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Part I - The Integration of the Jost, Luttinger and Slotnick Formula for σ_T

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The Total Pair Production Cross-Section in Hydrogen and Helium

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Abstract

The total pair production cross-section is evaluated using the formula of Jost, Luttinger, and Slotnick, for the elements Hydrogen and Helium. The accuracy of the work is 0.1%.

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Introduction

The absorption of photons by pair production has been treated by several authors.[1] Regrettably the theoretically most accurate work, that of Jost, Luttinger, and Slotnick (JLS) has up to now, not been evaluated. The JLS calculation involves no approximations, and is good for all photon energies. The formula of Bethe and Heitler, neglects electron screening of the nucleus, and is therefore good only at small photon energies (below 50 MeV). The formula of Bethe has approximations of the order 1/k and is thus only good at high energies (greater than 10 GeV).

In this paper we present numerical evaluations of the JLS formula, for the cases of Hydrogen and Helium. The precision of this work is 1 part in 1000. In both the high energy, and low energy limits the values obtained agree with the Bethe and Bethe-Heitler results respectively providing a valuable check on the work. In the intermediate region of photon energies the JLS formula should provide the most accurate values for the total pair cross-sections currently available. This formula has been recently verified to ± 0.3% precision in the region 1 to 4 GeV photon energy.[2]

EVALUATION OF THE JOST, LUTTINGER AND SLOTNICK FORMULA

The Basic Formula

JLS, by a covariant calculation, utilizing the unitarity of the S matrix, obtain for the total pair production cross-section

$$\sigma(k_o) = \int_{K - (K^2 - 1)^{1/2}}^{K + (K^2 - 1)^{1/2}} dQ\, P(Q, K) \qquad (1)$$

with $K = k_o/2m$

k_o = incident γ energy in MeV

$m = m_e c^2$, the electron rest mass.

The integrand is

$$P(Q, K) = 2 \frac{Z^2 r_o^2}{137 K^2} \left(\{1 - F(Q)\}^2 \frac{I(Q, K)}{Q^3}\right) \qquad (2)$$

where

Z = atomic charge

r_o = classical electron radius = $e^2/m_e c^2$

$F(Q)$ = coherent atomic scattering function

Q = momentum transfer in units of $2m_e c$

K = energy, in units of $2m_e c^2$

and finally, $I(Q, K)$ is given by JLS as:

$$I(Q, K) = (1 - 2Q^2) J_1 + (1 - 4Q^2 - 8QK + \frac{4Q^2 - 1}{3KQ}) \times$$

$$\ln\left[(y)^{1/2} + (y - 1)^{1/2}\right] + (3 + \frac{2K}{3Q} + \frac{2Q^2 - 1}{3KQ}) \left(y(y - 1)\right)^{1/2} \times$$

$$+ \{-2(1 + Q^2) + \frac{2K^2}{3} (-4 + \frac{1}{Q^2})\} \times \left(\frac{1}{1 + \frac{1}{Q^2}}\right)^{1/2} \times$$

$$\ln \frac{(1 + 1/Q^2)^{1/2} - (1 - 1/y)^{1/2}}{(1 + 1/Q^2)^{1/2} + (1 - 1/y)^{1/2}} \quad . \qquad (3)$$

With

$$J_1 = -R(1/Z\lambda) - R(\lambda/Z) + \frac{\pi^2}{6} + \frac{1}{2}[\ln (\lambda)]^2$$

$$+ \frac{1}{2}(\ln Z)^2 - (\ln Z) (\ln 8KQ)$$

$$Z = [(y - 1)^{1/2} + y^{1/2}]^2$$

$$\lambda = [Q + (Q^2 + 1)^{1/2}]^2$$

$$y = 2KQ - Q^2$$

$$R = R(t) = \int_o^t \ln(1 + x) \frac{dx}{x}$$

The Approximate Expression for R(t)

We see that the expansion of the integrand of R(t) for small x and integration yield the formula

$$R(t) = t - t^2/4 + t^3/9 - t^4/16 + \text{etc.} \tag{4}$$

By evaluating Eq.(1) for 2, 3 and 4 terms in R(t), it was possible to determine the error introduced by the approximate R equation. These results are presented in Table I. The error in the use of R(t) will be the dominant error in the value of σ_T at low photon energies, where a value of as much as 5% error could be obtained at 2 MeV. However, above 4 MeV the error introduced is less than 1%, and above 20 MeV less than 0.1%. Thus we are justified in the use of a 4-term equation for R(t) in the subsequent work; bearing in mind the precision quoted above.

The Approximate Expression for I (Q, K) when Q << K

In the region of $Q \ll K$, JLS give an approximate formula:

$$I\,(Q,\,K) = (1 - 2Q^2)\,J_1 + \frac{1}{2}\,(1 - 4y - \frac{2}{3y})\,\ell n\,Z + $$

$$+ \frac{y^2(1 - 1/y)^{1/2}}{3}\left[11 - \frac{13}{3}\,(1 - \frac{1}{y}) - 2(1 - \frac{1}{y})^2\right] \tag{5}$$

This is of value in checking any numerical evalutation of I(Q, K) as formula (3) is difficult to compute accurately for $Q \ll K$ due to the cancellation of terms. By evaluation of both Eq.(3) and (5) over the entire Q range using 16 significant figures in each calculation, it was determined that for Q of the order of 4/K that Eq.(3) and (5) agreed to at least 5 significant figures. Thus Eq.(3) only was used in a numerical integration of Eq.(1) with full confidence it accurately represents the JLS recoil momentum distribution.

Tables of σ_T from the JLS Formula

The end result of this work is to prepare tables of the cross-section predicted by the JLS formula for various energies, along with an indication of the calculational error. In order to make a realistic comparison with other formulae and with experimental data we include the screening effect, in particular we shall use the exact Hydrogen atom screening function, and two different screening functions for Helium. These are the radially correlated and uncorrelated wave functions explained in detail in Reference (3).

The Equation (1) was integrated numerically by applying the definition of the integral as a summation, and increasing the number of terms in the summation until the precision was at least 1 part in 1000 for the value σ_t . An equal number of steps were taken in variable Q for Q values below 1 MeV/c, and above this point. This is because the integrand is roughly constant up to $Q = 1$ MeV/c, but falls approximately as $1/Q^2$ beyond 1 MeV/c. Figure 1 presents the error with step no. for various cases. Table II presents the values of σ_T at various energies, for atomic Hydrogen and Helium. It is interesting to note that at high energies the JLS evolution agrees with a value of σ_T obtained for the case of coherent and incoherent atomic Hydrogen screening, respectively obtained from the integrated Bethe formula.[2] The precision of the comparison is 1 part per 1000.

The results of the Bethe formula, which is exact in the high energy limit, were obtained in two different ways, once by exact evaluation of the Bethe formula at 10^9 MeV by computer calculation, and once by an analytical integration of the Bethe formula in the high energy limit. Both methods agreed to much better than one part per thousand. At lower energies the JLS and Bethe formulae give different results. As the Bethe formula is not expected to be highly accurate at lower energies we attribute this error solely to the Bethe formula. In fact the error goes approximately as 1/k as mentioned by Bethe. At very low energies the screening terms are entirely negligible and comparison of the JLS with the integrated Bethe-Heitler formula without screening can be done. To within about .1% at

3 MeV these formulae do agree. We thus conclude that the JLS formula provides a method of obtaining reliable total cross section values. The Table II is useful, however, only to compare theoretical predictions. Certain small corrections have yet to be discussed which are necessary for comparison with experimental data. Part II of this paper will be devoted to the discussion of these points and will allow total cross sections to be computed to approximately 0.3% precision for comparison with experimental data. However above 1 GeV photon energy, the given σ_T values are precise to 1% if the radiation correction (computed by Mork and Olsen[4]) of + 0.9% of σ_T is added to the values of Table II.

References

1. H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London), A146, 83 (1934).
H. A. Bethe, Proc. Camb. Phil. Soc. 30, 524 (1934)
J. Jost, J. M. Luttinger, and M. Slotnick, Phys. Rev. 80, 189 (1950).

2. T. M. Knasel, Ph. D. Thesis Harvard University (unpublished).
T. M. Knasel and J. K. Walker, Phys. Let. 27B, 103 (1968)

3. T. M. Knasel, Phys. Rev. 171, 1643 (1968) and Phys. Rev. 179, 1632 (1968)

4. K. Mork and H. Olsen, Phys. Rev. 140, B1661 (1965), and Phys. Rev. 166, 1862 (1968).

TABLE I

CROSS SECTION EVALUATED FOR THE CASE OF COHERENT PAIR PRODUCTION IN HYDROGEN

(Exact atomic wave function used in screening correction.) The Jost *et al* formula with 2, 3 and 4-term expansion of $R(t)$ are compared. $\Delta(ij) = |\sigma(i \text{ term } R) - \sigma(j \text{ term } R)|$.

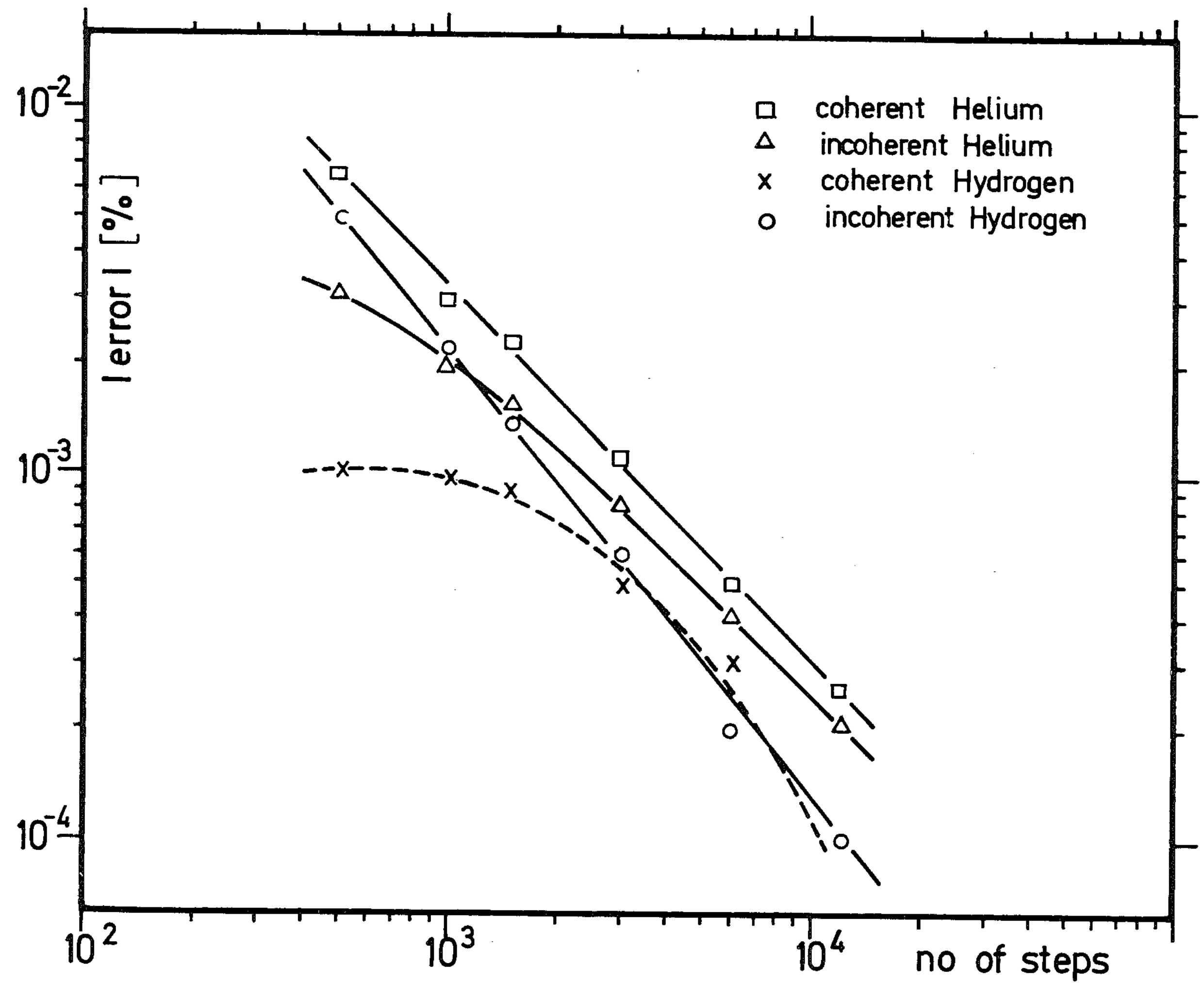
Energy MeV	$\Delta(32)/\sigma(3)$	$\Delta(43)/\sigma(4)$
2	.08	.05
4	.02	.007
6	.01	.004
10	.005	.002
20	.003	.001
40	.002	.0008
100	.0006	.0002
1000	$<10^{-6}$	$<10^{-6}$

Table Captions

Table II: Total cross section in millibarns for Hydrogen (H) [T = total cross section = coherent (C) + incoherent (I)] and Helium (He1) correlated wave function, and (He2) uncorrelated wave function. K is photon energy.

Figure Caption

% error on step size for integration of JLS formula for various cases.



coherent Helium
incoherent Helium
coherent Hydrogen
incoherent Hydrogen
| error | [%]
10^{-2}
10^{-3}
10^{-4}
10^{2}
10^{3}
10^{4}
no of steps

units=millibarns/atom

k MeV	$\sigma_T H$	$\sigma_T He(1)$	$\sigma_I He(2)$	$\sigma_C H$	$\sigma_I H$	$\sigma_C He(1)$	$\sigma_I He(1)$	$\sigma_C He(2)$	$\sigma_I He(1)$
2.0	0.394	1.183	1.183	0.197	0.197	0.789	0.394	0.789	0.394
3.0	1.011	3.034	3.034	0.506	0.506	2.023	1.011	2.023	1.011
4.0	1.630	4.890	4.890	0.815	0.815	3.260	1.630	3.260	1.630
5.0	2.189	6.566	6.566	1.094	1.094	4.377	2.189	4.377	2.189
6.0	2.685	8.056	8.056	1.343	1.343	5.371	2.685	5.371	2.685
7.0	3.129	9.386	9.386	1.564	1.564	6.257	3.129	6.257	3.129
8.0	3.527	10.582	10.582	1.764	1.764	7.055	3.527	7.055	3.527
9.0	3.889	11.666	11.666	1.944	1.944	7.777	3.889	7.777	3.889
10.0	4.219	12.656	12.656	2.109	2.109	8.437	4.219	8.437	4.219
20.0	6.509	19.521	19.522	3.254	3.254	13.012	6.509	13.013	6.509
30.0	7.909	23.707	23.711	3.954	3.955	15.798	7.910	15.801	7.910
40.0	8.917	26.703	26.710	4.458	4.460	17.784	8.918	17.792	8.919
50.0	9.704	29.018	29.030	4.850	4.854	19.312	9.706	19.323	9.707
60.0	10.348	30.889	30.907	5.170	5.178	20.539	10.350	20.555	10.352
70.0	10.892	32.449	32.470	5.440	5.452	21.554	10.894	21.574	10.897
80.0	11.361	33.775	33.801	5.671	5.690	22.412	11.364	22.434	11.367
90.0	11.774	34.923	34.952	5.873	5.900	23.148	11.775	23.172	11.780
100.0	12.140	35.928	35.960	6.052	6.088	23.788	12.141	23.814	12.146
200.0	14.447	41.894	41.928	7.142	7.305	27.468	14.426	27.494	14.433
300.0	15.651	44.744	44.764	7.677	7.974	29.140	15.604	29.158	15.606
400.0	16.415	46.459	46.465	8.001	8.414	30.114	16.344	30.125	16.340
500.0	16.949	47.619	47.613	8.220	8.729	30.759	16.860	30.765	16.848
600.0	17.347	48.464	48.447	8.380	8.967	31.221	17.242	31.222	17.225
700.0	17.657	49.109	49.084	8.502	9.156	31.570	17.539	31.567	17.517
800.0	17.907	49.621	49.588	8.598	9.309	31.844	17.777	31.837	17.750
900.0	18.112	50.037	49.997	8.676	9.436	32.065	17.973	32.056	17.942
1000.0	18.284	50.384	50.338	8.741	9.543	32.247	18.137	32.236	18.102
2000.0	19.181	52.138	52.058	9.068	10.113	33.152	18.985	33.128	18.929
3000.0	19.545	52.827	52.732	9.195	10.350	33.499	19.328	33.470	19.262
4000.0	19.747	53.204	53.100	9.264	10.483	33.686	19.518	33.654	19.446
5000.0	19.878	53.444	53.335	9.308	10.570	33.804	19.640	33.770	19.564
6000.0	19.970	53.612	53.498	9.339	10.631	33.886	19.726	33.851	19.647
7000.0	20.038	53.737	53.620	9.362	10.676	33.947	19.790	33.911	19.709
8000.0	20.091	53.833	53.713	9.379	10.712	33.994	19.840	33.957	19.757
9000.0	20.134	53.910	53.788	9.393	10.741	34.031	19.879	33.993	19.795
100000.0	20.503	54.565	54.425	9.511	10.992	34.342	20.223	34.299	20.126
1000000.0	20.552	54.649	54.506	9.526	11.026	34.380	20.268	34.337	20.169
10000000.0	20.559	54.660	54.516	9.528	11.030	34.385	20.274	34.341	20.175
100000000.0	20.559	54.661	54.517	9.528	11.031	34.386	20.275	34.342	20.175
1000000000.0	20.559	54.661	54.518	9.529	11.031	34.386	20.275	34.342	20.176

TABLE 2