678N-612, \quad (866)$$
$$P_{696} = N^7-N^6-11N^5+9N^4-66N^3-68N^2+168N-96, \quad (867)$$
$$P_{697} = 2N^7+7N^6-48N^5-160N^4-120N^3+29N^2+138N+72, \quad (868)$$
$$P_{698} = 6N^7+24N^6+19N^5-37N^4-73N^3-59N^2-36N-36, \quad (869)$$
$$P_{699} = 13N^7+13N^6-103N^5-113N^4+74N^3+172N^2+520N-864, \quad (870)$$
$$P_{700} = 17N^7+56N^6+216N^5+554N^4+891N^3+854N^2+748N+120, \quad (871)$$
$$P_{701} = 21N^7+80N^6-74N^5-468N^4-343N^3+164N^2+316N+48, \quad (872)$$
$$P_{702} = 23N^7+75N^6-152N^5-473N^4-109N^3+136N^2-156N+144, \quad (873)$$
$$P_{703} = 27N^7+42N^6-68N^5-51N^4+109N^3+15N^2-554N-384, \quad (874)$$
$$P_{704} = 31N^7+156N^6+722N^5+1524N^4+1595N^3+960N^2+1828N+1248, \quad (875)$$
$$P_{705} = 32N^7+146N^6-269N^5-972N^4-149N^3+416N^2-612N+144, \quad (876)$$
$$P_{706} = 77N^7+278N^6-260N^5-1501N^4-725N^3+1145N^2+1466N+384, \quad (877)$$
$$P_{707} = 14N^8+11N^7-137N^6-85N^5+514N^4-109N^3-880N^2+828N-144, \quad (878)$$
$$P_{708} = 29N^8+112N^7-42N^6-484N^5-463N^4-12N^3+300N^2+112N+192, \quad (879)$$
$$P_{709} = 49N^8+190N^7-244N^6-1346N^5-1213N^4+4N^3+1096N^2+696N+288, \quad (880)$$
$$P_{710} = 67N^8+334N^7-73N^6-1247N^5-406N^4+1645N^3+1708N^2-12N-72, \quad (881)$$
$$P_{711} = 358N^8+1078N^7-397N^6-2729N^5-529N^4+2743N^3+2944N^2-12N -216, \quad (882)$$
$$P_{712} = 91N^{10}+417N^9-128N^8-2686N^7-6937N^6-9783N^5-4234N^4+2324N^3 +424N^2-1440N-576, \quad (883)$$
$$P_{713} = 281N^{10}+1405N^9-105N^8-7986N^7-4629N^6+13197N^5+7549N^4 -15904N^3-2448N^2+16848N+2160, \quad (884)$$
$$P_{714} = 18N^{11}+68N^{10}+179N^9-304N^8-1767N^7-2263N^6-230N^5+1039N^4 +76N^3-1424N^2-1040N-240, \quad (885)$$
$$P_{715} = 59N^{11}+161N^{10}-392N^9-1310N^8+31N^7+2625N^6+1478N^5-1220N^4 -152N^3-1088N^2+64N+768, \quad (886)$$
$$P_{716} = 93N^{11}+543N^{10}+650N^9-1782N^8-5427N^7-4905N^6-900N^5+4272N^4 +7024N^3+5616N^2+7488N+3456, \quad (887)$$
$$P_{717} = 581N^{11}+1653N^{10}-3824N^9-13086N^8+1281N^7+26757N^6+12850N^5 -15036N^4-808N^3-5184N^2-6336N+3456, \quad (888)$$
$$P_{718} = 51N^{12}+423N^{11}-314N^{10}-9178N^9-21183N^8+4305N^7+82314N^6 +134826N^5+92180N^4+11968N^3-18048N^2-11232N-1728, \quad (889)$$

$$P_{719} = 1347N^{12}+5385N^{11}-13244N^{10}-70178N^9-33309N^8+167445N^7 +305262N^6+206148N^5-13096N^4-53200N^3+70560N^2+100800N +31104, \quad (890)$$

$$P_{720} = 1820N^{13}+12757N^{12}+15253N^{11}-73492N^{10}-200208N^9-38667N^8 +354250N^7+351875N^6-74335N^5-203477N^4+26664N^3+91944N^2 +58320N+19440, \quad (891)$$

$$P_{721} = 12N^{15}+134N^{14}+501N^{13}+453N^{12}-1684N^{11}-5780N^{10}-10781N^9 -18749N^8-8256N^7+28470N^6+10840N^5-52984N^4-74944N^3 -83808N^2-58752N-17280, \quad (892)$$

$$P_{722} = 96N^{15}+266N^{14}-5889N^{13}-31881N^{12}-27580N^{11}+147580N^{10}+402667N^9 +282187N^8-247206N^7-501336N^6-287480N^5-222400N^4-437536N^3 -461952N^2-206208N-34560, \quad (893)$$

$$P_{723} = 150N^{16}+1614N^{15}-1820N^{14}-61808N^{13}-210369N^{12}-122317N^{11} +723938N^{10}+1759218N^9+1253237N^8-923191N^7-2589744N^6 -2736284N^5-2333312N^4-1942608N^3-1354752N^2-565056N-103680. \quad (894)$$

The d_{abc} contribution read

$$C_{F_2,g}^{d_{abc},(3)} = \frac{d_{abc}d^{abc}N_F^2}{N_A}\Biggl\{ -\frac{256S_{-2,1}P_{725}}{45N(1+N)(2+N)^2}+\frac{128S_3P_{726}}{(N-2)N(1+N)(2+N)^2(3+N)} -\frac{256S_4P_{734}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}+\frac{512S_{3,1}P_{734}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)} +\frac{128(N-1)S_5P_{724}}{45N(1+N)(2+N)}+\frac{32P_{742}}{225(N-2)(N-1)^3N^3(1+N)^5(2+N)^5(3+N)^3} +\Biggl(-\frac{64P_{740}}{225(N-2)(N-1)^2N^2(1+N)^4(2+N)^4(3+N)^3}-\frac{1024(N-4)(N-1)S_{3,1}}{N(1+N)(2+N)} -\frac{128S_3P_{733}}{(N-2)N^2(1+N)^2(2+N)^2(3+N)}+\frac{512(N-4)(N-1)S_4}{N(1+N)(2+N)} +\frac{512\bigl(8-3N+3N^2\bigr)S_{-2,1}}{3N(1+N)(2+N)}\Biggr)S_1+\Biggl(\frac{128\bigl(19-30N+3N^2\bigr)}{3N(1+N)(2+N)}+\frac{256(N-4)(N-1)S_3}{N(1+N)(2+N)}\Biggr) \times S_1^2-\frac{768(N-4)(N-1)S_2S_3}{N(1+N)(2+N)}+\Biggl(\frac{256S_1P_{736}}{45(N-2)N^2(1+N)^2(2+N)^2(3+N)} +\frac{64P_{738}}{45(N-2)N^2(1+N)^3(2+N)^3(3+N)}-\frac{256\bigl(8-3N+3N^2\bigr)S_1^2}{3N(1+N)(2+N)} +\frac{256}{45}(N-1)\bigl(21-2N+N^2\bigr)S_3+\frac{256}{45}(N-1)\bigl(21-2N+N^2\bigr)S_{-2,1}\Biggr)S_{-2} -\frac{128(N-1)S_{-2}^2}{(1+N)(2+N)}+\Biggl(\frac{128P_{725}}{45N(1+N)(2+N)^2}-\frac{256\bigl(8-3N+3N^2\bigr)S_1}{3N(1+N)(2+N)}\Biggr)S_{-3}$$

$$
-\frac{256\big(4-3N+3N^2\big)S_{-4}}{3N(1+N)(2+N)}+\frac{128}{45}(N-1)\big(21-2N+N^2\big)S_{-5}
+\frac{768(N-4)(N-1)S_{2,3}}{N(1+N)(2+N)}-\frac{768(N-4)(N-1)S_{4,1}}{N(1+N)(2+N)}+\frac{256\big(8-3N+3N^2\big)S_{-2,2}}{3N(1+N)(2+N)}
-\frac{256}{45}(N-1)\big(21-2N+N^2\big)S_{-2,3}+\frac{256\big(8-3N+3N^2\big)S_{-3,1}}{3N(1+N)(2+N)}
+\frac{1536(N-4)(N-1)S_{3,1,1}}{N(1+N)(2+N)}-\frac{512\big(8-3N+3N^2\big)S_{-2,1,1}}{3N(1+N)(2+N)}
-\frac{256}{45}(N-1)\big(21-2N+N^2\big)S_{-2,1,-2}+\Bigg(-\frac{64P_{732}}{45(N-2)N(1+N)(2+N)^2(3+N)}
+\frac{512S_1P_{735}}{3(N-2)N^2(1+N)^2(2+N)^2(3+N)}-\frac{512(N-4)(N-1)S_1^2}{N(1+N)(2+N)}
+\frac{256}{45}(N-1)\big(21-2N+N^2\big)S_{-2}\Bigg)\zeta_3+\frac{64(N-1)P_{724}}{9N(1+N)(2+N)}\zeta_5\Bigg\}, \tag{895}
$$

$$
C_{F_L,g}^{d_{abc},(3)} = \frac{d_{abc}d^{abc}N_F^2}{N_A}\Bigg\{
\frac{256S_4P_{729}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}-\frac{512S_{3,1}P_{729}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}
-\frac{256P_{741}}{225(N-2)(N-1)^3N^3(1+N)^5(2+N)^5(3+N)^3}+\Bigg(\frac{6144S_{3,1}}{(1+N)(2+N)}-\frac{3072S_4}{(1+N)(2+N)}
+\frac{512S_3P_{728}}{(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}-\frac{1024S_{-2,1}}{(1+N)(2+N)}
+\frac{256P_{739}}{225(N-2)(N-1)^2N^2(1+N)^4(2+N)^4(3+N)^3}\Bigg)S_1+\Bigg(-\frac{1408}{(1+N)(2+N)}
-\frac{1536S_3}{(1+N)(2+N)}\Bigg)S_1^2-\frac{1536\big(1+N+N^2\big)\big(-10+3N+3N^2\big)S_3}{(N-2)(N-1)(1+N)(2+N)^2(3+N)}+\frac{4608S_2S_3}{(1+N)(2+N)}
-\frac{128(N-4)(5+N)\big(18+N+N^2\big)S_5}{15(1+N)(2+N)}+\Bigg(\frac{512S_1^2}{(1+N)(2+N)}-\frac{256}{15}(N-1)NS_3
-\frac{256S_1P_{731}}{15(N-2)(N-1)N(1+N)^2(2+N)^2(3+N)}-\frac{256}{15}(N-1)NS_{-2,1}
-\frac{128P_{737}}{15(N-2)(N-1)N(1+N)^3(2+N)^3(3+N)}\Bigg)S_{-2}+\frac{256S_{-2}^2}{(1+N)(2+N)}
+\Bigg(-\frac{128\big(10+N+N^2\big)\big(30+N+N^2\big)}{15(N-1)(1+N)(2+N)^2}+\frac{512S_1}{(1+N)(2+N)}\Bigg)S_{-3}+\frac{512S_{-4}}{(1+N)(2+N)}
-\frac{128}{15}(N-1)NS_{-5}-\frac{4608S_{2,3}}{(1+N)(2+N)}+\frac{256\big(10+N+N^2\big)\big(30+N+N^2\big)S_{-2,1}}{15(N-1)(1+N)(2+N)^2}
+\frac{4608S_{4,1}}{(1+N)(2+N)}-\frac{512S_{-2,2}}{(1+N)(2+N)}+\frac{256}{15}(N-1)NS_{-2,3}-\frac{512S_{-3,1}}{(1+N)(2+N)}
$$

$$
-\frac{9216 S_{3,1,1}}{(1+N)(2+N)}+\frac{1024 S_{-2,1,1}}{(1+N)(2+N)}+\frac{256}{15}(N-1) N S_{-2,1,-2}+\Biggl(\frac{3072 S_1^2}{(1+N)(2+N)}
-\frac{256}{15}(N-1) N S_{-2}+\frac{128 P_{727}}{15(N-2)(N-1)(1+N)(2+N)^2(3+N)}
-\frac{512 S_1 P_{730}}{(N-2)(N-1) N(1+N)^2(2+N)^2(3+N)}\Biggr) \zeta_3-\frac{64(N-4)(5+N)\bigl(18+N+N^2\bigr)}{3(1+N)(2+N)} \zeta_5\Biggr\}, \tag{896}
$$

with the polynomials

$$
\begin{aligned}
P_{724} &= N^5+N^4+17N^3+59N^2-138N+720, && (897)\\
P_{725} &= N^5+N^4+59N^3+59N^2+225N-450, && (898)\\
P_{726} &= 5N^5-16N^4-58N^3+25N^2+32N+60, && (899)\\
P_{727} &= N^6+3N^5+1162N^4+2319N^3-1661N^2-2820N-4860, && (900)\\
P_{728} &= 11N^6+33N^5-15N^4-85N^3-50N^2-2N+36, && (901)\\
P_{729} &= 21N^6+63N^5-25N^4-155N^3-104N^2-16N+72, && (902)\\
P_{730} &= 23N^6+69N^5-35N^4-185N^3-96N^2+8N+72, && (903)\\
P_{731} &= 45N^6+135N^5+215N^4+205N^3-1340N^2-1420N+624, && (904)\\
P_{732} &= 2N^7+4N^6+873N^5-3466N^4-11255N^3+6582N^2+7020N+16200, && (905)\\
P_{733} &= 5N^7-19N^6-105N^5+23N^4+240N^3+104N^2-8N-96, && (906)\\
P_{734} &= 6N^7-33N^6-154N^5+41N^4+358N^3+202N^2+36N-144, && (907)\\
P_{735} &= 9N^7-24N^6-161N^5+28N^4+362N^3+110N^2-60N-144, && (908)\\
P_{736} &= 45N^7+90N^6+50N^5-130N^4-1895N^3-740N^2+984N-1944, && (909)\\
P_{737} &= N^{10}+7N^9-58N^8-438N^7-1017N^6-987N^5-130N^4+674N^3\\
&\quad +3940N^2+9960N+6480, && (910)\\
P_{738} &= 2N^{11}+12N^{10}-315N^9-2240N^8-4508N^7+2140N^6+22169N^5\\
&\quad +36208N^4+30892N^3+10680N^2-23040N-21600, && (911)\\
P_{739} &= 1065N^{14}+14910N^{13}+91950N^{12}+334830N^{11}+786240N^{10}+1070370N^9\\
&\quad +148290N^8-2319270N^7-3721785N^6-1334280N^5+2039760N^4\\
&\quad +2233440N^3+654480N^2, && (912)\\
P_{740} &= 220N^{15}-960N^{14}-28415N^{13}-151530N^{12}-369410N^{11}-556890N^{10}\\
&\quad -716080N^9-97590N^8+2725310N^7+4299810N^6-627745N^5-5315880N^4\\
&\quad -2040120N^3+1823040N^2+1056240N, && (913)\\
P_{741} &= 405N^{18}+6480N^{17}+43335N^{16}+152280N^{15}+272970N^{14}+103680N^{13}\\
&\quad -541890N^{12}-1017360N^{11}-393255N^{10}+810000N^9+1042875N^8\\
&\quad +217080N^7-366120N^6-272160N^5-58320N^4, && (914)\\
P_{742} &= 555N^{19}+6195N^{18}+17955N^{17}-54135N^{16}-471780N^{15}-1092330N^{14}\\
&\quad -398730N^{13}+2590050N^{12}+4158675N^{11}-126225N^{10}-5426505N^9\\
&\quad -3556395N^8+1998870N^7+2874360N^6+341280N^5-641520N^4\\
&\quad -220320N^3. && (915)
\end{aligned}
$$

Despite the fact that the above equations contain terms in which the power of N is larger in the numerator than the denominator, the whole expressions behave maximally $\propto \ln(N)/N$ for large values of N for the Wilson coefficient of F_2 and $\propto \ln(N)/N^2$ for that of F_L.

6 The three–loop Wilson coefficients for the structure function $xF_3(x, Q^2)$

The Wilson coefficient for the structure function $xF_3(x, Q^2)$ has already received the renormalizations of the coupling and those of the axial-vector coupling, as well the collinear singularities have been removed, to arrive at $\mathbb{C}^{(3)}_{\hat{F}_3,q}$, which still requires the finite renormalization to restore the Ward identities performed by

$$\mathbb{C}^{(3)}_{F_3,q} = Z_5 \mathbb{C}^{(3)}_{\hat{F}_3,q}, \tag{916}$$

with, [17],

$$\begin{aligned}
Z_5 = {} & 1 + a_s C_F \Big[-4 - 10\varepsilon + \varepsilon^2\big(-22 + 2\zeta_2\big)\Big] + a_s^2\Big[-\frac{107}{9} C_A C_F + 22 C_F^2 + \frac{4}{9} C_F T_F N_F \\
& + \varepsilon\Big(\frac{662}{27} C_F T_F N_F + C_F^2\big(132 - 48\zeta_3\big) + C_A C_F\Big(-\frac{7229}{54} + 48\zeta_3\Big)\Big)\Big] \\
& + a_s^3\Big[\frac{208}{81} C_F T_F^2 N_F^2 + C_A C_F^2\Big(\frac{5834}{27} - 160\zeta_3\Big) + 2 C_F^2 T_F N_F\Big(-\frac{62}{27} - \frac{32\zeta_3}{3}\Big) \\
& + 2 C_A C_F T_F N_F\Big(\frac{356}{81} + \frac{32\zeta_3}{3}\Big) + C_A^2 C_F\Big(-\frac{2147}{27} + 56\zeta_3\Big) + C_F^3\Big(-\frac{370}{3} + 96\zeta_3\Big)\Big].
\end{aligned} \tag{917}$$

One obtains

$$\mathbb{C}^{(3)}_{F_3,q} = C^{\mathrm{NS},(3)}_{F_3,q} + C^{d_{abc},(3)}_{F_3,q} \tag{918}$$

with the contributions

$$\begin{aligned}
C^{\mathrm{NS},(3)}_{F_3} = {} & C_F^2\Big\{C_A\Big[-\frac{20\zeta_5 P_{754}}{3N(1+N)} + \frac{8S_{-2,1,-2}P_{758}}{3N(1+N)} + \frac{16S_{-2,3}P_{760}}{9N(1+N)} + \frac{36\zeta_2^2 P_{764}}{5N^2(1+N)^2} \\
& - \frac{4S_5 P_{766}}{9N(1+N)} - \frac{4S_{2,1,1}P_{776}}{3N^2(1+N)^2} + \frac{S_2^2 P_{788}}{27N^2(1+N)^2} - \frac{4S_{-3,1}P_{815}}{9(-1+N)N^2(1+N)^2(2+N)} \\
& + \frac{8S_{-2,2}P_{821}}{27(-1+N)N^2(1+N)^2(2+N)} + \frac{4S_4 P_{837}}{27(-1+N)N^2(1+N)^2(2+N)} \\
& + \frac{4S_{3,1}P_{838}}{27(-1+N)N^2(1+N)^2(2+N)} - \frac{8S_{-2,1,1}P_{839}}{27(-1+N)N^2(1+N)^2(2+N)} \\
& + \frac{\zeta_3 P_{850}}{3(-1+N)N^3(1+N)^3(2+N)} + \frac{P_{884}}{648(-1+N)^2N^6(1+N)^6(2+N)} \\
& + \Big(\frac{2\zeta_3 P_{828}}{3(-1+N)N^2(1+N)^2(2+N)} - \frac{2S_3 P_{820}}{27(-1+N)N^2(1+N)^2(2+N)}
\end{aligned}$$

$$
+\frac{8S_{-2,1}P_{836}}{27(-1+N)N^2(1+N)^2(2+N)}+\frac{P_{878}}{648(-1+N)^2N^5(1+N)^5(2+N)}
+\frac{4S_{2,1}P_{782}}{9N^2(1+N)^2}+\Biggl(\frac{P_{843}}{162N^3(1+N)^3(2+N)}-\frac{2960}{3}S_3+\frac{160}{3}S_{2,1}-\frac{15104}{3}S_{-2,1}
+480\zeta_3\Biggr)S_2+\frac{8\bigl(15+236N+236N^2\bigr)S_2^2}{9N(1+N)}+\frac{8\bigl(-99+734N+734N^2\bigr)S_4}{9N(1+N)}
+256S_5+\frac{2080}{3}S_{2,3}+3920S_{2,-3}-\frac{8\bigl(-1212+869N+869N^2\bigr)S_{3,1}}{9N(1+N)}+240S_{4,1}
-\frac{8\bigl(1662+61N+61N^2\bigr)S_{-2,2}}{9N(1+N)}-3840S_{-2,3}-\frac{32\bigl(107+50N+50N^2\bigr)S_{-3,1}}{3N(1+N)}
+64S_{-2,-3}-\frac{4000}{3}S_{-4,1}-\frac{40\bigl(6+13N+13N^2\bigr)S_{2,1,1}}{3N(1+N)}-\frac{11552}{3}S_{2,1,-2}-\frac{32}{3}S_{2,2,1}
+1984S_{3,1,1}-\frac{32\bigl(-1020+239N+239N^2\bigr)S_{-2,1,1}}{9N(1+N)}-\frac{11744}{3}S_{-2,1,-2}-\frac{12224}{3}S_{-2,2,1}
-\frac{6784}{3}S_{-3,1,1}-128S_{2,1,1,1}+11776S_{-2,1,1,1}+560\zeta_5\Biggr)S_1+\Biggl(-\frac{4S_2P_{786}}{9N^2(1+N)^2}
+\frac{P_{868}}{36(-1+N)N^4(1+N)^4(2+N)}-\frac{40}{3}S_2^2+\frac{4\bigl(-744+733N+733N^2\bigr)S_3}{9N(1+N)}-88S_4
+\frac{16\bigl(30+29N+29N^2\bigr)S_{2,1}}{9N(1+N)}-\frac{2464}{3}S_{3,1}+\frac{208\bigl(-54+19N+19N^2\bigr)S_{-2,1}}{9N(1+N)}
+\frac{7216}{3}S_{-2,2}+\frac{6304}{3}S_{-3,1}+80S_{2,1,1}-\frac{16256}{3}S_{-2,1,1}+\frac{4\bigl(94+35N+35N^2\bigr)\zeta_3}{N(1+N)}\Biggr)S_1^2
+\Biggl(\frac{P_{842}}{54N^3(1+N)^3}-\frac{4\bigl(108+491N+491N^2\bigr)S_2}{27N(1+N)}+\frac{1648}{9}S_3-\frac{320}{9}S_{2,1}+1120S_{-2,1}
-96\zeta_3\Biggr)S_1^3+\Biggl(\frac{-110+433N+433N^2}{9N(1+N)}+8S_2\Biggr)S_1^4+\frac{44}{9}S_1^5+\Biggl(-\frac{616}{3}S_4+880S_{-3,1}
+\frac{P_{873}}{324(-1+N)N^4(1+N)^4(2+N)}-\frac{16\bigl(-1143+1159N+1159N^2\bigr)S_3}{27N(1+N)}
-\frac{3760}{3}S_{-2,2}-\frac{20\bigl(4+9N+9N^2\bigr)S_{2,1}}{3N(1+N)}+\frac{3136}{3}S_{3,1}-\frac{8\bigl(-446+79N+79N^2\bigr)S_{-2,1}}{3N(1+N)}
-\frac{544}{3}S_{2,1,1}+\frac{12736}{3}S_{-2,1,1}-\frac{12\bigl(18+77N+77N^2\bigr)\zeta_3}{N(1+N)}\Biggr)S_2+\frac{64}{3}S_2^3
+\Biggl(\frac{P_{849}}{81(-1+N)N^3(1+N)^3(2+N)}-\frac{1072}{9}S_{2,1}+2656S_{-2,1}+256\zeta_3\Biggr)S_3
+\frac{3392}{9}S_3^2+224S_6+\Biggl(-\frac{16\zeta_3P_{753}}{3N(1+N)}+\frac{4S_2P_{832}}{27(-1+N)N^2(1+N)^2(2+N)}
-\frac{8S_{-2,1}P_{759}}{3N(1+N)}-\frac{8S_3P_{768}}{27N(1+N)}-\frac{2P_{876}}{81(-1+N)^2N^4(1+N)^4(2+N)}
$$

$$
\begin{aligned}
&+\Bigg(-\frac{4P_{867}}{81(-1+N)^2N^3(1+N)^3(2+N)}+\frac{8\big(24+185N+185N^2\big)S_2}{3N(1+N)}+608S_3\\
&+\frac{11264}{3}S_{2,1}+\frac{1312}{3}S_{-2,1}+1248\zeta_3\Bigg)S_1+\Bigg(-\frac{4P_{823}}{9(-1+N)N^2(1+N)^2(2+N)}\\
&-112S_2\Bigg)S_1^2-\frac{16\big(81+139N+139N^2\big)S_1^3}{27N(1+N)}+32S_1^4-\frac{32\big(150+71N+71N^2\big)S_{2,1}}{9N(1+N)}\\
&+\frac{128}{3}S_2^2+2224S_4-3136S_{3,1}-1728S_{-2,2}-2048S_{-3,1}+4480S_{-2,1,1}+\frac{288}{5}\zeta_2^2\big)S_{-2}\\
&+\Bigg(\frac{2P_{818}}{3(-1+N)N^2(1+N)^2(2+N)}+\frac{4\big(-222+161N+161N^2\big)S_1}{3N(1+N)}+392S_1^2\\
&-168S_2\Bigg)S_{-2}^2-\frac{32}{3}S_{-2}^3+\big(\frac{4P_{859}}{81(-1+N)N^3(1+N)^3(2+N)}+\Bigg(\frac{2176}{3}S_2\\
&+\frac{4P_{822}}{27(-1+N)N^2(1+N)^2(2+N)}\Bigg)S_1+\frac{8\big(1044+335N+335N^2\big)S_1^2}{9N(1+N)}-\frac{2416}{3}S_1^3\\
&-\frac{4\big(-678+773N+773N^2\big)S_2}{9N(1+N)}-2720S_3+\Bigg(\frac{8\big(-212+N+N^2\big)}{3N(1+N)}+\frac{4064S_1}{3}\Bigg)S_{-2}\\
&-80S_{2,1}+3584S_{-2,1}-64\zeta_3\big)S_{-3}-1184S_{-3}^2+\Bigg(\frac{2P_{840}}{27(-1+N)N^2(1+N)^2(2+N)}\\
&-\frac{4\big(-126+31N+31N^2\big)S_1}{9N(1+N)}-456S_1^2+\frac{1400}{3}S_2+960S_{-2}\Bigg)S_{-4}+\Bigg(-\frac{4P_{765}}{9N(1+N)}\\
&+\frac{3920S_1}{3}\Bigg)S_{-5}-\frac{544}{3}S_{-6}+\Bigg(\frac{4P_{846}}{9N^3(1+N)^3(2+N)}-288\zeta_3\Bigg)S_{2,1}+40S_{2,1}^2\\
&+\frac{16\big(-303+431N+431N^2\big)S_{2,3}}{9N(1+N)}-\frac{4\big(2898+487N+487N^2\big)S_{2,-3}}{9N(1+N)}\\
&-\frac{4\big(-162+1807N+1807N^2\big)S_{4,1}}{9N(1+N)}+552S_{4,2}-2080S_{4,-2}+\frac{5888}{3}S_{5,1}\\
&+\Bigg(\frac{8P_{855}}{81(-1+N)N^3(1+N)^3(2+N)}+96S_{2,1}-640\zeta_3\Bigg)S_{-2,1}-2688S_{-2,1}^2\\
&+\frac{16(-1+N)(2+N)S_{-2,-3}}{N(1+N)}+3456S_{-3,3}-\frac{8\big(-484+189N+189N^2\big)S_{-4,1}}{3N(1+N)}\\
&-2384S_{-4,2}-1216S_{-4,-2}+288S_{-5,1}+\frac{8\big(654+293N+293N^2\big)S_{2,1,-2}}{9N(1+N)}\\
&-\frac{4\big(-4+65N+65N^2\big)S_{2,2,1}}{3N(1+N)}-\frac{1760}{3}S_{2,3,1}+\frac{32\big(-387+361N+361N^2\big)S_{3,1,1}}{9N(1+N)}\\
&-2160S_{2,-3,1}+2784S_{3,1,-2}+48S_{3,2,1}+\frac{320}{3}S_{4,1,1}+\frac{16\big(390+133N+133N^2\big)S_{-2,2,1}}{9N(1+N)}\\
&+704S_{-2,2,2}+64S_{-2,2,-2}+2816S_{-2,3,1}+\frac{16\big(-40+127N+127N^2\big)S_{-3,1,1}}{3N(1+N)}
\end{aligned}
$$

$$
+960S_{-3,1,-2} - 112S_{-3,2,1} - 704S_{-3,-2,1} + 5504S_{-4,1,1} + \frac{32\big(6+13N+13N^2\big)S_{2,1,1,1}}{3N(1+N)}
$$
$$
+\frac{208}{3}S_{2,2,1,1} - 1792S_{2,-2,1,1} - \frac{6368}{3}S_{3,1,1,1} + \frac{160\big(-180+23N+23N^2\big)S_{-2,1,1,1}}{9N(1+N)}
$$
$$
-288S_{-2,1,1,2} + 192S_{-2,2,1,1} - 1408S_{-2,-2,1,1} - 896S_{-3,1,1,1} + \frac{640}{3}S_{2,1,1,1,1}
$$
$$
-9728S_{-2,1,1,1,1}\Bigg]
$$
$$
+T_F N_F\Bigg[-\frac{128S_{-2,1}P_{770}}{81N^2(1+N)^2} - \frac{16S_{2,1}P_{789}}{9N^2(1+N)^2(2+N)} + \frac{4S_3P_{792}}{81N^2(1+N)^2(2+N)}
$$
$$
+\frac{16\zeta_3P_{817}}{9(-1+N)N^2(1+N)^2(2+N)} + \frac{P_{879}}{162(-1+N)^2N^5(1+N)^5(2+N)}
$$
$$
+\Bigg(\frac{2S_2P_{793}}{81N^2(1+N)^2(2+N)} + \frac{P_{871}}{162(-1+N)N^4(1+N)^4(2+N)} - \frac{752}{9}S_2^2
$$
$$
-\frac{8\big(-210+61N+61N^2\big)S_3}{27N(1+N)} - \frac{1664}{9}S_4 - \frac{16\big(-10+13N+13N^2\big)S_{2,1}}{9N(1+N)} + \frac{1088}{9}S_{3,1}
$$
$$
-\frac{512\big(-3+10N+10N^2\big)S_{-2,1}}{27N(1+N)} + \frac{256}{9}S_{-2,2} + \frac{512}{3}S_{-3,1} + \frac{320}{3}S_{2,1,1} + \frac{2048}{9}S_{-2,1,1}
$$
$$
+\frac{64}{5}\zeta_2^2 - \frac{16\big(42+73N+73N^2\big)\zeta_3}{9N(1+N)}\Bigg)S_1 + \Big(\frac{16\big(-23+82N+82N^2\big)S_2}{9N(1+N)}
$$
$$
+\frac{P_{805}}{9N^3(1+N)^3} - \frac{464}{9}S_3 - \frac{352}{9}S_{2,1} - \frac{1280}{9}S_{-2,1} - 32\zeta_3\Big)S_1^2 + \Bigg(-\frac{2P_{783}}{27N^2(1+N)^2}
$$
$$
+\frac{832S_2}{27}\Bigg)S_1^3 - \frac{20\big(-2+7N+7N^2\big)S_1^4}{9N(1+N)} - \frac{16}{9}S_1^5 + \Bigg(\frac{P_{844}}{81N^3(1+N)^3(2+N)}
$$
$$
+\frac{4976}{27}S_3 + 32S_{2,1} + \frac{256}{3}S_{-2,1} + \frac{352}{3}\zeta_3\Bigg)S_2 - \frac{4\big(-210+167N+167N^2\big)S_2^2}{27N(1+N)}
$$
$$
-\frac{16\big(-3+170N+170N^2\big)S_4}{27N(1+N)} + \frac{2368}{9}S_5
$$
$$
+\Bigg(\frac{32P_{866}}{81(-1+N)^2N^3(1+N)^3(2+N)} + \Bigg(\frac{128P_{806}}{81(-1+N)N^2(1+N)^2(2+N)}
$$
$$
-\frac{512}{3}S_2\Bigg)S_1 + \frac{256\big(-1+5N+5N^2\big)S_1^2}{9N(1+N)}
$$
$$
+\frac{1280}{27}S_1^3 - \frac{128\big(-9+20N+20N^2\big)S_2}{27N(1+N)} + \frac{3712}{27}S_3 + \frac{512}{9}S_{2,1} + \frac{896}{3}\zeta_3\Bigg)S_{-2}
$$
$$
+\Bigg(-\frac{32\big(-2+13N+13N^2\big)}{3N(1+N)} - \frac{256}{3}S_1\Bigg)S_{-2}^2 - \Bigg(\frac{64P_{795}}{81(-1+N)N^2(1+N)^2(2+N)}
$$
$$
+\frac{512\big(-3+5N+5N^2\big)S_1}{27N(1+N)} + \frac{896}{9}S_1^2 - \frac{640}{9}S_2 - \frac{512}{3}S_{-2}\Bigg)S_{-3} + \Bigg(-\frac{256}{9}S_1
$$

$$-\frac{32\big(42+197N+197N^2\big)}{27N(1+N)}\Bigg)S_{-4}-\frac{16\big(48+431N+431N^2\big)S_{3,1}}{27N(1+N)}+\frac{3008}{9}S_{-5}$$
$$-\frac{1984}{9}S_{2,3}+\frac{512}{9}S_{2,-3}+\frac{3008}{9}S_{4,1}+\frac{128\big(-3+10N+10N^2\big)S_{-2,2}}{27N(1+N)}-\frac{896}{9}S_{-2,3}$$
$$+\frac{128\big(-3+10N+10N^2\big)S_{-3,1}}{9N(1+N)}+\frac{16\big(-4+5N+5N^2\big)S_{2,1,1}}{3N(1+N)}-\frac{512}{9}S_{2,1,-2}$$
$$+\frac{64}{3}S_{2,2,1}-\frac{2944}{9}S_{3,1,1}+\frac{256\big(-3+10N+10N^2\big)S_{-2,1,1}}{27N(1+N)}+\frac{512}{3}S_{-2,1,-2}$$
$$-\frac{512}{9}S_{-2,2,1}-\frac{512}{3}S_{-3,1,1}-\frac{256}{3}S_{2,1,1,1}-\frac{1024}{9}S_{-2,1,1,1}-\frac{16\big(2+3N+3N^2\big)\zeta_2^2}{5N(1+N)}\Bigg]\Bigg\}$$
$$+C_F\Bigg\{C_AT_FN_F\Bigg[-\frac{8S_{3,1}P_{773}}{27(-1+N)N(1+N)(2+N)}+\frac{4S_4P_{787}}{27(-1+N)N(1+N)(2+N)}$$
$$+\frac{64S_{-2,1}P_{770}}{81N^2(1+N)^2}+\frac{8S_{2,1}P_{791}}{27N^2(1+N)^2(2+N)}-\frac{8S_3P_{830}}{27(-1+N)N^2(1+N)^2(2+N)}$$
$$-\frac{4\zeta_3P_{834}}{27(-1+N)N^2(1+N)^2(2+N)}+\frac{P_{881}}{729(-1+N)^2N^5(1+N)^5(2+N)}$$
$$+\Bigg(-\frac{32S_3P_{772}}{27(-1+N)N(1+N)(2+N)}+\frac{16\zeta_3P_{778}}{27(-1+N)N(1+N)(2+N)}$$
$$+\frac{8S_2P_{790}}{27N^2(1+N)^2(2+N)}-\frac{4P_{872}}{729(-1+N)N^4(1+N)^4(2+N)}+\frac{224}{9}S_2^2+\frac{1312}{9}S_4$$
$$+\frac{16\big(-13+40N+40N^2\big)S_{2,1}}{9N(1+N)}-96S_{3,1}+\frac{256\big(-3+10N+10N^2\big)S_{-2,1}}{27N(1+N)}$$
$$-\frac{128}{9}S_{-2,2}-\frac{256}{3}S_{-3,1}-\frac{160}{3}S_{2,1,1}-\frac{1024}{9}S_{-2,1,1}-\frac{64}{5}\zeta_2^2\Bigg)S_1+\Bigg(-\frac{2P_{841}}{81N^3(1+N)^3}$$
$$+\frac{8\big(11+2N+2N^2\big)S_2}{9N(1+N)}-\frac{16}{9}S_3+\frac{304}{9}S_{2,1}+\frac{640}{9}S_{-2,1}+32\zeta_3\Bigg)S_1^2+\Bigg(-\frac{224}{27}S_2$$
$$-\frac{16\big(-33+194N+194N^2\big)}{81N(1+N)}\Bigg)S_1^3-\frac{88}{27}S_1^4+\Bigg(\frac{2P_{848}}{81N^3(1+N)^3(2+N)}-\frac{1360}{27}S_3$$
$$-16S_{2,1}-\frac{128}{3}S_{-2,1}-\frac{352}{3}\zeta_3\Bigg)S_2-\frac{4\big(16+109N+109N^2\big)S_2^2}{9N(1+N)}-\frac{1312}{9}S_5$$
$$+\Bigg(-\frac{16P_{866}}{81(-1+N)^2N^3(1+N)^3(2+N)}+\Bigg(-\frac{64P_{806}}{81(-1+N)N^2(1+N)^2(2+N)}$$
$$+\frac{256}{3}S_2\Bigg)S_1-\frac{128\big(-1+5N+5N^2\big)S_1^2}{9N(1+N)}-\frac{640}{27}S_1^3+\frac{64\big(-9+20N+20N^2\big)S_2}{27N(1+N)}$$
$$-\frac{1856}{27}S_3-\frac{256}{9}S_{2,1}-\frac{448}{3}\zeta_3\Bigg)S_{-2}+\Bigg(\frac{16\big(-2+13N+13N^2\big)}{3N(1+N)}+\frac{128}{3}S_1\Bigg)S_{-2}^2$$
$$+\Bigg(\frac{32P_{795}}{81(-1+N)N^2(1+N)^2(2+N)}+\frac{256\big(-3+5N+5N^2\big)S_1}{27N(1+N)}+\frac{448}{9}S_1^2-\frac{320}{9}S_2$$

$$
-\frac{256}{3}S_{-2}\Biggr)S_{-3}+\Biggl(\frac{16\big(42+197N+197N^2\big)}{27N(1+N)}+\frac{128}{9}S_1\Biggr)S_{-4}-\frac{1504}{9}S_{-5}+\frac{320}{3}S_{2,3}
-\frac{256}{9}S_{2,-3}-\frac{64\big(-3+10N+10N^2\big)S_{-2,2}}{27N(1+N)}-\frac{64\big(-3+10N+10N^2\big)S_{-3,1}}{9N(1+N)}
+\frac{448}{9}S_{-2,3}-\frac{1120}{9}S_{4,1}-\frac{32\big(-3+8N+8N^2\big)S_{2,1,1}}{9N(1+N)}+\frac{256}{9}S_{2,1,-2}-\frac{64}{3}S_{2,2,1}
+\frac{1696}{9}S_{3,1,1}-\frac{128\big(-3+10N+10N^2\big)S_{-2,1,1}}{27N(1+N)}-\frac{256}{3}S_{-2,1,-2}+\frac{256}{9}S_{-2,2,1}
+\frac{256}{3}S_{-3,1,1}+\frac{64}{9}S_{2,1,1,1}+\frac{512}{9}S_{-2,1,1,1}+\frac{16\big(2+3N+3N^2\big)\zeta_2^2}{5N(1+N)}\Biggr]
$$

$$
+T_F^2N_F^2\Biggl[-\frac{8S_2P_{784}}{81N^2(1+N)^2}-\frac{2P_{861}}{729N^4(1+N)^4}+\Biggl(-\frac{16\big(-6+29N+29N^2\big)S_2}{27N(1+N)}
+\frac{8P_{835}}{729N^3(1+N)^3}+\frac{128}{27}S_3+\frac{128}{27}\zeta_3\Biggr)S_1+\Biggl(\frac{8P_{775}}{81N^2(1+N)^2}-\frac{32S_2}{9}\Biggr)S_1^2
+\frac{16\big(-6+29N+29N^2\big)S_1^3}{81N(1+N)}+\frac{16}{27}S_1^4+\frac{80}{9}S_2^2+\frac{32\big(-6+247N+247N^2\big)S_3}{81N(1+N)}
-\frac{992}{27}S_4-\frac{1216}{27}S_{2,1}+\frac{256}{9}S_{3,1}-\frac{128}{9}S_{2,1,1}-\frac{32\big(2+3N+3N^2\big)\zeta_3}{27N(1+N)}\Biggr]
$$

$$
+C_A^2\Biggl[-\frac{4S_{-2,1,-2}P_{745}}{3N(1+N)}-\frac{4S_{-2,3}P_{761}}{9N(1+N)}+\frac{2S_5P_{763}}{9N(1+N)}-\frac{12\zeta_2^2P_{764}}{5N^2(1+N)^2}+\frac{S_2^2P_{779}}{9N^2(1+N)^2}
-\frac{8S_{-2,2}P_{811}}{27(-1+N)N^2(1+N)^2(2+N)}+\frac{4S_{-3,1}P_{813}}{9(-1+N)N^2(1+N)^2(2+N)}
-\frac{4S_{3,1}P_{814}}{27(-1+N)N^2(1+N)^2(2+N)}+\frac{8S_{-2,1,1}P_{824}}{27(-1+N)N^2(1+N)^2(2+N)}
-\frac{2S_4P_{831}}{9(-1+N)N^2(1+N)^2(2+N)}+\frac{2\zeta_3P_{860}}{27(-1+N)N^3(1+N)^3(2+N)}
+\frac{P_{882}}{5832(-1+N)^2N^6(1+N)^6(2+N)}+\Biggl(-\frac{8S_{-2,1}P_{825}}{27(-1+N)N^2(1+N)^2(2+N)}
+\frac{4S_3P_{829}}{27(-1+N)N^2(1+N)^2(2+N)}-\frac{8\zeta_3P_{833}}{27(-1+N)N^2(1+N)^2(2+N)}
-\frac{4S_{2,1}P_{781}}{9N^2(1+N)^2}+\frac{P_{877}}{729(-1+N)N^5(1+N)^5(2+N)}+\Biggl(\frac{P_{845}}{27N^3(1+N)^3(2+N)}
+\frac{1184}{3}S_3-32S_{2,1}+1376S_{-2,1}-64\zeta_3\Biggr)S_2-\frac{4\big(-6+145N+145N^2\big)S_2^2}{9N(1+N)}
-\frac{4\big(-162+893N+893N^2\big)S_4}{9N(1+N)}-1040S_{2,-3}+\frac{16\big(-73+48N+48N^2\big)S_{3,1}}{3N(1+N)}
+\frac{448}{3}S_5-\frac{1120}{3}S_{2,3}-\frac{176}{3}S_{4,1}+\frac{32\big(105+11N+11N^2\big)S_{-2,2}}{9N(1+N)}+\frac{2912}{3}S_{-2,3}
$$

$$
-\frac{64}{3}S_{-2,-3}+\frac{64\big(12+11N+11N^2\big)S_{-3,1}}{3N(1+N)}+\frac{1184}{3}S_{-4,1}+\frac{8\big(12+55N+55N^2\big)S_{2,1,1}}{3N(1+N)}
$$

$$
+992S_{2,1,-2}+64S_{2,2,1}-\frac{2816}{3}S_{3,1,1}+\frac{128\big(-69+22N+22N^2\big)S_{-2,1,1}}{9N(1+N)}
$$

$$
+\frac{3424}{3}S_{-2,1,-2}+\frac{3200}{3}S_{-2,2,1}+\frac{1600}{3}S_{-3,1,1}+128S_{2,1,1,1}-3200S_{-2,1,1,1}+240\zeta_5\bigg)S_1
$$

$$
+\bigg(\frac{2S_2P_{774}}{9N^2(1+N)^2}+\frac{P_{863}}{162N^4(1+N)^4}-\frac{8}{3}S_2^2+\frac{4\big(240+11N+11N^2\big)S_3}{9N(1+N)}-72S_4
$$

$$
-\frac{4\big(12+209N+209N^2\big)S_{2,1}}{9N(1+N)}+\frac{1168}{3}S_{3,1}-\frac{352\big(-9+5N+5N^2\big)S_{-2,1}}{9N(1+N)}
$$

$$
-\frac{1888}{3}S_{-2,2}-512S_{-3,1}-32S_{2,1,1}+\frac{4480}{3}S_{-2,1,1}-\frac{16\big(4+11N+11N^2\big)\zeta_3}{N(1+N)}\bigg)S_1^2
$$

$$
+\bigg(\frac{-726+4649N+4649N^2}{81N(1+N)}+\frac{616}{27}S_2-\frac{640}{9}S_3+\frac{32}{9}S_{2,1}-320S_{-2,1}\bigg)S_1^3+\frac{121}{27}S_1^4
$$

$$
+\bigg(\frac{P_{864}}{54N^4(1+N)^4(2+N)}+\frac{296\big(-18+13N+13N^2\big)S_3}{27N(1+N)}+\frac{424}{3}S_4
$$

$$
+\frac{16\big(1+3N+3N^2\big)S_{2,1}}{N(1+N)}-\frac{1168}{3}S_{3,1}+\frac{16\big(-57+22N+22N^2\big)S_{-2,1}}{3N(1+N)}+352S_{-2,2}
$$

$$
-\frac{1040}{3}S_{-3,1}+\frac{272}{3}S_{2,1,1}-1088S_{-2,1,1}+\frac{32\big(1+18N+18N^2\big)\zeta_3}{N(1+N)}\bigg)S_2-\frac{56}{9}S_2^3
$$

$$
+\bigg(\frac{2P_{862}}{81(-1+N)N^3(1+N)^3(2+N)}+\frac{880}{9}S_{2,1}-\frac{2080}{3}S_{-2,1}-128\zeta_3\bigg)S_3-\frac{1280}{9}S_3^2
$$

$$
+\frac{80}{9}S_6+\bigg(\frac{4S_{-2,1}P_{752}}{3N(1+N)}+\frac{4\zeta_3P_{756}}{3N(1+N)}+\frac{4S_3P_{767}}{27N(1+N)}-\frac{4S_2P_{785}}{27N^2(1+N)^2}
$$

$$
+\frac{2P_{875}}{81(-1+N)^2N^4(1+N)^4(2+N)}+\bigg(\frac{4P_{858}}{81(-1+N)N^3(1+N)^3(2+N)}
$$

$$
-\frac{32\big(-3+22N+22N^2\big)S_2}{3N(1+N)}-\frac{320}{3}S_3-960S_{2,1}-160S_{-2,1}-320\zeta_3\bigg)S_1
$$

$$
+\bigg(\frac{4P_{780}}{9N^2(1+N)^2}-32S_2\bigg)S_1^2+\frac{1760}{27}S_1^3-\frac{16}{3}S_2^2+\frac{32\big(27+22N+22N^2\big)S_{2,1}}{9N(1+N)}
$$

$$
-\frac{1648}{3}S_4+768S_{3,1}+\frac{1280}{3}S_{-2,2}+\frac{1600}{3}S_{-3,1}-\frac{3584}{3}S_{-2,1,1}-\frac{96}{5}\zeta_2^2\bigg)S_{-2}
$$

$$
+\bigg(-\frac{8P_{800}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{8\big(-38+47N+47N^2\big)S_1}{3N(1+N)}-\frac{400}{3}S_1^2
$$

$$
+\frac{208}{3}S_2\bigg)S_{-2}^2+\frac{32}{9}S_{-2}^3+\bigg(-\frac{2P_{857}}{81(-1+N)N^3(1+N)^3(2+N)}
$$

$$
+\Biggl(-\frac{4P_{819}}{27(-1+N)N^2(1+N)^2(2+N)}-144S_2\Biggr)S_1-\frac{16\bigl(117+77N+77N^2\bigr)S_1^2}{9N(1+N)}
+\frac{544}{3}S_1^3+\frac{40\bigl(-27+22N+22N^2\bigr)S_2}{9N(1+N)}+\frac{2080}{3}S_3+\Biggl(\frac{8\bigl(40+13N+13N^2\bigr)}{3N(1+N)}
-\frac{832}{3}S_1\Biggr)S_{-2}+\frac{80}{3}S_{2,1}-\frac{3136}{3}S_{-2,1}+32\zeta_3\Bigr)S_{-3}+\frac{1072}{3}S_{-3}^2+\Biggl(\frac{320}{3}S_1^2
-\frac{4P_{826}}{27(-1+N)N^2(1+N)^2(2+N)}-\frac{8\bigl(12+53N+53N^2\bigr)S_1}{9N(1+N)}-\frac{320}{3}S_2-\frac{736}{3}S_{-2}\Biggr)
\times S_{-4}+\Biggl(\frac{2P_{762}}{9N(1+N)}-208S_1\Biggr)S_{-5}+\frac{544}{9}S_{-6}+\Biggl(-\frac{4P_{847}}{27N^3(1+N)^3(2+N)}
+96\zeta_3\Biggr)S_{2,1}-\frac{16\bigl(-35+52N+52N^2\bigr)S_{2,3}}{3N(1+N)}+\frac{8\bigl(369+106N+106N^2\bigr)S_{2,-3}}{9N(1+N)}
-\frac{56}{3}S_{2,1}^2+\frac{8\bigl(33+367N+367N^2\bigr)S_{4,1}}{9N(1+N)}-\frac{952}{3}S_{4,2}+\frac{1600}{3}S_{4,-2}-\frac{2144}{3}S_{5,1}
+\Biggl(-\frac{4P_{856}}{81(-1+N)N^3(1+N)^3(2+N)}-32S_{2,1}+192\zeta_3\Biggr)S_{-2,1}+\frac{32S_{-2,-3}}{3N(1+N)}
+\frac{2240}{3}S_{-2,1}^2-\frac{2848}{3}S_{-3,3}+\frac{8\bigl(-140+57N+57N^2\bigr)S_{-4,1}}{3N(1+N)}+\frac{2096}{3}S_{-4,2}
-224S_{-5,1}+\frac{4\bigl(-84+607N+562N^2\bigr)S_{2,1,1}}{9N(1+N)}-\frac{16\bigl(63+44N+44N^2\bigr)S_{2,1,-2}}{9N(1+N)}
+\frac{1216}{3}S_{-4,-2}+\frac{4\bigl(-24+29N+29N^2\bigr)S_{2,2,1}}{3N(1+N)}+\frac{640}{3}S_{2,3,1}+\frac{2096}{3}S_{2,-3,1}
-\frac{16\bigl(-264+287N+287N^2\bigr)S_{3,1,1}}{9N(1+N)}-\frac{1952}{3}S_{3,1,-2}-\frac{208}{3}S_{3,2,1}+\frac{320}{3}S_{4,1,1}
-\frac{64\bigl(21+11N+11N^2\bigr)S_{-2,2,1}}{9N(1+N)}-\frac{832}{3}S_{-2,2,2}-\frac{64}{3}S_{-2,2,-2}-\frac{1984}{3}S_{-2,3,1}
-\frac{352\bigl(-1+2N+2N^2\bigr)S_{-3,1,1}}{3N(1+N)}-320S_{-3,1,-2}+\frac{112}{3}S_{-3,2,1}+\frac{704}{3}S_{-3,-2,1}
-\frac{4480}{3}S_{-4,1,1}-\frac{16\bigl(36+11N+11N^2\bigr)S_{2,1,1,1}}{9N(1+N)}-\frac{272}{3}S_{2,2,1,1}+384S_{2,-2,1,1}
+\frac{2720}{3}S_{3,1,1,1}-\frac{64\bigl(-117+22N+22N^2\bigr)S_{-2,1,1,1}}{9N(1+N)}+96S_{-2,1,1,2}-64S_{-2,2,1,1}
+\frac{1408}{3}S_{-2,-2,1,1}+384S_{-3,1,1,1}-\frac{640}{3}S_{2,1,1,1,1}+2560S_{-2,1,1,1,1}
+\frac{5(-5+N)(6+N)\bigl(8+N+N^2\bigr)\zeta_5}{3N(1+N)}\Biggr]\Biggr\}
$$

$$
+C_F^3\Biggl\{\frac{S_2^2P_{743}}{3N^2(1+N)^2}-\frac{16S_{-2,1,-2}P_{746}}{3N(1+N)}-\frac{16S_{-2,3}P_{748}}{3N(1+N)}+\frac{8S_5P_{755}}{3N(1+N)}+\frac{20\zeta_5P_{757}}{3N(1+N)}
$$

$$
\begin{aligned}
&-\frac{24\zeta_2^2 P_{764}}{5N^2(1+N)^2}+\frac{8S_{2,1,1}P_{769}}{N^2(1+N)^2}-\frac{8S_{-3,1}P_{798}}{3(-1+N)N^2(1+N)^2(2+N)}\\
&-\frac{16S_{-2,2}P_{801}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{8S_{3,1}P_{810}}{3(-1+N)N^2(1+N)^2(2+N)}\\
&+\frac{16S_{-2,1,1}P_{812}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{2S_4P_{816}}{3(-1+N)N^2(1+N)^2(2+N)}\\
&-\frac{4\zeta_3P_{853}}{3(-1+N)N^3(1+N)^3(2+N)}+\frac{P_{883}}{24(-1+N)^2N^6(1+N)^6(2+N)}\\
&+\Biggl(\frac{2S_3P_{804}}{3(-1+N)N^2(1+N)^2(2+N)}-\frac{16S_{-2,1}P_{809}}{3(-1+N)N^2(1+N)^2(2+N)}\\
&-\frac{4S_{2,1}P_{771}}{3N^2(1+N)^2}-\frac{8\zeta_3P_{803}}{(N-1)N^2(1+N)^2(2+N)}+\frac{P_{880}}{24(N-1)^2N^5(1+N)^5(2+N)}\\
&+\Biggl(\frac{P_{827}}{6N^3(1+N)^3}+\frac{976}{3}S_3-\frac{64}{3}S_{2,1}+\frac{13696}{3}S_{-2,1}-416\zeta_3\Biggr)S_2\\
&+\frac{2\bigl(-54+25N+25N^2\bigr)S_2^2}{N(1+N)}-\frac{4\bigl(-2+85N+85N^2\bigr)S_4}{N(1+N)}-\frac{2560}{3}S_5-64S_{2,3}\\
&-3680S_{2,-3}+\frac{8\bigl(-106+109N+109N^2\bigr)S_{3,1}}{N(1+N)}-\frac{16\bigl(-274+9N+9N^2\bigr)S_{-2,2}}{3N(1+N)}\\
&-\frac{1696}{3}S_{4,1}+\frac{11392}{3}S_{-2,3}-\frac{128}{3}S_{-2,-3}+\frac{64\bigl(59+6N+6N^2\bigr)S_{-3,1}}{3N(1+N)}+1088S_{-4,1}\\
&-\frac{24\bigl(-2+3N+3N^2\bigr)S_{2,1,1}}{N(1+N)}+\frac{11200}{3}S_{2,1,-2}+\frac{64\bigl(-52+7N+7N^2\bigr)S_{-2,1,1}}{N(1+N)}\\
&-\frac{1984}{3}S_{3,1,1}+3264S_{-2,1,-2}+\frac{11648}{3}S_{-2,2,1}+\frac{7168}{3}S_{-3,1,1}-10752S_{-2,1,1,1}-1120\zeta_5\\
&-\frac{160}{3}S_{2,2,1}\Biggr)S_1+\Biggl(\frac{2S_2P_{777}}{3N^2(1+N)^2}+\frac{P_{870}}{12(N-1)N^4(1+N)^4(2+N)}+108S_2^2\\
&-\frac{4\bigl(-46+75N+75N^2\bigr)S_3}{N(1+N)}+344S_4+\frac{24\bigl(-2+3N+3N^2\bigr)S_{2,1}}{N(1+N)}+336S_{3,1}\\
&-\frac{32\bigl(-34+3N+3N^2\bigr)S_{-2,1}}{N(1+N)}-\frac{6880}{3}S_{-2,2}-\frac{6464}{3}S_{-3,1}-48S_{2,1,1}+4864S_{-2,1,1}\\
&-\frac{352\zeta_3}{N(1+N)}\Biggr)S_1^2+\Biggl(\frac{P_{794}}{6N^3(1+N)^3}-\frac{12\bigl(-6+7N+7N^2\bigr)S_2}{N(1+N)}-48S_3+32S_{2,1}\\
&-960S_{-2,1}+96\zeta_3\Biggr)S_1^3+\Biggl(\frac{P_{744}}{N^2(1+N)^2}-36S_2\Biggr)S_1^4+\frac{2\bigl(-2+3N+3N^2\bigr)S_1^5}{N(1+N)}\\
&+\frac{4}{3}S_1^6+\Biggl(\frac{P_{869}}{12(-1+N)N^4(1+N)^4(2+N)}+\frac{4\bigl(-398+513N+513N^2\bigr)S_3}{3N(1+N)}+40S_4\\
&-\frac{8\bigl(-4+15N+15N^2\bigr)S_{2,1}}{3N(1+N)}-432S_{3,1}-\frac{16\bigl(218+9N+9N^2\bigr)S_{-2,1}}{3N(1+N)}+\frac{3296}{3}S_{-2,2}\\
&-\frac{1120}{3}S_{-3,1}+\frac{272}{3}S_{2,1,1}-\frac{12416}{3}S_{-2,1,1}+\frac{32\bigl(5+12N+12N^2\bigr)\zeta_3}{N(1+N)}\Biggr)S_2-\frac{148}{9}S_2^3
\end{aligned}
$$

$$+\left(\frac{P_{854}}{3(-1+N)N^3(1+N)^3(2+N)}+\frac{64}{3}S_{2,1}-\frac{7616}{3}S_{-2,1}-\frac{320}{3}\zeta_3\right)S_3-\frac{512}{3}S_3^2$$

$$-\frac{1664}{9}S_6+\left(\frac{16\zeta_3P_{747}}{3N(1+N)}+\frac{16S_3P_{749}}{3N(1+N)}-\frac{8S_2P_{802}}{3(-1+N)N^2(1+N)^2(2+N)}\right.$$

$$+\frac{16S_{-2,1}P_{750}}{3N(1+N)}+\frac{4P_{874}}{3(-1+N)^2N^4(1+N)^4(2+N)}-\left(\frac{2368}{3}S_3+\frac{11008}{3}S_{2,1}\right.$$

$$\left.+\frac{8P_{865}}{3(N-1)^2N^3(1+N)^3(2+N)}+\frac{16\big(16+3N+3N^2\big)S_2}{N(1+N)}+\frac{704}{3}S_{-2,1}+1216\zeta_3\right)S_1$$

$$+\left(\frac{8P_{807}}{3(-1+N)N^2(1+N)^2(2+N)}+352S_2\right)S_1^2-\frac{96\big(-1+N+N^2\big)S_1^3}{N(1+N)}-64S_1^4$$

$$-64S_2^2-\frac{6752}{3}S_4+\frac{64\big(32+9N+9N^2\big)S_{2,1}}{3N(1+N)}+3200S_{3,1}+\frac{5248}{3}S_{-2,2}+\frac{5888}{3}S_{-3,1}$$

$$\left.-\frac{12544}{3}S_{-2,1,1}-\frac{192}{5}\zeta_2^2\right)S_{-2}+\left(-\frac{752}{3}S_1^2-\frac{4P_{799}}{3(-1+N)N^2(1+N)^2(2+N)}\right.$$

$$\left.+\frac{176}{3}S_2+\frac{8\big(70+27N+27N^2\big)S_1}{3N(1+N)}\right)S_{-2}^2+\frac{64}{9}S_{-2}^3+\left(\left(\left(-\frac{2624}{3}S_2\right.\right.\right.$$

$$\left.+\frac{8P_{797}}{3(-1+N)N^2(1+N)^2(2+N)}\right)S_1-\frac{8P_{852}}{3(-1+N)N^3(1+N)^3(2+N)}$$

$$-\frac{16\big(64+3N+3N^2\big)S_1^2}{N(1+N)}+\frac{2656}{3}S_1^3+\frac{8\big(-46+111N+111N^2\big)S_2}{3N(1+N)}+\frac{8000}{3}S_3$$

$$\left.+\left(-\frac{16\big(-44+9N+9N^2\big)}{N(1+N)}-1600S_1\right)S_{-2}+\frac{160}{3}S_{2,1}-\frac{8960}{3}S_{-2,1}\right)S_{-3}$$

$$+\frac{2816}{3}S_{-3}^2+\left(-\frac{4P_{796}}{3(-1+N)N^2(1+N)^2(2+N)}+\frac{8\big(-26+81N+81N^2\big)S_1}{3N(1+N)}\right.$$

$$\left.+\frac{1456}{3}S_1^2-\frac{1520}{3}S_2-\frac{2816}{3}S_{-2}\right)S_{-4}+\left(\frac{8P_{751}}{3N(1+N)}-\frac{5344S_1}{3}\right)S_{-5}+\frac{1088}{9}S_{-6}$$

$$+\left(-\frac{4P_{808}}{3N^3(1+N)^3}+192\zeta_3\right)S_{2,1}-\frac{64}{3}S_{2,1}^2-\frac{32\big(-13+17N+17N^2\big)S_{2,3}}{N(1+N)}$$

$$+\frac{8\big(158+7N+7N^2\big)S_{2,-3}}{N(1+N)}+\frac{8\big(-38+135N+135N^2\big)S_{4,1}}{3N(1+N)}-\frac{1088}{3}S_{4,2}$$

$$+\frac{6080}{3}S_{4,-2}-1120S_{5,1}+\left(\frac{16P_{851}}{3(-1+N)N^3(1+N)^3(2+N)}-64S_{2,1}+512\zeta_3\right)$$

$$\times S_{-2,1}-\frac{32\big(-2+3N+3N^2\big)S_{-2,-3}}{3N(1+N)}+\frac{16\big(-68+25N+25N^2\big)S_{-4,1}}{N(1+N)}$$

$$-\frac{9344}{3}S_{-3,3}+\frac{5920}{3}S_{-4,2}+\frac{2432}{3}S_{-4,-2}-\frac{16\big(134+39N+39N^2\big)S_{2,1,-2}}{3N(1+N)}$$

$$+320S_{-5,1}+\frac{80S_{2,2,1}}{3N(1+N)}+\frac{352}{3}S_{2,3,1}-\frac{16\big(-206+195N+195N^2\big)S_{3,1,1}}{3N(1+N)}$$
$$+\frac{4576}{3}S_{2,-3,1}-\frac{8896}{3}S_{3,1,-2}+\frac{64}{3}S_{3,2,1}+\frac{128}{3}S_{4,1,1}-\frac{32\big(74+15N+15N^2\big)S_{-2,2,1}}{3N(1+N)}$$
$$-\frac{896}{3}S_{-2,2,2}-\frac{128}{3}S_{-2,2,-2}-\frac{8960}{3}S_{-2,3,1}-\frac{32\big(4+39N+39N^2\big)S_{-3,1,1}}{3N(1+N)}$$
$$-640S_{-3,1,-2}+\frac{224}{3}S_{-3,2,1}+\frac{1408}{3}S_{-3,-2,1}-\frac{15104}{3}S_{-4,1,1}+\frac{64}{3}S_{2,2,1,1}$$
$$+2048S_{2,-2,1,1}+704S_{3,1,1,1}-\frac{192\big(-16+N+N^2\big)S_{-2,1,1,1}}{N(1+N)}+192S_{-2,1,1,2}$$
$$\left.-128S_{-2,2,1,1}+\frac{2816}{3}S_{-2,-2,1,1}+256S_{-3,1,1,1}+9216S_{-2,1,1,1,1}+\frac{7168}{3}S_{-2,1}^2\right\} \tag{919}$$

with the polynomials

$$P_{743} = -617N^4-1180N^3-485N^2+90N+52, \tag{920}$$
$$P_{744} = -9N^4-12N^3-9N^2+22N+20, \tag{921}$$
$$P_{745} = N^4+2N^3-293N^2-294N+356, \tag{922}$$
$$P_{746} = N^4+2N^3-120N^2-121N+258, \tag{923}$$
$$P_{747} = N^4+2N^3+15N^2+14N+60, \tag{924}$$
$$P_{748} = N^4+2N^3+15N^2+14N+194, \tag{925}$$
$$P_{749} = N^4+2N^3+24N^2+23N+68, \tag{926}$$
$$P_{750} = N^4+2N^3+42N^2+41N+10, \tag{927}$$
$$P_{751} = N^4+2N^3+90N^2+89N+166, \tag{928}$$
$$P_{752} = N^4+2N^3+111N^2+110N+36, \tag{929}$$
$$P_{753} = N^4+2N^3+115N^2+114N+54, \tag{930}$$
$$P_{754} = N^4+2N^3+195N^2+194N-156, \tag{931}$$
$$P_{755} = N^4+2N^3+204N^2+203N+40, \tag{932}$$
$$P_{756} = N^4+2N^3+215N^2+214N+48, \tag{933}$$
$$P_{757} = N^4+2N^3+303N^2+302N-144, \tag{934}$$
$$P_{758} = 2N^4+4N^3-413N^2-415N+614, \tag{935}$$
$$P_{759} = 2N^4+4N^3+153N^2+151N+46, \tag{936}$$
$$P_{760} = 3N^4+6N^3+100N^2+97N+495, \tag{937}$$
$$P_{761} = 3N^4+6N^3+155N^2+152N+408, \tag{938}$$
$$P_{762} = 3N^4+6N^3+1321N^2+1318N-252, \tag{939}$$
$$P_{763} = 3N^4+6N^3+1435N^2+1432N-480, \tag{940}$$
$$P_{764} = 5N^4+10N^3+9N^2+4N+4, \tag{941}$$
$$P_{765} = 6N^4+12N^3+1591N^2+1585N+246, \tag{942}$$
$$P_{766} = 6N^4+12N^3+1871N^2+1865N-216, \tag{943}$$
$$P_{767} = 9N^4+18N^3+925N^2+916N+144, \tag{944}$$
$$P_{768} = 18N^4+36N^3+1141N^2+1123N+756, \tag{945}$$
$$P_{769} = 29N^4+51N^3+16N^2-20N-6, \tag{946}$$

$$P_{770} = 83N^4 + 166N^3 + 239N^2 + 192N + 63, \quad (947)$$
$$P_{771} = 85N^4 + 134N^3 + 57N^2 - 80N - 52, \quad (948)$$
$$P_{772} = 97N^4 + 194N^3 - 121N^2 - 218N - 6, \quad (949)$$
$$P_{773} = 131N^4 + 262N^3 - 23N^2 - 154N + 216, \quad (950)$$
$$P_{774} = 147N^4 + 294N^3 + 38N^2 - 121N + 6, \quad (951)$$
$$P_{775} = 235N^4 + 524N^3 + 211N^2 + 30N + 72, \quad (952)$$
$$P_{776} = 277N^4 + 536N^3 + 200N^2 - 143N - 36, \quad (953)$$
$$P_{777} = 291N^4 + 488N^3 + 217N^2 - 304N - 256, \quad (954)$$
$$P_{778} = 361N^4 + 722N^3 - 433N^2 - 794N - 72, \quad (955)$$
$$P_{779} = 423N^4 + 792N^3 + 413N^2 + 44N - 90, \quad (956)$$
$$P_{780} = 536N^4 + 1075N^3 + 601N^2 + 62N + 162, \quad (957)$$
$$P_{781} = 536N^4 + 1087N^3 + 414N^2 - 149N + 6, \quad (958)$$
$$P_{782} = 575N^4 + 1084N^3 + 420N^2 - 365N - 150, \quad (959)$$
$$P_{783} = 683N^4 + 1510N^3 + 479N^2 + 68N + 292, \quad (960)$$
$$P_{784} = 1055N^4 + 2164N^3 + 1031N^2 + 30N + 72, \quad (961)$$
$$P_{785} = 1108N^4 + 2261N^3 + 1099N^2 - 126N + 288, \quad (962)$$
$$P_{786} = 1124N^4 + 2224N^3 + 847N^2 - 331N - 57, \quad (963)$$
$$P_{787} = 1687N^4 + 3374N^3 - 1657N^2 - 3344N + 372, \quad (964)$$
$$P_{788} = 4513N^4 + 9026N^3 + 2815N^2 - 1698N + 612, \quad (965)$$
$$P_{789} = 112N^5 + 313N^4 + 188N^3 - 73N^2 + 8N + 52, \quad (966)$$
$$P_{790} = 328N^5 + 1513N^4 + 2150N^3 + 959N^2 - 102N - 120, \quad (967)$$
$$P_{791} = 1103N^5 + 4010N^4 + 4309N^3 + 1096N^2 - 60N + 240, \quad (968)$$
$$P_{792} = 7241N^5 + 26102N^4 + 28297N^3 + 8428N^2 - 540N - 384, \quad (969)$$
$$P_{793} = 9187N^5 + 33508N^4 + 33155N^3 + 2138N^2 + 2844N + 8136, \quad (970)$$
$$P_{794} = -279N^6 - 705N^5 - 663N^4 - 159N^3 - 402N^2 - 816N - 304, \quad (971)$$
$$P_{795} = 25N^6 + 75N^5 - 161N^4 - 303N^3 + 901N^2 + 705N + 54, \quad (972)$$
$$P_{796} = 33N^6 + 9N^5 - 669N^4 - 1433N^3 - 1180N^2 + 368N - 8, \quad (973)$$
$$P_{797} = 34N^6 + 114N^5 + 695N^4 + 1472N^3 + 1051N^2 - 430N - 632, \quad (974)$$
$$P_{798} = 36N^6 + 138N^5 + 969N^4 + 2040N^3 + 1355N^2 - 682N - 400, \quad (975)$$
$$P_{799} = 37N^6 + 49N^5 + 161N^4 + 491N^3 + 242N^2 - 188N - 216, \quad (976)$$
$$P_{800} = 49N^6 + 148N^5 + 110N^4 - 4N^3 - 24N^2 - 61N - 74, \quad (977)$$
$$P_{801} = 61N^6 + 190N^5 + 530N^4 + 926N^3 + 617N^2 - 320N - 276, \quad (978)$$
$$P_{802} = 74N^6 + 206N^5 + 5N^4 - 328N^3 - 203N^2 + 94N + 8, \quad (979)$$
$$P_{803} = 79N^6 + 215N^5 + 73N^4 - 143N^3 - 52N^2 + 36N - 96, \quad (980)$$
$$P_{804} = 81N^6 + 103N^5 - 651N^4 - 1335N^3 - 570N^2 + 748N + 568, \quad (981)$$
$$P_{805} = 83N^6 - 293N^5 + 15N^4 - 23N^3 + 170N^2 + 1040N + 704, \quad (982)$$
$$P_{806} = 83N^6 + 249N^5 + 164N^4 - 33N^3 + 14N^2 - 63N - 90, \quad (983)$$
$$P_{807} = 96N^6 + 250N^5 + 49N^4 - 300N^3 - 333N^2 + 78N + 16, \quad (984)$$
$$P_{808} = 100N^6 + 349N^5 + 301N^4 + 31N^3 + 79N^2 + 128N + 76, \quad (985)$$
$$P_{809} = 158N^6 + 454N^5 + 651N^4 + 780N^3 + 471N^2 - 290N - 496, \quad (986)$$

$$P_{810} = 173N^6 + 466N^5 - 273N^4 - 1264N^3 - 472N^2 + 462N + 380, \quad (987)$$
$$P_{811} = 182N^6 + 528N^5 + 2678N^4 + 5724N^3 + 4385N^2 - 1383N - 1746, \quad (988)$$
$$P_{812} = 208N^6 + 622N^5 + 1151N^4 + 1664N^3 + 1113N^2 - 598N - 704, \quad (989)$$
$$P_{813} = 248N^6 + 771N^5 - 1216N^4 - 4599N^3 - 3616N^2 + 828N + 672, \quad (990)$$
$$P_{814} = 379N^6 + 1065N^5 - 3293N^4 - 9309N^3 - 5402N^2 + 2304N + 3240, \quad (991)$$
$$P_{815} = 388N^6 + 1128N^5 - 5339N^4 - 15318N^3 - 11297N^2 + 3702N + 2544, \quad (992)$$
$$P_{816} = 403N^6 + 1191N^5 + 633N^4 - 599N^3 - 440N^2 - 116N - 16, \quad (993)$$
$$P_{817} = 417N^6 + 900N^5 - 253N^4 - 1298N^3 - 458N^2 + 20N - 192, \quad (994)$$
$$P_{818} = 429N^6 + 1233N^5 + 1041N^4 + 459N^3 + 50N^2 - 676N - 808, \quad (995)$$
$$P_{819} = 676N^6 + 2109N^5 - 2858N^4 - 10617N^3 - 5747N^2 + 2811N + 3258, \quad (996)$$
$$P_{820} = 781N^6 + 1227N^5 - 5909N^4 - 14895N^3 - 12548N^2 + 5496N + 8568, \quad (997)$$
$$P_{821} = 913N^6 + 2766N^5 + 10126N^4 + 19782N^3 + 14323N^2 - 5646N - 5976, \quad (998)$$
$$P_{822} = 1046N^6 + 3192N^5 - 11971N^4 - 34482N^3 - 20953N^2 + 9492N + 12204, \quad (999)$$
$$P_{823} = 1360N^6 + 3972N^5 + 1355N^4 - 3874N^3 - 2955N^2 + 310N - 600, \quad (1000)$$
$$P_{824} = 1472N^6 + 4425N^5 + 7064N^4 + 9099N^3 + 6692N^2 - 3048N - 4968, \quad (1001)$$
$$P_{825} = 1540N^6 + 4629N^5 + 4822N^4 + 3087N^3 + 1945N^2 - 1785N - 3870, \quad (1002)$$
$$P_{826} = 1540N^6 + 4701N^5 + 1444N^4 - 5937N^3 - 7331N^2 - 1671N - 522, \quad (1003)$$
$$P_{827} = 1545N^6 + 4511N^5 + 3537N^4 + 737N^3 + 1222N^2 + 2384N + 784, \quad (1004)$$
$$P_{828} = 1873N^6 + 5331N^5 + 2385N^4 - 3299N^3 - 2470N^2 + 44N - 1368, \quad (1005)$$
$$P_{829} = 1964N^6 + 5856N^5 + 68N^4 - 10116N^3 - 6055N^2 + 1461N + 1314, \quad (1006)$$
$$P_{830} = 2251N^6 + 6552N^5 + 1826N^4 - 6688N^3 - 3669N^2 + 292N - 132, \quad (1007)$$
$$P_{831} = 2284N^6 + 6951N^5 + 3143N^4 - 5233N^3 - 2970N^2 - 53N - 450, \quad (1008)$$
$$P_{832} = 2882N^6 + 8592N^5 + 2333N^4 - 10050N^3 - 5899N^2 + 1926N - 1080, \quad (1009)$$
$$P_{833} = 3098N^6 + 9240N^5 + 3107N^4 - 8898N^3 - 5665N^2 - 18N - 1188, \quad (1010)$$
$$P_{834} = 3153N^6 + 7047N^5 - 1489N^4 - 9635N^3 - 5120N^2 - 148N - 720, \quad (1011)$$
$$P_{835} = 4357N^6 + 16149N^5 + 16977N^4 + 9091N^3 + 3798N^2 - 1404N - 1944, \quad (1012)$$
$$P_{836} = 4502N^6 + 13344N^5 + 15503N^4 + 13194N^3 + 8129N^2 - 6180N - 12204, \quad (1013)$$
$$P_{837} = 4591N^6 + 13935N^5 + 6934N^4 - 8871N^3 - 5873N^2 - 1464N - 612, \quad (1014)$$
$$P_{838} = 4645N^6 + 13341N^5 - 4124N^4 - 31599N^3 - 16541N^2 + 8430N + 8568, \quad (1015)$$
$$P_{839} = 4816N^6 + 14448N^5 + 24487N^4 + 33174N^3 + 23401N^2 - 11478N - 16272, \quad (1016)$$
$$P_{840} = 6457N^6 + 18885N^5 - 245N^4 - 36645N^3 - 39944N^2 - 3372N - 2160, \quad (1017)$$
$$P_{841} = 7531N^6 + 23619N^5 + 23253N^4 + 7825N^3 + 2064N^2 + 1080N - 252, \quad (1018)$$
$$P_{842} = 8425N^6 + 26643N^5 + 24471N^4 + 6857N^3 + 2520N^2 + 1484N - 864, \quad (1019)$$
$$P_{843} = -116957N^7 - 556705N^6 - 889137N^5 - 480947N^4 - 48370N^3 - 93780N^2 - 76536N + 2592, \quad (1020)$$
$$P_{844} = -28023N^7 - 129045N^6 - 212881N^5 - 149635N^4 - 36908N^3 - 4628N^2 - 17376N - 9216, \quad (1021)$$
$$P_{845} = -3221N^7 - 19897N^6 - 43683N^5 - 43139N^4 - 15736N^3 + 3828N^2 + 3720N + 720, \quad (1022)$$
$$P_{846} = 1511N^7 + 6361N^6 + 8811N^5 + 3500N^4 - 61N^3 + 1842N^2 + 1772N + 552, \quad (1023)$$

$$P_{847} = 3691N^7 + 16568N^6 + 25248N^5 + 13189N^4 + 236N^3 + 1188N^2 + 1860N +360, \quad (1024)$$

$$P_{848} = 32287N^7 + 161273N^6 + 288291N^5 + 220675N^4 + 64886N^3 + 4416N^2 +1188N - 792, \quad (1025)$$

$$P_{849} = -97249N^8 - 360322N^7 - 305212N^6 + 226586N^5 + 292481N^4 - 76528N^3 -50508N^2 + 100320N + 63072, \quad (1026)$$

$$P_{850} = -5021N^8 - 15158N^7 - 6690N^6 + 17488N^5 + 14927N^4 - 3058N^3 + 1392N^2 +6920N + 4560, \quad (1027)$$

$$P_{851} = 25N^8 + 105N^7 + 101N^6 + 52N^5 + 1332N^4 + 2547N^3 + 890N^2 - 932N -664, \quad (1028)$$

$$P_{852} = 79N^8 + 323N^7 + 315N^6 + 22N^5 + 1546N^4 + 3031N^3 + 1088N^2 - 1076N -720, \quad (1029)$$

$$P_{853} = 307N^8 + 1174N^7 + 1142N^6 - 608N^5 - 2451N^4 - 2370N^3 - 250N^2 +888N + 824, \quad (1030)$$

$$P_{854} = 575N^8 + 2226N^7 + 1974N^6 - 1440N^5 + 15N^4 + 5006N^3 + 1172N^2 - 3320N -1984, \quad (1031)$$

$$P_{855} = 1288N^8 + 4747N^7 + 10549N^6 + 7126N^5 - 99269N^4 - 195599N^3 - 68574N^2 +70380N + 51624, \quad (1032)$$

$$P_{856} = 1963N^8 + 7582N^7 + 13276N^6 + 8530N^5 - 63305N^4 - 126830N^3 - 44544N^2 +45216N + 33696, \quad (1033)$$

$$P_{857} = 3761N^8 + 15422N^7 + 6284N^6 - 22930N^5 + 78905N^4 + 182846N^3 + 61872N^2 -51408N - 36288, \quad (1034)$$

$$P_{858} = 4258N^8 + 16762N^7 + 18841N^6 - 1469N^5 - 28625N^4 - 39305N^3 - 12672N^2 +15102N + 12636, \quad (1035)$$

$$P_{859} = 5894N^8 + 24143N^7 + 14789N^6 - 22336N^5 + 120647N^4 + 264683N^3 +91248N^2 - 80460N - 55728, \quad (1036)$$

$$P_{860} = 17529N^8 + 56958N^7 + 34840N^6 - 52470N^5 - 72773N^4 - 21696N^3 - 4220N^2 -10440N - 7344, \quad (1037)$$

$$P_{861} = 28551N^8 + 92280N^7 + 118370N^6 + 46548N^5 - 11961N^4 + 10748N^3 +12600N^2 - 14112N - 9936, \quad (1038)$$

$$P_{862} = 43949N^8 + 169784N^7 + 155918N^6 - 99922N^5 - 174643N^4 - 20848N^3 +15276N^2 - 15534N - 7884, \quad (1039)$$

$$P_{863} = 50689N^8 + 208318N^7 + 323478N^6 + 236812N^5 + 86401N^4 + 16554N^3 -2304N^2 - 2772N - 540, \quad (1040)$$

$$P_{864} = -71503N^9 - 425688N^8 - 980826N^7 - 1101180N^6 - 615171N^5 -147456N^4 - 4292N^3 + 4428N^2 + 2940N + 72, \quad (1041)$$

$$P_{865} = 24N^9 + 82N^8 - N^7 - 166N^6 + 590N^5 + 262N^4 - 1321N^3 - 1098N^2 +52N + 424, \quad (1042)$$

$$P_{866} = 261N^9 + 783N^8 - 127N^7 - 1760N^6 - 493N^5 - 1289N^4 - 1879N^3 -758N^2 + 114N - 36, \quad (1043)$$

$$
\begin{aligned}
P_{867} &= 7868N^9+22794N^8+4185N^7-36138N^6-70242N^5-28434N^4\\
&\quad +88933N^3+85194N^2-6336N-36720, && (1044)\\
P_{868} &= -5563N^{10}-22141N^9-36762N^8-34822N^7+5681N^6+48719N^5\\
&\quad +42980N^4+28996N^3+19056N^2-7936N-10560, && (1045)\\
P_{869} &= -4253N^{10}-19223N^9-23706N^8+10042N^7+39207N^6+16757N^5\\
&\quad -6072N^4+1800N^3+6168N^2-528N-1760, && (1046)\\
P_{870} &= 561N^{10}+2523N^9+4538N^8+4510N^7+1005N^6-5025N^5-8992N^4\\
&\quad -7960N^3-8936N^2-1968N+1312, && (1047)\\
P_{871} &= 6009N^{10}+21117N^9+38586N^8+68306N^7+61705N^6+28925N^5\\
&\quad -2468N^4-31900N^3+38584N^2+73824N+29088, && (1048)\\
P_{872} &= 80453N^{10}+440578N^9+771262N^8+284116N^7-538565N^6-656852N^5\\
&\quad -328702N^4-44730N^3+83592N^2+19008N-16848, && (1049)\\
P_{873} &= 463143N^{10}+2173209N^9+3165914N^8+374654N^7-3058933N^6\\
&\quad -2659411N^5-640204N^4-29428N^3-117840N^2+26496N+53568, && (1050)\\
P_{874} &= 245N^{11}+944N^{10}+786N^9-975N^8-1959N^7-1998N^6-1756N^5\\
&\quad -695N^4+748N^3+228N^2-112N-64, && (1051)\\
P_{875} &= 10683N^{11}+42732N^{10}+25931N^9-78468N^8-83703N^7-19116N^6\\
&\quad -61407N^5-68436N^4-10052N^3+14856N^2-468N-648, && (1052)\\
P_{876} &= 27981N^{11}+110952N^{10}+73084N^9-183261N^8-220299N^7\\
&\quad -92178N^6-170226N^5-155637N^4+92N^3+35868N^2-3960N-3024, && (1053)\\
P_{877} &= 599375N^{12}+3815193N^{11}+8947106N^{10}+8383052N^9-1037899N^8\\
&\quad -9404623N^7-8442036N^6-2234074N^5+1430550N^4+814536N^3\\
&\quad -384156N^2-321408N-50544, && (1054)\\
P_{878} &= -458487N^{13}-2189691N^{12}-3476151N^{11}-706147N^{10}\\
&\quad +5181991N^9+6607155N^8-353013N^7-6891201N^6-4645260N^5\\
&\quad +669484N^4+2725928N^3+920640N^2-844704N-521856, && (1055)\\
P_{879} &= -3069N^{13}+2556N^{12}+79368N^{11}+221328N^{10}+43770N^9-547648N^8\\
&\quad -435404N^7-15640N^6-194549N^5-563692N^4-68324N^3+170136N^2\\
&\quad +10848N-26784, && (1056)\\
P_{880} &= 3003N^{13}+13255N^{12}+27059N^{11}+32543N^{10}+397N^9-54447N^8-26095N^7\\
&\quad +141989N^6+192852N^5+62644N^4-92576N^3-56336N^2+29952N\\
&\quad +20672, && (1057)\\
P_{881} &= 428649N^{13}+1845381N^{12}+2026189N^{11}-1935731N^{10}-5520669N^9\\
&\quad -1634037N^8+3712287N^7+2628591N^6+266216N^5+1213892N^4\\
&\quad +461088N^3-542160N^2-82944N+119232, && (1058)\\
P_{882} &= -5729259N^{15}-30802914N^{14}-53071856N^{13}-478888N^{12}\\
&\quad +104531506N^{11}+96490716N^{10}-31073520N^9-90914040N^8-63564367N^7\\
&\quad -54366650N^6-31332904N^5+18604224N^4+17762832N^3-2637792N^2\\
&\quad -4240512N-559872, && (1059)\\
P_{883} &= -7255N^{15}-40527N^{14}-54850N^{13}+77690N^{12}+228204N^{11}+45864N^{10}
\end{aligned}
$$

$$
\begin{aligned}
& -240778N^9 - 240422N^8 - 203989N^7 - 270809N^6 - 72500N^5 + 144732 \\
& N^4 + 110144N^3 - 17904N^2 - 37248N - 10176, \qquad (1060) \\
P_{884} =& \; 494694N^{15} + 2631933N^{14} + 3703662N^{13} - 3856542N^{12} - 13210842N^{11} \\
& -3078616N^{10} + 14028102N^9 + 13282506N^8 + 9865380N^7 + 14654523N^6 \\
& +7299204N^5 - 6176300N^4 - 4977096N^3 + 807456N^2 + 1390176N \\
& +300672. \qquad (1061)
\end{aligned}
$$

and

$$
\begin{aligned}
C_{F_3}^{d_{abc}} =& \frac{d_{abc}d^{abc}N_F}{N_c}\Biggl\{-\frac{64S_{-2,1}P_{887}}{3(N-1)N^3(1+N)^3(2+N)} - \frac{32S_3P_{888}}{3(N-1)N^3(1+N)^3(2+N)} \\
& +\frac{64S_{2,1}P_{885}}{3N^3(1+N)^3} + \frac{16S_2P_{890}}{3(N-1)N^4(1+N)^4(2+N)} - \frac{80S_1^2P_{891}}{3(N-1)N^4(1+N)^4(2+N)} \\
& +\frac{64P_{894}}{3(N-1)^2N^6(1+N)^6(2+N)^2} - \Biggl[\frac{16P_{895}}{3(N-1)^2N^5(1+N)^5(2+N)^2} \\
& +\frac{32(2+N+N^2)(18+5N+5N^2)S_3}{3(N-1)N^2(1+N)^2(2+N)} + \frac{64(2+N+N^2)(17N^2+17N-10)S_{-2,1}}{3(N-1)N^2(1+N)^2(2+N)} \\
& +\frac{32S_2P_{885}}{3N^3(1+N)^3}\Biggr]S_1 - \frac{16(2+N+N^2)S_2^2}{3N^2(1+N)^2} + \frac{16(2+N+N^2)(-14+3N+3N^2)S_4}{(N-1)N^2(1+N)^2(2+N)} \\
& +\Biggl[-\frac{64S_1P_{892}}{3(N-1)^2N^3(1+N)^3(2+N)^2} - \frac{32P_{893}}{3(N-1)^2N^4(1+N)^4(2+N)^2} \\
& -\frac{320(2+N+N^2)^2S_1^2}{3(N-1)N^2(1+N)^2(2+N)} + \frac{256(2+N+N^2)S_2}{3(N-1)N^2(1+N)^2(2+N)}\Biggr]S_{-2} \\
& -\frac{64(2+N+N^2)S_{-2}^2}{3N^2(1+N)^2} + \Biggl[\frac{32(2+N+N^2)(18+35N+35N^2)S_1}{3(N-1)N^2(1+N)^2(2+N)} \\
& +\frac{32P_{889}}{3(N-1)N^3(1+N)^3(2+N)}\Biggr]S_{-3} + \frac{32(2+N+N^2)(6+N+N^2)S_{-4}}{(N-1)N^2(1+N)^2(2+N)} \\
& +\frac{256(2+N+N^2)S_{3,1}}{(N-1)N^2(1+N)^2(2+N)} - \frac{64(2+N+N^2)(2+3N+3N^2)S_{-2,2}}{(N-1)N^2(1+N)^2(2+N)} \\
& -\frac{128(2+N+N^2)^2S_{-3,1}}{(N-1)N^2(1+N)^2(2+N)} + \frac{512(2+N+N^2)S_{-2,1,1}}{(N-1)N(1+N)(2+N)} \\
& +\Biggl[\frac{64P_{886}}{(N-1)N^3(1+N)^3(2+N)} + \frac{256(2+N+N^2)(4+N+N^2)S_1}{(N-1)N^2(1+N)^2(2+N)}\Biggr]\zeta_3\Biggr\}, \qquad (1062)
\end{aligned}
$$

with the polynomials

$$
\begin{aligned}
P_{885} &= N^4 + 2N^3 - 5N^2 - 6N - 4, \qquad (1063) \\
P_{886} &= 3N^6 + 9N^5 - 9N^4 - 33N^3 - 74N^2 - 56N - 32, \qquad (1064) \\
P_{887} &= 7N^6 + 12N^5 - 73N^4 - 170N^3 - 104N^2 + 56N + 80, \qquad (1065) \\
P_{888} &= 11N^6 + 30N^5 - 26N^4 - 106N^3 - 113N^2 - 24N + 4, \qquad (1066)
\end{aligned}
$$

$$
\begin{aligned}
P_{889} &= 13N^6+28N^5-139N^4-326N^3-288N^2-40N+48, &(1067)\\
P_{890} &= N^8+6N^7+20N^6+28N^5+32N^4+44N^3+73N^2+40N+12, &(1068)\\
P_{891} &= 3N^8+12N^7+16N^6+6N^5+30N^4+64N^3+73N^2+40N+12, &(1069)\\
P_{892} &= 8N^8+24N^7-51N^6-281N^5-459N^4-383N^3-190N^2+76N+104, &(1070)\\
P_{893} &= 9N^{10}+37N^9+10N^8+2N^7+491N^6+1291N^5+1566N^4+926N^3\\
&\quad +300N^2-8N-16, &(1071)\\
P_{894} &= 4N^{12}+21N^{11}+20N^{10}-158N^9-683N^8-1644N^7-2947N^6\\
&\quad -3115N^5-1578N^4+92N^3+484N^2+240N+48, &(1072)\\
P_{895} &= 43N^{12}+232N^{11}+319N^{10}-336N^9-1485N^8-2782N^7-5229N^6\\
&\quad -7186N^5-5296N^4-304N^3+1896N^2+1344N+352. &(1073)
\end{aligned}
$$

7 The three–loop Wilson coefficients for the structure function $g_1(x,Q^2)$

The non–singlet Wilson coefficient for the polarized structure function $g_1(x,Q^2)$ has the representation

$$
\begin{aligned}
\Delta\mathbb{C}_{g_1,q}^{(3)} &= \Delta C_{g_1,q}^{\rm NS,(3)} + \Delta C_{g_1,q}^{d_{abc},(3)} + \Delta C_{g_1,q}^{\rm PS,(3)}, &(1074)\\
\Delta\mathbb{C}_{g_1,g}^{(3)} &= \Delta C_{g_1,g}^{(3)}, &(1075)
\end{aligned}
$$

with

$$
\Delta C_{g_1,q}^{\rm NS,(3),M} = C_{F_3,q}^{\rm NS,(3)} \qquad (1076)
$$

in the $\overline{\rm MS}$ scheme similar to the observation at one– and two–loop order in Eqs. (134, 135) and (390, 400). $\Delta C_{g_1,q}^{\rm NS,(3),L}$ is given by

$$
\Delta C_{g_1,q}^{\rm NS,(3),M}(N,a_s) = Z_5^{-1}(N,a_s)\,\Delta C_{g_1,q}^{\rm NS,(3),L}(N,a_s) \qquad (1077)
$$

The Z-factor $Z_5^{-1}(N,a_s)$ providing the finite renormalization is calculated in Appendix A.

The function $\Delta C_{g_1,q}^{d_{abc},(3)}$ is given by

$$
\begin{aligned}
\Delta C_{g_1,q}^{d_{abc},(3)} = \frac{d_{abc}d^{abc}}{N_C}N_F\Biggl\{&\frac{64P_{896}}{(N-1)N(1+N)(2+N)} - \frac{512P_{897}}{(N-1)N(1+N)(2+N)}S_{-2,1}\\
&-\frac{256P_{898}}{(N-1)N^2(1+N)^2(2+N)}S_4 + \frac{512P_{898}}{(N-1)N^2(1+N)^2(2+N)}S_{3,1}\\
&+\frac{32P_{900}}{3(N-1)N(1+N)(2+N)}\zeta_3 + \Biggl(-\frac{128\bigl(-16+5N+5N^2\bigr)}{(N-1)N(1+N)(2+N)}\\
&-\frac{256P_{898}}{(N-1)N^2(1+N)^2(2+N)}S_3 - \frac{1024}{(N-1)(2+N)}S_{-2,1}\\
&-\frac{1024\bigl(2+N+N^2\bigr)}{(N-1)N^2(1+N)^2(2+N)}\zeta_3\Biggr)S_1 - \frac{128\bigl(2-N+2N^3+N^4\bigr)}{(N-1)N(1+N)(2+N)}S_3
\end{aligned}
$$

$$+\Biggl(-\frac{64P_{899}}{(N-1)N(1+N)(2+N)}-\frac{256\bigl(-1+2N+2N^2\bigr)}{(N-1)N(1+N)(2+N)}S_1$$
$$+\frac{1024}{N(1+N)}\zeta_3\Biggr)S_{-2}+64S_{-2}^2+\Biggl(\frac{256P_{897}}{(N-1)N(1+N)(2+N)}$$
$$+\frac{512}{(N-1)(2+N)}S_1\Biggr)S_{-3}+\frac{64\bigl(6+N+N^2\bigr)}{(N-1)(2+N)}S_{-4}-\frac{1024}{(N-1)(2+N)}S_{-2,2}$$
$$-\frac{512}{N(1+N)}S_{-2,3}-\frac{1024}{(N-1)(2+N)}S_{-3,1}+\frac{512}{N(1+N)}S_{-4,1}$$
$$+\frac{2048}{(N-1)(2+N)}S_{-2,1,1}-\frac{1280(N-2)(3+N)}{3N(1+N)}\zeta_5\Biggr\}, \quad (1078)$$

with

$$P_{896} = N^4+2N^3-7N^2-8N+30, \quad (1079)$$
$$P_{897} = N^4+2N^3-4N^2-5N+2, \quad (1080)$$
$$P_{898} = N^4+2N^3-N^2-2N-4, \quad (1081)$$
$$P_{899} = 5N^4+10N^3-19N^2-24N+10, \quad (1082)$$
$$P_{900} = 7N^4+14N^3-19N^2-26N+216. \quad (1083)$$

The Wilson coefficient $\Delta C_{g_1,q}^{d_{abc},(3)}$ obeys

$$\Delta C_{g_1,q}^{d_{abc},(3)}(N=1)=0 \quad (1084)$$

and does not contribute to the Bjorken sum rule [46,93]. Four-index d_{abcd} structures do, however, contribute in four–loop order. Let us also present this term in z–space. One obtains

$$\Delta C_{g_1,q}^{d_{abc},(3),\delta(1-z)}(z) = \frac{d_{abc}d^{abc}N_F}{N_C}\Biggl[64+160\zeta_2+\frac{224}{3}\zeta_3-\frac{32}{5}\zeta_2^2-\frac{1280}{3}\zeta_5\Biggr]\delta(1-z) \quad (1085)$$
$$\Delta C_{g_1,q}^{d_{abc},(3),+}(z) = 0 \quad (1086)$$
$$\Delta C_{g_1,q}^{d_{abc},(3),reg}(z) = \frac{d_{abc}d^{abc}N_F}{N_C}\Biggl\{-768(1-z)+\frac{1}{1+z}\Biggl[\frac{1}{1-z}\Biggl(32\bigl(2+z+3z^2-3z^3+z^4\bigr)\mathrm{H}_0^2$$
$$-\frac{64}{3}z^4\mathrm{H}_0^3+\frac{128}{3}\bigl(-6-11z+3z^2+11z^3+6z^4\bigr)\mathrm{H}_0\zeta_2\Biggr)+64\bigl(-3$$
$$+3z+11z^2\bigr)\mathrm{H}_0-128\bigl(1+6z+z^2\bigr)\mathrm{H}_{0,1}+64\bigl(-5+5z+3z^2$$
$$+z^3\bigr)\zeta_2\Biggr]+(1-z)\Biggl[-896\mathrm{H}_1+\frac{64}{3}\mathrm{H}_0^2\mathrm{H}_1+\frac{64\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_0^2\mathrm{H}_1^2$$
$$-\frac{128}{3}\mathrm{H}_0\mathrm{H}_{0,1}-\frac{256\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_0\mathrm{H}_1\mathrm{H}_{0,1}+\frac{128\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_{0,1}^2$$
$$+\frac{256\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_0\mathrm{H}_{0,1,1}-\frac{256\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_{0,0,1,1}+\frac{512}{3}z\mathrm{H}_{0,-1,0,1}$$
$$+\frac{256\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_0\mathrm{H}_1\zeta_2-\frac{256\bigl(1+7z+z^2\bigr)}{3z}\mathrm{H}_{0,1}\zeta_2-\frac{512}{3}z\mathrm{H}_{0,-1}\zeta_2$$

$$
\begin{aligned}
&-\frac{512\big(1+4z+z^2\big)}{3z}\mathrm{H}_1\zeta_3\Bigg]+(1+z)\Bigg[\frac{(1-z)^2}{z}\big(64\mathrm{H}_{-1}^2\mathrm{H}_0-128\mathrm{H}_{-1}\mathrm{H}_{0,-1}\\
&+128\mathrm{H}_{0,-1,-1}\big)+\frac{512(1-z)}{3z}\big(\mathrm{H}_{0,-1,0,1}-\mathrm{H}_{0,-1}\zeta_2\big)+\frac{64\big(1-8z+z^2\big)}{z}\\
&\times\mathrm{H}_{-1}\mathrm{H}_0-\frac{64\big(3+4z+3z^2\big)}{3z}\mathrm{H}_{-1}\mathrm{H}_0^2+\frac{128\big(1-z+z^2\big)}{3z}\mathrm{H}_{-1}^2\mathrm{H}_0^2\\
&+256\mathrm{H}_{-1}\mathrm{H}_0^2\mathrm{H}_{0,1}-\frac{128\big(3+14z+3z^2\big)}{3z}\mathrm{H}_{-1}\mathrm{H}_{0,1}+\frac{512\big(1-z+z^2\big)}{3z}\\
&\times\mathrm{H}_{-1}^2\mathrm{H}_{0,1}-256\mathrm{H}_0\mathrm{H}_{0,1}^2-\frac{64\big(1-8z+z^2\big)}{z}\mathrm{H}_{0,-1}-\frac{512\big(1-z+z^2\big)}{3z}\\
&\mathrm{H}_{-1}\mathrm{H}_0H_{0,-1}+\frac{128}{3}(7+3z)\mathrm{H}_{0,0,1}-1024\mathrm{H}_{-1}\mathrm{H}_0\mathrm{H}_{0,0,1}-\frac{512\big(1-z+z^2\big)}{3z}\\
&\times\mathrm{H}_{-1}\mathrm{H}_{0,0,1}+\frac{512\big(1-z+z^2\big)}{3z}\mathrm{H}_{-1}\mathrm{H}_{0,0,-1}+\frac{128\big(3+14z+3z^2\big)}{3z}\mathrm{H}_{0,1,-1}\\
&+256(\mathrm{H}_{0,1,1}-\mathrm{H}_{0,1,-1})\mathrm{H}_0^2-\frac{1024\big(1-z+z^2\big)}{3z}\mathrm{H}_{-1}\mathrm{H}_{0,1,-1}-256\mathrm{H}_{0,-1,1}\mathrm{H}_0^2\\
&+\frac{128\big(3+14z+3z^2\big)}{3z}\mathrm{H}_{0,-1,1}-\frac{1024\big(1-z+z^2\big)}{3z}\mathrm{H}_{-1}\mathrm{H}_{0,-1,1}\\
&+\frac{512\big(1-z+z^2\big)}{3z}\mathrm{H}_0\mathrm{H}_{0,-1,-1}+1024\mathrm{H}_{-1}\mathrm{H}_{0,0,0,1}+\frac{512\big(1-z+z^2\big)}{3z}\\
&\times\mathrm{H}_{0,0,1,-1}+1024\mathrm{H}_0\mathrm{H}_{0,0,1,-1}+\frac{512\big(1-z+z^2\big)}{3z}\mathrm{H}_{0,0,-1,1}+1024\mathrm{H}_0\mathrm{H}_{0,0,-1,1}\\
&-\frac{512\big(1-z+z^2\big)}{3z}\mathrm{H}_{0,0,-1,-1}+\frac{1024\big(1-z+z^2\big)}{3z}\mathrm{H}_{0,1,-1,-1}+512\mathrm{H}_0\\
&\times\mathrm{H}_{0,-1,0,1}+\frac{1024\big(1-z+z^2\big)}{3z}\mathrm{H}_{0,-1,1,-1}+\frac{1024\big(1-z+z^2\big)}{3z}\mathrm{H}_{0,-1,-1,1}\\
&+1536\mathrm{H}_{0,0,0,1,1}-1024\mathrm{H}_{0,0,0,1,-1}-1024\mathrm{H}_{0,0,0,-1,1}+1024\mathrm{H}_{0,0,1,0,1}\\
&-512\mathrm{H}_{0,0,-1,0,1}+\frac{512\big(1-z+z^2\big)}{3z}\mathrm{H}_{-1}\mathrm{H}_0\zeta_2+256\mathrm{H}_{-1}\mathrm{H}_0^2\zeta_2\\
&+\frac{64\big(9+22z+9z^2\big)}{3z}\mathrm{H}_{-1}\zeta_2-\frac{512\big(1-z+z^2\big)}{3z}\mathrm{H}_{-1}^2\zeta_2+512\mathrm{H}_0\mathrm{H}_{0,1}\zeta_2\\
&-512\mathrm{H}_0\mathrm{H}_{0,-1}\zeta_2-1024\mathrm{H}_{0,0,1}\zeta_2+512\mathrm{H}_{0,0,-1}\zeta_2-\frac{2048}{5}\mathrm{H}_{-1}\zeta_2^2\\
&-512\mathrm{H}_{-1}\mathrm{H}_0\zeta_3+\frac{256\big(1-z+z^2\big)}{z}\mathrm{H}_{-1}\zeta_3-512\mathrm{H}_{0,1}\zeta_3+512\mathrm{H}_{0,-1}\zeta_3\Bigg]\\
&-\frac{128}{3}z(6+z)\mathrm{H}_0^2\mathrm{H}_{0,1}-\frac{128\big(-3-4z-6z^2+10z^3\big)}{3(1-z)z}\mathrm{H}_0\mathrm{H}_{0,-1}\\
&-\frac{256}{3}z^2\mathrm{H}_0^2\mathrm{H}_{0,-1}+\frac{512}{3}z(6+z)\mathrm{H}_0\mathrm{H}_{0,0,1}+\frac{128\big(-3-z-13z^2+3z^3\big)}{3z}\\
&\times\mathrm{H}_{0,0,-1}+\frac{1024}{3}z^2\mathrm{H}_0\mathrm{H}_{0,0,-1}-1024z\mathrm{H}_{0,0,0,1}-512z^2\mathrm{H}_{0,0,0,-1}\\
&-128z(2+z)\mathrm{H}_0^2\zeta_2+\frac{64(1-z)(1+z)^2}{z}\mathrm{H}_1\zeta_2+\frac{512}{5}z(4+z)\zeta_2^2
\end{aligned}
$$

$$+\left(-\frac{64}{3}\left(79-25z+15z^2\right)+256\zeta_2\right)\zeta_3-\frac{256}{3}(-6+z)z\mathrm{H}_0\zeta_3+2560\zeta_5$$
$$-\left(\frac{256\mathrm{H}_{0,0,-1}-192\zeta_3}{1-z}\right)\Biggr\}. \tag{1087}$$

Illustrating three–loop heavy flavor effects on the polarized non–singlet structure function $g_1^{\rm NS}(x,Q^2)$ in Ref. [53] it was assumed that there is no contribution by $\Delta C_{g_1,q}^{d_{abc},(3)}$, despite this function is only known now. This is correct for the three massless quark flavors dealt with there, but changes if four or five massless flavors are considered at very high virtualities or energies. One may generally assume some suppression due to (1084), but there is a finite contribution. Note that $\Delta C_{g_1,q}^{d_{abc},(3)}$ is independent of the scheme employed for the treatment of γ_5, since its contribution is finite.

The other Wilson coefficients are calculated in the Larin scheme. The Wilson coefficient $\Delta C_{g_1,q}^{\rm PS,(3),L}$ reads

$$\Delta C_{g_1,q}^{\rm PS,(3),L} =$$
$$C_F\Biggl\{T_F^2N_F^2\Biggl[\frac{128\zeta_3 P_{902}}{9(N-1)N^2(1+N)^2(2+N)}-\frac{32P_{936}}{243(N-1)^2N^5(1+N)^5(2+N)}$$
$$+\left(-\frac{32(2+N)P_{919}}{81N^4(1+N)^4}-\frac{32(N-1)(2+N)S_2}{9N^2(1+N)^2}\right)S_1+\frac{64(N-1)(2+N)S_1^3}{27N^2(1+N)^2}$$
$$+\frac{16(2+N)\left(21+17N-6N^2+16N^3\right)S_1^2}{27N^3(1+N)^3}+\frac{16(2+N)\left(21+5N-24N^2+46N^3\right)}{27N^3(1+N)^3}$$
$$\times S_2-\frac{160(N-1)(2+N)S_3}{27N^2(1+N)^2}+\frac{512\left(-5-12N+5N^2\right)S_{-2}}{9(N-1)^2N(1+N)^2(2+N)}$$
$$-\frac{1024S_{-3}}{3(N-1)N(1+N)(2+N)}\Biggr]$$
$$+C_AT_FN_F\Biggl[\frac{64S_{-3,1}P_{901}}{3(N-1)N^2(1+N)^2(2+N)}-\frac{20S_4P_{903}}{3(N-1)N^2(1+N)^2(2+N)}$$
$$-\frac{16S_{-2,2}P_{905}}{(N-1)N^2(1+N)^2(2+N)}+\frac{16S_{2,1}P_{906}}{3N^3(1+N)^3}+\frac{64S_{-2,1,1}P_{907}}{3(N-1)N^2(1+N)^2(2+N)}$$
$$+\frac{8S_1^3P_{910}}{27N^3(1+N)^3}-\frac{8S_{-4}P_{915}}{3(N-1)N^2(1+N)^2(2+N)}-\frac{16S_{-2,1}P_{923}}{3(N-1)N^3(1+N)^3(2+N)}$$
$$+\frac{8S_3P_{927}}{27(N-1)N^3(1+N)^3(2+N)}+\frac{8P_{938}}{243(N-1)^2N^6(1+N)^6(2+N)}$$
$$-\left(\frac{8S_2P_{913}}{9N^3(1+N)^3}+\frac{16S_{-2,1}P_{911}}{3(N-1)N^2(1+N)^2(2+N)}-\frac{8S_3P_{912}}{9(N-1)N^2(1+N)^2(2+N)}\right.$$
$$\left.-\frac{8P_{934}}{81(N-1)N^5(1+N)^5(2+N)}\right)S_1+\left(\frac{4P_{926}}{27N^4(1+N)^4}-\frac{80(N-1)(2+N)S_2}{3N^2(1+N)^2}\right)$$
$$\times S_1^2+\frac{28(N-1)(2+N)S_1^4}{9N^2(1+N)^2}-\frac{16\left(5-2N-2N^2\right)S_2^2}{3N^2(1+N)^2}-\left(\frac{16\left(22+N+N^2\right)S_1^2}{3N^2(1+N)^2}\right.$$

$$
+\frac{16S_1P_{922}}{3(N-1)N^3(1+N)^3(2+N)}+\frac{16P_{933}}{9(N-1)^2N^4(1+N)^4(2+N)}
$$

$$
-\frac{32\big(8+N+N^2\big)S_2}{3N^2(1+N)^2}\Bigg)S_{-2}-\frac{16\big(-2+3N+3N^2\big)S_{-2}^2}{3N^2(1+N)^2}-\frac{4S_2P_{928}}{27N^4(1+N)^4}
$$

$$
+\Bigg(\frac{8P_{925}}{3(N-1)N^3(1+N)^3(2+N)}+\frac{8S_1P_{916}}{3(N-1)N^2(1+N)^2(2+N)}\Bigg)S_{-3}
$$

$$
+\frac{320\big(2+N+N^2\big)S_{3,1}}{(N-1)N^2(1+N)^2(2+N)}+\frac{32(N-1)(2+N)S_{2,1,1}}{3N^2(1+N)^2}-\frac{96(N-1)(2+N)\zeta_2^2}{5N^2(1+N)^2}
$$

$$
+\Bigg(-\frac{16S_1P_{908}}{3(N-1)N^2(1+N)^2(2+N)}-\frac{16P_{924}}{9(N-1)N^3(1+N)^3(2+N)}\Bigg)\zeta_3\Bigg]\Bigg\}
$$

$$
+C_F^2T_FN_F\Bigg\{-\frac{64S_{3,1}P_{904}}{(N-1)N^2(1+N)^2(2+N)}+\frac{8S_4P_{909}}{3(N-1)N^2(1+N)^2(2+N)}
$$

$$
-\frac{8S_3P_{920}}{9(N-1)N^3(1+N)^3(2+N)}-\frac{4S_2P_{931}}{3(N-1)N^4(1+N)^4(2+N)}
$$

$$
-\frac{8P_{937}}{3(N-1)^2N^6(1+N)^6(2+N)}+\Bigg(\frac{16S_3P_{914}}{9(N-1)N^2(1+N)^2(2+N)}
$$

$$
-\frac{8S_2P_{917}}{3N^3(1+N)^3}+\frac{8P_{935}}{3(N-1)^2N^5(1+N)^5(2+N)}
$$

$$
+\frac{256\big(2+N+N^2\big)S_{-2,1}}{(N-1)N^2(1+N)^2(2+N)}\Bigg)S_1
$$

$$
+\Bigg(\frac{4P_{932}}{3(N-1)N^4(1+N)^4(2+N)}-\frac{112(N-1)(2+N)S_2}{3N^2(1+N)^2}\Bigg)S_1^2
$$

$$
+\frac{16(2+N)\big(22+2N-15N^2+27N^3\big)S_1^3}{9N^3(1+N)^3}+\frac{68(N-1)(2+N)S_2^2}{3N^2(1+N)^2}
$$

$$
+\Bigg(-\frac{32P_{929}}{(N-1)^2N^3(1+N)^3(2+N)}-\frac{32S_1P_{930}}{(N-1)^2N^3(1+N)^3(2+N)}
$$

$$
+\frac{128\big(-4+N+N^2\big)S_1^2}{(N-1)N^2(1+N)^2(2+N)}-\frac{128\big(-4+N+N^2\big)S_2}{(N-1)N^2(1+N)^2(2+N)}\Bigg)S_{-2}
$$

$$
+\Bigg(\frac{64P_{918}}{(N-1)N^3(1+N)^2(2+N)}-\frac{128(N-2)(3+N)S_1}{(N-1)N^2(1+N)^2(2+N)}\Bigg)S_{-3}
$$

$$
-\frac{128\big(-6+5N+5N^2\big)S_{-4}}{(N-1)N^2(1+N)^2(2+N)}-\frac{64\big(4+3N-3N^2+2N^3\big)S_{2,1}}{3N^3(1+N)^3}
$$

$$
-\frac{256\big(6-N-2N^2+N^3\big)S_{-2,1}}{(N-1)N^3(1+N)^2(2+N)}+\frac{256\big(-2+3N+3N^2\big)S_{-2,2}}{(N-1)N^2(1+N)^2(2+N)}
$$

$$
+\frac{1024\big(-1+N+N^2\big)S_{-3,1}}{(N-1)N^2(1+N)^2(2+N)}-\frac{32(N-1)(2+N)S_{2,1,1}}{3N^2(1+N)^2}
$$

$$
+\frac{96(N-1)(2+N)\zeta_2^2}{5N^2(1+N)^2}-\frac{1024S_{-2,1,1}}{(N-1)N(1+N)(2+N)}+\frac{44(N-1)(2+N)S_1^4}{9N^2(1+N)^2}
$$

$$+\left(\frac{32P_{921}}{3(N-1)N^3(1+N)^3(2+N)}-\frac{128\big(7+4N+4N^2\big)S_1}{3N^2(1+N)^2}\right)\zeta_3\Bigg\}, \tag{1088}$$

with the polynomials

$$P_{901} = N^4+2N^3-63N^2-64N+28, \tag{1089}$$
$$P_{902} = N^4+2N^3-39N^2-40N+4, \tag{1090}$$
$$P_{903} = N^4+2N^3-15N^2-16N+124, \tag{1091}$$
$$P_{904} = N^4+2N^3+5N^2+4N+20, \tag{1092}$$
$$P_{905} = N^4+2N^3+73N^2+72N-20, \tag{1093}$$
$$P_{906} = N^4+10N^3-5N^2+18N+12, \tag{1094}$$
$$P_{907} = 5N^4+10N^3+93N^2+88N-4, \tag{1095}$$
$$P_{908} = 7N^4+14N^3-81N^2-88N+100, \tag{1096}$$
$$P_{909} = 19N^4+38N^3-105N^2-124N+556, \tag{1097}$$
$$P_{910} = 23N^4+N^3-107N^2+167N+222, \tag{1098}$$
$$P_{911} = 25N^4+50N^3+137N^2+112N+60, \tag{1099}$$
$$P_{912} = 37N^4+74N^3-375N^2-412N-44, \tag{1100}$$
$$P_{913} = 37N^4+77N^3-97N^2+151N+246, \tag{1101}$$
$$P_{914} = 43N^4+86N^3+87N^2+44N+316, \tag{1102}$$
$$P_{915} = 47N^4+94N^3-393N^2-440N+308, \tag{1103}$$
$$P_{916} = 63N^4+126N^3+127N^2+64N+4, \tag{1104}$$
$$P_{917} = 65N^4+86N^3-49N^2+66N+96, \tag{1105}$$
$$P_{918} = N^5+2N^4+7N^3-6N+12, \tag{1106}$$
$$P_{919} = 161N^5+23N^4+55N^3+55N^2-12N-18, \tag{1107}$$
$$P_{920} = 3N^6+27N^5+157N^4+461N^3+1212N^2+380N+64, \tag{1108}$$
$$P_{921} = 3N^6+33N^5+71N^4+115N^3+294N^2+4N-88, \tag{1109}$$
$$P_{922} = 10N^6+24N^5+7N^4-30N^3-135N^2-64N+92, \tag{1110}$$
$$P_{923} = 21N^6+34N^5-191N^4-248N^3-40N^2-184N-160, \tag{1111}$$
$$P_{924} = 58N^6+237N^5-811N^4-1741N^3+873N^2+652N-852, \tag{1112}$$
$$P_{925} = 63N^6+138N^5+23N^4+228N^3+308N^2-456N-240, \tag{1113}$$
$$P_{926} = 79N^6+78N^5-1526N^4-1632N^3-794N^2-3351N-1710, \tag{1114}$$
$$P_{927} = 199N^6+588N^5+20N^4-208N^3+3177N^2+1612N-1068, \tag{1115}$$
$$P_{928} = 953N^6+2298N^5-664N^4-2490N^3+980N^2-1455N-1422, \tag{1116}$$
$$P_{929} = N^7+N^6-10N^5-14N^4-35N^3+9N^2+20N-4, \tag{1117}$$
$$P_{930} = N^7+2N^6+20N^5+10N^4-25N^3-92N^2-12N+32, \tag{1118}$$
$$P_{931} = 56N^8+205N^7+154N^6+40N^5-84N^4-977N^3-942N^2+308N+472, \tag{1119}$$
$$P_{932} = 156N^8+501N^7+164N^6-374N^5-242N^4-1139N^3-958N^2+524N +600, \tag{1120}$$
$$P_{933} = 45N^9+57N^8+386N^7+232N^6-1896N^5-2008N^4+937N^3+63N^2 -228N-84, \tag{1121}$$
$$P_{934} = 2701N^{10}+11348N^9+6606N^8-13281N^7+8301N^6+20247N^5+980N^4$$

$$+32572N^3+10626N^2-17460N-10800, \quad (1122)$$

$$P_{935} = 96N^{11}+302N^{10}-111N^9-890N^8-689N^7+706N^6+2477N^5+2544N^4 -497N^3-1854N^2+324N+664, \quad (1123)$$

$$P_{936} = 2042N^{11}+6095N^{10}-2575N^9-15042N^8-2412N^7+15972N^6+22057N^5 +18358N^4+2452N^3-8427N^2+468N+2484, \quad (1124)$$

$$P_{937} = 260N^{13}+1033N^{12}+579N^{11}-2050N^{10}-2768N^9-582N^8+146N^7-680N^6 +752N^5+1237N^4-765N^3-950N^2+356N+360, \quad (1125)$$

$$P_{938} = 30520N^{13}+123245N^{12}+50588N^{11}-274955N^{10}-241017N^9+129972N^8 +331898N^7+404113N^6+59461N^5-137557N^4-25818N^3+116766N^2 +1080N-29160. \quad (1126)$$

Finally, the Wilson coefficient $\Delta C_{g_1,g}^{(3)}$ is given by

$$\Delta C_{g_1,g}^{(3)} = \Delta C_{g_1,g}^{(3),a} + \Delta C_{g_1,g}^{(3),d_{abc}}, \quad (1127)$$

with

$$\Delta C_{g_1,g}^{(3),a} = C_F \Biggl\{ T_F^2 N_F^2 \Biggl[-\frac{256 P_{941}}{9N(1+N)^2(2+N)^2} S_{2,1} - \frac{64 P_{1002}}{9(N-1)N^3(1+N)^3(2+N)} \zeta_3 + \frac{16(N-1)P_{946}}{81N^3(1+N)^3} S_1^3 + \frac{32 P_{1007}}{81N^3(1+N)^3(2+N)^2} S_3 - \frac{4 P_{1029}}{27N^4(1+N)^4(2+N)^2} \times S_2 + \frac{P_{1045}}{243(N-1)^2N^6(1+N)^6(2+N)^2} + \Biggl(-\frac{16 P_{1003}}{27N^3(1+N)^3(2+N)^2} \times S_2 - \frac{8 P_{1039}}{243(N-1)N^5(1+N)^5(2+N)^2} + \frac{32\bigl(74+35N+35N^2\bigr)}{27N(1+N)(2+N)} S_3 - \frac{128\bigl(2+N+N^2\bigr)}{3N(1+N)(2+N)} S_{2,1} - \frac{512}{3N(1+N)(2+N)} S_{-2,1} + \frac{64\bigl(-46+5N+5N^2\bigr)}{9N(1+N)(2+N)} \zeta_3 \Biggr) S_1 + \frac{68(N-1)}{27N(1+N)} S_1^4 + \Biggl(\frac{4 P_{1025}}{81N^4(1+N)^4(2+N)} - \frac{8\bigl(-62+7N+7N^2\bigr)}{9N(1+N)(2+N)} S_2 \Biggr) S_1^2 + \frac{4\bigl(-226+17N+17N^2\bigr)}{9N(1+N)(2+N)} S_2^2 - \frac{56\bigl(14+17N+17N^2\bigr)}{9N(1+N)(2+N)} S_4 + \Biggl(-\frac{64 P_{960}}{9(N-1)N^2(1+N)^2(2+N)^2} S_1 - \frac{32 P_{1028}}{9(N-1)^2N^3(1+N)^3(2+N)^2} + \frac{512}{3N(1+N)(2+N)} S_1^2 - \frac{512}{3N(1+N)(2+N)} S_2 \Biggr) S_{-2} + \Biggl(\frac{128 P_{961}}{9(N-1)N^2(1+N)^2(2+N)^2} - \frac{256}{N(1+N)(2+N)} S_1 \Biggr) S_{-3} - \frac{256 S_{-2}^2}{3N(1+N)(2+N)} - \frac{1024}{3N(1+N)(2+N)} S_{-4} + \frac{64\bigl(-10+3N+3N^2\bigr)}{3N(1+N)(2+N)} S_{3,1} - \frac{512(-3+5N)}{9N^2(1+N)(2+N)} S_{-2,1}$$

$$
\begin{aligned}
&+\frac{512}{3N(1+N)(2+N)}S_{-2,2}+\frac{1024}{3N(1+N)(2+N)}S_{-3,1}+\frac{64\big(10+N+N^2\big)}{3N(1+N)(2+N)}\\
&\times S_{2,1,1}\Bigg]+{\color{blue}C_AT_FN_F}\Bigg[-\frac{32P_{965}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,1,1}\\
&-\frac{2P_{949}}{9N^2(1+N)^2(2+N)}S_2^2+\frac{16P_{966}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,2}\\
&+\frac{16P_{967}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-3,1}-\frac{16P_{978}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{3,1}\\
&+\frac{4P_{997}}{9(N-1)N^2(1+N)^2(2+N)^2}S_4+\frac{8P_{1006}}{9N^3(1+N)^3(2+N)^2}S_{2,1}\\
&-\frac{16P_{1017}}{9(N-1)N^3(1+N)^3(2+N)^2}S_{-2,1}-\frac{8P_{1023}}{9(N-1)N^3(1+N)^3(2+N)^2}\zeta_3\\
&-\frac{8P_{1027}}{81(N-1)N^3(1+N)^3(2+N)^2}S_3+\frac{P_{1047}}{972(N-1)^2N^6(1+N)^6(2+N)^3}\\
&+\Bigg(\frac{8P_{944}}{3N^2(1+N)^2(2+N)}S_{2,1}-\frac{32P_{959}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,1}\\
&-\frac{8P_{992}}{27(N-1)N^2(1+N)^2(2+N)^2}S_3-\frac{8P_{999}}{9(N-1)N^2(1+N)^2(2+N)^2}\zeta_3\\
&+\frac{4P_{1010}}{27N^3(1+N)^3(2+N)^2}S_2+\frac{2P_{1043}}{243(N-1)^2N^5(1+N)^5(2+N)^3}\\
&-\frac{80(N-1)}{N(1+N)}S_2^2+\frac{8\big(-382+47N+47N^2\big)}{3N(1+N)(2+N)}S_4+\frac{64\big(130+7N+7N^2\big)}{3N(1+N)(2+N)}S_{3,1}\\
&+\frac{128\big(-54+5N+5N^2\big)}{N(1+N)(2+N)}S_{-2,2}+\frac{16\big(-1378+113N+113N^2\big)}{3N(1+N)(2+N)}S_{-3,1}\\
&-\frac{80(N-1)}{N(1+N)}S_{2,1,1}-\frac{32\big(-1214+127N+127N^2\big)}{3N(1+N)(2+N)}S_{-2,1,1}-\frac{96(N-1)\zeta_2^2}{5N(1+N)}\Bigg)S_1\\
&+\Bigg(\frac{4P_{948}}{9N^2(1+N)^2(2+N)}S_2+\frac{P_{1030}}{81(N-1)N^4(1+N)^4(2+N)^2}\\
&-\frac{8\big(310+13N+13N^2\big)}{3N(1+N)(2+N)}S_3-\frac{32(N-1)}{N(1+N)}S_{2,1}+\frac{32\big(-286+35N+35N^2\big)}{3N(1+N)(2+N)}\\
&\times S_{-2,1}+\frac{32\big(166+19N+19N^2\big)}{3N(1+N)(2+N)}\zeta_3\Bigg)S_1^2+\Bigg(-\frac{4P_{1001}}{81N^3(1+N)^3(2+N)}\\
&+\frac{136(N-1)}{3N(1+N)}S_2\Bigg)S_1^3+\Bigg(\frac{P_{1035}}{27(N-1)N^4(1+N)^4(2+N)^2}+\frac{8\big(898+7N+7N^2\big)}{3N(1+N)(2+N)}\\
&\times S_3-\frac{128\big(-72+7N+7N^2\big)}{N(1+N)(2+N)}S_{-2,1}-\frac{96\big(-2+3N+3N^2\big)}{N(1+N)(2+N)}\zeta_3\Bigg)S_2\\
&-\frac{2\big(369-23N-96N^2+206N^3\big)}{27N^2(1+N)^2}S_1^4+\frac{256\big(11-N-N^2\big)}{3N(1+N)(2+N)}S_5\\
&+\Bigg(-\frac{8P_{981}}{3(N-1)N^2(1+N)^2(2+N)^2}S_2
\end{aligned}
$$

$$+\frac{8P_{985}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1^2-\frac{8P_{1037}}{9(N-1)^2N^4(1+N)^4(2+N)^3}$$

$$+\Biggl(-\frac{16P_{1031}}{9(N-1)^2N^3(1+N)^3(2+N)^3}-\frac{48\bigl(-18+N+N^2\bigr)}{N(1+N)(2+N)}S_2\Biggr)S_1$$

$$-\frac{16\bigl(178-17N-17N^2\bigr)}{9N(1+N)(2+N)}S_1^3-\frac{64\bigl(2+35N+35N^2\bigr)}{9N(1+N)(2+N)}$$

$$\times S_3+\frac{64\bigl(-146+13N+13N^2\bigr)}{N(1+N)(2+N)}S_{2,1}-\frac{1024\bigl(-4+N+N^2\bigr)}{N(1+N)(2+N)}\zeta_3$$

$$-\frac{128(N-1)}{3N(1+N)}S_{-2,1}\Biggr)S_{-2}+\Biggl(\frac{32P_{971}}{3(N-1)N^2(1+N)^2(2+N)^2}+\frac{256\bigl(-11+N+N^2\bigr)}{3N(1+N)(2+N)}$$

$$\times S_1\Biggr)S_{-2}^2+\Biggl(-\frac{16P_{977}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1+\frac{8P_{1022}}{9(N-1)N^3(1+N)^3(2+N)^2}$$

$$-\frac{16\bigl(-490+41N+41N^2\bigr)}{3N(1+N)(2+N)}S_1^2+\frac{128\bigl(-107+10N+10N^2\bigr)}{3N(1+N)(2+N)}S_2$$

$$-\frac{32\bigl(22+5N+5N^2\bigr)}{N(1+N)(2+N)}S_{-2}\Biggr)S_{-3}+\Biggl(\frac{8P_{980}}{3(N-1)N^2(1+N)^2(2+N)^2}$$

$$-\frac{16\bigl(-198+19N+19N^2\bigr)}{N(1+N)(2+N)}S_1\Biggr)S_{-4}-\frac{16\bigl(562+31N+31N^2\bigr)}{3N(1+N)(2+N)}S_{-5}$$

$$+\frac{64\bigl(-38+N+N^2\bigr)}{N(1+N)(2+N)}S_{2,3}-\frac{64\bigl(-134+13N+13N^2\bigr)}{3N(1+N)(2+N)}S_{4,1}$$

$$+\frac{160\bigl(-86+7N+7N^2\bigr)}{3N(1+N)(2+N)}S_{2,-3}-\frac{256\bigl(-50+N+N^2\bigr)}{3N(1+N)(2+N)}S_{-2,3}$$

$$+\frac{256\bigl(-7+N+N^2\bigr)}{N(1+N)(2+N)}S_{-4,1}-\frac{16\bigl(128+67N-26N^2+5N^3\bigr)}{3N(1+N)^2(2+N)}S_{2,1,1}$$

$$-\frac{64\bigl(-434+37N+37N^2\bigr)}{3N(1+N)(2+N)}S_{2,1,-2}-\frac{96\bigl(46+N+N^2\bigr)}{N(1+N)(2+N)}S_{3,1,1}$$

$$-\frac{16(N-1)}{3N(1+N)}S_{2,2,1}-\frac{64\bigl(-118+11N+11N^2\bigr)}{3N(1+N)(2+N)}S_{-2,1,-2}-\frac{2560\bigl(-11+N+N^2\bigr)}{3N(1+N)(2+N)}$$

$$\times S_{-2,2,1}-\frac{160\bigl(-58+5N+5N^2\bigr)}{N(1+N)(2+N)}S_{-3,1,1}+\frac{320\bigl(-178+17N+17N^2\bigr)}{3N(1+N)(2+N)}S_{-2,1,1,1}$$

$$+\frac{160(N-1)}{N(1+N)}S_{2,1,1,1}+\frac{96(N-1)\bigl(5-3N-3N^2\bigr)}{5N^2(1+N)^2}\zeta_2^2-\frac{80\bigl(-118+N+N^2\bigr)}{N(1+N)(2+N)}$$

$$\times\zeta_5\Biggr]\Biggr\}+C_AT_F^2N_F^2\Biggl\{\frac{128P_{954}}{27N^2(1+N)^2(2+N)^2}S_{2,1}+\frac{32P_{955}}{9(N-1)N^2(1+N)^2(2+N)}\zeta_3$$

$$-\frac{64P_{956}}{81N^2(1+N)^2(2+N)^2}S_3+\frac{8P_{1041}}{243(N-1)^2N^5(1+N)^5(2+N)^2}$$

$$-\frac{16P_{1004}}{81N^3(1+N)^3(2+N)^2}S_2+\Biggl[\frac{8P_{1033}}{243(N-1)N^4(1+N)^4(2+N)^2}$$

$$
-\frac{32P_{957}}{27N^2(1+N)^2(2+N)^2}S_2+\frac{32\big(-110+19N+19N^2\big)}{27N(1+N)(2+N)}S_3+\frac{256\big(1+N+N^2\big)}{9N(1+N)(2+N)}
$$

$$
\times S_{2,1}+\frac{256}{3N(1+N)(2+N)}S_{-2,1}-\frac{32\big(-62+13N+13N^2\big)}{9N(1+N)(2+N)}\zeta_3\Bigg]S_1
$$

$$
+\Bigg[\frac{16P_{996}}{81N^3(1+N)^3(2+N)}-\frac{8\big(-14+19N+19N^2\big)}{9N(1+N)(2+N)}S_2\Bigg]S_1^2+\frac{28(N-1)}{27N(1+N)}S_1^4
$$

$$
+\frac{32\big(27-22N-9N^2+19N^3\big)}{81N^2(1+N)^2}S_1^3+\frac{4\big(34+31N+31N^2\big)}{9N(1+N)(2+N)}S_2^2+\frac{56\big(10+7N+7N^2\big)}{9N(1+N)(2+N)}
$$

$$
\times S_4+\Bigg[-\frac{64P_{964}}{9(N-1)N^2(1+N)^2(2+N)^2}S_1-\frac{32P_{1021}}{81(N-1)^2N^2(1+N)^3(2+N)^2}
$$

$$
-\frac{32\big(6+N+N^2\big)}{3N(1+N)(2+N)}S_1^2+\frac{32\big(6+N+N^2\big)}{3N(1+N)(2+N)}S_2\Bigg]S_{-2}+\frac{128}{3N(1+N)(2+N)}S_{-2}^2
$$

$$
+\Bigg[\frac{128P_{972}}{27(N-1)N^2(1+N)^2(2+N)^2}+\frac{128\big(1+N+N^2\big)}{3N(1+N)(2+N)}S_1\Bigg]S_{-3}
$$

$$
-\frac{64\big(-38+7N+7N^2\big)}{9N(1+N)(2+N)}S_{-4}-\frac{128\big(-13+5N+5N^2\big)}{9N(1+N)(2+N)}S_{3,1}+\frac{256(-3+5N)}{9N^2(1+N)(2+N)}
$$

$$
\times S_{-2,1}-\frac{256}{3N(1+N)(2+N)}S_{-2,2}-\frac{512}{3N(1+N)(2+N)}S_{-3,1}
$$

$$
-\frac{64\big(16+N+N^2\big)}{9N(1+N)(2+N)}S_{2,1,1}\Bigg\}+C_A^2T_FN_F\Bigg\{\frac{P_{939}}{9N^2(1+N)^2(2+N)}S_2^2
$$

$$
+\frac{16P_{943}}{9N^2(1+N)^2(2+N)}S_{2,1,1}-\frac{32P_{983}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,1,1}
$$

$$
+\frac{16P_{984}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,2}+\frac{16P_{986}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-3,1}
$$

$$
+\frac{8P_{987}}{9(N-1)N^2(1+N)^2(2+N)^2}S_{3,1}-\frac{2P_{998}}{9(N-1)N^2(1+N)^2(2+N)^2}S_4
$$

$$
-\frac{8P_{1008}}{27N^3(1+N)^3(2+N)^2}S_{2,1}+\frac{4P_{1014}}{9(N-1)N^3(1+N)^3(2+N)^2}\zeta_3
$$

$$
-\frac{32P_{1015}}{9(N-1)N^3(1+N)^3(2+N)^2}S_{-2,1}+\frac{8P_{1026}}{81(N-1)N^3(1+N)^3(2+N)^2}S_3
$$

$$
-\frac{4P_{1046}}{243(N-1)^2N^6(1+N)^6(2+N)^3}+\Bigg[-\frac{16P_{945}}{9N^2(1+N)^2(2+N)}S_{2,1}
$$

$$
+\frac{16P_{982}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,1}+\frac{8P_{994}}{9(N-1)N^2(1+N)^2(2+N)^2}\zeta_3
$$

$$
-\frac{8P_{995}}{27(N-1)N^2(1+N)^2(2+N)^2}S_3+\frac{8P_{1009}}{27N^3(1+N)^3(2+N)^2}S_2
$$

$$
-\frac{4P_{1044}}{243(N-1)^2N^5(1+N)^5(2+N)^3}-\frac{124(N-1)}{3N(1+N)}S_2^2-\frac{48(N-1)}{N(1+N)}S_4
$$

$$
-\frac{32\big(70+N+N^2\big)}{3N(1+N)(2+N)}S_{3,1}-\frac{80\big(-106+5N+5N^2\big)}{3N(1+N)(2+N)}S_{-3,1}-\frac{16(N-1)}{3N(1+N)}S_{2,1,1}
$$

$$
-\frac{32\big(-226+5N+5N^2\big)}{3N(1+N)(2+N)}S_{-2,2}-\frac{32\big(374+5N+5N^2\big)}{3N(1+N)(2+N)}S_{-2,1,1}+\frac{96(N-1)}{5N(1+N)}\zeta_2^2\Bigg]
$$

$$
\times S_1+\Bigg[\frac{2P_{950}}{9N^2(1+N)^2(2+N)}S_2-\frac{2P_{1036}}{81(N-1)N^4(1+N)^4(2+N)^2}
$$

$$
-\frac{8\big(-154+29N+29N^2\big)}{3N(1+N)(2+N)}S_3+\frac{32(N-1)}{3N(1+N)}S_{2,1}+\frac{32\big(22+3N+3N^2\big)}{N(1+N)(2+N)}S_{-2,1}
$$

$$
+\frac{16\big(-190+17N+17N^2\big)}{3N(1+N)(2+N)}\zeta_3\Bigg]S_1^2+\Bigg[-\frac{8P_{1000}}{81N^3(1+N)^3(2+N)}+\frac{128(N-1)}{9N(1+N)}S_2\Bigg]
$$

$$
\times S_1^3+\frac{\big(-528+221N+216N^2-293N^3\big)}{27N^2(1+N)^2}S_1^4-\frac{4(N-1)}{3N(1+N)}S_1^5
$$

$$
+\Bigg[\frac{2P_{1034}}{81(N-1)N^4(1+N)^4(2+N)^2}-\frac{8\big(658+31N+31N^2\big)}{9N(1+N)(2+N)}S_3+\frac{48\big(2+N+N^2\big)}{N(1+N)(2+N)}
$$

$$
\times\zeta_3+\frac{128\big(-27+2N+2N^2\big)}{N(1+N)(2+N)}S_{-2,1}\Bigg]S_2+\frac{32\big(-26+7N+7N^2\big)}{3N(1+N)(2+N)}S_5
$$

$$
+\Bigg[\frac{8P_{975}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1^2-\frac{8P_{979}}{3(N-1)N^2(1+N)^2(2+N)^2}S_2
$$

$$
+\frac{8P_{1040}}{81(N-1)^2N^4(1+N)^4(2+N)^3}+\Bigg(\frac{16P_{1032}}{9(N-1)^2N^3(1+N)^3(2+N)^3}
$$

$$
-\frac{16\big(50+11N+11N^2\big)}{3N(1+N)(2+N)}S_2\Bigg)S_1-\frac{16\big(-26+N+N^2\big)}{3N(1+N)(2+N)}S_1^3+\frac{128\big(-5+4N+4N^2\big)}{3N(1+N)(2+N)}
$$

$$
\times S_3+\frac{1024\big(11-N-N^2\big)}{3N(1+N)(2+N)}S_{2,1}+\frac{128(N-1)}{3N(1+N)}S_{-2,1}-\frac{128\big(7-2N-2N^2\big)}{N(1+N)(2+N)}
$$

$$
\times\zeta_3\Bigg]S_{-2}+\Bigg[-\frac{8P_{976}}{3(N-1)N^2(1+N)^2(2+N)^2}-\frac{16\big(-18+N+N^2\big)}{N(1+N)(2+N)}S_1\Bigg]S_{-2}^2
$$

$$
+\Bigg[-\frac{8P_{991}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1-\frac{16P_{1024}}{27(N-1)N^3(1+N)^3(2+N)^2}
$$

$$
-\frac{16\big(122+29N+29N^2\big)}{3N(1+N)(2+N)}S_1^2-\frac{32\big(-64+9N+9N^2\big)}{N(1+N)(2+N)}S_2
$$

$$
+\frac{32\big(10+13N+13N^2\big)}{3N(1+N)(2+N)}S_{-2}\Bigg]S_{-3}+\Bigg[-\frac{8P_{989}}{9(N-1)N^2(1+N)^2(2+N)^2}
$$

$$
+\frac{128\big(-13+2N+2N^2\big)}{N(1+N)(2+N)}S_1\Bigg]S_{-4}+\frac{16\big(238+N+N^2\big)}{3N(1+N)(2+N)}S_{-5}
$$

$$
-\frac{48\big(-14+N+N^2\big)}{N(1+N)(2+N)}S_{2,3}+\frac{128\big(34+N+N^2\big)}{3N(1+N)(2+N)}S_{2,-3}-\frac{256\big(19+N+N^2\big)}{3N(1+N)(2+N)}S_{-2,3}
$$

$$
+\frac{16\big(-146+19N+19N^2\big)}{3N(1+N)(2+N)}S_{4,1}-\frac{32\big(-142+35N+35N^2\big)}{3N(1+N)(2+N)}S_{-4,1}
$$

$$
\begin{aligned}
&+\frac{1024\big(-11+N+N^2\big)}{3N(1+N)(2+N)}S_{2,1,-2}+\frac{176(N-1)}{3N(1+N)}S_{2,2,1}+\frac{1152}{N(1+N)(2+N)}S_{3,1,1}\\
&-\frac{64\big(22+7N+7N^2\big)}{3N(1+N)(2+N)}S_{-2,1,-2}+\frac{64\big(-178+17N+17N^2\big)}{3N(1+N)(2+N)}S_{-2,2,1}\\
&+\frac{32\big(-130+17N+17N^2\big)}{N(1+N)(2+N)}S_{-3,1,1}-\frac{64\big(-322+17N+17N^2\big)}{3N(1+N)(2+N)}S_{-2,1,1,1}\\
&+\frac{48(N-1)\big(-8+3N+3N^2\big)}{5N^2(1+N)^2}\zeta_2^2+\frac{160\big(-14+N+N^2\big)}{N(1+N)(2+N)}\zeta_5\Bigg\}\\
&+C_F^2T_FN_F\Bigg\{\frac{P_{940}}{3N^2(1+N)^2(2+N)}S_2^2+\frac{64P_{952}}{3(N-1)N(1+N)^2(2+N)^2}S_{-2,1,1}\\
&-\frac{16P_{953}}{3N^3(1+N)^2(2+N)}S_{2,1}-\frac{64P_{962}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-2,2}\\
&+\frac{32P_{968}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{3,1}-\frac{32P_{969}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{-3,1}\\
&-\frac{2P_{988}}{3(N-1)N^2(1+N)^2(2+N)^2}S_4+\frac{4P_{1005}}{9(N-1)N^3(1+N)^2(2+N)^2}S_3\\
&+\frac{32P_{1013}}{3(N-1)N^3(1+N)^3(2+N)^2}S_{-2,1}+\frac{16P_{1019}}{3(N-1)N^3(1+N)^3(2+N)^2}\zeta_3\\
&+\frac{P_{1042}}{12(N-1)N^6(1+N)^6(2+N)^2}+\Bigg[\frac{32P_{942}}{3N^2(1+N)^2(2+N)}S_{2,1}\\
&-\frac{64P_{951}}{3(N-1)N(1+N)^2(2+N)^2}S_{-2,1}+\frac{16P_{990}}{3(N-1)N^2(1+N)^2(2+N)^2}\zeta_3\\
&+\frac{2P_{974}}{3N^3(1+N)^3(2+N)}S_2-\frac{8P_{993}}{9(N-1)N^2(1+N)^2(2+N)^2}S_3\\
&-\frac{2P_{1038}}{3(N-1)N^5(1+N)^5(2+N)^2}-\frac{8\big(-766+95N+95N^2\big)}{3N(1+N)(2+N)}S_4-\frac{188(N-1)}{3N(1+N)}\\
&\times S_2^2-\frac{32\big(262+13N+13N^2\big)}{3N(1+N)(2+N)}S_{3,1}+\frac{128\big(14+N+N^2\big)}{N(1+N)(2+N)}S_{-2,2}\\
&+\frac{64\big(70+13N+13N^2\big)}{3N(1+N)(2+N)}S_{-3,1}+\frac{256(N-1)}{3N(1+N)}S_{2,1,1}+\frac{128\big(-98+N+N^2\big)}{3N(1+N)(2+N)}S_{-2,1,1}\\
&\Bigg]S_1+\Bigg[\frac{2P_{947}}{3N^2(1+N)^2(2+N)}S_2+\frac{48\big(18-N-N^2\big)}{N(1+N)(2+N)}S_3\\
&+\frac{P_{1011}}{3N^4(1+N)^4(2+N)}+\frac{64(N-1)}{3N(1+N)}S_{2,1}-\frac{256\big(-14+N+N^2\big)}{3N(1+N)(2+N)}S_{-2,1}\\
&-\frac{128\big(14+5N+5N^2\big)}{3N(1+N)(2+N)}\zeta_3\Bigg]S_1^2+\Bigg[-\frac{2P_{958}}{9N^3(1+N)^3}+\frac{472(N-1)}{9N(1+N)}S_2\Bigg]S_1^3\\
&-\frac{20(N-1)}{3N(1+N)}S_1^5-\frac{(N-1)\big(-226+97N+249N^2\big)}{9N^2(1+N)^2}S_1^4+\Bigg[\frac{P_{1018}}{3N^4(1+N)^4(2+N)}\\
&+\frac{16\big(-1462+83N+83N^2\big)}{9N(1+N)(2+N)}S_3+\frac{256\big(-38+N+N^2\big)}{3N(1+N)(2+N)}S_{-2,1}+\frac{192(N-1)}{N(1+N)}\zeta_3\Bigg]S_2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{64\big(-82+17N+17N^2\big)}{3N(1+N)(2+N)}S_5+\Biggl[\frac{8P_{1020}}{3(N-1)N^2(1+N)^4(2+N)^2}\\
&+\Biggl(\frac{32P_{1012}}{3(N-1)N^3(1+N)^3(2+N)^2}-\frac{640(N-1)}{3N(1+N)}S_2\Biggr)S_1+\frac{640(N-1)}{9N(1+N)}S_1^3\\
&-\frac{64\big(-3+10N+26N^2+6N^3\big)}{3N^2(1+N)^2(2+N)}S_1^2+\frac{64\big(-5+N+13N^2+6N^3\big)}{3N^2(1+N)^2(2+N)}S_2\\
&+\frac{512\big(-5+7N+7N^2\big)}{9N(1+N)(2+N)}S_3+\frac{3072}{N(1+N)(2+N)}S_{2,1}-\frac{128(N-1)}{3N(1+N)}S_{-2,1}\\
&+\frac{512\big(-9+2N+2N^2\big)}{N(1+N)(2+N)}\zeta_3\Biggr]S_{-2}+\Biggl[-\frac{32P_{973}}{3(N-1)N^2(1+N)^2(2+N)^2}\\
&-\frac{256\big(-8+N+N^2\big)}{3N(1+N)(2+N)}S_1\Biggr]S_{-2}^2+\Biggl[\frac{32P_{963}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1\\
&-\frac{16P_{1016}}{3(N-1)N^3(1+N)^3(2+N)^2}-\frac{64\big(2+11N+11N^2\big)}{3N(1+N)(2+N)}S_1^2-\frac{64\big(-74+N+N^2\big)}{3N(1+N)(2+N)}\\
&\times S_2+\frac{512\big(-1+N+N^2\big)}{N(1+N)(2+N)}S_{-2}\Biggr]S_{-3}+\Biggl[-\frac{32P_{970}}{3(N-1)N^2(1+N)^2(2+N)^2}\\
&-\frac{512S_1}{N(1+N)(2+N)}\Biggr]S_{-4}+\frac{32\big(10+67N+67N^2\big)}{3N(1+N)(2+N)}S_{-5}-\frac{128(-4+N)(5+N)}{N(1+N)(2+N)}\\
&\times S_{2,3}+\frac{192\big(6+N+N^2\big)}{N(1+N)(2+N)}S_{2,-3}+\frac{32\big(-98+13N+13N^2\big)}{N(1+N)(2+N)}S_{4,1}\\
&-\frac{64\big(-2+25N+25N^2\big)}{3N(1+N)(2+N)}S_{-2,3}-\frac{64\big(2+3N+3N^2\big)}{N(1+N)(2+N)}S_{-4,1}-\frac{160(N-1)}{3N(1+N)}S_{2,2,1}\\
&-\frac{32\big(-8-5N+19N^2+12N^3\big)}{3N^2(1+N)^2(2+N)}S_{2,1,1}-\frac{128\big(70+N+N^2\big)}{3N(1+N)(2+N)}S_{2,1,-2}\\
&+\frac{4608}{N(1+N)(2+N)}S_{3,1,1}-\frac{128\big(22+N+N^2\big)}{3N(1+N)(2+N)}S_{-2,1,-2}-\frac{3072}{N(1+N)(2+N)}S_{-2,2,1}\\
&-\frac{64\big(142+N+N^2\big)}{3N(1+N)(2+N)}S_{-3,1,1}-\frac{160(N-1)}{N(1+N)}S_{2,1,1,1}+\frac{128\big(146-N-N^2\big)}{3N(1+N)(2+N)}\\
&\times S_{-2,1,1,1}+\frac{48(N-1)\big(-2+3N+3N^2\big)}{5N^2(1+N)^2}\zeta_2^2-\frac{160\big(66+N+N^2\big)}{N(1+N)(2+N)}\zeta_5\Biggr\}
\end{aligned}
\tag{1128}
$$

and

$$P_{939} = -305N^4-514N^3-19N^2+142N-1416, \tag{1129}$$
$$P_{940} = -117N^4+14N^3+791N^2+676N-116, \tag{1130}$$
$$P_{941} = 2N^4+8N^3-6N^2-35N-20, \tag{1131}$$
$$P_{942} = 3N^4+15N^3+14N^2+2N+8, \tag{1132}$$
$$P_{943} = 11N^4-14N^3+148N^2+191N-48, \tag{1133}$$
$$P_{944} = 17N^4-20N^3+51N^2+112N-36, \tag{1134}$$
$$P_{945} = 44N^4+61N^3+55N^2+74N-6, \tag{1135}$$

$$P_{946} = 124N^4 + 233N^3 + 49N^2 - 60N + 54, \quad (1136)$$
$$P_{947} = 177N^4 + 146N^3 - 731N^2 - 492N + 324, \quad (1137)$$
$$P_{948} = 349N^4 + 680N^3 - 550N^2 - 245N + 1038, \quad (1138)$$
$$P_{949} = 368N^4 + 1036N^3 - 1145N^2 - 1741N + 954, \quad (1139)$$
$$P_{950} = 497N^4 + 646N^3 - 785N^2 + 482N + 1848, \quad (1140)$$
$$P_{951} = 6N^5 + 12N^4 + 65N^3 + 138N^2 + 225N + 130, \quad (1141)$$
$$P_{952} = 12N^5 + 36N^4 + 191N^3 + 318N^2 + 369N + 226, \quad (1142)$$
$$P_{953} = 16N^5 - 47N^3 + 19N^2 - 4N + 12, \quad (1143)$$
$$P_{954} = 19N^5 + 76N^4 + 45N^3 - 52N^2 + 20N + 36, \quad (1144)$$
$$P_{955} = 42N^5 + 15N^4 + 23N^3 - 141N^2 - 153N + 70, \quad (1145)$$
$$P_{956} = 53N^5 + 158N^4 + 9N^3 - 119N^2 + 34N - 108, \quad (1146)$$
$$P_{957} = 67N^5 + 277N^4 + 216N^3 - 145N^2 + 44N + 180, \quad (1147)$$
$$P_{958} = 127N^5 + N^4 - 421N^3 + 267N^2 - 114N - 436, \quad (1148)$$
$$P_{959} = 3N^6 - 14N^5 - 551N^4 - 838N^3 - 644N^2 - 572N - 264, \quad (1149)$$
$$P_{960} = 3N^6 + 9N^5 - N^4 + 7N^3 + 82N^2 + 20N + 24, \quad (1150)$$
$$P_{961} = 3N^6 + 9N^5 + 49N^4 + 77N^3 - 28N^2 - 50N + 84, \quad (1151)$$
$$P_{962} = 6N^6 + 24N^5 + 113N^4 + 158N^3 + 159N^2 + 112N + 4, \quad (1152)$$
$$P_{963} = 6N^6 + 36N^5 + 147N^4 + 158N^3 + 87N^2 + 78N + 64, \quad (1153)$$
$$P_{964} = 8N^6 + 27N^5 + 18N^4 - 25N^3 - 80N^2 - 44N + 24, \quad (1154)$$
$$P_{965} = 9N^6 + 71N^5 + 981N^4 + 1675N^3 + 1936N^2 + 1224N - 136, \quad (1155)$$
$$P_{966} = 9N^6 + 95N^5 + 955N^4 + 1477N^3 + 2132N^2 + 1620N - 528, \quad (1156)$$
$$P_{967} = 9N^6 + 119N^5 + 929N^4 + 1279N^3 + 2328N^2 + 2016N - 920, \quad (1157)$$
$$P_{968} = 12N^6 + 50N^5 + 565N^4 + 958N^3 + 433N^2 + 86N + 488, \quad (1158)$$
$$P_{969} = 12N^6 + 60N^5 + 261N^4 + 314N^3 + 267N^2 + 222N + 16, \quad (1159)$$
$$P_{970} = 15N^6 + 23N^5 - 101N^4 - 131N^3 - 32N^2 - 82N + 20, \quad (1160)$$
$$P_{971} = 15N^6 + 47N^5 + 107N^4 + 139N^3 + 108N^2 + 24N - 8, \quad (1161)$$
$$P_{972} = 19N^6 + 63N^5 - 28N^4 - 154N^3 - 63N^2 - 35N - 18, \quad (1162)$$
$$P_{973} = 21N^6 + 57N^5 + 29N^4 + 3N^3 + 126N^2 + 76N - 24, \quad (1163)$$
$$P_{974} = 23N^6 - 201N^5 - 955N^4 - 583N^3 + 828N^2 - 48N - 536, \quad (1164)$$
$$P_{975} = 23N^6 + 61N^5 + 93N^4 - 117N^3 - 692N^2 - 256N + 600, \quad (1165)$$
$$P_{976} = 27N^6 + 87N^5 + 179N^4 + 213N^3 + 94N^2 - 40N + 16, \quad (1166)$$
$$P_{977} = 30N^6 + 163N^5 + 788N^4 + 865N^3 + 1322N^2 + 1408N - 544, \quad (1167)$$
$$P_{978} = 33N^6 + 113N^5 + 815N^4 + 1319N^3 + 1140N^2 + 708N + 1632, \quad (1168)$$
$$P_{979} = 35N^6 + 93N^5 - 17N^4 - 97N^3 - 238N^2 - 320N + 256, \quad (1169)$$
$$P_{980} = 39N^6 + 13N^5 - 439N^4 - 421N^3 - 2156N^2 - 2412N + 1344, \quad (1170)$$
$$P_{981} = 39N^6 + 161N^5 + 45N^4 - 345N^3 + 600N^2 + 908N - 832, \quad (1171)$$
$$P_{982} = 60N^6 + 126N^5 - 491N^4 - 724N^3 - 271N^2 - 584N - 420, \quad (1172)$$
$$P_{983} = 60N^6 + 132N^5 - 367N^4 - 652N^3 - 711N^2 - 782N + 16, \quad (1173)$$
$$P_{984} = 66N^6 + 140N^5 - 337N^4 - 550N^3 - 835N^2 - 1024N + 236, \quad (1174)$$
$$P_{985} = 69N^6 + 203N^5 - 153N^4 + 21N^3 + 1536N^2 + 308N - 1408, \quad (1175)$$

$$
\begin{align}
P_{986} &= 72N^6+148N^5-307N^4-448N^3-959N^2-1266N+456, \tag{1176}\\
P_{987} &= 85N^6+243N^5+443N^4+425N^3+2592N^2+2644N+3072, \tag{1177}\\
P_{988} &= 93N^6+39N^5+3465N^4+7681N^3+5118N^2+924N+3416, \tag{1178}\\
P_{989} &= 143N^6+231N^5+298N^4+1495N^3-4263N^2-6796N+3708, \tag{1179}\\
P_{990} &= 147N^6+453N^5+811N^4+755N^3+606N^2+500N+760, \tag{1180}\\
P_{991} &= 176N^6+390N^5-829N^4-1288N^3-429N^2-1208N-268, \tag{1181}\\
P_{992} &= 205N^6+225N^5-7345N^4-15529N^3-14712N^2-4748N-9936, \tag{1182}\\
P_{993} &= 213N^6+703N^5+3205N^4+4849N^3+3122N^2+1340N+2120, \tag{1183}\\
P_{994} &= 233N^6+591N^5-713N^4-683N^3+7632N^2+5372N-4656, \tag{1184}\\
P_{995} &= 263N^6+771N^5-848N^4-1349N^3+9819N^2+7532N-1932, \tag{1185}\\
P_{996} &= 274N^6+624N^5-278N^4-615N^3+625N^2+108N-540, \tag{1186}\\
P_{997} &= 434N^6+810N^5+5245N^4+10912N^3+3351N^2-2956N+16764, \tag{1187}\\
P_{998} &= 485N^6+975N^5+2077N^4+4777N^3+666N^2-3820N+13848, \tag{1188}\\
P_{999} &= 641N^6+1995N^5+3601N^4+5437N^3+11178N^2+6116N-1320, \tag{1189}\\
P_{1000} &= 764N^6+1896N^5-592N^4-1797N^3+413N^2-3762N-3600, \tag{1190}\\
P_{1001} &= 1721N^6+3837N^5+29N^4+909N^3+4838N^2-5394N-5508, \tag{1191}\\
P_{1002} &= 30N^7+51N^6-5N^5-173N^4-63N^3-22N^2-118N+12, \tag{1192}\\
P_{1003} &= 140N^7+643N^6+902N^5+N^4-676N^3+94N^2+264N-216, \tag{1193}\\
P_{1004} &= 200N^7+1357N^6+2729N^5+1069N^4-481N^3+1174N^2-324N-1080, \tag{1194}\\
P_{1005} &= 293N^7+513N^6+2089N^5+4287N^4+11362N^3+9848N^2+3496N-784, \tag{1195}\\
P_{1006} &= 383N^7+1702N^6+689N^5-4466N^4-4516N^3-176N^2-312N-336, \tag{1196}\\
P_{1007} &= 388N^7+1781N^6+1660N^5-2725N^4-5366N^3-2458N^2-192N-216, \tag{1197}\\
P_{1008} &= 1067N^7+5191N^6+5663N^5-2933N^4-2176N^3+4672N^2-288N-1008, \tag{1198}\\
P_{1009} &= 1427N^7+6712N^6+6878N^5-5021N^4-4822N^3+280N^2-12132N-8856, \tag{1199}\\
P_{1010} &= 1471N^7+7367N^6+12949N^5+9107N^4+6508N^3+4334N^2-12144N \nonumber\\
&\quad -10440, \tag{1200}\\
P_{1011} &= -205N^8-904N^7+88N^6+3218N^5+2309N^4+254N^3-944N^2-2584N \nonumber\\
&\quad -1424, \tag{1201}\\
P_{1012} &= 13N^8+50N^7+104N^6+130N^5+127N^4+136N^3+100N^2+36N-24, \tag{1202}\\
P_{1013} &= 35N^8+142N^7+16N^6-424N^5+341N^4+1418N^3+704N^2+24N+48, \tag{1203}\\
P_{1014} &= 39N^8+390N^7-2192N^6-4350N^5+10577N^4+8384N^3-28776N^2-9800N \nonumber\\
&\quad +13632, \tag{1204}\\
P_{1015} &= 42N^8+141N^7+284N^6+351N^5-1843N^4-2925N^3-730N^2-1272N \nonumber\\
&\quad -960, \tag{1205}\\
P_{1016} &= 71N^8+290N^7+108N^6-624N^5+437N^4+1838N^3+384N^2 \nonumber\\
&\quad -248N+48, \tag{1206}\\
P_{1017} &= 90N^8+417N^7-155N^6-1779N^5+7189N^4+14142N^3+6376N^2+4776N \nonumber\\
&\quad +3504, \tag{1207}\\
P_{1018} &= 93N^8+1048N^7+2200N^6+1622N^5+1891N^4+890N^3-936N^2+376N \nonumber\\
&\quad +624, \tag{1208}
\end{align}
$$

$$P_{1019} = 114N^8 + 345N^7 - 222N^6 - 1074N^5 - 1960N^4 - 3335N^3 - 2280N^2 - 68N + 416, \quad (1209)$$

$$P_{1020} = 153N^8 + 725N^7 + 380N^6 - 2494N^5 - 3431N^4 - 399N^3 - 510N^2 - 1672N - 1008, \quad (1210)$$

$$P_{1021} = 191N^8 + 684N^7 - 217N^6 - 1949N^5 - 1246N^4 + 2305N^3 + 4728N^2 + 1120N - 432, \quad (1211)$$

$$P_{1022} = 240N^8 + 1089N^7 - 917N^6 - 6465N^5 + 6181N^4 + 19980N^3 + 8596N^2 + 3840N + 3168, \quad (1212)$$

$$P_{1023} = 339N^8 + 843N^7 - 2092N^6 - 2274N^5 + 1147N^4 - 14645N^3 - 26082N^2 + 116N + 6360, \quad (1213)$$

$$P_{1024} = 430N^8 + 1951N^7 - 340N^6 - 7385N^5 + 2891N^4 + 11059N^3 + 970N^2 + 8136N + 3888, \quad (1214)$$

$$P_{1025} = 1565N^8 + 5906N^7 + 3476N^6 - 7570N^5 - 4705N^4 + 152N^3 - 8328N^2 + 3888, \quad (1215)$$

$$P_{1026} = 1973N^8 + 5921N^7 + 1402N^6 - 1417N^5 + 14062N^4 + 21596N^3 + 41243N^2 + 14832N - 14076, \quad (1216)$$

$$P_{1027} = 4724N^8 + 15473N^7 + 3469N^6 - 22813N^5 + 48721N^4 + 140300N^3 + 112538N^2 + 15108N - 6480, \quad (1217)$$

$$P_{1028} = 19N^9 + 63N^8 + 192N^7 + 162N^6 - 513N^5 - 513N^4 + 230N^3 - 720N^2 - 216N + 144, \quad (1218)$$

$$P_{1029} = 651N^9 + 3056N^8 + 1492N^7 - 12986N^6 - 24959N^5 - 15422N^4 - 6056N^3 - 5680N^2 + 1296N + 2592, \quad (1219)$$

$$P_{1030} = -12043N^{10} - 52199N^9 - 68592N^8 - 22926N^7 + 115557N^6 + 458949N^5 + 428806N^4 - 302176N^3 - 337584N^2 + 149040N + 140832, \quad (1220)$$

$$P_{1031} = 63N^{10} + 291N^9 + 581N^8 + 1646N^7 + 2335N^6 + 4289N^5 + 13213N^4 + 5150N^3 - 12616N^2 + 552N + 5232, \quad (1221)$$

$$P_{1032} = 226N^{10} + 1061N^9 + 860N^8 - 1831N^7 - 2444N^6 + 2045N^5 + 8496N^4 + 6381N^3 - 4438N^2 - 2268N + 2280, \quad (1222)$$

$$P_{1033} = 5399N^{10} + 21865N^9 + 4191N^8 - 66642N^7 - 36909N^6 + 68193N^5 + 32287N^4 + 4016N^3 + 45360N^2 + 3888N - 19440, \quad (1223)$$

$$P_{1034} = 5459N^{10} + 35950N^9 + 52398N^8 - 48294N^7 - 52359N^6 + 143592N^5 + 37432N^4 - 26596N^3 + 134946N^2 - 70632N - 87480, \quad (1224)$$

$$P_{1035} = 9429N^{10} + 33109N^9 - 26912N^8 - 199686N^7 - 184155N^6 + 5745N^5 + 39702N^4 + 97328N^3 + 107792N^2 - 12816N - 35424, \quad (1225)$$

$$P_{1036} = 13789N^{10} + 60242N^9 + 31062N^8 - 128262N^7 - 48825N^6 + 174492N^5 + 110408N^4 + 144844N^3 + 66474N^2 - 183816N - 115992, \quad (1226)$$

$$P_{1037} = 205N^{12} + 1122N^{11} - 2485N^{10} - 21156N^9 - 27453N^8 + 17874N^7 + 42893N^6 - 5568N^5 - 25832N^4 + 2784N^3 + 10512N^2 + 576N - 384, \quad (1227)$$

$$P_{1038} = 254N^{12} + 1391N^{11} + 413N^{10} - 8542N^9 - 8007N^8 + 22639N^7 + 38167N^6 + 10124N^5 - 9715N^4 - 4980N^3 + 2736N^2 + 3344N + 1328, \quad (1228)$$

$$P_{1039} = 1538N^{12} + 8703N^{11} + 20491N^{10} + 20280N^9 - 17340N^8 - 20277N^7$$

$$+105025N^6+77580N^5-83254N^4+66642N^3+131004N^2-18792N$$
$$-42768, \qquad (1229)$$
$$P_{1040} = 3805N^{12}+23880N^{11}+31020N^{10}-63109N^9-153501N^8-33090N^7+131168N^6 +144597N^5-48240N^4-172070N^3+49356N^2+44280N+10800, \qquad (1230)$$
$$P_{1041} = 4920N^{13}+22305N^{12}+218N^{11}-95044N^{10}-22165N^9+174177N^8+27618N^7 -156150N^6-99101N^5-98972N^4-72698N^3+67716N^2+20376N-22032 \qquad (1231)$$
$$P_{1042} = 617N^{14}+4175N^{13}+14349N^{12}+30747N^{11}+2959N^{10}-140311N^9-226729N^8 -109247N^7-131196N^6-283508N^5-248768N^4-18848N^3+45408N^2 +6528N-2944, \qquad (1232)$$
$$P_{1043} = 32968N^{14}+214291N^{13}+230166N^{12}-1212607N^{11}-2507302N^{10}+1955271N^9 +5076722N^8-7562245N^7-17306682N^6-3182438N^5+7389328N^4+339888N^3 -2572992N^2+515808N+673920, \qquad (1233)$$
$$P_{1044} = 34763N^{14}+211157N^{13}+229596N^{12}-789770N^{11}-1434236N^{10}+1014978N^9 +2087956N^8-1947938N^7-3974499N^6-2470711N^5+66548N^4+2887812N^3 +1290168N^2-982368N-702432, \qquad (1234)$$
$$P_{1045} = -10701N^{15}-42606N^{14}-112294N^{13}-235574N^{12}-177548N^{11}-133306N^{10} -388110N^9+1317318N^8+3159337N^7+1521824N^6+1661012N^5+2875000N^4 -65712N^3-1912320N^2+39744N+466560, \qquad (1235)$$
$$P_{1046} = 23379N^{16}+176637N^{15}+336908N^{14}-418342N^{13}-1616541N^{12}+367147N^{11} +3663481N^{10}+1241253N^9-2594354N^8-1895204N^7+1153767N^6 +3040589N^5-340384N^4-2307312N^3-1311840N^2+714096N+513216, \qquad (1236)$$
$$P_{1047} = 295317N^{16}+2034216N^{15}+4013522N^{14}-2161486N^{13}-8694264N^{12} +30856978N^{11}+88306930N^{10}+17465238N^9-110878589N^8-41117522N^7 +85727340N^6+17006960N^5-58178704N^4-9306528N^3+13384512N^2 -1838592N-3027456. \qquad (1237)$$

$$\Delta C_{g_1,g}^{(3),d_{abc}} = \frac{d_{abc}d^{abc}N_F^2}{N_A}\Biggl\{-\frac{128(N-2)(3+N)P_{1048}}{45N(1+N)(2+N)}S_5-\frac{64(N-2)(3+N)P_{1048}}{9N(1+N)(2+N)}\zeta_5 +\frac{128P_{1049}}{(N-1)N(1+N)(2+N)^2}S_3+\frac{32P_{1052}}{45(N-1)N(1+N)(2+N)^2} -\frac{256P_{1053}}{45N^2(1+N)^2(2+N)}S_{-2,1}-\frac{512P_{1055}}{3(N-1)N^2(1+N)^2(2+N)^2}S_4 +\frac{1024P_{1055}}{3(N-1)N^2(1+N)^2(2+N)^2}S_{3,1}-\frac{64P_{1058}}{45(N-1)N^2(1+N)^2(2+N)^2}\zeta_3 +\Biggl[-\frac{128\bigl(-4+N+N^2\bigr)P_{1050}}{(N-1)N^2(1+N)^2(2+N)^2}S_3-\frac{64P_{1051}}{45(N-1)N(1+N)(2+N)^2} +\frac{512P_{1056}}{3(N-1)N^2(1+N)^2(2+N)^2}\zeta_3+\frac{512\bigl(-4+N+N^2\bigr)}{N(1+N)(2+N)}S_4 -\frac{1024\bigl(-4+N+N^2\bigr)S_{3,1}}{N(1+N)(2+N)}+\frac{512\bigl(-8+3N+3N^2\bigr)}{3N(1+N)(2+N)}S_{-2,1}\Biggr]S_1$$

$$+\Biggl[\frac{128\bigl(-19+3N+3N^2\bigr)}{3N(1+N)(2+N)}+\frac{256\bigl(-4+N+N^2\bigr)}{N(1+N)(2+N)}S_3-\frac{512\bigl(-4+N+N^2\bigr)}{N(1+N)(2+N)}$$

$$\times\zeta_3\Biggr]S_1^2-\frac{768\bigl(-4+N+N^2\bigr)}{N(1+N)(2+N)}S_2S_3+\Biggl[-\frac{64P_{1054}}{45(N-1)N(1+N)(2+N)^2}$$

$$+\frac{256P_{1057}}{45(N-1)N^2(1+N)^2(2+N)^2}S_1-\frac{256\bigl(-8+3N+3N^2\bigr)}{3N(1+N)(2+N)}S_1^2$$

$$-\frac{256}{45}(N-1)\bigl(-21+N+N^2\bigr)[S_3+S_{-2,1}+\zeta_3]\Biggr]S_{-2}-\frac{128}{2+N}S_{-2}^2$$

$$+\Biggl[\frac{128P_{1053}}{45N^2(1+N)^2(2+N)}-\frac{256\bigl(-8+3N+3N^2\bigr)}{3N(1+N)(2+N)}S_1\Biggr]S_{-3}$$

$$-\frac{256\bigl(-4+3N+3N^2\bigr)}{3N(1+N)(2+N)}S_{-4}-\frac{128}{45}(N-1)\bigl(-21+N+N^2\bigr)S_{-5}$$

$$+\frac{768\bigl(-4+N+N^2\bigr)}{N(1+N)(2+N)}[S_{2,3}-S_{4,1}]+\frac{256\bigl(-8+3N+3N^2\bigr)}{3N(1+N)(2+N)}S_{-2,2}$$

$$+\frac{256}{45}(N-1)\bigl(-21+N+N^2\bigr)[S_{-2,3}+S_{-2,1,-2}]+\frac{256\bigl(-8+3N+3N^2\bigr)}{3N(1+N)(2+N)}S_{-3,1}$$

$$+\frac{1536\bigl(-4+N+N^2\bigr)}{N(1+N)(2+N)}S_{3,1,1}-\frac{512\bigl(-8+3N+3N^2\bigr)}{3N(1+N)(2+N)}S_{-2,1,1}\Biggr\} \qquad (1238)$$

and

$$P_{1048} = N^4+2N^3-16N^2-17N+120, \qquad (1239)$$
$$P_{1049} = 5N^4+10N^3-16N^2-21N+10, \qquad (1240)$$
$$P_{1050} = 5N^4+10N^3+3N^2-2N-4, \qquad (1241)$$
$$P_{1051} = 46N^4+92N^3+1049N^2+1003N-3342, \qquad (1242)$$
$$P_{1052} = 65N^4+130N^3-593N^2-658N+5232, \qquad (1243)$$
$$P_{1053} = N^6+3N^5+24N^4+43N^3-142N^2-163N-156, \qquad (1244)$$
$$P_{1054} = 2N^6+6N^5+315N^4+620N^3-1391N^2-1700N+60, \qquad (1245)$$
$$P_{1055} = 3N^6+9N^5-4N^4-23N^3-14N^2-N+12, \qquad (1246)$$
$$P_{1056} = 9N^6+27N^5-13N^4-71N^3-26N^2+14N+24, \qquad (1247)$$
$$P_{1057} = 45N^6+135N^5-140N^4-505N^3+35N^2+310N+156, \qquad (1248)$$
$$P_{1058} = 2N^8+8N^7+815N^6+2417N^5-1509N^4-7037N^3-1780N^2 +2140N+624. \qquad (1249)$$

The above Wilson coefficients complete the massless parts of the polarized single and two mass heavy flavor Wilson coefficients still missing in Refs. [49–53].

8 The small- and large x expansions of the Wilson coefficients

We will represent the small x and large x behaviour of the different Wilson coefficients first in Mellin N space, where the former corresponds to the expansion around $N = 0$ and $N = 1$, while the latter corresponds to the expansion in the limit $N \to \infty$. Finally, we also consider the large N_F limit.

The non–singlet Wilson coefficients receive their leading small x contributions from the expansion around $N = 0$, retaining the pole terms. The singlet ones also receive contributions from the pole terms at $N = 1$. This is due to the Mellin transforms

$$\frac{1}{(N-1)^k} = \frac{(-1)^{k-1}}{(k-1)!}\mathbf{M}\left[\frac{\ln^{k-1}(x)}{x}\right](N) \quad (1250)$$

$$\frac{1}{N^k} = \frac{(-1)^{k-1}}{(k-1)!}\mathbf{M}[\ln^{k-1}(x)](N). \quad (1251)$$

In the large x region we will retain all terms up to $\propto 1/N$ in the asymptotic expansion. Here, terms of the kind

$$L_a^l \equiv (\ln(N) + \gamma_E)^l \quad (1252)$$

contribute, which we will replace by

$$\ln(N) + \gamma_E = S_1(N) - \frac{1}{2N} + O\left(\frac{1}{N^2}\right), \quad (1253)$$

to derive in a more direct way the inverse Mellin transforms. One obtains asymptotically up to terms of $O(L_a^k/N^2)$, $k \geq 0$, and smaller

$$\mathbf{M}^{-1}[S_1(N)](x) \simeq -\frac{1}{1-x} + 1, \quad (1254)$$

$$\mathbf{M}^{-1}[S_1^2(N)](x) \simeq \frac{2\ln(1-x)}{1-x} - 2\ln(1-x) - \zeta_2\delta(1-x) + 1, \quad (1255)$$

$$\mathbf{M}^{-1}[S_1^3(N)](x) \simeq -\frac{3\ln^2(1-x)}{1-x} + \frac{3\zeta_2}{1-x} + 3\ln^2(1-x) - 3\ln(1-x) - 2\zeta_3\delta(1-x)$$
$$-3\zeta_2, \quad (1256)$$

$$\mathbf{M}^{-1}[S_1^4(N)](x) \simeq \frac{4\ln^3(1-x)}{1-x} - \frac{12\zeta_2\ln(1-x)}{1-x} + \frac{8\zeta_3}{1-x} - 4\ln^3(1-x) + 6\ln^2(1-x)$$
$$+12\zeta_2\ln(1-x) + \frac{3}{5}\zeta_2^2\delta(1-x) - 6\zeta_2 - 8\zeta_3, \quad (1257)$$

$$\mathbf{M}^{-1}[S_1^5(N)](x) \simeq -\frac{5\ln^4(1-x)}{1-x} + \frac{30\zeta_2\ln^2(1-x)}{1-x} - \frac{40\zeta_3\ln(1-x)}{1-x} - \frac{3\zeta_2^2}{1-x}$$
$$+5\ln^4(1-x) - 10\ln^3(1-x) - 30\zeta_2\ln^2(1-x) + (30\zeta_2 + 40\zeta_3)$$
$$\times\ln(1-x) + (20\zeta_2\zeta_3 - 24\zeta_5)\delta(1-x) + 3\zeta_2^2 - 20\zeta_3, \quad (1258)$$

$$\mathbf{M}^{-1}[S_1^6(N)](x) \simeq \frac{6\ln^5(1-x)}{1-x} - \frac{60\zeta_2\ln^3(1-x)}{1-x} + \frac{120\zeta_3\ln^2(1-x)}{1-x} + \frac{18\zeta_2^2\ln(1-x)}{1-x}$$
$$-(120\zeta_2\zeta_3 - 144\zeta_5)\frac{1}{1-x} - 6\ln^5(1-x) + 15\ln^4(1-x)$$
$$+60\zeta_2\ln^3(1-x) - (90\zeta_2 + 120\zeta_3)\ln^2(1-x) + (120\zeta_3 - 18\zeta_2^2)\ln(1-x)$$
$$+\left(40\zeta_3^2 - \frac{45}{7}\zeta_2^3\right)\delta(1-x) + 9\zeta_2^2 + 120\zeta_2\zeta_3 - 144\zeta_5, \quad (1259)$$

and

$$\mathbf{M}^{-1}\left[\frac{S_1(N)}{N}\right](x) \simeq -\ln(1-x), \quad (1260)$$

$$\mathbf{M}^{-1}\left[\frac{S_1^2(N)}{N}\right](x) \simeq \ln^2(1-x)-\zeta_2, \tag{1261}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^3(N)}{N}\right](x) \simeq -\ln^3(1-x)+3\zeta_2\ln(1-x)-2\zeta_3, \tag{1262}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^4(N)}{N}\right](x) \simeq \ln^4(1-x)-6\zeta_2\ln^2(1-x)+8\zeta_3\ln(1-x)+\frac{3}{5}\zeta_2^2, \tag{1263}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^5(N)}{N}\right](x) \simeq -\ln^5(1-x)+10\zeta_2\ln^3(1-x)-20\zeta_3\ln^2(1-x)-3\zeta_2^2\ln(1-x) +20\zeta_2\zeta_3-24\zeta_5, \tag{1264}$$

$$\mathbf{M}^{-1}\left[\frac{S_1^6(N)}{N}\right](x) \simeq \ln^6(1-x)-15\zeta_2\ln^4(1-x)+40\zeta_3\ln^3(1-x)+9\zeta_2^2\ln^2(1-x) +(144\zeta_5-120\zeta_2\zeta_3)\ln(1-x)-\frac{45}{7}\zeta_2^3+40\zeta_3^2. \tag{1265}$$

We will now present the asymptotic behaviour of the Wilson coefficients in the small x and large x region. For the pure singlet and gluonic Wilson coefficients of the structure function $g_1(x,Q^2)$ we refer to the Larin scheme.[4]

8.1 The small x limit

In this section we will show the small x expansion of the Wilson coefficients calculated above. The non–singlet unpolarized Wilson coefficients have leading singularities at $N=0$, $\propto \ln^k(x)$ and the unpolarized pure singlet and gluonic Wilson coefficients have their leading singularity at $N=1$, i.e. terms as $\ln(x)/x$ and $1/x$. Here we also list the logarithmic contributions up to the constant terms. One obtains

$$C_{F_2,q}^{(1),\rm NS} \simeq C_F[3-2\ln(x)], \tag{1266}$$

$$C_{F_2,q}^{(2),\rm NS} \simeq C_F\Biggl\{T_FN_F\left[\frac{10}{3}\ln^2(x)+8\ln(x)+\frac{178}{27}-\frac{16}{3}\zeta_2\right]+C_A\left[-\frac{55}{6}\ln^2(x)-32\ln(x) -\frac{1693}{54}+\frac{44}{3}\zeta_2-12\zeta_3\right]\Biggr\}+C_F^2\Biggl\{-\frac{5}{3}\ln^3(x)+9\ln^2(x)+(17+24\zeta_2)\ln(x)-8\zeta_2 +\frac{3}{2}+56\zeta_3\Biggr\}, \tag{1267}$$

$$C_{F_2,q}^{(3),\rm NS} \simeq -\frac{1}{2}C_F^3\ln^5(x)+\left[\frac{67}{12}C_F^3+C_F^2\left(-\frac{1001}{108}C_A+\frac{91}{27}T_FN_F\right)\right]\ln^4(x) +\Biggl[C_F\left(\frac{2024}{81}C_AT_FN_F-\frac{368}{81}T_F^2N_F^2+\left(-\frac{2783}{81}+20\zeta_2\right)C_A^2\right)+C_F^2\Biggl(\frac{10}{27}N_FT_F +\left(-\frac{835}{54}-64\zeta_2\right)C_A\Biggr)+C_F^3\left(5+\frac{262}{3}\zeta_2\right)\Biggr]\ln^3(x)+\Biggl[C_F\Biggl(-\frac{1984}{81}T_F^2N_F^2 +\left(\frac{14392}{81}-16\zeta_2\right)C_AT_FN_F+\left(-\frac{23062}{81}+84\zeta_2\right)C_A^2\Biggr)+C_F^2\Biggl(T_FN_F\Biggl(-\frac{2630}{81}$$

[4] To one- and two-loop order it is known, that the gluonic Wilson coefficients are the same in the Larin and M-scheme [26].

$$-\frac{532}{9}\zeta_2\Biggr) + C_A\Biggl(\frac{17315}{162} - \frac{265\zeta_2}{9} - 64\zeta_3\Biggr)\Biggr) + \Biggl(-\frac{113}{6} + \frac{299}{3}\zeta_2$$

$$+\frac{646}{3}\zeta_3\Biggr)C_F^3\Biggr]\ln^2(x) + \Biggl[C_F\Biggl[T_F^2N_F^2\Biggl(-\frac{4816}{81} + \frac{64}{3}\zeta_2) + C_A^2\Biggl(-\frac{78338}{81} + \frac{3058}{9}\zeta_2$$

$$+32\zeta_3 - 24\zeta_2^2\Biggr) + C_AT_FN_F\Biggl(\frac{14152}{27} - \frac{1376}{9}\zeta_2 + \frac{256}{3}\zeta_3\Biggr)\Biggr] + C_F^2\Biggl[T_FN_F\Biggl($$

$$-\frac{2999}{81} - \frac{964}{9}\zeta_2 - 264\zeta_3\Biggr) + C_A\Biggl(\frac{106801}{324} + \frac{599}{9}\zeta_2 + \frac{418}{3}\zeta_3 + \frac{656}{15}\zeta_2^2\Biggr)\Biggr]$$

$$+C_F^3\Biggl(\frac{1619}{12} + \frac{764}{3}\zeta_2 + 154\zeta_3 - \frac{556}{3}\zeta_2^2\Biggr)\Biggr]\ln(x)$$

$$+\Biggl[C_F^2\Biggl(C_A\Biggl(\frac{193961}{648} + \frac{6014}{81}\zeta_2 + \frac{13189}{27}\zeta_3 - \frac{3434}{45}\zeta_2^2 + 520\zeta_2\zeta_3 + \frac{1970}{3}\zeta_5\Biggr)$$

$$+T_FN_F\Biggl(-\frac{2881}{162} + \frac{944}{81}\zeta_2 - \frac{11804}{27}\zeta_3 - \frac{848}{45}\zeta_2^2\Biggr)\Biggr) + C_F\Biggl(C_A^2\Biggl(-\frac{1779023}{1458}$$

$$+\frac{14917}{27}\zeta_2 - \frac{1960}{27}\zeta_3 - \frac{148}{3}\zeta_2^2 - \frac{436}{3}\zeta_2\zeta_3 - \frac{152}{3}\zeta_5\Biggr) + T_F^2N_F^2\Biggl(-\frac{44680}{729}$$

$$+\frac{928\zeta_2}{27} + \frac{128\zeta_3}{27}\Biggr) + C_AT_FN_F\Biggl(\frac{448438}{729} - \frac{9040}{27}\zeta_2 + \frac{704}{15}\zeta_2^2 + \frac{2824}{27}\zeta_3\Biggr)\Biggr)$$

$$+C_F^3\Biggl(\frac{5603}{24} + \frac{533}{3}\zeta_2 + \frac{1730}{3}\zeta_3 - \frac{68}{15}\zeta_2^2 - 904\zeta_2\zeta_3 - 872\zeta_5\Biggr)\Biggr], \qquad (1268)$$

$$C_{F_2,q}^{(3,d_{abc})} \simeq 0, \qquad (1269)$$

$$C_{F_2,q}^{(2),\mathrm{PS}} \simeq C_FT_FN_F\Biggl[\frac{1}{x}\Biggl(\frac{688}{27} - \frac{32}{3}\zeta_2\Biggr) + \frac{20}{3}\ln^3(x) - 2\ln^2(x) + \bigl(112 - 32\zeta_2\bigr)\ln(x)$$

$$+12 - 16\zeta_3\Biggr], \qquad (1270)$$

$$C_{F_2,q}^{(3),\mathrm{PS}} \simeq C_AC_FT_FN_F\Biggl[-\frac{78976}{243} + \frac{832}{9}\zeta_2 - \frac{256}{9}\zeta_3\Biggr]\frac{\ln(x)}{x} + \Biggl\{C_F\Biggl[T_F^2N_F^2\Biggl(\frac{88448}{729} - \frac{128}{9}\zeta_2$$

$$+\frac{512}{27}\zeta_3\Biggr) + C_AT_FN_F\Biggl(-\frac{1942568}{729} + \frac{30080}{81}\zeta_2 + \frac{1504}{9}\zeta_3 + \frac{7936}{45}\zeta_2^2\Biggr)\Biggr]$$

$$+C_F^2T_FN_F\Biggl(\frac{2180}{27} - 32\zeta_2 - \frac{1600}{9}\zeta_3 + \frac{384}{5}\zeta_2^2\Biggr)\Biggr\}\frac{1}{x} + \Biggl[-4C_FC_AT_FN_F$$

$$+4C_F^2T_FN_F\Biggr]\ln^5(x) + \Biggl[-\frac{110}{9}C_F^2T_FN_F + C_F\Biggl(\frac{854}{27}C_AT_FN_F - \frac{184}{27}T_F^2N_F^2\Biggr)\Biggr]$$

$$\times\ln^4(x) + \Biggl[C_F\Biggl(-\frac{3568}{81}T_F^2N_F^2 + C_AT_FN_F\Biggl(-\frac{16984}{81} + \frac{320}{9}\zeta_2\Biggr)\Biggr)$$

$$+C_F^2T_FN_F\left(\frac{572}{3}-\frac{1328}{9}\zeta_2\right)\Bigg]\ln^3(x)+\Bigg[C_F\Bigg(T_F^2N_F^2\left(-\frac{22912}{81}+\frac{64}{9}\zeta_2\right)$$

$$+C_AT_FN_F\left(\frac{101192}{81}-\frac{920}{9}\zeta_2-\frac{1024}{3}\zeta_3\right)\Bigg)+C_F^2T_FN_F\Bigg(-\frac{218}{3}-\frac{512}{3}\zeta_2$$

$$-\frac{640}{3}\zeta_3\Bigg)\Bigg]\ln^2(x)+\Bigg[C_F\Bigg(C_AT_FN_F\left(-\frac{304304}{243}+\frac{30872}{27}\zeta_2-\frac{640}{9}\zeta_3-\frac{752}{5}\zeta_2^2\right)$$

$$+T_F^2N_F^2\left(-\frac{198560}{243}+\frac{3392}{27}\zeta_2-\frac{64}{9}\zeta_3\right)\Bigg)+C_F^2T_FN_F\Bigg(692-2200\zeta_2-\frac{160}{3}\zeta_3$$

$$+\frac{2896}{15}\zeta_2^2\Bigg)\Bigg]\ln(x)+C_F\Bigg[C_AT_FN_F\Bigg(\frac{1102604}{243}-\frac{71896}{81}\zeta_2-\frac{32864}{27}\zeta_3+\frac{1856}{15}\zeta_2^2$$

$$+592\zeta_2\zeta_3-1240\zeta_5\Bigg)+T_F^2N_F^2\left(-\frac{225376}{243}+\frac{6176}{81}\zeta_2+\frac{1312}{27}\zeta_3+\frac{224}{15}\zeta_2^2\right)\Bigg]$$

$$+C_F^2T_FN_F\left(-248-\frac{4936}{3}\zeta_2-880\zeta_3+\frac{736}{5}\zeta_2^2+288\zeta_2\zeta_3+352\zeta_5\right), \qquad (1271)$$

$$C_{F_2,g}^{(1)} \simeq -4T_FN_F(\ln(x)+1), \qquad (1272)$$

$$C_{F_2,g}^{(2)} \simeq C_FT_FN_F\left[-\frac{10}{3}\ln^3(x)-3\ln^2(x)-32(1-\zeta_2)\ln(x)-86+32\zeta_2+64\zeta_3\right]$$

$$+C_AT_FN_F\Bigg[\frac{1}{x}\left(\frac{688}{27}-\frac{32}{3}\zeta_2\right)+\frac{20}{3}\ln^3(x)-2\ln^2(x)+(116-16\zeta_2)\ln(x)+30$$

$$+16\zeta_2+8\zeta_3\Bigg], \qquad (1273)$$

$$C_{F_2,g}^{(3),a} \simeq C_A^2T_FN_F\left[-\frac{78976}{243}+\frac{832}{9}\zeta_2-\frac{256}{9}\zeta_3\right]\frac{\ln(x)}{x}+\Bigg\{C_F\Bigg[C_AT_FN_F\Bigg(\frac{2180}{27}-32\zeta_2$$

$$-\frac{1600}{9}\zeta_3+\frac{384}{5}\zeta_2^2\Bigg)+T_F^2N_F^2\left(\frac{181472}{729}-\frac{2048}{27}\zeta_2+\frac{512}{27}\zeta_3\right)\Bigg]$$

$$+C_AT_F^2N_F^2\left(-\frac{2288}{729}+\frac{640}{27}\zeta_2+\frac{256}{27}\zeta_3\right)+C_A^2T_FN_F\Bigg(-\frac{2004664}{729}+\frac{32192}{81}\zeta_2$$

$$+\frac{7936}{45}\zeta_2^2+\frac{4384}{27}\zeta_3\Bigg)\Bigg\}\frac{1}{x}+\Bigg[-4C_A^2T_FN_F-C_F^2T_FN_F+C_F\Bigg(2C_AT_FN_F$$

$$+4T_F^2N_F^2\Bigg)\Bigg]\ln^5(x)+\Bigg[\frac{359}{27}C_A^2T_FN_F+\frac{11}{9}C_F^2T_FN_F-\frac{4}{27}C_AT_F^2N_F^2$$

$$+C_F\left(-\frac{409}{27}C_AT_FN_F+\frac{818}{27}T_F^2N_F^2\right)\Bigg]\ln^4(x)+\Bigg[-\frac{1064}{81}C_AT_F^2N_F^2+C_F\Bigg($$

$$C_AT_FN_F\left(\frac{4589}{81}-\frac{656}{9}\zeta_2\right)+T_F^2N_F^2\left(\frac{35564}{81}-\frac{176}{3}\zeta_2\right)\Bigg)+C_A^2T_FN_F\Bigg($$

$$
-\frac{27338}{81}+\frac{56}{3}\zeta_2\Big)+C_F^2T_FN_F\Big(-\frac{178}{9}+\frac{524}{9}\zeta_2\Big)\Big]\ln^3(x)+\Big[C_F\Big(C_AT_FN_F\Big(
-\frac{20815}{81}+\frac{284}{3}\zeta_2-\frac{320}{3}\zeta_3\Big)+T_F^2N_F^2\Big(\frac{156500}{81}-\frac{872}{3}\zeta_2-\frac{304}{3}\zeta_3\Big)\Big)
+C_AT_F^2N_F^2\Big(-\frac{1024}{81}-\frac{16}{3}\zeta_2\Big)+C_A^2T_FN_F\Big(\frac{28580}{81}-32\zeta_2-\frac{1136}{3}\zeta_3\Big)
+C_F^2T_FN_F\Big(-88+\frac{284}{3}\zeta_2+284\zeta_3\Big)\Big]\ln^2(x)+\Big[C_F\Big(T_F^2N_F^2\Big(\frac{1683904}{243}
-\frac{17632}{9}\zeta_2-\frac{5072}{9}\zeta_3+\frac{224}{5}\zeta_2^2\Big)+C_AT_FN_F\Big(-\frac{147806}{243}-\frac{1516}{9}\zeta_2+\frac{2632}{9}\zeta_3
-\frac{2176}{15}\zeta_2^2\Big)\Big)+C_A^2T_FN_F\Big(-\frac{935806}{243}+\frac{10912}{9}\zeta_2-\frac{2432}{15}\zeta_2^2-\frac{832}{9}\zeta_3\Big)
+C_AT_F^2N_F^2\Big(-\frac{53512}{243}+\frac{112}{9}\zeta_2-\frac{256}{9}\zeta_3\Big)+C_F^2T_FN_F\Big(\frac{368}{3}+\frac{550}{3}\zeta_2+600\zeta_3
-\frac{56}{15}\zeta_2^2\Big)\Big]\ln(x)+C_F\Big[T_F^2N_F^2\Big(\frac{2179673}{243}-\frac{226144}{81}\zeta_2-\frac{41744}{27}\zeta_3-\frac{3632}{45}\zeta_2^2
+128\zeta_2\zeta_3-96\zeta_5\Big)+C_AT_FN_F\Big(-\frac{2310157}{972}+\frac{91424}{81}\zeta_2+\frac{1840}{27}\zeta_3-\frac{13064}{45}\zeta_2^2
-352\zeta_2\zeta_3+\frac{1852}{3}\zeta_5\Big)\Big]+C_A^2T_FN_F\Big[\frac{359827}{243}-\frac{4138}{9}\zeta_2-\frac{7348}{9}\zeta_3+\frac{908}{15}\zeta_2^2
+\frac{1288}{3}\zeta_2\zeta_3-\frac{5224}{3}\zeta_5\Big]+C_F^2T_FN_F\Big[\frac{1925}{4}-\frac{572}{3}\zeta_2+\frac{4478}{3}\zeta_3+\frac{2176}{15}\zeta_2^2
-320\zeta_2\zeta_3-248\zeta_5\Big]+C_AT_F^2N_F^2\Big[-\frac{52712}{243}-\frac{400}{9}\zeta_2-\frac{1072}{9}\zeta_3+\frac{224}{15}\zeta_2^2\Big], \tag{1274}
$$

$$
C_{F_2,g}^{(3),d_{abc}} \simeq 0, \tag{1275}
$$

$$
\tag{1276}
$$

$$
C_{F_L,q}^{(1),\mathrm{NS}} \simeq 0, \tag{1277}
$$

$$
C_{F_L,q}^{(2),\mathrm{NS}} \simeq C_F\Big[-C_A\frac{44}{3}+T_FN_F\frac{16}{3}\Big]-4C_F^2\Big[2\ln(x)-3\Big], \tag{1278}
$$

$$
C_{F_L,q}^{(3),\mathrm{NS}} \simeq -\frac{20}{3}C_F^3\ln^3(x)+\Big[32C_F^3-66C_F^2C_A+24C_F^2T_FN_F\Big]\ln^2(x)+\Big[C_F\Big(\frac{704}{9}C_AT_FN_F
-\frac{128}{9}T_F^2N_F^2+\Big(-\frac{968}{9}+120\zeta_2\Big)C_A^2\Big)+C_F^2\Big(\frac{448}{9}T_FN_F-\Big(\frac{1832}{9}+384\zeta_2\Big)C_A\Big)
+\big(168+416\zeta_2\big)C_F^3\Big]\ln(x)+C_F\Big[-\frac{1216}{27}T_F^2N_F^2+\Big(\frac{8368}{27}-32\zeta_2\Big)C_AT_FN_F
$$

$$+\left(-\frac{13060}{27}+244\zeta_2\right)C_A^2\Bigg]+C_F^2\Bigg(\left(\left(-\frac{2288}{27}-\frac{64}{3}\zeta_2\right)T_FN_F+\left(\frac{5800}{27}-\frac{1696}{3}\zeta_2\right.\right.$$

$$\left.\left.-96\zeta_3\right)C_A\right)+\left(608\zeta_2+288\zeta_3\right)C_F^3, \tag{1279}$$

$$\tag{1280}$$

$$C_{F_L,q}^{(3),d_{abc}} \simeq 0, \tag{1281}$$

$$C_{F_L,q}^{(2),\mathrm{PS}} \simeq 32C_FT_FN_F\left[-\frac{1}{9x}+\ln(x)\right], \tag{1282}$$

$$C_{F_L,q}^{(3),\mathrm{PS}} \simeq C_F\Bigg\{T_F^2N_F^2\left[\frac{1}{x}\left(\frac{13568}{81}-\frac{256}{9}\zeta_2\right)-\frac{128}{3}\ln^2(x)-\frac{2048}{9}\ln(x)-\frac{11264}{27}\right]$$
$$+C_AT_FN_F\Bigg[\frac{1}{x}\left(-\frac{28768}{27}+224\zeta_2+\frac{640}{3}\zeta_3\right)-\frac{160}{3}\ln^3(x)+\frac{496}{3}\ln^2(x)$$
$$+\left(-\frac{800}{9}+\frac{1}{x}\left(-\frac{4352}{27}+\frac{128}{3}\zeta_2\right)+224\zeta_2\right)\ln(x)+\frac{35200}{27}+64\zeta_2-704\zeta_3\Bigg]\Bigg\}$$
$$+C_F^2T_FN_F\Bigg[\frac{1}{x}\left(\frac{3584}{27}-\frac{64}{3}\zeta_2-\frac{256}{3}\zeta_3\right)-576\zeta_2+\left(160-512\zeta_2\right)\ln(x)$$
$$-16\ln^2(x)+\frac{160}{3}\ln^3(x)\Bigg], \tag{1283}$$

$$C_{F_L,g}^{(1)} \simeq 0, \tag{1284}$$

$$C_{F_L,g}^{(2)} \simeq 32C_AT_FN_F\left[-\frac{1}{9x}+\ln(x)\right]-16C_FT_FN_F(\ln(x)+1), \tag{1285}$$

$$C_{F_L,g}^{(3),a} \simeq C_A^2T_FN_F\left[-\frac{4352}{27}+\frac{128}{3}\zeta_2\right]\frac{\ln(x)}{x}+\Bigg\{C_F\Bigg[T_F^2N_F^2\left(\frac{7744}{81}-\frac{256}{9}\zeta_2\right)+C_AT_FN_F\Bigg($$
$$\frac{3584}{27}-\frac{64}{3}\zeta_2-\frac{256}{3}\zeta_3\Bigg)\Bigg]+C_AT_F^2N_F^2\left(\frac{3232}{27}-\frac{128}{9}\zeta_2\right)+C_A^2T_FN_F\Bigg(-\frac{88192}{81}$$
$$+\frac{2080}{9}\zeta_2+\frac{640}{3}\zeta_3\Bigg)\Bigg\}\frac{1}{x}+\Bigg[-\frac{160}{3}C_A^2T_FN_F-\frac{40}{3}C_F^2T_FN_F+C_F\Bigg(\frac{80}{3}C_AT_FN_F$$
$$+\frac{160}{3}T_F^2N_F^2\Bigg)\Bigg]\ln^3(x)+\Bigg[\frac{56}{3}C_A^2T_FN_F-20C_F^2T_FN_F+\frac{32}{3}C_AT_F^2N_F^2$$
$$+C_F\left(-\frac{152}{3}C_AT_FN_F+\frac{928}{3}T_F^2N_F^2\right)\Bigg]\ln^2(x)+\Bigg[-\frac{320}{9}C_AT_F^2N_F^2+C_F\Bigg($$
$$C_AT_FN_F\left(-\frac{616}{9}-288\zeta_2\right)+T_F^2N_F^2\left(\frac{13664}{9}-256\zeta_2\right)\Bigg)+C_F^2T_FN_F\Bigg(32$$
$$+192\zeta_2\Bigg)+C_A^2T_FN_F\left(-\frac{7280}{9}+240\zeta_2\right)\Bigg]\ln(x)+C_F\Bigg[T_F^2N_F^2\Bigg(\frac{20864}{9}$$

$$
\begin{aligned}
& -\frac{1664}{3}\zeta_2 - 128\zeta_3\Big) + C_A T_F N_F\Big(-\frac{2776}{9} - \frac{512}{3}\zeta_2 + 64\zeta_3\Big)\Big] + C_A^2 T_F N_F\Big(\\
& \frac{5168}{27} + 200\zeta_2 - 704\zeta_3\Big) + C_F^2 T_F N_F\Big(-24 + 256\zeta_2 + 384\zeta_3\Big) \\
& -\frac{3520}{27} C_A T_F^2 N_F^2, \qquad (1286)
\end{aligned}
$$

$$
C_{F_L,g}^{(3),d_{abc}} \simeq 0, \qquad (1287)
$$

$$
C_{F_3,q}^{(1),\mathrm{NS}} \simeq C_F\big[1 - 2\log(x)\big], \qquad (1288)
$$

$$
\begin{aligned}
C_{F_3,q}^{(2),\mathrm{NS}} \simeq\ & C_F\Big\{C_A\Big[-2\ln^3(x) - \frac{103}{6}\ln^2(x) + \Big(-\frac{122}{3} + 8\zeta_2\Big)\ln(x) - \frac{1327}{54} + \frac{80}{3}\zeta_2 \\
& +8\zeta_3\Big] + T_F N_F\Big[\frac{10}{3}\ln^2(x) + \frac{40}{3}\ln(x) + \frac{262}{27} - \frac{16}{3}\zeta_2\Big]\Big\} + C_F^2\Big[\frac{7}{3}\ln^3(x) \\
& +21\ln^2(x) + (21 + 8\zeta_2)\ln(x) - \frac{59}{2} - 20\zeta_2 + 16\zeta_3\Big], \qquad (1289)
\end{aligned}
$$

$$
\begin{aligned}
C_{F_3,q}^{(3),\mathrm{NS}} \simeq\ & C_F^2\Big\{T_F N_F\Big[-\frac{31}{9}\ln^4(x) - \frac{4358}{81}\ln^3(x) + \Big(-\frac{5374}{27} - \frac{124}{3}\zeta_2\Big)\ln^2(x) + \Big(-\frac{271}{81} \\
& -\frac{1436}{27}\zeta_2 - \frac{2056}{9}\zeta_3\Big)\ln(x) + \frac{4123}{54} + \frac{8344}{81}\zeta_2 + \frac{752}{9}\zeta_2^2 - \frac{9884}{27}\zeta_3\Big] \\
& +C_A\Big[-\frac{29}{15}\ln^5(x) + \frac{43}{36}\ln^4(x) + \Big(\frac{24245}{162} + \frac{220}{3}\zeta_2\Big)\ln^3(x) + \Big(\frac{13297}{54} + 401\zeta_2 \\
& +328\zeta_3\Big)\ln^2(x) + \Big(-\frac{404119}{324} + \frac{16273}{27}\zeta_2 + \frac{9206}{9}\zeta_3 + \frac{3416}{15}\zeta_2^2\Big)\ln(x) - \frac{496687}{216} \\
& +\frac{54412}{81}\zeta_2 + \frac{1252}{45}\zeta_2^2 + \Big(-\frac{1865}{27} - 648\zeta_2\Big)\zeta_3 + \frac{5306}{3}\zeta_5\Big]\Big\} \\
& +C_F\Big\{T_F^2 N_F^2\Big[-\frac{368}{81}\ln^3(x) - \frac{2848}{81}\ln^2(x) + \Big(-\frac{6736}{81} + \frac{64}{3}\zeta_2\Big)\ln(x) - \frac{35896}{729} \\
& +\frac{1312}{27}\zeta_2 + \frac{128}{27}\zeta_3\Big] + C_A T_F N_F\Big[\frac{92}{27}\ln^4(x) + \frac{1544}{27}\ln^3(x) + \Big(\frac{8432}{27} - \frac{224}{9}\zeta_2\Big) \\
& \times\ln^2(x) + \Big(\frac{52168}{81} - \frac{6224}{27}\zeta_2 + \frac{608}{9}\zeta_3\Big)\ln(x) + \frac{433462}{729} - \frac{37480}{81}\zeta_2 + \frac{1288}{27}\zeta_3 \\
& -\frac{64}{15}\zeta_2^2\Big] + C_A^2\Big[\frac{2}{5}\ln^5(x) - \frac{166}{27}\ln^4(x) + \Big(-\frac{9799}{81} - \frac{64}{9}\zeta_2\Big)\ln^3(x) + \Big(-\frac{38642}{81} \\
& +\frac{226}{9}\zeta_2 - \frac{212}{3}\zeta_3\Big)\ln^2(x) + \Big(-\frac{53650}{81} + \frac{11260}{27}\zeta_2 - \frac{416}{5}\zeta_2^2 - \frac{1696}{9}\zeta_3\Big)\ln(x) \\
& -\frac{1010495}{1458} + \frac{52214}{81}\zeta_2 - \frac{1258}{15}\zeta_2^2 + \Big(\frac{9932}{27} + \frac{292}{3}\zeta_2\Big)\zeta_3 - \frac{896}{3}\zeta_5\Big]\Big\}
\end{aligned}
$$

$$+C_F^3\Bigg[\frac{53}{30}\ln^5(x)+\frac{89}{12}\ln^4(x)+\left(-\frac{49}{3}-\frac{710}{9}\zeta_2\right)\ln^3(x)+\left(\frac{1085}{6}-\frac{1373}{3}\zeta_2\right.$$
$$\left.-286\zeta_3\right)\ln^2(x)+\left(\frac{5553}{4}-\frac{2308}{3}\zeta_2-\frac{1978}{3}\zeta_3-\frac{4748}{15}\zeta_2^2\right)\ln(x)+\frac{71777}{24}$$
$$-\frac{2561}{3}\zeta_2-\frac{2248}{15}\zeta_2^2+\left(\frac{710}{3}+\frac{1384}{3}\zeta_2\right)\zeta_3-2104\zeta_5\Bigg], \tag{1290}$$

$$C_{F_3,q}^{(3),d_{abc}} \simeq \frac{d_{abc}d^{abc}N_F}{N_C}\Bigg[-\frac{32}{15}\ln^5(x)+\left(-\frac{704}{9}+\frac{256}{3}\zeta_2\right)\ln^3(x)+\left(-1600-\frac{2176}{5}\zeta_2^2\right.$$
$$\left.+\frac{1024}{3}\zeta_2+\frac{1088}{3}\zeta_3\right)\ln(x)+\left(-1056+96\zeta_2+\frac{1216}{3}\zeta_3\right)\ln^2(x)-\frac{11456}{3}$$
$$+2560\zeta_2-\frac{3104}{15}\zeta_2^2+\left(\frac{3904}{3}-\frac{1856\zeta_2}{3}\right)\zeta_3-\frac{3584}{3}\zeta_5\Bigg], \tag{1291}$$

$$\Delta C_{g_1,q}^{(1),\mathrm{NS,L}} \simeq -C_F(7+2\ln(x)), \tag{1292}$$

$$\Delta C_{g_1,q}^{(2),\mathrm{NS,L}} \simeq C_F\Bigg\{C_A\left[-2\ln^3(x)-\frac{151}{6}\ln^2(x)+\left(-\frac{238}{3}+8\zeta_2\right)\ln(x)-\frac{6391}{54}+\frac{128}{3}\zeta_2+8\zeta_3\right]$$
$$+T_FN_F\left[\frac{10}{3}\ln^2(x)+\frac{56}{3}\ln(x)+\frac{502}{27}-\frac{16}{3}\zeta_2\right]\Bigg\}+C_F^2\Bigg[\frac{7}{3}\ln^3(x)+21\ln^2(x)$$
$$+\big(21+8\zeta_2\big)\ln(x)+\frac{117}{2}-20\zeta_2+16\zeta_3\Bigg], \tag{1293}$$

(1294)

$$\Delta C_{g_1,q}^{(2),\mathrm{PS,L}} \simeq C_FT_FN_F\left[\frac{20}{3}\ln^3(x)+58\ln^2(x)+\big(172-32\zeta_2\big)\ln(x)+280-96\zeta_2-16\zeta_3\right],$$

(1295)

$$\Delta C_{g_1,q}^{(3),\mathrm{PS,L}} \simeq C_F\Bigg\{T_F^2N_F^2\Bigg[-\frac{184}{27}\ln^4(x)-\frac{7312}{81}\ln^3(x)+\left(-\frac{37744}{81}+\frac{64}{9}\zeta_2\right)\ln^2(x)$$
$$+\left(-\frac{245504}{243}+\frac{1088}{27}\zeta_2-\frac{64}{9}\zeta_3\right)\ln(x)-\frac{292640}{243}+\frac{14528}{81}\zeta_2-\frac{32}{27}\zeta_3+\frac{224}{15}\zeta_2^2\Bigg]$$
$$+C_FT_FN_F\Bigg[4\ln^5(x)+\frac{632}{9}\ln^4(x)+\left(\frac{5156}{9}-\frac{1328}{9}\zeta_2\right)\ln^3(x)+\Bigg(2708$$
$$-\frac{4096\zeta_2}{3}-\frac{256\zeta_3}{3}\Bigg)\ln^2(x)+\left(\frac{23360}{3}-5064\zeta_2-224\zeta_3+\frac{3856}{15}\zeta_2^2\right)\ln(x)$$
$$+\frac{31712}{3}-6856\zeta_2+\left(-\frac{6160}{3}+32\zeta_2\right)\zeta_3+\frac{15008}{15}\zeta_2^2+352\zeta_5\Bigg]$$
$$+C_AT_FN_F\Bigg[4\ln^5(x)+\frac{2450}{27}\ln^4(x)+\left(\frac{67472}{81}-\frac{688}{9}\zeta_2\right)\ln^3(x)+\Bigg(\frac{325580}{81}$$

$$-\frac{6992\zeta_2}{9} - 352\zeta_3\Big) \ln^2(x) + \left(\frac{2395192}{243} - \frac{74464}{27}\zeta_2 - \frac{21136}{9}\zeta_3 + \frac{3088}{15}\zeta_2^2\right) \ln(x)$$

$$+\frac{2166976}{243} - \frac{315832}{81}\zeta_2 + \left(-\frac{93968}{27} + \frac{1808}{3}\zeta_2\right)\zeta_3 + \frac{10312}{15}\zeta_2^2 - 264\zeta_5\Big]\Big\}, \quad (1296)$$

$$\Delta C_{g_1,g}^{(1)} \simeq 4T_F N_F[3 + \ln(x)], \quad (1297)$$

$$\Delta C_{g_1,g}^{(2)} \simeq C_F T_F N_F \left[\frac{10}{3}\ln^3(x) + 41\ln^2(x) + \left(160 - 32\zeta_2\right)\ln(x) + 134 - 88\zeta_2\right]$$

$$+C_A T_F N_F \Big[\frac{28}{3}\ln^3(x) + 90\ln^2(x) + \left(304 - 64\zeta_2\right)\ln(x) + 484 - 168\zeta_2$$

$$-48\zeta_3\Big], \quad (1298)$$

$$\Delta C_{g_1,g}^{(3)} \simeq T_F N_F \Big\{\left[\frac{44}{5}C_A^2 - \frac{23}{45}C_F^2 + C_F\left(\frac{146}{45}C_A - 4T_F N_F\right)\right]\ln^5(x) + \Big[\frac{1627}{9}C_A^2 - 21C_F^2$$

$$-\frac{68}{9}C_A T_F N_F + C_F\left(\frac{2143}{27}C_A - \frac{2654}{27}T_F N_F\right)\Big]\ln^4(x) + \Big[-\frac{12056}{81}C_A T_F N_F$$

$$+C_F\left(C_A\left(\frac{61699}{81} - \frac{976}{9}\zeta_2\right) + T_F N_F\left(-\frac{79220}{81} + \frac{176}{3}\zeta_2\right)\right) + C_A^2\Big(\frac{132454}{81}$$

$$-\frac{656}{3}\zeta_2\Big) + C_F^2\left(-\frac{602}{3} - \frac{332}{9}\zeta_2\right)\Big]\ln^3(x) + \Big[C_F\Big(T_F N_F\Big(-\frac{422240}{81} + \frac{2600}{3}\zeta_2$$

$$+\frac{304}{3}\zeta_3\Big) + C_A\left(\frac{221032}{81} - 1188\zeta_2 + \frac{824}{3}\zeta_3\right)\Big) + C_A T_F N_F\left(-\frac{30296}{27} + 80\zeta_2\right)$$

$$+C_A^2\left(\frac{236002}{27} - \frac{6844}{3}\zeta_2 - \frac{2216}{3}\zeta_3\right) + C_F^2\left(-\frac{877}{3} - \frac{824}{3}\zeta_2 - 252\zeta_3\right)\Big]\ln^2(x)$$

$$+\Big[C_F\Big(T_F N_F\left(-\frac{3711364}{243} + \frac{37264\zeta_2}{9} + \frac{10064\zeta_3}{9} - \frac{224\zeta_2^2}{5}\right) + C_A\Big(\frac{728285}{243}$$

$$-\frac{46892}{9}\zeta_2 + \frac{2944}{15}\zeta_2^2 + \frac{27296}{9}\zeta_3\Big)\Big) + C_A^2\Big(\frac{6175024}{243} - \frac{25564}{3}\zeta_2 + \frac{9968}{15}\zeta_2^2$$

$$-\frac{52064}{9}\zeta_3\Big) + C_F^2\left(\frac{2755}{3} - \frac{1450}{3}\zeta_2 - \frac{5632}{3}\zeta_3 + \frac{5464}{15}\zeta_2^2\right)$$

$$+C_A T_F N_F\left(-\frac{970424}{243} + \frac{2144}{3}\zeta_2 + \frac{1408}{9}\zeta_3\right)\Big]\ln(x) + C_F^2\Big(\frac{39635}{12} - 1280\zeta_2$$

$$+\frac{2494}{3}\zeta_3 + \frac{9568}{15}\zeta_2^2 + \frac{640}{3}\zeta_2\zeta_3 - \frac{11032}{3}\zeta_5\Big) + C_A^2\Big(\frac{2218820}{81} - \frac{1035208}{81}\zeta_2$$

$$-\frac{83068}{9}\zeta_3 + \frac{86584}{45}\zeta_2^2 + 1544\zeta_2\zeta_3 - 664\zeta_5\Big) + C_A T_F N_F\Big(-\frac{495448}{81} + \frac{135320}{81}\zeta_2$$

$$+\frac{5696}{9}\zeta_3-\frac{1376}{45}\zeta_2^2\Biggr)+C_F\Biggl[T_FN_F\Biggl(-\frac{1456493}{81}+\frac{536656}{81}\zeta_2+\frac{75680}{27}\zeta_3$$
$$-\frac{22288}{45}\zeta_2^2-128\zeta_2\zeta_3+96\zeta_5\Biggr)+C_A\Biggl(-\frac{367691}{324}-\frac{488996}{81}\zeta_2-\frac{7432}{27}\zeta_3$$
$$+\frac{72272}{45}\zeta_2^2-\frac{1936}{3}\zeta_2\zeta_3+\frac{12844}{3}\zeta_5\Biggr)\Biggr]\Biggr\}, \quad (1299)$$

$$\Delta C_{g_1,g}^{(3),d_{abc}} \simeq \frac{d_{abc}d^{abc}N_F^2}{N_A}\Biggl[-\frac{13952}{15}-\frac{64}{3}\zeta_2+640\zeta_3+\frac{8192}{75}\zeta_2^2\Biggr]. \quad (1300)$$

Note, that the expansion has to be performed in z–space, since the Mellin space expression is either applicable for even or odd moments. There are predictions of the leading series small x behaviour of the Wilson coefficients of the structure functions $F_2(x,Q^2)$ and $F_L(x,Q^2)$ from Ref. [94], to which we agree to three–loop order except of the overall sign. A small x expansion of the non–singlet Wilson coefficient of the structure function $g_1(x,Q^2)$ has been considered in [95] based on [96], however, in a different not fully specified scheme.

8.2 The large x limit

In the large x limit one obtains the following leading term behaviour for the Wilson coefficients calculated in the present paper. Here we dropped the terms $O(L_a^k/N^2),\quad k\geq 0$.

$$C_{F_2,q}^{(1),\mathrm{NS}} \simeq C_F\Biggl[2S_1^2+3S_1-9-2\zeta_2+\frac{9}{N}\Biggr], \quad (1301)$$

$$C_{F_2,q}^{(2),\mathrm{NS}} \simeq C_F\Biggl\{T_FN_F\Biggl[-\frac{8}{9}S_1^3-\frac{58}{9}S_1^2-\Biggl(\frac{494}{27}-\frac{8}{3}\zeta_2+\frac{52}{3N}\Biggr)S_1+\frac{457}{18}+\frac{170}{9}\zeta_2+\frac{8}{9}\zeta_3-\frac{484}{9N}\Biggr]$$
$$+C_A\Biggl[\frac{22}{9}S_1^3+\Biggl(\frac{367}{18}-4\zeta_2\Biggr)S_1^2+\Biggl(\frac{3155}{54}-\frac{22}{3}\zeta_2-40\zeta_3+\frac{161}{3N}-\frac{16}{N}\zeta_2\Biggr)S_1-\frac{5465}{72}$$
$$-\frac{1139}{18}\zeta_2+\frac{464}{9}\zeta_3+\frac{51}{5}\zeta_2^2+\frac{1202}{9N}+\frac{8}{N}\zeta_2-\frac{24}{N}\zeta_3\Biggr]\Biggr\}+C_F^2\Biggl\{2S_1^4+6S_1^3$$
$$+\Biggl(-\frac{27}{2}+\frac{30}{N}-4\zeta_2\Biggr)S_1^2+\Biggl(-\frac{51}{2}-18\zeta_2+24\zeta_3-\frac{11}{N}+\frac{32\zeta_2}{N}\Biggr)S_1+\frac{331}{8}$$
$$+\frac{111}{2}\zeta_2-66\zeta_3+\frac{4}{5}\zeta_2^2-\frac{95}{N}-\frac{38}{N}\zeta_2+\frac{48}{N}\zeta_3\Biggr\}, \quad (1302)$$

$$C_{F_2,q}^{(3),\mathrm{NS}} \simeq C_F\Biggl\{T_F^2N_F^2\Biggl[\frac{16}{27}S_1^4+\frac{464}{81}S_1^3+\Biggl(\frac{1880}{81}-\frac{32}{9}\zeta_2+\frac{208}{9N}\Biggr)S_1^2+\Biggl(\frac{34856}{729}-\frac{464}{27}\zeta_2$$
$$+\frac{256}{27}\zeta_3+\frac{4192}{27N}\Biggr)S_1-\frac{19034}{243}-\frac{8440}{81}\zeta_2+\frac{320}{81}\zeta_3-\frac{1168}{135}\zeta_2^2+\frac{24608}{81N}-\frac{208}{9N}\zeta_2\Biggr]$$
$$+C_AT_FN_F\Biggl[-\frac{88}{27}S_1^4+\Biggl(-\frac{3104}{81}+\frac{32}{9}\zeta_2\Biggr)S_1^3+\Biggl(-\frac{15062}{81}-\frac{1252}{9N}+\frac{112}{3}\zeta_2$$

$$+16\zeta_3 + \frac{32}{N}\zeta_2\Big) S_1^2 + \Big(-\frac{321812}{729} + \frac{10528}{81}\zeta_2 + \frac{3952}{27}\zeta_3 - \frac{256}{15}\zeta_2^2 - \frac{29908}{27N} + \frac{488}{3N}\zeta_2$$

$$+\frac{512}{3N}\zeta_3\Big) S_1 + \frac{142883}{243} + \frac{66662}{81}\zeta_2 - \frac{42836}{81}\zeta_3 + \frac{328}{135}\zeta_2^2 - \frac{128}{9}\zeta_2\zeta_3 + \frac{16}{3}\zeta_5$$

$$-\frac{55130}{27N} + \frac{2140\zeta_2}{27N} + \frac{2552\zeta_3}{9N} + \frac{96\zeta_2^2}{5N}\Big] + C_A^2\Big[\frac{121}{27}S_1^4 + \Big(\frac{4649}{81} - \frac{88}{9}\zeta_2\Big)S_1^3$$

$$+\Big(\frac{50689}{162} - \frac{778}{9}\zeta_2 - 132\zeta_3 + \frac{88}{5}\zeta_2^2 + \frac{2011}{9N} - \frac{178}{3N}\zeta_2 - \frac{32}{N}\zeta_3\Big)S_1^2 + \Big(\frac{599375}{729}$$

$$-\frac{18179}{81}\zeta_2 - \frac{6688}{9}\zeta_3 + \frac{212}{15}\zeta_2^2 + \frac{176}{3}\zeta_2\zeta_3 + 232\zeta_5 + \frac{50410}{27N} - \frac{568}{N}\zeta_2 - \frac{904}{3N}\zeta_3$$

$$+\frac{128}{5N}\zeta_2^2\Big)S_1 - \frac{1909753}{1944} - \frac{78607}{54}\zeta_2 + \frac{115010}{81}\zeta_3 + \frac{13151}{135}\zeta_2^2 + \frac{3496}{9}\zeta_2\zeta_3 - \frac{416}{3}\zeta_5$$

$$-\frac{12016}{315}\zeta_2^3 - \frac{248}{3}\zeta_3^2 + \frac{534221}{162N} + \frac{4399}{27N}\zeta_2 + \frac{518}{15N}\zeta_2^2 - \frac{12844}{9N}\zeta_3 - \frac{112}{N}\zeta_2\zeta_3$$

$$-\frac{160}{N}\zeta_5\Big]\Big\} + C_F^2\Big\{T_F N_F\Big[-\frac{16}{9}S_1^5 - \frac{140}{9}S_1^4 + \Big(-\frac{1366}{27} + \frac{64\zeta_2}{9} - \frac{1504}{27N}\Big)S_1^3$$

$$+\Big(\frac{83}{9} + \frac{224}{3}\zeta_2 + \frac{16}{9}\zeta_3 - \frac{602}{3N} - \frac{64}{N}\zeta_2\Big)S_1^2 + \Big(\frac{2003}{54} + \frac{4354}{27}\zeta_2 - \frac{40}{9}\zeta_3 - \frac{16}{3}\zeta_2^2$$

$$+\frac{19690}{81N} - \frac{1456}{9N}\zeta_2 - \frac{640}{3N}\zeta_3\Big)S_1 - \frac{341}{18} - \frac{10733}{27}\zeta_2 + \frac{21532}{27}\zeta_3 - \frac{21604}{135}\zeta_2^2$$

$$-\frac{80}{3}\zeta_2\zeta_3 - \frac{1568}{9}\zeta_5 + \frac{47101}{54N} + \frac{10610\zeta_2}{27N} - \frac{16136\zeta_3}{27N} + \frac{192\zeta_2^2}{5N}\Big] + C_A\Big[\frac{44}{9}S_1^5$$

$$+\Big(\frac{433}{9} - 8\zeta_2\Big)S_1^4 + \Big(\frac{8425}{54} - \frac{284}{9}\zeta_2 - 80\zeta_3 + \frac{4460}{27N} - \frac{32}{N}\zeta_2\Big)S_1^3 + \Big(-\frac{5563}{36}$$

$$-\frac{592}{3}\zeta_2 + \frac{142}{5}\zeta_2^2 + \frac{640}{9}\zeta_3 + \frac{3127}{6N} - \frac{8}{3N}\zeta_2 + \frac{80}{N}\zeta_3\Big)S_1^2 + \Big(-\frac{16981}{24} - \frac{28495}{54}\zeta_2$$

$$+752\zeta_3 + \frac{299}{3}\zeta_2^2 + 96\zeta_2\zeta_3 + 120\zeta_5 - \frac{152317}{162N} + \frac{7532}{9N}\zeta_2 - \frac{1496}{3N}\zeta_3 + \frac{672}{5N}\zeta_2^2\Big)S_1$$

$$+\frac{191545}{108}\zeta_2 - \frac{49346}{27}\zeta_3 + \frac{11419}{27}\zeta_2^2 - \frac{3896}{9}\zeta_5 - 828\zeta_2\zeta_3 - \frac{23098}{315}\zeta_2^3 + \frac{536}{3}\zeta_3^2$$

$$-\frac{751331}{216N} - \frac{88043}{54N}\zeta_2 + \frac{78052}{27N}\zeta_3 - \frac{1721}{5N}\zeta_2^2 + \frac{432}{N}\zeta_2\zeta_3 + \frac{1200}{N}\zeta_5 + \frac{9161}{12}\Big]\Big\}$$

$$+C_F^3\Big[\frac{4}{3}S_1^6 + 6S_1^5 + \Big(-9 - 4\zeta_2 + \frac{42}{N}\Big)S_1^4 + \Big(-\frac{93}{2} - 36\zeta_2 + 48\zeta_3 + \frac{14}{N} + \frac{64}{N}\zeta_2\Big)$$

$$\times S_1^3 + \Big(\frac{187}{4} + 66\zeta_2 - 60\zeta_3 + \frac{8}{5}\zeta_2^2 - \frac{1237}{6N} - \frac{4}{N}\zeta_2 - \frac{32}{N}\zeta_3\Big)S_1^2 + \Big(\frac{1001}{8} + \frac{579}{2}\zeta_2$$

$$-346\zeta_3+84\zeta_2^2-80\zeta_2\zeta_3-240\zeta_5+\frac{119}{6N}+\frac{984}{N}\zeta_3-\frac{998}{3N}\zeta_2-\frac{1856}{5N}\zeta_2^2\Bigg)S_1-\frac{7255}{24}$$

$$-\frac{6197}{12}\zeta_2-411\zeta_3-\frac{1791}{5}\zeta_2^2+556\zeta_2\zeta_3+1384\zeta_5+\frac{8144}{315}\zeta_2^3-\frac{176}{3}\zeta_3^2+\frac{16889}{24N}$$

$$+\frac{5029}{6N}\zeta_2-\frac{650}{N}\zeta_3+\frac{1780}{3N}\zeta_2^2-\frac{416}{N}\zeta_2\zeta_3-\frac{1760}{N}\zeta_5\Bigg], \quad (1303)$$

$$C_{F_2,q}^{(3),d_{abc}} \simeq \frac{d_{abc}d^{abc}N_F}{N_C}\Bigg[64+160\zeta_2$$

$$+\frac{224}{3}\zeta_3-\frac{32}{5}\zeta_2^2-\frac{1280}{3}\zeta_5-\frac{128}{N}-\frac{448}{N}\zeta_2+\frac{64}{5N}\zeta_2^2-\frac{704}{N}\zeta_3+\frac{128}{N}\zeta_2\zeta_3$$

$$+\frac{1280}{N}\zeta_5\Bigg], \quad (1304)$$

$$C_{F_2,q}^{(2),\mathrm{PS}} \simeq 0, \quad (1305)$$

$$C_{F_2,q}^{(3),\mathrm{PS}} \simeq 0, \quad (1306)$$

$$C_{F_2,g}^{(1)} \simeq -4T_FN_F\frac{1}{N}[S_1+1], \quad (1307)$$

$$C_{F_2,g}^{(2)} \simeq \frac{1}{N}\Bigg\{C_AT_FN_F\Bigg[-\frac{4}{3}S_1^3-8S_1^2+\Bigg(-28+12\zeta_2\Bigg)S_1-16-\frac{92}{3}\zeta_3\Bigg]$$

$$+C_FT_FN_F\Bigg[-\frac{20}{3}S_1^3-18S_1^2+\Bigg(4+12\zeta_2\Bigg)S_1-4+10\zeta_2+\frac{176}{3}\zeta_3\Bigg]\Bigg\}, \quad (1308)$$

$$C_{F_2,g}^{(3),a} \simeq \frac{1}{N}\Bigg\{C_F\Bigg\{T_F^2N_F^2\Bigg[\frac{68}{27}S_1^4+\frac{1984}{81}S_1^3+\Bigg(\frac{6260}{81}-\frac{56}{9}\zeta_2\Bigg)S_1^2+\Bigg(-\frac{12304}{243}-\frac{1952}{27}\zeta_2$$

$$-\frac{224}{27}\zeta_3\Bigg)S_1-\frac{1189}{27}-\frac{188}{3}\zeta_2-\frac{16672}{81}\zeta_3+\frac{1028}{45}\zeta_2^2\Bigg]+C_AT_FN_F\Bigg[-\frac{412}{27}S_1^4$$

$$+\Bigg(-\frac{6884}{81}+\frac{272}{9}\zeta_2\Bigg)S_1^3+\Bigg(-\frac{12043}{81}+\frac{568}{9}\zeta_2+\frac{104}{3}\zeta_3\Bigg)S_1^2+\Bigg(\frac{65936}{243}+\frac{7396}{27}\zeta_2$$

$$-\frac{10796}{27}\zeta_3+\frac{976}{15}\zeta_2^2\Bigg)S_1+\frac{32813}{108}+\frac{1321}{3}\zeta_2-\frac{11908}{81}\zeta_3-\frac{1172}{9}\zeta_2^2+\frac{352}{9}\zeta_2\zeta_3$$

$$+\frac{1496}{3}\zeta_5\Bigg]\Bigg\}+C_AT_F^2N_F^2\Bigg[\frac{28}{27}S_1^4+\frac{608}{81}S_1^3+\Bigg(\frac{4384}{81}-\frac{104}{9}\zeta_2\Bigg)S_1^2+\Bigg(\frac{43192}{243}$$

$$-\frac{1376}{27}\zeta_2+\frac{32}{27}\zeta_3\Bigg)S_1+\frac{13120}{81}-\frac{16}{9}\zeta_2+\frac{17824}{81}\zeta_3-\frac{4}{5}\zeta_2^2\Bigg]+C_F^2T_FN_F\Bigg[-\frac{20}{3}S_1^5$$

$$-\frac{83}{3}S_1^4+\Bigg(-\frac{254}{9}+\frac{152}{9}\zeta_2\Bigg)S_1^3+\Bigg(-\frac{205}{3}+118\zeta_2+\frac{32}{3}\zeta_3\Bigg)S_1^2+\Bigg(-\frac{508}{3}$$

$$-54\zeta_2+\frac{2072}{3}\zeta_3-\frac{2732}{15}\zeta_2^2\Bigg)S_1+\frac{617}{12}-173\zeta_2+\frac{5564}{9}\zeta_3+\frac{193}{5}\zeta_2^2-\frac{1040}{9}\zeta_2\zeta_3$$

$$-\frac{2384}{3}\zeta_5\Bigg] + C_A^2 T_F N_F\Bigg[-\frac{4}{3}S_1^5 - \frac{293}{27}S_1^4 + \Bigg(-\frac{6112}{81} + \frac{152}{9}\zeta_2\Bigg)S_1^3 + \Bigg(-\frac{27578}{81} + \frac{718}{9}\zeta_2 + \frac{272}{3}\zeta_3\Bigg)S_1^2 + \Bigg(-\frac{139052}{243} + \frac{5992}{27}\zeta_2 + \frac{3368}{27}\zeta_3 - \frac{136}{3}\zeta_2^2\Bigg)S_1 - \frac{31172}{81} - \frac{478}{9}\zeta_2 - \frac{10988}{81}\zeta_3 + \frac{163}{5}\zeta_2^2 - \frac{344}{9}\zeta_2\zeta_3 - 16\zeta_5\Bigg]\Bigg\}, \quad (1309)$$

$$C_{F_2,g}^{(3),d_{abc}} \simeq \frac{d_{abc}d^{abc}N_F^2}{N_A}\frac{1}{N}\Bigg[\Bigg(128 + 128\zeta_2 - 256\zeta_3\Bigg)S_1^2 + \Bigg(-64 - 128\zeta_2 - \frac{1536}{5}\zeta_2^2 + 768\zeta_3\Bigg)S_1 + 80 + 256\zeta_2 - 512\zeta_3 + \frac{1728}{5}\zeta_2^2 - 640\zeta_5\Bigg], \quad (1310)$$

$$C_{F_L,q}^{(1),\rm NS} \simeq C_F\frac{4}{N}, \quad (1311)$$

$$C_{F_L,q}^{(2),\rm NS} \simeq \frac{1}{N}\Bigg\{C_F\Bigg[T_F N_F\Bigg(-\frac{16S_1}{3} - \frac{152}{9}\Bigg) + C_A\Bigg(\Bigg(\frac{92}{3} - 16\zeta_2\Bigg)S_1 + \frac{430}{9} + 16\zeta_2 - 24\zeta_3\Bigg)\Bigg] + C_F^2\Bigg[8S_1^2 + \Bigg(-36 + 32\zeta_2\Bigg)S_1 - 34 - 40\zeta_2 + 48\zeta_3\Bigg]\Bigg\}, \quad (1312)$$

$$C_{F_L,q}^{(3),\rm NS} \simeq \frac{1}{N}\Bigg\{C_F\Bigg\{T_F^2 N_F^2\Bigg[\frac{64}{9}S_1^2 + \frac{1216}{27}S_1 + \frac{6496}{81} - \frac{64}{9}\zeta_2\Bigg] + C_A T_F N_F\Bigg[\Bigg(-\frac{640}{9} + 32\zeta_2\Bigg) \times S_1^2 + \Bigg(-\frac{12880}{27} + \frac{640\zeta_2}{9} + \frac{512\zeta_3}{3}\Bigg)S_1 - \frac{42976}{81} - 64\zeta_2 + \frac{128}{3}\zeta_3 + \frac{96}{5}\zeta_2^2\Bigg] + C_A^2\Bigg[\Bigg(\frac{1276}{9} - 56\zeta_2 - 32\zeta_3\Bigg)S_1^2 + \Bigg(\frac{25756}{27} - \frac{3008\zeta_2}{9} + \frac{128\zeta_2^2}{5} - \frac{880\zeta_3}{3}\Bigg)S_1 + \frac{67312}{81} + \frac{3820}{9}\zeta_2 - \frac{1240}{3}\zeta_3 - 112\zeta_2\zeta_3 - 160\zeta_5\Bigg]\Bigg\} + C_F^2\Bigg\{T_F N_F\Bigg[-\frac{128}{9}S_1^3 + \Bigg(\frac{184}{9} - 64\zeta_2\Bigg)S_1^2 + \Bigg(\frac{9472}{27} - \frac{1088\zeta_2}{9} - \frac{640\zeta_3}{3}\Bigg)S_1 + \frac{2312}{9}\zeta_2 - \frac{2656}{9}\zeta_3 + 158 + \frac{192}{5}\zeta_2^2\Bigg] + C_A\Bigg[\Bigg(-\frac{32732}{27} + \frac{6640}{9}\zeta_2 - \frac{472}{3}\zeta_3 + \frac{672}{5}\zeta_2^2\Bigg)S_1 + \Bigg(-\frac{530}{9} + 80\zeta_2 + 80\zeta_3\Bigg)S_1^2 + \Bigg(\frac{640}{9} - 32\zeta_2\Bigg)S_1^3 - \frac{5255}{6} - \frac{11806}{9}\zeta_2 + \frac{10832}{9}\zeta_3 - 436\zeta_2^2 + 432\zeta_2\zeta_3 + 1200\zeta_5\Bigg]\Bigg\} + C_F^3\Bigg\{8S_1^4 + \Bigg(-72 + 64\zeta_2\Bigg)S_1^3 + \Bigg(-34 + 16\zeta_2 - 32\zeta_3\Bigg)S_1^2 + \Bigg(264 - 232\zeta_2 + 816\zeta_3 - \frac{1856}{5}\zeta_2^2\Bigg)S_1 + \frac{1937}{6} + 506\zeta_2 - 200\zeta_3 + 688\zeta_2^2 - 416\zeta_2\zeta_3 - 1760\zeta_5\Bigg\}\Bigg\}, \quad (1313)$$

$$C_{F_L,q}^{(3),d_{abc}} \simeq \frac{d_{abc}d^{abc}N_F}{N_C}\frac{1}{N}\Bigg[-128-448\zeta_2-704\zeta_3+\frac{64}{5}\zeta_2^2+128\zeta_2\zeta_3+1280\zeta_5\Bigg], \tag{1314}$$

$$C_{F_L,q}^{(2),\mathrm{PS}} \simeq 0, \tag{1315}$$

$$C_{F_L,q}^{(3),\mathrm{PS}} \simeq 0, \tag{1316}$$

$$C_{F_L,g}^{(1)} \simeq 0, \tag{1317}$$

$$C_{F_L,g}^{(2)} \simeq 0, \tag{1318}$$

$$C_{F_L,g}^{(3),a} \simeq 0, \tag{1319}$$

$$C_{F_L,g}^{(3),d_{abc}} \simeq 0, \tag{1320}$$

$$C_{F_3,q}^{(1),\mathrm{NS}} \simeq C_F\Bigg[2S_1^2+3S_1-9-2\zeta_2+\frac{5}{N}\Bigg], \tag{1321}$$

$$\begin{aligned}C_{F_3,q}^{(2),\mathrm{NS}} \simeq\ & C_F\Bigg\{T_FN_F\Bigg[-\frac{8}{9}S_1^3-\frac{58}{9}S_1^2+\Bigg(-\frac{494}{27}+\frac{8}{3}\zeta_2-\frac{12}{N}\Bigg)S_1+\frac{457}{18}+\frac{170}{9}\zeta_2+\frac{8}{9}\zeta_3\\&-\frac{332}{9N}\Bigg]+C_A\Bigg[\frac{22}{9}S_1^3+\Bigg(\frac{367}{18}-4\zeta_2\Bigg)S_1^2+\Bigg(\frac{3155}{54}+\frac{23}{N}-\frac{22\zeta_2}{3}-40\zeta_3\Bigg)S_1\\&-\frac{5465}{72}-\frac{1139}{18}\zeta_2+\frac{464}{9}\zeta_3+\frac{51}{5}\zeta_2^2+\frac{772}{9N}-\frac{8\zeta_2}{N}\Bigg]\Bigg\}+C_F^2\Bigg[2S_1^4+6S_1^3\\&+\Bigg(-\frac{27}{2}-4\zeta_2+\frac{22}{N}\Bigg)S_1^2+\Bigg(-\frac{51}{2}-18\zeta_2+24\zeta_3+\frac{25}{N}\Bigg)S_1+\frac{331}{8}+\frac{111}{2}\zeta_2\\&-66\zeta_3+\frac{4}{5}\zeta_2^2-\frac{61}{N}+\frac{2\zeta_2}{N}\Bigg],\end{aligned} \tag{1322}$$

$$\begin{aligned}C_{F_3,q}^{(3),\mathrm{NS}} \simeq\ & C_F^2\Bigg\{C_A\Bigg[\frac{44}{9}S_1^5+\Bigg(\frac{433}{9}-8\zeta_2\Bigg)S_1^4+\Bigg(\frac{8425}{54}-\frac{284\zeta_2}{9}-80\zeta_3+\frac{2540}{27N}\Bigg)S_1^3\\&+\Bigg(-\frac{5563}{36}-\frac{592}{3}\zeta_2+\frac{640}{9}\zeta_3+\frac{142}{5}\zeta_2^2+\frac{10441}{18N}-\frac{248}{3N}\zeta_2\Bigg)S_1^2+\Bigg(-\frac{16981}{24}-\frac{28495}{54}\zeta_2\\&+752\zeta_3+\frac{299}{3}\zeta_2^2+96\zeta_2\zeta_3+120\zeta_5+\frac{44075}{162N}+\frac{892}{9N}\zeta_2-\frac{1024}{3N}\zeta_3\Bigg)S_1+\frac{9161}{12}\\&+\frac{191545}{108}\zeta_2-\frac{49418}{27}\zeta_3+\frac{11419}{27}\zeta_2^2-828\zeta_2\zeta_3-\frac{3896}{9}\zeta_5-\frac{23098}{315}\zeta_2^3+\frac{536}{3}\zeta_3^2\\&-\frac{562151}{216N}-\frac{17207}{54N}\zeta_2+\frac{45556}{27N}\zeta_3+\frac{459}{5N}\zeta_2^2\Bigg]+T_FN_F\Bigg[-\frac{16}{9}S_1^5-\frac{140}{9}S_1^4\\&+\Bigg(-\frac{1366}{27}-\frac{1120}{27N}+\frac{64}{9}\zeta_2\Bigg)S_1^3+\Bigg(\frac{83}{9}-\frac{1990}{9N}+\frac{224}{3}\zeta_2+\frac{16}{9}\zeta_3\Bigg)S_1^2\\&+\Bigg(\frac{2003}{54}-\frac{8726}{81N}+\frac{4354}{27}\zeta_2-\frac{368}{9N}\zeta_2-\frac{16}{3}\zeta_2^2-\frac{40}{9}\zeta_3\Bigg)S_1-\frac{341}{18}-\frac{10733}{27}\zeta_2\end{aligned}$$

$$+\frac{21532}{27}\zeta_3-\frac{21604}{135}\zeta_2^2-\frac{80}{3}\zeta_2\zeta_3-\frac{1568}{9}\zeta_5+\frac{38569}{54N}+\frac{3674}{27N}\zeta_2-\frac{8168}{27N}\zeta_3\Biggr]\Biggr\}$$

$$+C_F\Biggl\{T_F^2N_F^2\Biggl[\frac{16}{27}S_1^4+\frac{464}{81}S_1^3+\Biggl(\frac{1880}{81}-\frac{32}{9}\zeta_2+\frac{16}{N}\Biggr)S_1^2+\Biggl(\frac{34856}{729}-\frac{464}{27}\zeta_2$$

$$+\frac{256}{27}\zeta_3+\frac{992}{9N}\Biggr)S_1-\frac{19034}{243}-\frac{8440}{81}\zeta_2+\frac{320}{81}\zeta_3-\frac{1168}{135}\zeta_2^2+\frac{18112}{81N}-\frac{16}{N}\zeta_2\Biggr]$$

$$+C_A^2\Biggl[\frac{121}{27}S_1^4+\Biggl(\frac{4649}{81}-\frac{88}{9}\zeta_2\Biggr)S_1^3+\Biggl(\frac{50689}{162}-\frac{778}{9}\zeta_2-132\zeta_3+\frac{88}{5}\zeta_2^2$$

$$+\frac{245}{3N}-\frac{10}{3N}\zeta_2\Biggr)S_1^2+\Biggl(\frac{599375}{729}-\frac{18179}{81}\zeta_2-\frac{6688}{9}\zeta_3+\frac{212}{15}\zeta_2^2+\frac{176}{3}\zeta_2\zeta_3$$

$$+232\zeta_5+\frac{8218}{9N}-\frac{2104}{9N}\zeta_2-\frac{8}{N}\zeta_3\Biggr)S_1-\frac{1909753}{1944}-\frac{78607}{54}\zeta_2+\frac{115064}{81}\zeta_3$$

$$+\frac{13151}{135}\zeta_2^2+\frac{3496}{9}\zeta_2\zeta_3-\frac{416}{3}\zeta_5-\frac{12016}{315}\zeta_2^3-\frac{248}{3}\zeta_3^2+\frac{133199}{54N}-\frac{7061}{27N}\zeta_2$$

$$-\frac{9124}{9N}\zeta_3+\frac{518}{15N}\zeta_2^2\Biggr]+C_AT_FN_F\Biggl[-\frac{88}{27}S_1^4+\Biggl(-\frac{3104}{81}+\frac{32\zeta_2}{9}\Biggr)S_1^3+\Biggl(-\frac{15062}{81}$$

$$-\frac{68}{N}+\frac{112\zeta_2}{3}+16\zeta_3\Biggr)S_1^2+\Biggl(-\frac{321812}{729}-\frac{1892}{3N}+\frac{10528}{81}\zeta_2$$

$$+\frac{824}{9N}\zeta_2-\frac{256}{15}\zeta_2^2+\frac{3952}{27}\zeta_3\Biggr)S_1+\frac{142883}{243}-\frac{122414}{81N}+\frac{66662}{81}\zeta_2-\frac{42836}{81}\zeta_3+\frac{328}{135}\zeta_2^2$$

$$-\frac{128}{9}\zeta_2\zeta_3+\frac{16}{3}\zeta_5+\frac{3868\zeta_2}{27N}+\frac{2168\zeta_3}{9N}\Biggr]\Biggr\}+C_F^3\Biggl\{\frac{4}{3}S_1^6+6S_1^5+\Biggl(-\frac{93}{2}-36\zeta_2$$

$$+48\zeta_3+\frac{86}{N}\Biggr)S_1^3+\Biggl(-9-4\zeta_2+\frac{34}{N}\Biggr)S_1^4+\Biggl(\frac{187}{4}+66\zeta_2-60\zeta_3+\frac{8}{5}\zeta_2^2$$

$$-\frac{1033}{6N}-\frac{20}{N}\zeta_2\Biggr)S_1^2+\Biggl(\frac{1001}{8}+\frac{579}{2}\zeta_2-346\zeta_3+84\zeta_2^2-80\zeta_2\zeta_3-240\zeta_5$$

$$-\frac{1465}{6N}-\frac{302}{3N}\zeta_2+\frac{168}{N}\zeta_3\Biggr)S_1-\frac{7255}{24}-\frac{6197}{12}\zeta_2-\frac{1225}{3}\zeta_3-\frac{1791}{5}\zeta_2^2$$

$$+556\zeta_2\zeta_3+1384\zeta_5+\frac{8144}{315}\zeta_2^3-\frac{176}{3}\zeta_3^2+\frac{3047}{8N}+\frac{1993}{6N}\zeta_2-\frac{450}{N}\zeta_3-\frac{284}{3N}\zeta_2^2\Biggr\}, \quad (1323)$$

$$C_{F_3,q}^{(3),d_{abc}} \simeq 0, \quad (1324)$$

and

$$\Delta C_{g_1,q}^{(1),\mathrm{NS,L}} \simeq C_F\Biggl[2S_1^2+3S_1-9-2\zeta_2+\frac{5}{N}\Biggr], \quad (1325)$$

$$\Delta C_{g_1,q}^{(2),\mathrm{NS,L}} \simeq C_F\Biggl\{T_FN_F\Biggl[-\frac{58}{9}S_1^2-\frac{8}{9}S_1^3+\Biggl(-\frac{494}{27}+\frac{8\zeta_2}{3}-\frac{12}{N}\Biggr)S_1+\frac{457}{18}+\frac{170}{9}\zeta_2+\frac{8}{9}\zeta_3$$

$$
-\frac{332}{9N}\Biggr] + C_A\Biggl[\frac{22}{9}S_1^3 + \Biggl(\frac{367}{18} - 4\zeta_2\Biggr)S_1^2 + \Biggl(\frac{3155}{54} + \frac{23}{N} - \frac{22\zeta_2}{3} - 40\zeta_3\Biggr)S_1
-\frac{5465}{72} - \frac{1139}{18}\zeta_2 + \frac{464}{9}\zeta_3 + \frac{51}{5}\zeta_2^2 + \frac{772}{9N} - \frac{8}{N}\zeta_2\Biggr]\Biggr\} + C_F^2\Biggl[2S_1^4 + 6S_1^3
+\Biggl(-\frac{27}{2} + \frac{22}{N} - 4\zeta_2\Biggr)S_1^2 + \Biggl(-\frac{51}{2} + \frac{25}{N} - 18\zeta_2 + 24\zeta_3\Biggr)S_1 + \frac{331}{8} + \frac{111}{2}\zeta_2
-66\zeta_3 + \frac{4}{5}\zeta_2^2 - \frac{61}{N} + \frac{2\zeta_2}{N}\Biggr], \tag{1326}
$$

$$(1327)$$

$$
\Delta C_{g_1,q}^{(2),\mathrm{PS,L}} \simeq 0, \tag{1328}
$$

$$
\Delta C_{g_1,q}^{(3),\mathrm{PS,L}} \simeq 0, \tag{1329}
$$

$$
\Delta C_{g_1,g}^{(1)} \simeq -\frac{4}{N}T_F N_F\bigl(1 + S_1\bigr), \tag{1330}
$$

$$
\Delta C_{g_1,g}^{(2)} \simeq \frac{1}{N}\Biggl\{C_A T_F N_F\Biggl[-\frac{4}{3}S_1^3 - 8S_1^2 + \Biggl(-28 + 12\zeta_2\Biggr)S_1 - 16 - \frac{92}{3}\zeta_3\Biggr]
+C_F T_F N_F\Biggl[-\frac{20}{3}S_1^3 - 18S_1^2 + \Biggl(4 + 12\zeta_2\Biggr)S_1 - 4 + 10\zeta_2 + \frac{176}{3}\zeta_3\Biggr]\Biggr\}, \tag{1331}
$$

$$
\Delta C_{g_1,g}^{(3)} \simeq \frac{1}{N}T_F N_F\Biggl\{-\Biggl[\frac{4}{3}C_A^2 + \frac{20}{3}C_F^2\Biggr]S_1^5 + \Biggl[-\frac{293}{27}C_A^2 - \frac{83}{3}C_F^2 + \frac{28}{27}C_A T_F N_F
+C_F\Biggl(-\frac{412}{27}C_A + \frac{68}{27}T_F N_F\Biggr)\Biggr]S_1^4 + \Biggl[\frac{608}{81}C_A T_F N_F + C_F\Biggl(\frac{1984}{81}T_F N_F + C_A\Biggl(
-\frac{6884}{81} + \frac{272}{9}\zeta_2\Biggr)\Biggr) + C_A^2\Biggl(-\frac{6112}{81} + \frac{152}{9}\zeta_2\Biggr) + C_F^2\Biggl(-\frac{254}{9} + \frac{152}{9}\zeta_2\Biggr)\Biggr]S_1^3
+\Biggl[C_F\Biggl(T_F N_F\Biggl(\frac{6260}{81} - \frac{56}{9}\zeta_2\Biggr) + C_A\Biggl(-\frac{12043}{81} + \frac{568}{9}\zeta_2 + \frac{104}{3}\zeta_3\Biggr)\Biggr) + C_A T_F N_F
\times\Biggl(\frac{4384}{81} - \frac{104}{9}\zeta_2\Biggr) + C_F^2\Biggl(-\frac{205}{3} + 118\zeta_2 + \frac{32}{3}\zeta_3\Biggr) + C_A^2\Biggl(-\frac{27578}{81} + \frac{718}{9}\zeta_2
+\frac{272}{3}\zeta_3\Biggr)\Biggr]S_1^2 + \Biggl[C_F\Biggl(C_A\Biggl(\frac{65936}{243} + \frac{7396}{27}\zeta_2 - \frac{10796}{27}\zeta_3 + \frac{976}{15}\zeta_2^2\Biggr)
+T_F N_F\Biggl(-\frac{12304}{243} - \frac{1952}{27}\zeta_2 - \frac{224}{27}\zeta_3\Biggr)\Biggr) + C_A T_F N_F\Biggl(\frac{43192}{243} - \frac{1376}{27}\zeta_2
+\frac{32}{27}\zeta_3\Biggr) + C_A^2\Biggl(-\frac{139052}{243} + \frac{5992}{27}\zeta_2 + \frac{3368}{27}\zeta_3 - \frac{136}{3}\zeta_2^2\Biggr) + C_F^2\Biggl(-\frac{508}{3} - 54\zeta_2
+\frac{2072}{3}\zeta_3 - \frac{2732}{15}\zeta_2^2\Biggr)\Biggr]S_1 + C_F\Biggl\{C_A\Biggl[\frac{32813}{108} + \frac{1321}{3}\zeta_2 - \frac{11908}{81}\zeta_3 - \frac{1172}{9}\zeta_2^2
$$

$$+\frac{352}{9}\zeta_2\zeta_3+\frac{1496}{3}\zeta_5\Bigg]+T_FN_F\left(-\frac{1189}{27}-\frac{188}{3}\zeta_2-\frac{16672}{81}\zeta_3+\frac{1028}{45}\zeta_2^2\right)\Bigg\}$$

$$+C_F^2\Bigg[\frac{617}{12}-173\zeta_2+\frac{5564}{9}\zeta_3+\frac{193}{5}\zeta_2^2-\frac{1040}{9}\zeta_2\zeta_3-\frac{2384}{3}\zeta_5\Bigg]+C_A^2\Bigg[-\frac{31172}{81}$$

$$-\frac{478}{9}\zeta_2-\frac{10988}{81}\zeta_3+\frac{163}{5}\zeta_2^2-\frac{344}{9}\zeta_2\zeta_3-16\zeta_5\Bigg]+C_AT_FN_F\Bigg[\frac{13120}{81}-\frac{16}{9}\zeta_2$$

$$+\frac{17824}{81}\zeta_3-\frac{4}{5}\zeta_2^2\Bigg]\Bigg\}, \tag{1332}$$

$$\Delta C_{g_1,g}^{(3),d_{abc}} \simeq \frac{1}{N}\frac{d_{abc}d^{abc}N_F^2}{N_A}\Bigg\{\Bigg[128+128\zeta_2-256\zeta_3\Bigg]S_1^2+48+\Bigg[-64$$

$$-128\zeta_2+768\zeta_3-\frac{1536}{5}\zeta_2^2\Bigg]S_1+256\zeta_2-512\zeta_3+\frac{1728}{5}\zeta_2^2-640\zeta_5\Bigg\}. \tag{1333}$$

8.3 The large N_F limit

There are also some predictions on the large N_F behaviour of deep–inelastic Wilson coefficients. The $O(C_F T_F^2 N_F^2)$ contribution to the flavor non–singlet Wilson coefficient of the structure function $F_2(x,Q^2)$ agrees with Eq. (15a) of [97], which can be brought into a more compact form.

For the structure function $F_L^{\rm NS}(N,a_s)$ the generating functional of the Wilson coefficients reads [98].

$$G_L(g)=C_F\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial^k}{\partial x^k}\left[\frac{g e^{5/3x}\Gamma(x+N)}{(1-x)(2-x)(1+x+n)\Gamma(1+x)\Gamma(N)}\right]_{x=(4/3)T_FN_Fg}. \tag{1334}$$

The $O(a_s^k)$ Wilson coefficient is given by its kth expansion coefficient in the variable g, $f_L^{(k)}(g=a_s)$ to be multiplied by $(-1)^{k+1}$. This prediction agrees with the corresponding contributions in our direct calculation to three loop order and implies a modification of [98].

9 Conclusions

We have calculated the unpolarized massless three–loop Wilson coefficients of the structure functions $F_2(x,Q^2)$ and $F_L(x,Q)$ for pure photon exchange and for the charged current structure function $xF_3(x,Q^2)$ to three–loop order. In the polarized case, we calculated the Wilson coefficients to the structure function $g_1(x,Q^2)$ to three–loop order in the Larin scheme and for the non–singlet case also in the $\overline{\sf MS}$ scheme. We applied the forward Compton amplitude for the respective scattering cross section and followed the other technical calculation steps having been described in our previous calculations of the three–loop anomalous dimensions in Refs. [27, 28]. By using the method of arbitrarily high Mellin moments [61] we have obtained a sufficiently large basis to compute the Wilson coefficients without any reference to further structural assumptions. In parallel, we also used the method of differential equations [80] since we face only first order factorizable problems, which, even further, relate to harmonic polylogarithms only. This method works for any basis of master integrals. The Wilson coefficients depend on 60 harmonic sums, weighted by rational functions in the Mellin variable N, after applying the algebraic relations, and on 31 harmonic sums by also applying the structural relations. We confirm former results on F_2, F_L and xF_3 at three–loop order in Refs. [15, 17] for the first time and all earlier results at one– and two–loop order, also for the structure function $g_1(x,Q^2)$. The three–loop results for the structure function $g_1(x,Q^2)$ are new. Concerning the polarized case we would like to remark that one may perfectly work in the Larin scheme, expressing the Wilson coefficients, the anomalous dimensions and the parton densities at the starting scale in this scheme, representing the polarized deep–inelastic structure functions or any other observables. For convenience, we also provide the expansions of the Wilson coefficients in the small x and large x region. The latter information may be of relevance studying the high energy Regge limit and the soft region, respectively. We also checked some predictions in the large N_F limit. The present results are of importance for the application in experimental and phenomenological analyzes of precision deep–inelastic data and in particular for precision measurements of the strong coupling constant.

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A $Z_5^{\rm NS}(N)$

In the following we calculate the function $Z_5^{\rm NS}(N)$ needed to transform the non–singlet Wilson coefficient of the structure function g_1 from the Larin into the $\overline{\sf MS}$ scheme to three–loop order. To two–loop order it has been calculated in [26] in unrenormalized form.[5] In [29] we provided the expansion coefficients needed to two loop order at the required depth in ε for the present calculation in the unrenormalized case.

The practical approach consists in deriving $Z_5^{\rm NS}(N)$ in renormalized form, since it provides a finite renormalization. It is given by

$$Z_5^{\rm NS}(N) = \frac{A_{qq}^{\rm NS,phys,ac,ren}}{A_{qq}^{\rm NS,phys,Larin,ren}} = 1 + a_s z_5^{(1),\rm NS} + a_s^2 z_5^{(2),\rm NS} + a_s^3 z_5^{(3),\rm NS} + O(\hat{a}_s^4), \qquad (1335)$$

where the OMEs are calculated for anticommuting γ_5 and in the Larin scheme. The gauge parameter ξ is defined in [27]. Note that in the Larin scheme one has to add the physical and EOM expansion coefficients projected in Ref. [29], although the two projections are orthogonal.

For odd integers N it is given by

$$z_5^{(1),\rm NS,odd} = -C_F \frac{8}{N(1+N)} \qquad (1336)$$

$$z_5^{(2),\rm NS,odd} = C_F T_F N_F \frac{16\big(-3-N+5N^2\big)}{9N^2(1+N)^2} + C_F^2\left[\frac{16(1+2N)}{N^2(1+N)^2}S_1 + \frac{16}{N(1+N)}S_2 + \frac{8Q_2}{N^3(1+N)^3} + \frac{32}{N(1+N)}S_{-2}\right] - C_A C_F\left[\frac{4Q_5}{9N^3(1+N)^3} + \frac{16}{N(1+N)}S_{-2}\right],$$

[5] Note that Eq. (A11) in Ref. [26] contains typographical errors.

(1337)

$$
\begin{aligned}
z_5^{(3),\mathrm{NS,odd}} = {} & C_F T_F^2 N_F^2 \frac{128 Q_1}{81 N^3 (1+N)^3} + C_A \Biggl\{ C_F T_F N_F \Biggl[-\frac{32 Q_3}{9 N^3 (1+N)^3} S_1 + \frac{16 Q_{13}}{81 N^4 (1+N)^4} \\
& - \frac{64}{3N(1+N)} S_3 + \frac{128\big(-3+4N+10N^2\big)}{27 N^2 (1+N)^2} S_{-2} - \frac{128}{9N(1+N)} S_{-3} \\
& - \frac{256}{9N(1+N)} S_{-2,1} + \frac{128}{3N(1+N)} \zeta_3 \Biggr] + C_F^2 \Biggl[\frac{64 Q_4}{27 N^3 (1+N)^3} S_2 \\
& + \frac{8 Q_{15}}{27 (N-1) N^5 (1+N)^5 (2+N)} + \Biggl(-\frac{16 Q_{11}}{27 N^4 (1+N)^4} - \frac{512}{3N(1+N)} S_3 \\
& - \frac{3584}{3N(1+N)} S_{-2,1} \Biggr) S_1 - \frac{32\big(-30-7N+5N^2\big)}{9N^2(1+N)^2} S_3 - \frac{320}{3N(1+N)} S_4 \\
& + \Biggl(\frac{32 Q_{12}}{27 (N-1) N^3 (1+N)^3 (2+N)} + \frac{64\big(-10+19N+11N^2\big) S_1}{3N^2(1+N)^2} \\
& + \frac{256}{3N(1+N)} S_2 \Biggr) S_{-2} - \frac{512}{3N(1+N)} S_{-2}^2 + \Biggl(-\frac{64\big(78-N+5N^2\big)}{9N^2(1+N)^2} \\
& + \frac{768}{N(1+N)} S_1 \Biggr) S_{-3} + \frac{1472}{3N(1+N)} S_{-4} - \frac{256\big(-33+16N+13N^2\big)}{9N^2(1+N)^2} S_{-2,1} \\
& + \frac{1024}{3N(1+N)} S_{3,1} - \frac{3712}{3N(1+N)} S_{-2,2} - \frac{1280}{N(1+N)} S_{-3,1} + \frac{7168}{3N(1+N)} S_{-2,1,1} \\
& - \frac{96\big(-2+5N+5N^2\big)}{N^2(1+N)^2} \zeta_3 \Biggr] \Biggr\} + C_F^2 T_F N_F \Biggl[\frac{8 Q_7}{27 N^4 (1+N)^4} + \frac{256\big(3+N-5N^2\big)}{27N^2(1+N)^2} \\
& \times S_2 + \frac{128\big(12+17N-14N^3+3N^4\big)}{27N^3(1+N)^3} S_1 + \frac{512}{9N(1+N)} S_3 + \frac{256}{9N(1+N)} S_{-3} \\
& - \frac{256\big(-3+4N+10N^2\big)}{27N^2(1+N)^2} S_{-2} + \frac{512}{9N(1+N)} S_{-2,1} - \frac{128}{3N(1+N)} \zeta_3 \Biggr] \\
& + C_A^2 C_F \Biggl[-\frac{4 Q_{16}}{81 (N-1) N^5 (1+N)^5 (2+N)} + \Biggl(\frac{16 Q_9}{9 N^4 (1+N)^4} + \frac{256}{3N(1+N)} S_3 \\
& + \frac{1024}{3N(1+N)} S_{-2,1} \Biggr) S_1 - \frac{16\big(8+N+N^2\big)}{3N^2(1+N)^2} S_3 + \frac{256}{3N(1+N)} S_4 \\
& + \Biggl(-\frac{64 Q_{10}}{27 (N-1) N^3 (1+N)^3 (2+N)} - \frac{256\big(-1+N+N^2\big)}{3N^2(1+N)^2} S_1 \Biggr) S_{-2} \\
& + \frac{64}{N(1+N)} S_{-2}^2 + \Biggl(\frac{64\big(21+N+N^2\big)}{9N^2(1+N)^2} - \frac{512}{3N(1+N)} S_1 \Biggr) S_{-3} - \frac{320}{3N(1+N)} S_{-4} \\
& - \frac{512}{3N(1+N)} S_{3,1} + \frac{128\big(-21+10N+10N^2\big)}{9N^2(1+N)^2} S_{-2,1} + \frac{1024}{3N(1+N)} S_{-2,2} \\
& + \frac{1024}{3N(1+N)} S_{-3,1} - \frac{2048}{3N(1+N)} S_{-2,1,1} + \frac{32\big(-2+5N+5N^2\big)}{N^2(1+N)^2} \zeta_3 \Biggr]
\end{aligned}
$$

$$
+C_F^3\Biggl[-\frac{8Q_{14}}{3(N-1)N^5(1+N)^5(2+N)}+\Biggl(\frac{16Q_8}{3N^4(1+N)^4}-\frac{256(1+2N)}{3N^2(1+N)^2}S_2
+\frac{1024}{N(1+N)}S_{-2,1}\Biggr)S_1-\frac{128\bigl(1+3N+3N^2\bigr)}{3N^3(1+N)^3}S_1^2-\frac{32\bigl(3+11N-4N^2+4N^3\bigr)}{3N^2(1+N)^3}S_2
-\frac{128}{3N(1+N)}S_2^2-\frac{64\bigl(2+5N+N^2\bigr)S_3}{3N^2(1+N)^2}-\frac{128}{3N(1+N)}S_4
+\Biggl(-\frac{64Q_6}{3(N-1)N^3(1+N)^3(2+N)}-\frac{128\bigl(-2+11N+3N^2\bigr)}{3N^2(1+N)^2}S_1
-\frac{512}{3N(1+N)}S_2\Biggr)S_{-2}+\frac{256}{3N(1+N)}S_{-2}^2+\Biggl(\frac{128\bigl(12-N+N^2\bigr)}{3N^2(1+N)^2}
-\frac{2560}{3N(1+N)}S_1\Biggr)S_{-3}-\frac{1664}{3N(1+N)}S_{-4}+\frac{512\bigl(-4+2N+N^2\bigr)}{3N^2(1+N)^2}S_{-2,1}
-\frac{512}{3N(1+N)}S_{3,1}+\frac{3328}{3N(1+N)}S_{-2,2}+\frac{3584}{3N(1+N)}S_{-3,1}-\frac{2048}{N(1+N)}S_{-2,1,1}
+\frac{64\bigl(-2+5N+5N^2\bigr)}{N^2(1+N)^2}\zeta_3\Biggr] \qquad (1338)
$$

with

$$Q_1 = N^4+12N^3+7N^2-4N-3, \qquad (1339)$$
$$Q_2 = 2N^4+N^3+8N^2+5N+2, \qquad (1340)$$
$$Q_3 = 3N^4+6N^3+5N^2+2N+2, \qquad (1341)$$
$$Q_4 = 85N^4+104N^3+13N^2-6N+18, \qquad (1342)$$
$$Q_5 = 103N^4+140N^3+58N^2+21N+36, \qquad (1343)$$
$$Q_6 = N^6-3N^5+9N^3-33N^2-6N+8, \qquad (1344)$$
$$Q_7 = 17N^6+207N^5-685N^4-691N^3-312N^2+80N+48, \qquad (1345)$$
$$Q_8 = 22N^6+50N^5+41N^4-132N^3-153N^2-104N-40, \qquad (1346)$$
$$Q_9 = 24N^6+72N^5+44N^4-32N^3-35N^2-7N-12, \qquad (1347)$$
$$Q_{10} = 85N^6+222N^5-38N^4-336N^3-92N^2+69N+36, \qquad (1348)$$
$$Q_{11} = 165N^6-185N^5-1034N^4-1285N^3-895N^2-726N-396, \qquad (1349)$$
$$Q_{12} = 349N^6+861N^5-152N^4-1263N^3-665N^2+222N+216, \qquad (1350)$$
$$Q_{13} = 485N^6+643N^5+253N^4+85N^3+326N^2-96N-144, \qquad (1351)$$
$$Q_{14} = 24N^{10}+99N^9+259N^8+308N^7-186N^6-853N^5-1153N^4-82N^3 +344N^2+328N+144, \qquad (1352)$$
$$Q_{15} = 845N^{10}+3292N^9+7545N^8+11366N^7-121N^6-19168N^5-19017N^4 -2522N^3+5420N^2+4008N+1440, \qquad (1353)$$
$$Q_{16} = 6087N^{10}+24679N^9+32532N^8+11838N^7-18471N^6-38727N^5 -37968N^4-12190N^3+11772N^2+8352N+1728. \qquad (1354)$$

To derive the above relations one has to apply the relations [60]

$$S_{n_1,\ldots,n_p}\left(\frac{N}{2}\right) = 2^{n_1+\ldots+n_p-p}\sum_{\pm} S_{\pm n_1,\ldots,\pm n_p}(N). \tag{1355}$$

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