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A QUADRUPOLE MAGNET WITH NON-CIRCULAR APERTURE AND LINEARIZED END FRINGING FIELD

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Abstract

A new type of quadrupole magnet cross section is suggested and compared to the standard circular aperture. It is shown that a significant improvement of acceptance in phase space is gained by this design. Formulae for the acceptance per unit cost are given for the two quadrupole types. A general investigation of the acceptance of a long periodic strong-focussing channel is used as an approach to optimizing standard quadrupole parameters.

Special end field configurations are suggested for quadrupole and for bending magnets. They eliminate to a large extent the nonlinear aberrations which are generally caused by the fringe fields.

Survey of different types of quadrupoles

We have considered several types of quadrupole cross sections, which are shown in figure 1.

No. 1 is a conventional type quadrupole with circular aperture, which has the advantage of a very low power consumption[1].

Since in a strong focussing system the beam cross section within lenses in general has a non-circular shape, one might expect lenses with non-circular aperture to be of economical advantage, because in this case the useful field region can be matched to the cross section of the beam.

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1) B. Langeseth, G. Pluym and B. deRaad, CERN PS/Int. EA 60-5 (1960)

Such a lens is the Panofsky-type quadrupole[2] No. 2, which has a rectangular cross section and a very high power demand due to its limitation in coil space and the unused field regions in its four corners.

No. 3 is the Panofsky-type lens cut off along a straight equipotential line[3], thus saving the unused corner fields and half of the coil cross section. For the same field gradient, this lens therefore needs only half the power of No. 2.

Continuing the line of this evolution, we suggested the lens type No. 4. It is a quadrupole with hyperbolic pole faces, in which the triangular shape of the coil exactly compensates for the distortion caused by cutting off the hyperbola tails. The power consumption is the same as in lens No. 3. The increased aperture allows transmission of the rectangular beam cross section in horizontal and in vertical orientation (see figure 2.).

Lens No. 5 is a modification, developed by Dr. H. Hultschig[4] out of No. 4. It has a simpler coil cross section*) and needs only half the power of No. 4, i. e. one quarter of the power of the Panofsky-type No. 2 for the same field gradient.

Among the lenses with non-circular cross sections, the types 4 and 5 have the advantage over types 2 and 3 of being better adapted to the varying beam cross section within lens systems. We would prefer type 5 as a standard lens because of its relatively low power and technical simplicity.

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2) L. N. Hand and W. K. H. Panofsky, Bull.Am.Phys.Soc. 3/421 (1958)

3) M. H. Blewett and H. S. Snyder, private communication

4) H. Hultschig, private communication

*) A lens with circular aperture and rectangular coil cross section has been designed for the BNL by Dr. M. H. Blewett, who kindly made available to us a set of her drawings.

In searching for an optimized standard lens design, we shall therefore restrict ourselves to a comparison between the

circular aperture, low power lens No. 1 and the
non-circular aperture, medium power lens No. 5.

In the following, we shall describe an attempt to quantitatively assess the relative merits of these lenses as a function of their parameters. Only some basic considerations will be given rather than numerical results, which depend on local conditions such as manufacturing and power cost, experimental floor facilities, maximum crane load etc. and therefore do not have much general meaning.

Figure of merit of a lens system

The figure of merit determining the economy of a lens system is its acceptance ε^2, divided by its cost C:

$$\text{Figure of merit:} \quad \frac{\varepsilon^2}{C}$$

ε^2 is the transverse phase volume accepted by the system, i. e. the maximum target cross section times the maximum solid angle.

The acceptance ε^2 depends strongly on the optics of the system under consideration. Therefore, it cannot be used as a characteristic number for a single lens unless it is referred to a special system which serves as a representative system for a number of applications.

Acceptance of a strong focussing channel of infinite length

We feel that the periodic FODO strong focussing channel of infinite length[5] shown in figure 2 might be such a representative system.

It has the advantage of being described by only three structure parameters and it turns out that its acceptance can be expressed as a simple function of these parameters. They are

2ℓ = length of lenses,

$k = \frac{eg}{p} \sim$ field gradient of lenses, divided by particle momentum,

$f = \frac{2\ell}{2\cdot\ell + L}$ = filling factor = ratio of lens length to total length.

Lens type 5:

With the restrictions that $k \cdot \ell^2 \lessapprox \frac{1}{2}$ and $f \lessapprox \frac{1}{3}$, the acceptance ε^2 of this system for the lens type 5 may be expressed to very good approximation by the following formula:

$$\varepsilon_5^2 = \frac{1}{4} a^4 \cdot k^2 \cdot \ell^2 \cdot (1 - \frac{2}{3} f) \qquad (1)$$

Here, a is the radius of the inscribed circle of the lens aperture. This formula has been derived from

5) M. G. N. Hine, CERN PS-MGNH/Note 22 (1958)

$$\varepsilon^2 = \frac{1}{4} a^4 k \cdot \frac{1 - tg\,\psi(\zeta \cdot tg\,\psi + \frac{L}{\ell}\,\psi)}{1 + \nu\, tg\,\psi(tg\,\psi + \frac{L}{\ell} \cdot \psi)}, \text{ where } \psi = \sqrt{k} \cdot \ell \quad (1a)$$

by expanding the trigonometric functions, including terms to ψ^6. Introducing the beat factor m, defined as the ratio of beam width to beam hieght in the centre of the horizontally focussing lenses, one has approximately

$$k \cdot \ell^2 \cdot \frac{1 - \frac{2}{3} f}{f} \approx \frac{1}{2} \cdot \frac{m^2 - 1}{m^2 + 1}$$

and therefore

$$\varepsilon_5^2 \approx \frac{1}{8} a^4 \cdot k \cdot f \cdot \frac{m^2 - 1}{m^2 + 1} \quad (1b)$$

$$= \frac{1}{8} a^4 \cdot k \cdot f \cdot \frac{M^2 - 1}{M^2 + 1} \quad \text{for } m = M,$$

where M is the aperture ratio (see fig. 2). The function

$$\frac{M^2 - 1}{M^2 + 1}$$

is shown graphically in figure 3. We conclude that M = 4 would be a reasonable choice for the aperture ratio, since a further increase of M would not significantly increase the acceptance any more.

Lens type 1:

In order to simplify computations, we shall assume for the lens type 1 from now on a quadratic aperture inscribed in the circle of radius a. Its acceptance is then given by

$$\varepsilon_1^2 = \frac{1}{m^2} \cdot \frac{1}{4} a^4 \cdot k^2 \cdot \ell^2 \left(1 - \frac{2}{3} f\right) \qquad (2)$$

$$\varepsilon_1^2 \approx \frac{1}{8} a^4 \cdot k \cdot f \cdot \frac{1}{m^2} \cdot \frac{m^2 - 1}{m^2 + 1} \qquad (2a)$$

The function

$$\frac{1}{m^2} \cdot \frac{m^2 - 1}{m^2 + 1}$$

has a maximum at $m = \sqrt{1 + \sqrt{2}} = 1.55$.

This means that for maximum acceptance with given parameters k and f the lens length $2 \cdot \ell$ should be chosen such as to produce a beat factor $m \approx 1.5$.

For constant ℓ and f, on the other hand, the acceptance ε_1^2 is proportional to

$$\frac{1}{m^2} \left(\frac{m^2 - 1}{m^2 + 1}\right)^2 ,$$

which has a maximum for $m = \sqrt{2 + \sqrt{5}} = 2.06$.

In order to achieve maximum acceptance in this case, the field gradient should be chosen according to a beat factor $m \approx 2$.

Thirdly, for constant k and ℓ and a variable filling factor f, one finds that the optimum value of the beat factor m depends on $k \cdot \ell^2$ and is for instance given by $m \approx 1.24$ for $k \cdot \ell^2 = 0.25$ and by $m \approx 1.35$ for $k \cdot \ell^2 = 0.5$.

Comparison:

The acceptances of the two kinds of lenses with equal parameters a, k and f, M = 4 and optimum ℓ_5 and ℓ_1, respectively, compare as follows:

$$\varepsilon_5^2 \approx 5 \cdot \varepsilon_1^2$$

This factor of five may be regarded to represent the average gain in acceptance due to increasing the lens aperture from the quadratic to the rectangular shape with an aperture ratio M = 4. One has to remember, that this factor would be somewhat smaller for a circular useful lens aperture instead of the square aperture assumed above for lens type No. 1.

Choice of quadrupole type

The cost per meter of the system may be written as

$$C = a^2 \cdot f \cdot (A + B\, g^2) \qquad (3)$$

with g being the field gradient, where A is the manufacturing cost per meter lens, normalized for a = 1, and B is the normalized cost of the power supply and of the consumed power, integrated over the life of the lens. A and B are assumed to be constants for a given type of lens. For the two lens types considered above, we have found approximately the ratios

$$\frac{A_5}{A_1} \approx 0.8 \quad ; \quad \frac{B_5}{B_1} \approx 3$$

which are based on current German cost figures.

Writing

$$k = \frac{3}{p} \cdot g \quad \text{with p in GeV/c and g in kGauss/cm,}$$

the figure of merit for each lens type can now be written as follows:

$$\frac{\mathcal{E}_5^2}{C_5} \approx \frac{3}{8} a^2 \frac{1}{p} \frac{M^2 - 1}{M^2 + 1} \cdot \frac{g}{A_5 + B_5 \cdot g^2} \tag{4}$$

$$\frac{\mathcal{E}_1^2}{C_1} \approx \frac{3}{8} a^2 \cdot \frac{1}{p} \cdot \frac{1}{m^2} \cdot \frac{m^2 - 1}{m^2 + 1} \cdot \frac{g}{A_1 + B_1 \cdot g^2} \tag{5}$$

In order to compare the economical merit of the two lens types, an assumption has to be made about the average gradient at which the lenses will be operated. Denoting this average gradient by $\overline{g}$, the ratio of the figures of merit is given by

$$\frac{\varepsilon_5^2}{C_5} \bigg/ \frac{\varepsilon_1^2}{C_1} \approx 5 \cdot \frac{A_1 + B_1 \cdot \bar{g}^2}{0.8\, A_1 + 3 \cdot B_1 \cdot \bar{g}^2} \qquad (6)$$

where $m_1 \approx 1.5$ and $M = 4$ has been assumed.

Inserting a $\bar{g}$ of 0.45 kGauss/cm in equation (6), the lens type 5 appears to be economically slightly superior in our case, although we have very high power costs.

There are some more arguments favoring lens type 5:

1. It is easier to achieve a precise field distribution over the entire aperture if M is increased.

2. The end fringing fields can be shaped more precisely for larger M.

3. A larger aperture in one dimension allows a more efficient use of bending magnets, since the dispersive power of such a magnet increases linearly with beam width.

Choice of quadrupole radius

The lens radius a should be chosen as large as possible for maximum economy, as is apparent from equation (4). Since the figure of merit is proportional to the square of a, it pays to increase the radius even into the region where the maximum achievable gradient becomes inversely proportional to a due to saturation in the iron. For M = 4, a radius of 14 cm allows transmission of a beam cross section of 10 x 40 cm^2 at a maximum gradient of close to 1 kGauss/cm, and it may not seem unreasonable to build even larger lens apertures at reduced maximum gradients for special purposes.

Choice of quadrupole length

In order to choose the length 2ℓ of the standard lens, one might start from an average lens strength

$$2 g \cdot \ell = s = \text{const}$$

and minimize the cost for achieving this lens strength. Since the cost is proportional to

$$\ell \cdot (A + Bg^2)$$

the cost minimum is given at

$$g^2 = \frac{s^2}{4\ell^2} = \frac{A}{B} . \qquad (7)$$

An alternative approach consists in finding the optimum length of a system with given optical properties.

When scaling all the lengths in the system proportionally, the optical properties remain unchanged if all the products $g \cdot \ell^2$ are kept constant. The acceptance of the system then changes proportional to $\frac{1}{\ell^2}$, i. e. proportional to g. Therefore, the figure of merit for such a system is proportional to

$$\frac{\mathcal{E}^2}{C} \sim \frac{g}{\ell \cdot (A + Bg^2)} \sim \frac{g^{3/2}}{A + Bg^2} \quad \text{for } g \cdot \ell^2 = \text{const},$$

where C now is the total cost of the system.

In this case, the optimum is given at

$$g^2 = 3\,\frac{A}{B} \qquad (8)$$

As compared to equation (7), this result recommends a larger gradient and therefore, in general, a shorter length of the quadrupoles. However, standard lenses should always be chosen long enough in order to keep fringe field distortions and power losses in the coil ends within permissible limits.

Shaping of end fringe fields

Shaping or shimming of magnet pole ends is done in order to avoid saturation[6] or to improve the optical properties of the fringe fields[1, 7, 8, 9]. For the latter purpose, one usually demands that

the "magnetic length" $\int B_z \, ds$ for a bending magnet and

the "gradient length" $\int \frac{\partial B_z}{\partial x} \, ds$ for a quadrupole,

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6) W. Hardt, DESY-Bericht A 1.5 (1959)
7) P. Crivet and A. Septier, CERN 58-25 (1958)
8) C. A. Ramm, Proceedings of the Berkeley Conference 1960
9) M. Foss, private communication

integrated through the fringe field, should be constant over the entire aperture.

It is possible, however, to design fringe fields according to a more stringent but in principle more satisfactory requirement, by demanding that throughout the fringe field region

1. for a bending magnet, the field component in every plane parallel to the magnet face should have a homogenous distribution, i. e.

$$B_z = \text{const.}\ f(s)\ ; \qquad (9a)$$

2. for a quadrupole, the field component in every plane parallel to the magnet face should have a quadrupole distribution, i. e.

$$\begin{aligned} B_x &= \text{const.}^* \cdot z \cdot h(s) \\ B_z &= \text{const.}^* \cdot x \cdot h(s) \end{aligned} \qquad (10a)$$

with $f(s)$ and $h(s)$ being functions of s only.

It is obvious that such linear transverse fringe field configurations automatically provide constant "magnetic length" and "gradient length", respectively.

It follows from Maxwells equations that the only fringe fields compatible with the above requirements are

$$\begin{cases} B_z = g \cdot s \\ B_s = g \cdot z \end{cases} \qquad (9)$$

for the bending magnet, and

$$\begin{cases} B_x = g^* \cdot s \cdot z \\ B_z = g^* \cdot s \cdot x \\ B_s = g^* \cdot x \cdot z \end{cases} \qquad (10)$$

for the quadrupole.

The field given by equation (9) is a two-dimensional quadrupole field with its axis parallel to the magnet face.

The field given by equation (10) is a three-dimensional quadrupole field with one of its six semi-axes coinciding with the axis of the quadrupole.

In these fields, the equations of motion assume an especially simple form since most of the nonlinear terms, which are usually introduced by the fringe fields, vanish. This means that non-linear aberrations caused by the fringe fields are to a large extent eliminated.

For example, the equations of motion within an ideal quadrupole are given by

$$\begin{aligned} \ddot{x} + v\,\dot{s}\,k\,x &= 0 \\ \ddot{s} - \frac{1}{2}\,v\,k\,\frac{d}{dt}\,(x^2 - z^2) &= 0 \end{aligned} \qquad (11)$$

where x, z = transverse deviations from quadrupole axis

$\ddot{x}$ = $\frac{d^2x}{dt^2}$

v = particle velocity

k = $\frac{e \cdot g}{p}$

p = momentum

g = field gradient

On the other hand, in the three-dimensional quadrupole fringe field (10) the equations of motion are

$$\ddot{x} + v\,\dot{s}\,k^*\,s\,x = \frac{e}{m} B_s \dot{z}$$
$$\ddot{s} - \frac{1}{2} v\,k^*\,s\,\frac{d}{dt}(x^2 - z^2) = 0 \qquad (12)$$

As compared to equations (11), the constant k has been replaced by (k*s) here, and only one additional nonlinear term has appeared, which is due to the longitudinal field component B_s in the fringe field region.

End configuration for bending magnets

The end configuration suggested for shaping the fringe field of a "window-frame" bending magnet is shown in figure 4. The pole ends are rounded off along an equipotential surface $z \cdot s$ = const. The two-dimensional quadrupole fringe field (9) is terminated on the outside by a magnetic mirror surface, located at $s \cong 0$, and on the inside by a break in the pole contour which leads from the hyperbolic part to the parallel part inside the magnet. This break as well as the central hole in the magnetic mirror plate, needed for passing the beam, causes some distortion of the ideal quadrupole end field configuration. However, preliminary analog measurements[4], using a stainless steel plate model[10], have shown that the field rises essentially linearly between the mirror plate and the pole break for a geometry as shown in figure 4. In this geometry, the arrangement of coil ends aids in generating a fairly undistorted quadrupole fringe field. Rounding the break in the pole contour would give some further improvement.

The end configuration of figure 4 closely resembles a design which has been developed by Dr. M. Foss using a more empirical approach[9].

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10) S. van der Meer, private communication

End configuration for quadrupole magnets

The end configuration suggested for shaping the end field of the quadrupole type 5 is shown in figure 5. The pole faces are rounded off along an equipotential surface $x \cdot z \cdot s = \text{const}$. The three-dimensional quadrupole fringe field (10) is terminated on the outside by a magnetic mirror surface, located at $s \cong 0$, and on the inside by a break in the pole contour which leads from the rounded pole end to the inner part of the quadrupole. This break in the pole as well as the central hole in the magnetic mirror plate, which has a cross section equal to the quadrupole aperture, again causes some distortion of the ideal three-dimensional quadrupole end field configuration. Rounding the break in the pole contour would give some improvement.

In the geometry of figure 5, the arrangement of coil ends aids in generating a fairly undistorted three-dimensional quadrupole fringe field.

Another advantage of this end configuration, applying to bending magnets as well, consists in a reduction of saturation effects due to the rounded pole ends.

Acknowledgements

We gratefully wish to acknowledge that the basic principle applied in designing the quadrupole cross section is due to discussions, informations and drawings of Dr. M. H. Blewett[11] of the Brookhaven National Laboratory. Using her method, computations on various types of quadrupole cross sections have been carried out by Dr. D. Lublow[12], whom we wish to thank for these contributions. We are indebted to Dr. W. Jentschke and Dr. P. Staehelin for their continuing interest and encouragement, and to Drs. T. Collins and P. Cooper of the CEA for stimulating discussions.

11) M.H. Blewett, BNL-Report MHB-8 (1957)
12) D. Lublow, DESY-Notiz A 2.78 (1961)

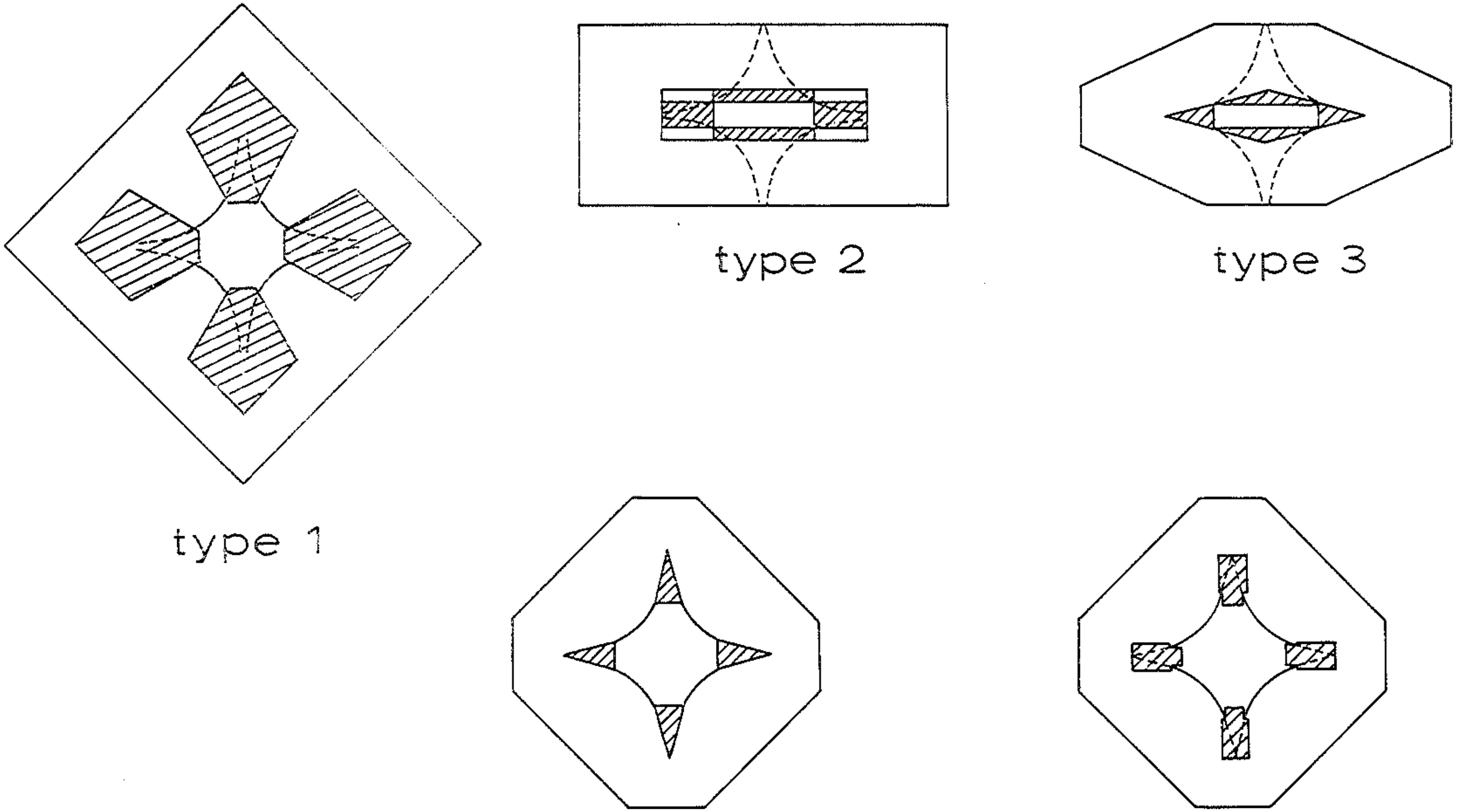
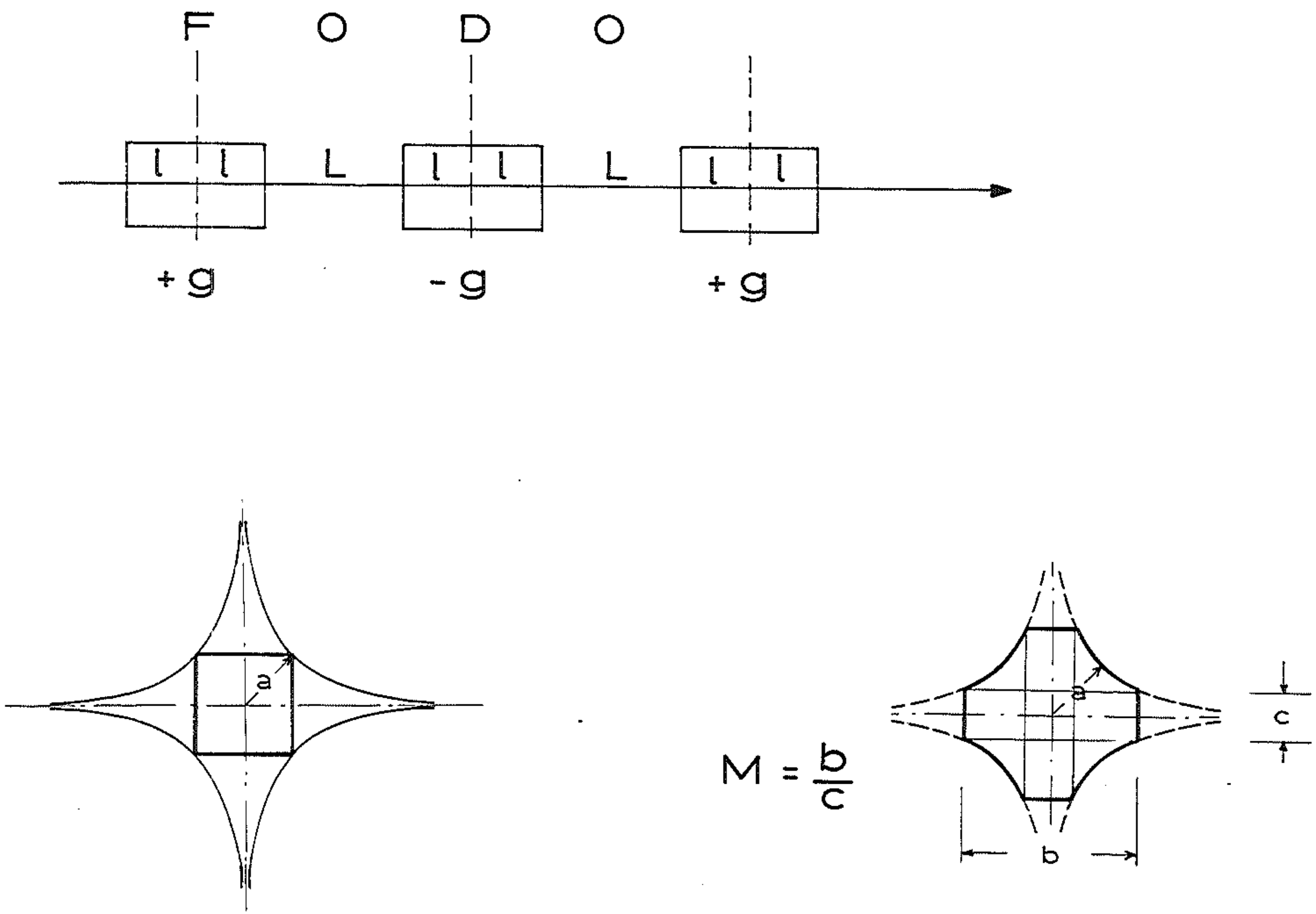


FIG. 1



type 1

type 5

FIG. 2

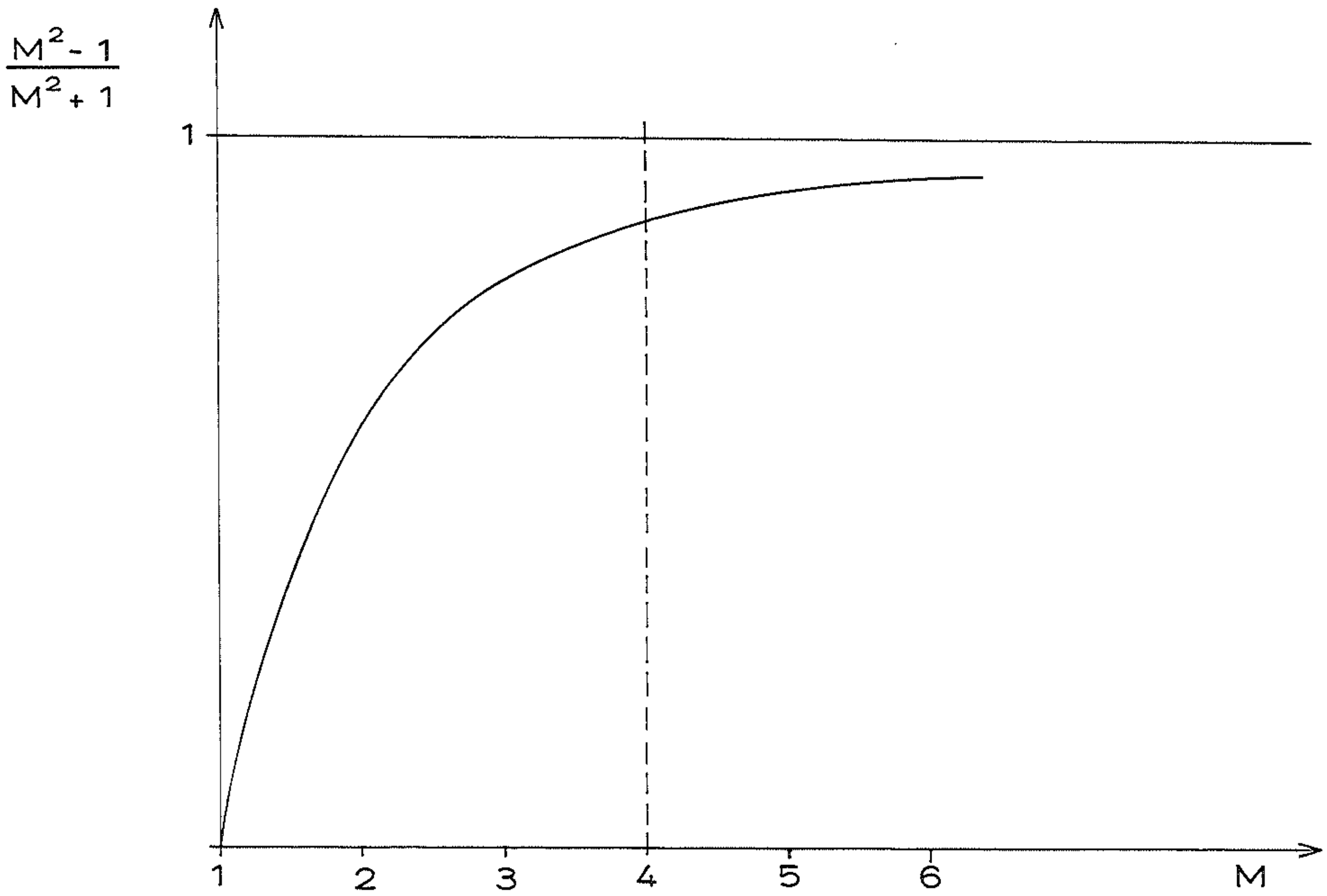
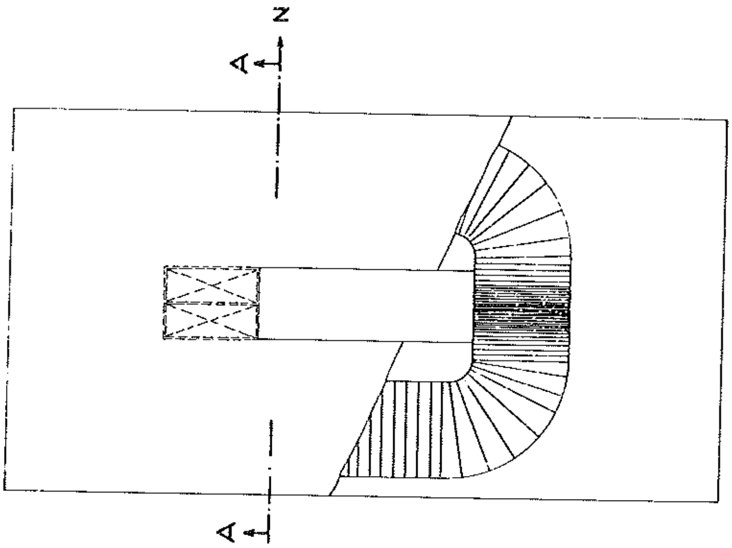
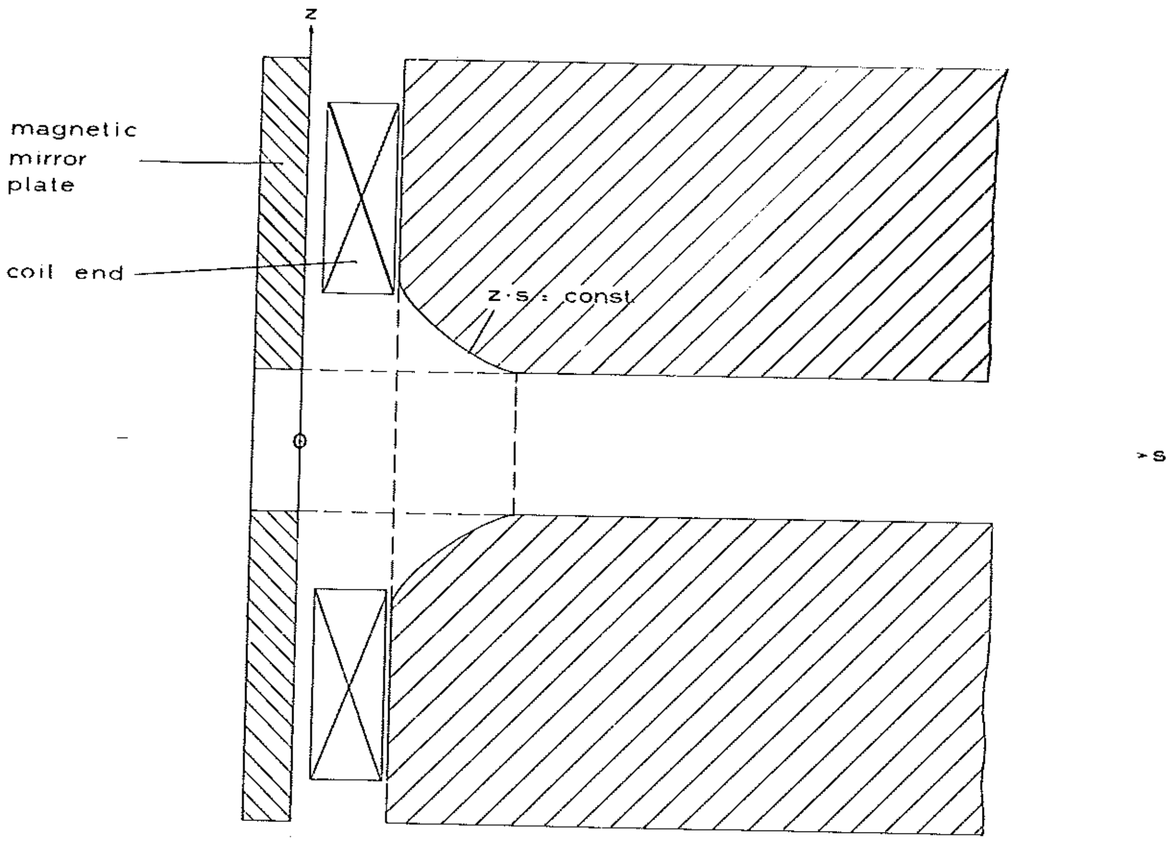


FIG. 3

FIG. 4

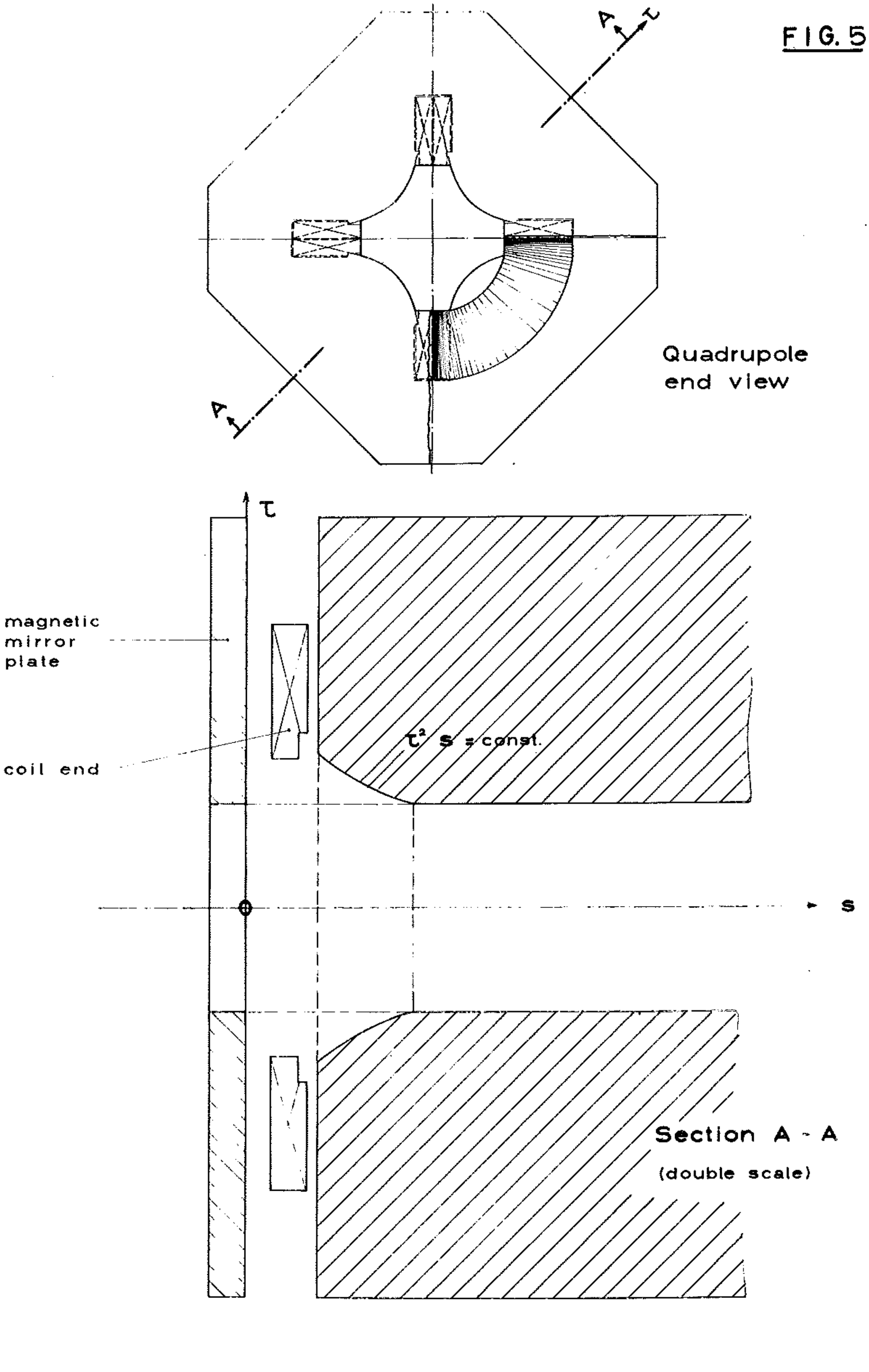


Magnet
end view



Section A-A
(double scale)

FIG. 5


A
τ
Quadrupole end view
τ
magnetic mirror plate
coil end
$\tau^2 s = const.$
s
Section A - A
(double scale)

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