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Orbit deviations generated by f-type distortions in an AG synchrotron

Contents:

For these types of deflecting fields particle trajectories are represented in a closed form by means of phase and amplitude functions.

The formulas allow a simple numerical evaluation and are provided for application in beam ejection and betatron frequency measurement problems.

Orbit deviations generated by f-type distortions in an AG synchrotron

I. General equations.

With $r(\sigma)$ being the radial or vertical deviation of a particle trajectory from the principal orbit and $F(\sigma)$ being the distortion produced by a homogeneous magnetic or electric deflecting field, the equation of motion for this component is given by

$$r''(\sigma) + K(\sigma) \cdot r(\sigma) = F(\sigma)$$

The general solution of this equation is given by the general solution of the homogeneous equation plus a special solution of the inhomogeneous equation and can be written as [1)]

$$r(s,k) = A\sqrt{\beta(\sigma)}\,\cos\left\{\phi(\sigma)+B\right\} - \sqrt{\beta(s)}\int_0^{\sigma} F(x)\sqrt{\beta(x)}\,\sin\left\{\phi(x)-\phi(\sigma)\right\}dx \quad (1)$$

with $\sigma = s + (k-1)\cdot L$

In order to distinguish between different particle revolutions, the arc length variable s, starting at a point s=0 and being measured along the principal orbit, is restricted to only one revolution in the above notation. L is the length of the principal orbit and k - 1 is the number of completed revolutions $\phi(\sigma)$ and $\beta(s)$ are the phase and amplitude functions, with

$$\beta(s) = \beta(\sigma) = \frac{1}{\phi'(\sigma)}$$

The function $F(\sigma)$ represents the distortion.
A and B are constants fixed by the initial conditions r(o) and r'(o). It shall be generally assumed A = 0 in the following, looking at particles only which move along the principal orbit when putting on the deflecting field.

We now shall restrict ourselves to the special case of a distortion F_k which is independent of σ within the regions $s_1 + (k-1)L \leq \sigma \leq s_2 + (k-1)L$ $(k=1,2,\ldots)$ and which is zero everywhere outside these regions.

Equation (1) with A = 0 can then be transformed into

$$r(s,k) = \sqrt{\beta(s)} \sum_{j=0}^{k-1} F_{k-j} \int_{s_1}^{s_2} \sqrt{\beta(x)} \cdot \sin\left\{\phi(s) - \phi(x) + 2\delta j\right\} dx \qquad (2)$$
$$\left(s + (k-1)L > s_2 + (k-1)L\right)$$

The phase shift 2δ is modulo 2π equal to the betatron phase shift per revolution

$$2\delta = 2\pi \cdot \Delta Q \quad \text{with} \quad \Delta Q = Q - q ,$$

Q being the number of betatron oscillations per revolution and q being the next smaller integer. Equation (2) is valid for the k^{th} revolution after the k^{th} traverse through the deflecting field. The corresponding equation which is valid for the k^{th} revolution before traversing the deflecting field is obtained from equation (2) by replacing k by k-1 and $\phi(s)$ by $\phi(s)+2$

$$r(s,k) = \sqrt{\beta(s)} \sum_{j=0}^{k-2} F_{k-j} \int_{s_1}^{s_2} \sqrt{\beta(x)} \cdot \sin\left\{\phi(s) - \phi(x) + 2\delta(j+1)\right\} dx \qquad (2a)$$
$$\left(s + (k-1)L < s_1 + (k-1)L\right)$$

Assuming the relative change of the distortion to be small during one revolution, the instantaneous closed orbit $\hat{r}(s,k)$ can be derived by applying to equation (1) the boundary conditions

$$r(s,k+1) = r(s,k) \quad \text{and}$$

$$r'(s,k+1) = r'(s,k) \quad \text{one gets}$$

$$\hat{r}(s,k) = \frac{\sqrt{\beta(s)}}{2\sin\delta} \cdot \int_{\sigma}^{\sigma+L} F(x) \sqrt{\beta(x)} \cos\left\{\phi(s) - \phi(x) + \delta\right\} dx \qquad (3)$$

with $\sigma = s + (k-1)L$

Assuming again a constant distortion F_k in the region $s_1 + (k-1)L \le \sigma \le s_2 + (k-1)L$, equation (3) can be written as

$$\hat{r}(s,k) = \frac{\sqrt{\beta(s)}}{2\sin\delta} \cdot F_k \int_{s_1}^{s_2} \sqrt{\beta(x)} \cos\left\{\phi(s) - \phi(x) - \delta\right\} dx \qquad 4)$$

Equations (2) and (4) are only valid outside the deflecting field. They get a much simpler form by introducing the following notations:

$$I_c = \frac{1}{l} \int_{s_1}^{s_2} \sqrt{\beta(x)} \cos \phi(x) dx \,; \qquad I_s = \frac{1}{l} \int_{s_1}^{s_2} \sqrt{\beta(x)} \sin \phi x \, dx$$

$$R^2 = I_c^2 + I_s^2 \,; \qquad \vartheta = \operatorname{arc\,tg} I_s/I_c$$

$l = s_2 - s_1$ is the length of the deflecting field.
The trajectory (2) of a particle starting with $r(0) = r'(0) = 0$ is then given by

$$r(s,k) = eR \sqrt{\beta(s)} \sum_{j=0}^{K-1} F_{K-j} \sin\left\{\phi(s) + 2\delta j - \vartheta\right\} \quad , \tag{5}$$

and the closed orbit by

$$\hat{r}(s,k) = \frac{eF_K R}{2 \sin\delta} \sqrt{\beta(s)} \cos\left\{\phi s - \delta - \vartheta\right\} \tag{6}$$

For a short deflecting field centered about s_d, the integrands in I_c and I_s can be assumed constant over the region of integration and one has

$$R \rightarrow \sqrt{\beta(s_d)} \qquad ; \qquad \vartheta \longrightarrow \phi(s_d)$$

For a deflecting field located in a straight section, approximations for the phase and amplitude functions can be used, allowing I_c and I_s to be evaluated in a closed form. Choosing $s = o$ and $\phi = 0$ at the beginning of the straight section, one gets for the integrals

$$I_c = \frac{1}{2\sqrt{\beta_o}} \left\{2 \beta_o - \alpha_o (s_1 + s_2)\right\} \tag{7a}$$

$$I_s = \frac{1}{2\sqrt{\beta_o}} (s_1 + s_2) \tag{7b}$$

β_o and α_o are the values of the functions $\beta(s)$ and $\alpha(s) = -\frac{1}{2} \beta'(s)$ at the beginning of the straight section.

II. Deflecting field constant in time

For a deflecting field B or E constant in time one has

$$\ell \cdot F = \varepsilon = r'(s_2) - r'(s_1) = \begin{cases} \ell \cdot \dfrac{eB}{p} \\[2ex] \ell \cdot \dfrac{eE}{pv} \end{cases}$$

respectively. ε is the change of particle direction due to the deflecting field. Evaluation of the sum in equation (5) leads to

$$r(s,k) = \varepsilon R \sqrt{\beta(s)} \; \frac{\sin k\delta}{\sin \delta} \; \sin\left\{ \phi(s) + (k-1)\delta - \vartheta \right\} \tag{8}$$

This means that a constant deflecting field, turned on suddenly, induces an oscillation which undergoes a phase shift of $-\delta = -\pi \cdot \Delta Q$ at each traverse through the field. The amplitude of this oscillation beats with a beating time of $k = \frac{1}{\Delta Q}$ revolutions.
Equation (8) can also be written as follows

$$r(s,k) = \frac{\varepsilon R \sqrt{\beta(s)}}{2 \sin\delta} \left[\cos\left\{ \phi(s) - \delta - \vartheta \right\} - \cos\left\{ \phi(s) + (k+1)2\delta + \delta - \vartheta \right\} \right] \tag{9}$$

The first term is the closed orbit (6): the motion can therefore be described as a betatron oscillation of constant amplitude and the normal phase increase 2δ per revolution around the new closed orbit.

III. Slowly rising deflecting field.

The distortion F may increase by ΔF each revolution. Then, with $F_k = k \cdot \Delta F = k \cdot \Delta\varepsilon / \ell$ one gets from equation (5)

$$r(s,k) = \frac{\Delta\varepsilon \cdot R}{2 \sin\delta} \sqrt{\beta(s)} \left[k \cdot \cos\left\{ \phi(s) - \delta - \vartheta \right\} - \frac{\sin k\delta}{\sin\delta} \cos\left\{ \phi(s) + k\delta - \vartheta \right\} \right] \tag{10}$$

The first term is the closed orbit corresponding to the momentary distortion $k\cdot \Delta F$. The second term can be neglected compared to the first one for $k \gg \frac{1}{\sin\delta}$. E.g. for $\Delta Q = \frac{1}{4}$ after 20 revolutions the second term amounts to less than 10 % of the first term already. This means, the particle substantially follows the equilibrium orbit.

IV. Rf deflecting field with constant frequency.

The influence of an rf deflecting field [2] can be investigated by inserting into equation (5) the distortion

$$F_k = F_o \cos\left\{\frac{\omega}{f}(k-1) + \varphi_o\right\}$$

with f being the orbital frequency and ω being the angular frequency of the rf. φ_o is the rf phase at the first traverse of the particle through the field (k=1). Particles consecutively arriving at the rf field are distinguished by different φ_o, with $\varphi_o = 0$ corresponding to a particle arriving at maximum field. Introducing $\mathcal{E}_o = 1\cdot F_o$, evaluation of equation (5) leads to

$$r(s,k) = \frac{1}{2}\mathcal{E}_o R\sqrt{\beta(s)}\left[\frac{\sin k(\delta+\frac{\omega}{2f})}{\sin(\delta+\frac{\omega}{2f})}\cdot\sin\left\{\phi(s)+(k-1)2\delta-(k-1)(\delta+\frac{\omega}{2f})-\vartheta-\varphi_o\right\}\right.$$
$$\left.+\frac{\sin k(\delta-\frac{\omega}{2f})}{\sin(\delta-\frac{\omega}{2f})}\cdot\sin\left\{\phi(s)+(k-1)2\delta-(k-1)(\delta-\frac{\omega}{2f})-\vartheta+\varphi_o\right\}\right] \qquad (11)$$

The rf field induces a superposition of two oscillations, beating with frequencies $\frac{2f\delta+\omega}{2\pi}$ and $\frac{2f\delta-\omega}{2\pi}$ respectively and undergoing phase shifts of $-(\frac{\omega}{2f}+\delta)$ and $(\frac{\omega}{2f}-\delta)$ respectively at each traverse through the field.

In the resonant case $\delta \pm \frac{\omega}{2f} = 0 \bmod \pi$, which is equivalent to $\omega/2\pi = f(mF \mp \Delta Q)$ with m=(0), 1,2, ..., equation (11) yields

$$r(s,k) = \frac{1}{2}\mathcal{E}_o R\sqrt{\beta(s)}\left[k\cdot\sin\left\{\phi(s)+(k-1)2\delta \mp \varphi_o - \vartheta\right\}\right.$$
$$\left.+\frac{\sin 2\delta k}{\sin 2\delta}\sin\left\{\phi(s) \pm \varphi_o - \vartheta\right\}\right] \qquad (12)$$

The second term in equation (12) can be neglected compared to the first one for $k \gg \frac{1}{\sin 2\delta}$. Therefore, in the resonant case for a large number k of revolutions the rf field induces a betatron oscillation with the normal phase shift 2δ per revolution and an amplitude linearly rising with time. In this case, the rf phase shift per revolution is also 2δ; the phase difference between rf and betatron oscillation therefore remains constant.

The following two questions might illustrate the behaviour of the beam under the influence of a resonant rf field for $K \gg \frac{1}{\sin 2\delta}$:

1) What is the shape of the beam center line around the synchrotron at a given instant t?

2) How does the beam center line move in a given observation point?

We assume that the rf field has been turned on at t = 0 with maximum amplitude. Those particles which have passed the field at this moment correspond to the initial rf phase $\varphi_o = 0$. The initial phase of the other particles follows from the distance between the particle and the rf field at t = 0:

$$\varphi_o = \frac{\omega}{v} \left[vt - \{ s + (k-1) L - s_d \} \right]$$

S_d is the location of the rf field, $s + (k-1)L - s_d$ the path length from the first traverse of the field to the point of observation and vt the path length covered between t = 0 and the instant t of observation. Inserting the above expression for φ_o into the first term of equation (12) yields the following equation for the center line r_c of the beam:

$$r_c(s,k) = k \cdot \frac{\varepsilon_o R}{2} \sqrt{\beta(s)} \sin\left\{ \phi(s) \mp \omega t \pm \frac{\omega}{v}(s - s_d) - \vartheta \right\} \qquad (13)$$

Here, the revolution number k is determined by s, t and the particle velocity v. At a given observation point the beam center line oscillates with the rf frequency $\frac{\omega}{2\pi}$ about the principal orbit, according to equation (13).

Introducing now in "smooth approximation"

$$\phi(s) = \frac{2\pi Q f}{V} \cdot (s - s_d) + \phi(s_d) \quad ,$$

equation (13) represents a sine wave, modulated by $\sqrt{\beta(s)}$

$$r_c(s,f) = k \cdot \frac{\varepsilon_o R}{2} \sqrt{\beta(s)} \; \sin\left\{ \mp \omega f + \frac{1}{V}(2\pi Q f \pm \omega)(s - s_d) - \vartheta + \phi(s_d) \right\}$$

The wavelength is $\frac{V}{Q f \pm \omega / 2\pi}$. In the special case $\omega = 2\pi Q f$ i.e. for an rf frequency equal to the betatron frequency, r_c does, in smooth approximation, not depend on s, and the beam center line is a "circle", breathing with the rf frequency $\frac{\omega}{2\pi}$.

Looking at the resonance term in equation (11) in the vicinity of a resonance, e.g. $\delta - \frac{\omega}{2f} = 0$, one gets for the amplitude

$$|r| = \frac{\varepsilon_o R}{2} \sqrt{\beta(s)} \left| \frac{\sin k(\delta - \frac{\omega}{2f})}{\sin(\delta - \frac{\omega}{2f})} \right|$$

The first beating maximum occurs at

$$k_{max} = \frac{f}{2 f \Delta Q - \frac{\omega}{\pi}}$$

with the amplitude

$$|r|_{max} = \varepsilon_o R \sqrt{\beta(s)} \; \frac{k_{max}}{\pi}$$

In the special case of an rf frequency $\frac{\omega}{2\pi}$ equal to the orbit frequency f modulo 2π, equation (11) reduces to

$$r(s,K) = \varepsilon_o \cos\varphi_o \, R \sqrt{\beta(s)} \cdot \frac{\sin k\delta}{\sin \delta} \cdot \sin\left\{ \phi(s) + (K-1)\delta - \vartheta \right\}$$

which is equation (8) with $\varepsilon = \varepsilon_o \cos\varphi_o$, since in this case a particle finds the same field proportional to $\cos\varphi_o$ at each traverse.

In smooth approximation, the resonant term in equation (12) can also be obtained by a different approach, as shown by K.W. Robinson[3].
At the k^{th} traverse of the particle through the rf deflecting field, the directional change ε_k imposed on its trajectory is, in the resonant case $\delta - \frac{\omega}{2f} = 0 \mod \pi$, given by

$$\varepsilon_K = \varepsilon_o \cos\left\{(K-1)2\delta + \varphi_o\right\} = \varepsilon_o \cos \vartheta_{rf}(k)$$

The betatron amplitude at the rf deflector may, in smooth approximation, be written as

$$r(k) = r_o(k) \cos\left\{(K-1)2\delta + \varphi_o + \Theta(k)\right\} = r_o(k) \cos \vartheta_\beta(k) \tag{14}$$

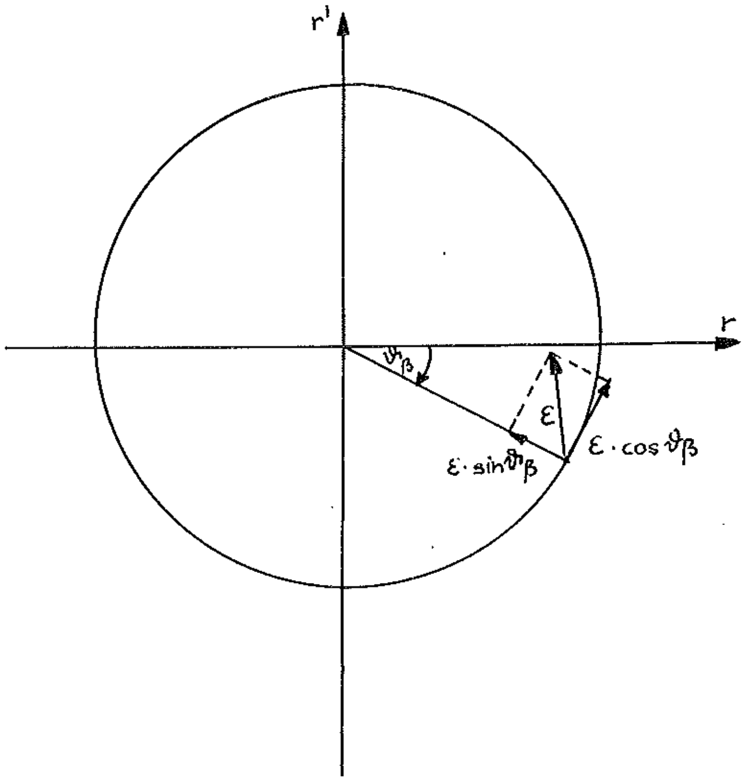
with $\Theta(k)$ being the difference between the betatron phase and the rf phase at the k^{th} traverse.

A directional change ε will induce a maximum amplitude increase

$$\Delta r_o = \varepsilon \cdot \overline{\beta} \qquad \text{with} \qquad \overline{\beta} = \frac{L}{2\pi Q}$$

$\overline{\beta}$ is the smooth approximation amplitude function.

Figure 1.



Assuming Δr_o to be small compared to r_o , it can be seen from figure 1 that the following equations hold:

$$\frac{dr_o}{dk} = -\varepsilon_o\bar{\beta}\,\cos\vartheta_{rf}\,\sin\vartheta_\beta$$

$$\frac{d\vartheta_\beta}{dk} = -\frac{\varepsilon_o\bar{\beta}}{r_o}\,\cos\vartheta_{rf}\,\cos\vartheta_\beta + 2\delta \qquad (15)$$

Replacing ϑ_β by $\vartheta_{rf} + \theta$ and averaging equations (15) over one period of $\cos\vartheta_{rf}$, one gets

$$\frac{dr_o}{dk} = -\frac{\varepsilon_o\bar{\beta}}{2}\,\sin\theta$$

$$\frac{d\theta}{dk} = \frac{\varepsilon_o\bar{\beta}}{2r_o}\,\cos\theta \qquad (16)$$

The solutions of (16) are given by

$$\theta(k) = \mathrm{arc\,tg}\left\{\mathrm{tg}\,\theta_1 - \frac{\varepsilon_o\bar{\beta}}{2}\,\frac{k}{r_1\cos\theta_1}\right\}$$

$$r_o(k) = \frac{r_1\cos\theta_1}{\cos\theta(k)} = \sqrt{r_1^2 + \left(\frac{\varepsilon_o\bar{\beta}}{2}\right)^2 k^2 - \varepsilon_o\bar{\beta}\,k\,r_1\sin\theta_1} \qquad (7)$$

r_1 and θ_1 are the initial values of r_o and θ for $k = 0$, i.e. before the first traverse through the rf field.

For a large number k of revolutions one has

$$\lim_{k\to\infty} \mathrm{tg}\,\theta(k) = \begin{cases} -\infty & \text{for } -\frac{\pi}{2} \le \theta_1 < \frac{\pi}{2} \\ +\infty & \text{for } -\frac{3\pi}{2} < \theta_1 \le -\frac{\pi}{2} \end{cases}$$

In both cases $\theta(k)$ therefore approaches $-\frac{\pi}{2}$.
Varying the initial phase θ_1 between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ (or between $-\frac{3\pi}{2}$ and $-\frac{\pi}{2}$), the amplitude $r_o(k)$ correspondingly varies between

$$r_o(k)_{min} = \frac{\varepsilon_o\bar{\beta}}{2}\cdot k - r_1 \qquad \text{and}$$

$$r_o(k)_{max} = \frac{\varepsilon_o\bar{\beta}}{2}\cdot k + r_1 \qquad (18)$$

These results agree with those obtained above in smooth approximation for a large k and a short deflecting field, since in this case with $R \to \sqrt{\beta(s_d)} \to \sqrt{\bar\beta}$ and $\phi(s_d) \to \vartheta$ equation (12) transforms into

$$r(s_d, k) = \frac{\varepsilon_o \bar\beta}{2} \cdot k \cdot \sin\left\{(k-1)2\delta + \varphi_o\right\} \qquad (19)$$

for $\delta - \frac{\omega}{2\ell} = 0 \mod \pi$.

Adding to this, according to equation (1), an initial betatron oscillation

$$r^*(s_d, k) = r_1 \cos\left\{(k-1)2\delta + \varphi_o + \theta_1\right\} \quad ,$$

one gets exactly the equations (17). The phase shift between r in equation (19) and the rf distortion $F_k = F_o \cos\left\{(k-1)2\delta + \varphi_o\right\}$ is always $-\frac{\pi}{2}$.

So far we have been dealing with an rf distortion turned on suddenly with its full amplitude. The case of a linearly rising rf amplitude according to

$$F_K = k \cdot \Delta F_o \cos\left\{\frac{\omega}{\ell}(k-1) + \varphi_o\right\}$$

shall only briefly be mentioned. Introducing F_k into equation (5), one gets again equation (11) for the dominating terms, with ε_o being replaced by $K\ell \cdot \Delta F$. The additional terms can be neglected for $k \gg \frac{1}{\sin 2\delta}$.

V. Rf deflecting field with linearly changing frequency

As has been previously shown by Geiger[4], the case of a linearly changing frequency can also be treated in a relatively simple way. Using again the formalism developed in part I, a formula will be derived which describes the particle trajectories in the vicinity of a resonance.

In the distortion

$$F_k = F_o \cos\left\{\frac{1}{f}\int_0^{k-1} \omega(\tau)\,d\tau + \varphi_o\right\} \tag{20}$$

the rf angular frequency ω may be linearly changing with the revolution number according to

$$\omega(\tau) = \omega_{res} + \Delta\omega(\tau - k_o) \tag{20a}$$

ω_{res} is the resonant frequency as given by $2\delta \pm \frac{\omega_{res}}{f} = 0 \mod 2\pi$, $\Delta\omega$ is the increase of the rf angular frequency per revolution, and the resonance is reached at the k_o^{th} traverse. Inserting equation (20) into equation (5), one gets

$$\begin{aligned} r(s,k) = \frac{\ell F_o R}{2} \sqrt{\beta(s)} \cdot \exp i\left\{\phi(s) + (k-1)2\delta - \vartheta - \frac{\pi}{2}\right\} \times \\ \times \Bigg[\exp\{-i\varphi_o\} \sum_{j=0}^{k-1} \exp\frac{-i}{f}\left\{\frac{\Delta\omega}{2} j^2 + (\omega_{res} + 2\delta f)j - \Delta\omega k_o j\right\} \\ + \exp\{i\varphi_o\} \sum_{j=0}^{k-1} \exp\frac{i}{f}\left\{\frac{\Delta\omega}{2} j^2 + (\omega_{res} - 2\delta f)j - \Delta\omega k_o j\right\}\Bigg] \end{aligned} \tag{21}$$

The complex notation has been chosen here for convenience; we mean only the real part of the expression on the right.

In passing the resonance $2\delta f + \omega_{res} = 0 \mod 2\pi$ the first term in equation (21) will be the dominating term, while the second term will be dominating in passing the resonance $2\delta f - \omega_{res} \mod 2\pi$, as can be seen form the result of part IV. Thus, for passing a resonance equation (21) can be written as

$$\begin{aligned} r(s,k) = \frac{\varepsilon_o R}{2} \sqrt{\beta(s)} \exp i\left\{\phi(s) + (k-1)2\delta - \vartheta \mp \varphi_o - \frac{\pi}{2}\right\} \times \\ \times \sum_{j=0}^{k-1} \exp\frac{\mp i\Delta\omega}{2f}\left\{j^2 - 2k_o j\right\} \end{aligned} \tag{22}$$

The sum in this equation can be approximated by an integral if the derivative of the exponent with respect to j is small compared to $\frac{\pi}{2}$, i.e. if

$$\left| \frac{\Delta\omega}{4} \; (j-k_0) \right| \ll \frac{\pi}{2} \qquad \text{eq.} \qquad \frac{1}{4} \left| \omega(k) - \omega_{res} \right| \ll \frac{\pi}{2}$$

Therefore, $\omega(k)$ has to stay close to the resonant frequency for this approximation. Equation (22) then transforms into

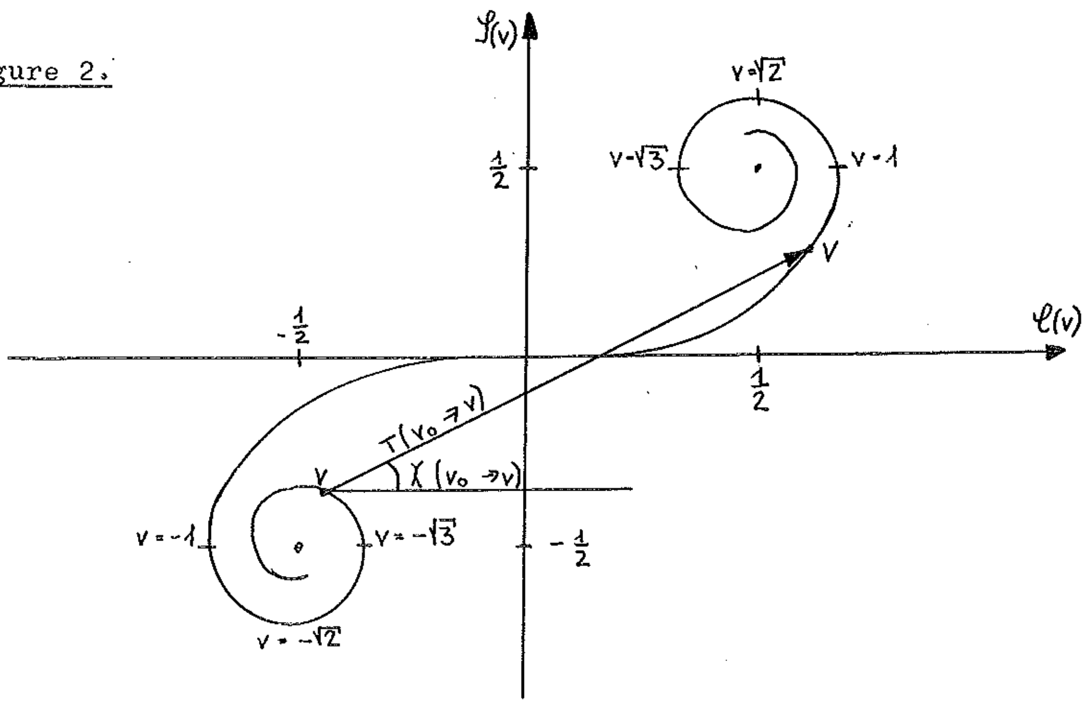
$$r(s,k) = \frac{\varepsilon_0 R}{2} \sqrt{\beta(s)} \exp i \left\{ \phi(s) + (k-1) 2\delta - \vartheta \mp \varphi_0 - \frac{\pi}{2} \right\} \cdot \int_0^{K-1} e^{F \frac{i\Delta\omega}{24} \{\tau^2 - 2K_0\tau\}} d\tau \tag{23}$$

The integral in equation (23) can be expressed by means of Fresnel's integrals

$$\mathcal{C}(v) = \int_0^v \cos \frac{\pi}{2} \tau^2 \, d\tau \qquad \text{and} \qquad \mathcal{S}(v) = \int_0^v \sin \frac{\pi}{2} \tau^2 \, d\tau \tag{25}$$

which e.g. are tabulated in Jahnke-Embde [5].
Plotting S(v) over C(v) in Cartesian coordinates, one gets Cornu's spiral which is shown in figure 2. In this graph, the argument v appears to be the arc length measured along the spiral.

Figure 2.



We now introduce the functions

$$T(v_0 \to v) = \sqrt{\left(\mathcal{C}(v) - \mathcal{C}(v_0)\right)^2 + \left(\mathcal{S}(v) - \mathcal{S}(v_0)\right)^2}$$

which is shown in figure 1 to be the distance between v_0 and v, and the "angle"

$$\chi(v_0 \to v) = \operatorname{arc\,tg} \frac{\mathcal{S}(v) - \mathcal{S}(v_0)}{\mathcal{C}(v) - \mathcal{C}(v_0)}$$

With the aid of this notation, equation (23) can finally be written

$$r(s,k) = \frac{\varepsilon_0 R}{2} \sqrt{\beta(s)} \sqrt{\frac{\pi f}{\Delta\omega}} \cdot T(v_0 \to v) \sin\left\{\phi(s) + (k-1)2\delta - \vartheta F \varphi_0 \pm \frac{\Delta\omega}{2f} k_0^2 F \chi(v_0 \to v)\right\} \qquad (26)$$

$$\text{with } v_0 = -\sqrt{\frac{(\Delta\omega)}{\pi \cdot f}} \cdot k_0 \quad \text{and}$$

$$v = \sqrt{\frac{(\Delta\omega)}{\pi \cdot f}} \cdot (k - 1 - k_0)$$

Equation (26) holds for $\Delta\omega > 0$, i.e. for increasing frequency. For decreasing frequency $(\Delta\omega < 0)$ the signs of v_0 and v and consequently of χ have to be reserved.

In the case of v and v_0 being located in the central part of Cornu's spiral, where it is approximately linear, one has the amplitude factor

$$\sqrt{\frac{\pi f}{(\Delta\omega)}} \cdot T(v_0 \to v) \approx \frac{\pi f}{(\Delta\omega)} \cdot (v - v_0) = k - 1 \quad ,$$

and the phase χ is practically zero. Equation (26) then transforms into the resonance term of equation (12).
The behaviour of the beam when passing a resonance shall be quantitatively illustrated by two examples.

For $\qquad \dfrac{k_0 \cdot \Delta\omega}{f} = \dfrac{1}{20} \qquad$ and $k \approx k_0 = 40$

one gets an amplitude factor $\sqrt{\frac{\pi \cdot f}{(\Delta\omega)}} \cdot T = 38{,}2$ instead of the factor $k - 1 = 39$ which one would have in the case of a constant resonant rf frequency.

For smaller frequency shifts per turn, i.e. for larger revolution numbers k_o, the deviation increases strongly.

For $\frac{k_o \cdot \Delta\omega}{f} = \frac{1}{20}$ and $k = k_o = 200$

one gets an amplitude factor $\sqrt{\frac{\pi f}{(\Delta\omega)}} \cdot T = 61{,}4$ instead of the factor $k - 1 = 199$ for a constant resonant rf frequency.

F. Brasse
K.G. Steffen

References

1) K.G. Steffen: CEA-45

2) see also e.g. M. Geiger: CERN-PS/Int. RF 59-2

3) K.W. Robinson: CEA-72

4) M. Geiger: private communication, to be published as a second part of 2)

5) Jahnke-Embde: Tafeln höherer Funktionen, p. 35