D e u t s c h e s E l e k t r o n e n - S y n c h r o t r o n
(DESY)
Hamburg-Gr.Flottbek 1, Flottbeker Drift 56

DESY A 2.70
Hamburg, August 1960

Use of Generalized Amplitude and Phase Functions in Designing Beam Transport Systems

by

K.G. Steffen, H. Hultschig and W. Kern
DESY, Hamburg

Contents:

Abstract:

The method of amplitude and phase functions used in AG synchrotron theory is generalized for application in high energy beam transport system.

The beam is represented by phase ellipses and the ellipse parameters are transformed through the system. A complete description of the beam as a whole rather than of single particle trajectories is thus obtained.

The method allows a quick estimate of the cost of a system and appears to be useful in looking for close-to-optimum systems satisfying given experimental requirements. Some suggestions are made concerning the use of an analog computer for treating this problem.

Introduction

One of the main problems in designing a beam transport system for high energy particles is to satisfy the conditions required by the experiment with a minimum of cost. For this purpose, an uncomfortably large number of parameters is usually available, making the problem of optimizing a beam transport system very difficult.

Unfortunately, the standard method of tracing individual particles by means of linear transformations (represented by two-by-two matrices) does in general not permit an easy evaluation of the cost of a system. One would like to have a formalism which yields the envelope of a whole beam rather than single particle trajectories, giving a direct display of the beam diameter along the entire flight path and thus allowing a quick estimate of the neccessary magnet apertures and cost.

The method of amplitude and phase functions used in AG synchrotron theory can be extended to serve this purpose and, moreover, allows ray tracing as well whenever desired.

Beam Representation by Phase Ellipses

A beam of monoenergetic particles with no appreciable coupling between horizontal and vertical motion can be adequately described by means of density distributions in the horizontal and vertical phase planes. All particles of interest are contained within an area bounded by a closed curve in each phase plane. The method considered here consists in transforming these envelope curves through the system under consideration.

The simplest envelope curves to be used for this purpose are parallelograms and ellipses, since they retain their character throughout the linear transformations involved. Here, we shall consider ellipses only, since they are better adapted to a transformation through a beam transport system designed for minimum cost than parallelograms with their sharp corners. In addition, they have a higher symmetry and can be described by only 3 parameter.

With s being the coordinate along the optical axis and y(s) and $y'(s) = \frac{dy}{ds}$ the transverse deviation and angular divergence of a particle respectively, we write all ellipses in a normalized form

$$\gamma \cdot y^2 + 2\alpha \cdot yy' + \beta \cdot y'^2 = \varepsilon \qquad (1)$$

$$\text{with } \beta\gamma - \alpha^2 = 1$$

Then, the constants $\varepsilon, \beta, \gamma$ have simple geometrical meanings (see figure 1):

<u>"emittance"</u> $\varepsilon = \frac{1}{\pi} \cdot F$

with F being the area of the ellipse;

<u>"half width"</u> $y_{max} = \sqrt{\varepsilon}\,\sqrt{\beta}$

<u>"half divergence"</u> $y'_{max} = \sqrt{\varepsilon}\,\sqrt{\gamma}$

The constant α also can be given a useful geometrical interpretation, if we assume the beam represented by the ellipse to be moving in a (field free) drift space. We call the point on the trajectory, in which the ellipse is on principal axes, a <u>"waist"</u>, since the beam width has a minimum there. The distance of the waist from the point under consideration is then given by

<u>"waist distance"</u> $t = \frac{\alpha}{\gamma}$,

where

$t > 0$: waist ahead, i.e. beam "converging"

$t < 0$: waist in rear, i.e. beam "diverging".

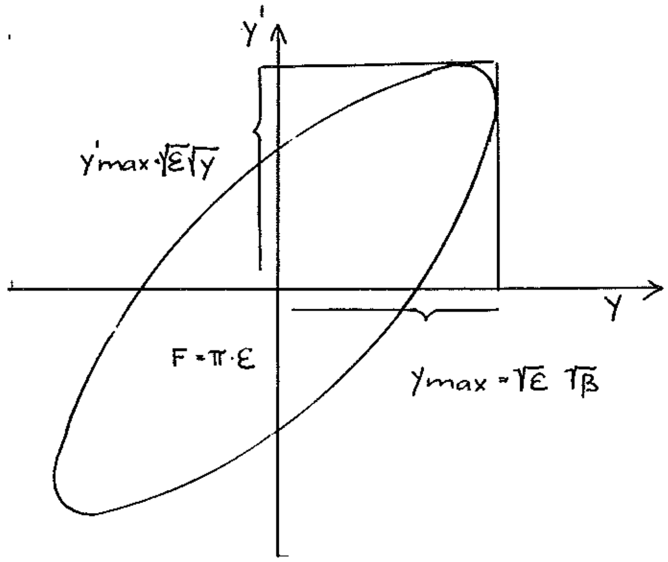

Fig. 1: Phase plane ellipse

Generalized Amplitude and Phase Functions

We shall now investigate how the ellipse parameters β, α and γ transform through the system. The trajectory of an individual particle is a solution of

$$y''(s) \pm K(s) \cdot y(s) = 0 \quad \text{with} \quad K(s) = \frac{e \, \partial B/\partial x}{p} \tag{2}$$

and may be given by the familiar transformation

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} \tag{3a}$$

where

$$c(s) = a'(s) = \frac{da}{ds}$$
$$d(s) = b'(s) = \frac{db}{ds} \tag{3b}$$

Using (3a), the transformation of the ellipse parameters can then be expressed by the linear transformation

$$\begin{pmatrix} \beta(s) \\ \alpha(s) \\ \gamma(s) \end{pmatrix} = \begin{pmatrix} a^2 & -2ab & b^2 \\ -ac & (ad+bc) & -bd \\ c^2 & -2cd & d^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix} \tag{4}$$

where β_0, α_0, γ_0 characterize the initial ellipse.

From (3b), one obtains

$$\alpha(s) = -\frac{1}{2} \cdot \beta'(s) \tag{5}$$

and from the normalization introduced in eq. (1)

$$\gamma(s) = \frac{1 + \alpha^2(s)}{\beta(s)} \tag{6}$$

Thus, all ellipse parameters can be expressed by the amplitude function β and its first derivative. The emittance ε is a constant of the motion, as follows from ad - bc = 1.

If we consider β, α and γ to be known functions of s, the trajectory y(s) of each particle starting from some point on the initial ellipse is a solution of eq. (1) and can be written in the form

$$y(s) = \sqrt{\varepsilon} \cdot \sqrt{\beta(s)} \cos\left(\phi(s) + C\right) \tag{7}$$

$$\text{with} \quad \phi(s) = \phi_0 + \int_{s_0}^{s} \frac{1}{\beta(\tau)} \, d\tau \tag{7a}$$

the constant C being determinded by the initial conditions. ϕ is called the phase function.
The direction of the trajectory is

$$y'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} \left\{ \sin\left(\phi(s)+C\right) + \alpha(s) \cdot \cos\left(\phi(s)+C\right) \right\} \tag{8}$$

Inserting y(s) and its second derivative into eq. (2) yields a differential equation for (s):

$$\gamma(s) + \alpha'(s) \mp K(s) \cdot \beta(s) = 0 \tag{9}$$

Eqs. (9) and (5) can also be obtained by differentiating eq. (1) and eliminating y" by means of eq. (2).

If we express the constants ε and C in eq. (7) by the initial conditions y_0 and y'_0 , the matrix elements a, b, c, d in eq. (3) can be derived in terms of ϕ, β and α :

$$\begin{pmatrix} y(s) \\ \\ y'(s) \end{pmatrix} = \frac{\sqrt{\beta}}{\sqrt{\beta_0}} \begin{pmatrix} \cos(\phi-\phi_0) + \alpha_0 \sin(\phi-\phi_0) & \beta_0 \sin(\phi-\phi_0) \\ \\ -\frac{1}{\beta}\left\{(\alpha-\alpha_0)\cos(\phi-\phi_0) + (1+\alpha\alpha_0)\sin(\phi-\phi_0)\right\} & \frac{\beta_0}{\beta}\left\{\cos(\phi-\phi_0) - \alpha\sin(\phi-\phi_0)\right\} \end{pmatrix} \begin{pmatrix} y_0 \\ \\ y'_0 \end{pmatrix} \tag{3c}$$

The functions ϕ, β, α as introduced here are a slight generalization of the corresponding functions used in AG synchrotron theory. They depend on s, β_0 and α_0 , whereas in the synchrotron they are functions of s only. In the latter case their dependence on β_0 and α_0 is eliminated by the condition that the synchrotron amplitude function β_{syn} be periodic with the machine structure.

In periodic systems, β_{syn} is proportional to the square of the maximum transmissible beam envelope. In non-periodic beam transport systems, however, the maximum ellipse accepted will, in general, be larger and have a shape and orientation quite different from the one defined by β_{syn}. It is found by varying the initial ellipse and using the more general function $\beta(s, \beta_0, \alpha_0)$ instead of β_{syn}, the latter being useful in dealing with periodic structures only.

Similarly,

$$\phi(s, \beta_0, \alpha_0) = \phi_0 + \int_{s_0}^{s} \frac{1}{\beta(\tau, \beta_0, \alpha_0)} d\tau$$

is the generalized phase function. Wherever needed, it can be used for getting information on individual particle trajectories, location of images etc. In general, however, we shall be mainly interested in the behaviour of the beam as a whole, as described by β and its derivative.

Optimizing procedures

The cost of a beam transport system is strongly dependent on the beam cross section within magnets. Special attention should therefore be given to obtaining the optimum shape of the beam envelope. A system satisfying the requirements of a particular experiment will be called optimized, if it is

a) conducting a given emittance at minimum cost
or b) conducting a maximum emittance through a given set of lenses.

For a conventional quadrupole lens with a pole tip diameter $2 \cdot r$, the maximum rectangular beam cross section accepted by the lens has an ares of $2r^2$. This is within limits independent of the ratio of beam width to beam height due to the hyperbolic shape of the pole faces (see figure 2).

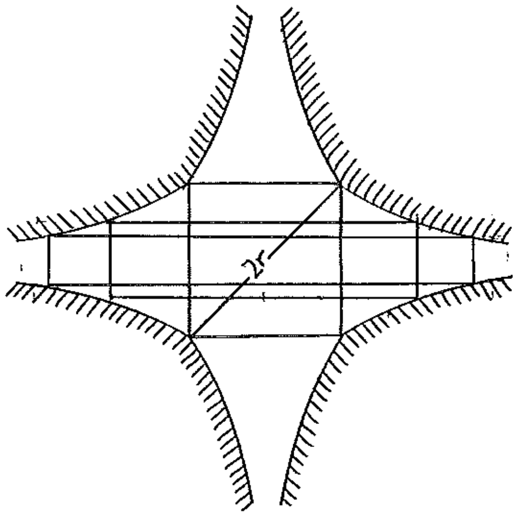

Fig. 2: Quadrupole magnet cross section

Therefore, the cost of the lens is a function of the product $(\beta^{(x)} \cdot \beta^{(z)})$ only, i.e. of the product of horizontal and vertical beam envelope. For the purpose of a rough estimate, the cost may be assumed to be proportional to the magnetic field volume V, normalized by setting $\varepsilon^{(x)} = \varepsilon^{(z)} = 1$.

The volume V is found by computing the maximum value of $\sqrt{\beta^{(x)} \cdot \beta^{(z)}}$ in each magnet, multiplying it by the length L of the magnet and adding these values up for all magnets in the system:

$$V = \sum_i L_i \cdot \left(\sqrt{\beta^{(x)} \cdot \beta^{(z)}} \right)_{max} \qquad (10)$$

V is a function not only of the lens parameters, but also of the initial conditions $\beta_0^{(x)}, \beta_0^{(z)}, \alpha_0^{(x)}, \alpha_0^{(z)}$ describing the beam.

In looking for a minimum value of the field volume V, ohne therefore has to vary the initial beam conditions as well as the lens parameters. In a system with a target as a beam source, the former corresponds to changing the target size and distance and the locations and sizes of beam limiting apertures, while the latter means changing the strengths and locations of lenses.

If the target size is given by y_{max} in one component, and if a certain emittance ε is required for reasons of intensity, the amplitude function at the target is determined by

$$\beta_0 = \frac{y^2_{max}}{\varepsilon}$$

Then,

$$\gamma_0 = \frac{1}{\beta_0} + \frac{\alpha_0^2}{\beta_0}$$

At the target, the divergence $\sqrt{\varepsilon} \cdot \sqrt{\gamma_0}$ has its minimum value for $\alpha_0 = 0$, i.e. for the ellipse being on principal axes. Varying α_0 for instance between the limits

$$-3 \leqq \alpha_0 \leqq +3$$

the divergence varies by more than a factor of three, and the corresponding variety of initial beam conditions can be expected to contain all cases of practical interest.

In principle, the optimizing procedures according to a) or b) can now be described as follows:

a) Vary the lens parameters and initial conditions, calculate the beam characteristics as displayed by $\beta^{(x)}(s)$ and $\beta^{(z)}(s)$ and find a number of solutions which fulfill the special requirements dictated by the experiment. Calculate the field volume V of each of these solutions and choose the one with minimal cost.

b) Proceed as described under a). To each of the solutions satisfying the requirements of the experiment compute the maximum product of horizontal and vertical emittances accepted by the system for the assumed initial conditions. This product is given by

$$\left(\varepsilon^{(x)} \cdot \varepsilon^{(z)}\right)_{max} = \text{minimum of } \left(\frac{r_i^{\,4}}{4\left(\beta_i^{(x)} \cdot \beta_i^{(z)}\right)}\right) \qquad (11)$$

The right hand side refers to the (i^{th}) lens which is the one limiting the aperture of the system. Choose the system with the maximum value of $(\varepsilon^{(x)} \cdot \varepsilon^{(z)})_{max}$.

Methods of computation

The optimizing procedure involves computation of the amplitude function for a large number of parameter combinations, since one does not want to unduly restrict the range of useful solutions by introducing arbitrary symmetry conditions etc. On the other hand one has to keep time and effort for finding practicable solutions within reasonable limits. Therefore, the aid of a computer seems indispensable.

Using an analog computer, a programm of the type described subsequently might be practical:

With the aid of two analog circuits, compute the amplitude functions $\beta^{(x)}$ and $\beta^{(z)}$ by solving the differential equations (9) for both components simultaneously. For this purpose, generate the function K(s), which represents the lens parameters, in a step function generator and feed it with opposite signs into the two circuits. Compute the product function ($\beta^{(x)} \cdot \beta^{(z)}$) by a multiplying circuit, and the phase functions $\phi^{(x)}$ and $\phi^{(z)}$, as defined in eq. (7a), by two additional integrators.

Operate the analog computer repetitively (e.g. a few runs per second) and display the functions $\beta^{(x)}, \beta^{(z)}, (\beta^{(x)} \cdot \beta^{(z)})$ and the lens structure K(s) in four lines on an oscilloscope screen. Alternatively, the functions α, γ and ϕ for both components may be displayed.

Vary the lens parameters and initial beam conditions and look for favourable solutions. It is thought that the displayed functions will allow an immediate estimate of the relative merits of a system.

A printout of lens parameters and initial conditions will permit storage of useful solutions and, at a later time, a recomputation with lower speed, higher accuracy and with a graphical display of the curves characterizing the system.

By this method a large number of parameter combinations can be surveyed quickly. Since the effects of changing a parameter are immediately visible, one expects that an operator with skill and intuition will be able to find close-to-optimum solutions to a given problem within a reasonable time.

With a digital computer, one would use the matrix formulation (4) instead of the differential equation (9) for computation of β s). Setting up a general digital computer program on beam optimizing appears to be rather cumbersome if one wants to include all the boundary conditions and the large number of parameters involved in a given problem. However, one might hope to learn from the analog computer how to introduce simplifying assumptions without loosing solutions of practical interest.