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Quantization of the Electromagnetic Potential in Asymptotically Flat Spacetimes

Diploma Thesis

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They knew many things but had no idea why. And strangely this made them more, rather than less, certain that they were right.

— Neal Stephenson, *Anathem*

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Abstract

We study the quantization of the vector potential in asymptotically flat, globally hyperbolic spacetimes. To this end, we will extend the notion of general local covariance to account for the conformal invariance of conformal quantum fields. Subsequently we discuss the field algebra of a locally conformally covariant quantum field. Moreover, we develop a generalized bulk to boundary correspondence in asymptotically flat spacetimes for an abstractly defined, conformal quantum field. Thereby we construct a Hadamard state in the bulk spacetime by pulling-back a state defined on past null infinity. Equipped with these general results, we quantize the vector potential as a locally conformally covariant quantum field and, using the bulk to boundary correspondence, construct a Hadamard state on the field algebra of the vector potential.

Zusammenfassung

Diese Arbeit beschäftigt sich mit der Quantisierung des Vektorpotentials in asymptotisch flachen, global hyperbolischen Raumzeiten. Zu diesem Zweck wird der Begriff der allgemeinen lokalen Kovarianz erweitert, um die konforme Kovarianz von konformen Quantenfeldern zu berücksichtigen. Anschließend wird die Feldalgebra eines lokal konform kovarianten Quantenfeldes behandelt. Außerdem wird eine verallgemeinerte holographische Korrespondenz in asymptotisch flachen Raumzeiten für ein abstrakt definiertes, konformes Quantenfeld erarbeitet. Damit erhält man einen Hadamardzustand auf der physikalischen Raumzeit durch Zurückziehen eines Zustandes, der im lichtartig Unendlichen definiert ist. Diese allgemeinen Resultate ermöglichen eine Quantisierung des Vektorpotentials als ein lokal konform kovariantes Quantenfeld. Unter Benutzung der holographischen Korrespondenz wird schließlich ein Hadamardzustand auf der Feldalgebra des Vektorpotentials konstruiert.

Contents

Introduction

A key ingredient of the usual approach to *quantum field theory* on Minkowski spacetime is the symmetry group of Minkowski spacetime, the Poincaré group. In particular, the Poincaré group is used to select a distinguished state, called the *vacuum*, which is the unique state invariant under the action of the Poincaré group [1]. However, even the slightest perturbation of the background spacetime can cause this picture to break down. Moreover, in a curved spacetime one may find surprising effects which cannot be described by standard quantum field theory like the famous Hawking radiation [2] and the Fulling-Davies-Unruh effect [3, 4, 5]. Hence, it is appropriate to ask ourselves if Minkowskian quantum field theory is a valid approximation even if quantum effects of gravity are negligible. Since we don't question the success of quantum field theory in general, as it confirms observations with an unprecedented accuracy [6], we strive to improve upon what quantum field theory has taught us in the past.

Following our argumentation that we should never neglect gravitational effects but might disregard quantum gravitational effects at least in a first approximation due to the weakness of the gravitational coupling, this thesis is attributed to *quantum field theory on curved spacetime*. In constrast to what a full-fledged theory of quantum gravity will need to accomplish, the background spacetime in quantum field theory on curved spacetime is fixed by hand. Since this spacetime will, in general, not even have a timelike Killing field, we cannot perform the standard construction to identify a global vacuum state [7]. Therefore, the notion of a quantum field has to be formulated without referring to a preferred state. This can be accomplished in the *algebraic approach* to quantum field theory in which one starts with an abstract algebra of local observables encoding the dynamics of the quantum field [1, 7].

Nevertheless, a state is still needed to obtain any concrete results which can then be understood in the usual probabilty interpretation of quantum theories. Not all possible states have physically reasonable properties. For a free field theory one demands that the states are of *Hadmard form*. Such states mimic the UV singularities of the Minkowski vacuum and may be renormalized to yield a finite energy density [8, 9, 10]. It was later found that Hadamard states can be characterized in terms of their wavefront set [11, 12]. This discovery lead to an improved understanding of Hadmard states and opened the doors for the development of a rigorous *perturbation theory* on curved spacetimes [13, 14, 15, 16].

Although the existence of Hadamard states was proven for various quantum fields on globally hyperbolic spacetimes using deformation arguments [10, 17, 18], an explicit construction is often notoriously difficult. In [19] Dappiaggi, Moretti and Pinamonti suggested a construction which yields a boundary state on the conformal boundary of *asymptotically flat spacetimes* that is invariant under the action of the symmetry group of the boundary manifold, the Bondi-Metzner-Sachs group, and thus can be interpreted as an *asymptotic vacuum state* as shown in [20]. Subsequently it was proven in [21] that pulling back this boundary state to the bulk spacetime yields indeed a Hadamard state. This construction, called the *bulk to boundary correspondence*, was initially done for the conformally coupled massless scalar field and later also applied to the Dirac field [22, 23]. *In this thesis we seek to apply the bulk to boundary correspondence to the vector potential.*

After an introduction into several mathematical topics in Chap. 1, in particular, category theory, $*$-algebras, differential geometry, distributions and wave equations, we delve into the subject of quantum field theory on curved spacetimes in Chap. 2. There, we will review the categorical framework of *general local covariance* [24] and an extension more appropriate to conformal quantum fields, namely, general local *conformal* covariance, already discussed in similar terms in [25]. Moreover, we will investigate the *(conformal) field algebra* of a class of bosonic quantum fields and recall some properties of Hadamard states. This leads us to the discussion of the bulk to boundary correspondence and the construction of a Hadamard state for an abstract conformal quantum field.

Equipped with these notions and results, we start our study of the electromagnetic field in Chap. 3. On the classical level the electromagnetic field is described by Maxwell's equations. If we make certain assumptions on the topology of the spacetime, namely, that the second Betti number is zero or equivalently that the second de Rham cohomology group is trivial, Maxwell's equations attain a particularly simple form in terms of the vector potential. Our treatment of the vector potential follows that of Dimock in [26]. Building on these results and that of [18, 27], we quantize the vector potential in the language of general local conformal covariance. Finally, we adjust the general results on the bulk to boundary correspondence obtained before to the special case of the vector potential to construct a Hadamard state for the vector potential in asymptotically flat spacetimes.

1

Mathematical Preliminaries

We will start off by reviewing the necessary notions and structures. Thereby, we will also introduce the notations and conventions used throughout this thesis. The style of these preliminaries is rather colloquial and brief. Only those definitions, propositions, theorems etc. which are non-standard or are of particular importance are distingushed as such. For a more detailed introduction to the topics studied below the reader is, as always, encouraged to consult the literature listed in the Bibliography.

1.1 Category Theory

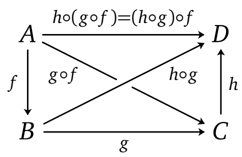
Many fundamental relationships between mathematical structures can be efficiently formulated within the language of category theory. In Sect. 2.1 we will use some basic notions of category theory to introduce the concept of a locally covariant quantum field theory.

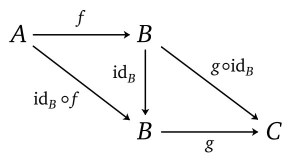
We will not present the most general approach to category theory here. Instead, we work with what are called small categories. For a thorough introduction to the theory of categories we refer to the book by Mac Lane [28].

A *categeory* C consists of a set $\mathrm{Obj}(\mathsf{C})$ of *objects* and a set of *arrows* $\mathrm{Hom}_{\mathsf{C}}(A,B)$ (also called *morphisms*) between any two objects A, B, where each arrow $f \in \mathrm{Hom}_{\mathsf{C}}(A,B)$ is represented diagramatically as an arrow

$$f : A \to B \qquad \text{or} \qquad A \xrightarrow{\;f\;} B\ .$$

Moreover, to each object A there exists a unique *identity arrow* $\mathrm{id}_A : A \to A$, and there is a composition of arrows which assigns to each pair of arrows $f : A \to B$ and $g : B \to C$ a *composite* arrow $g \circ f : A \to C$ which is *associative* and respects the *unit law*, i.e., we have that the diagrams





are both commutative.

The most basic example of a category is the category Set of small sets. It has as its objects all small sets and as its morphisms functions between them. Another example of a category is the category Top of small topological spaces. It is the category whose objects are small topological spaces and whose morphisms are the continuous maps between these.

Besides categories themselves the second most important concept in category theory are the morphisms of categories called *functors*. Given two categories

C and C', a functor $\mathscr{F} : \mathsf{C} \to \mathsf{C}'$ assigns to each object $A \in \mathrm{Obj}(\mathsf{C})$ an object $\mathscr{F}(A) \in \mathrm{Obj}(\mathsf{C}')$ and to each arrow $f : A \to B$ an arrow $\mathscr{F}(f) : \mathscr{F}(A) \to \mathscr{F}(B)$ such that it is compatible with the identity and the composition, i.e.,

$$\mathrm{id}_{\mathscr{A}} = \mathrm{id}_{\mathscr{F}(A)} \quad \text{and} \quad \mathscr{F}(f \circ g) = \mathscr{F}(f) \circ \mathscr{F}(g)$$

for all objects A and all composable morphisms f and g of C.

An often appearing functor is the so called *forgetful functor* which “forgets” some or all of the structure or properties of the category it is operating upon. Usually the forgetful functor maps into the category Set.

Another important concept in category is the notion of *natural transformations*. Given two functors $\mathscr{F}, \mathscr{G} : \mathsf{C} \to \mathsf{C}'$, a natural transformation τ between $\mathscr{F}$ and $\mathscr{G}$, written $\tau : \mathscr{F} \dot{\to} \mathscr{G}$, assigns to each object $A \in \mathrm{Obj}(\mathsf{C})$ an arrow $\tau_A : \mathscr{F}(A) \to \mathscr{G}(A)$ in C' such that the diagram

$$\begin{array}{ccc} \mathscr{F}(A) & \xrightarrow{\tau_A} & \mathscr{G}(A) \\ {\scriptstyle \mathscr{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathscr{G}(f)} \\ \mathscr{F}(B) & \xrightarrow[\tau_B]{} & \mathscr{G}(B) \end{array}$$

is commutative for all arrows $f : A \to B$ of C.

If the functors $\mathscr{F}$ and $\mathscr{G}$ appearing above are not acting on the same categories, we may still speak of natural transformations if we can apply forgetful functors to the source and target categories of both functors such that the categories can be made equal.

1.2 $*$-Algebras

In the algebraic approach to QFT the algebras of observables play a distinguished role. Since observables are identified with self-adjoint operators, we want a concept of operator algebras and a notion of taking adjoints within these algebra. Thus, we review some basic properties of topological[1] $*$-algebras within Sect. 1.2.1 and show the connection of $*$-algebras and operators on a Hilbert space via the so called GNS construction in Sect. 1.2.2.

The topics covered here and many more results on unbounded operator algebras can be found in [29] by Inoue. The treatment of bounded operator algebras, in particular C^*-algebras, is much more prominent in the literature and there exists an almost unbounded set of monographs on this topic, e.g. [30] by Bratelli and Robinson.

1.2.1 Fundamentals

Definition 1.1. A *$*$-algebra* is an algebra $\mathcal{A}$ over $\mathbb{C}$ together with an automorphism $* : \mathcal{A} \to \mathcal{A}, x \mapsto x^*$ called *involution* which is $\mathbb{C}$-antilinear and involutive, i.e., it satisfies

[1] In this and the forthcoming sections we are only concerned with *topological algebras* and hence ‘algebra’ shall always mean ‘topological algebra’, i.e., a linear associative algebra whose underlying vector space is a topological vector space such that its algebra multiplication is continuous.

(i) $(ax + by)^* = \overline{a}x^* + \overline{b}y^*$,

(ii) $(xy)^* = y^* x^*$,

(iii) $(x^*)^* = x$

for all $x, y \in \mathcal{A}$ and $a, b \in \mathbb{C}$. If, in addition, $\mathcal{A}$ has a unit element, denoted $\mathbf{1}$, we say that $\mathcal{A}$ is a *unital* $*$-algebra.

Let us also state some nomenclature for elements of (unital) $*$-algebras: Elements $x, y \in \mathcal{A}$ are called

$$\begin{aligned} &\textit{adjoint} && \text{if} \quad x^* = y, \\ &\textit{self-adjoint} && \text{if} \quad x^* = x, \\ &\textit{normal} && \text{if} \quad x^* x = x x^*, \\ &\textit{unitary} && \text{if} \quad x^* x = \mathbf{1} = x x^*. \end{aligned}$$

A $*$-subalgebra $\mathcal{I} \subset \mathcal{A}$ is called a left (right) $*$-*ideal* if yx (xy) is in $\mathcal{I}$ for all $y \in \mathcal{I}$ and $x \in \mathcal{A}$. A left and right $*$-ideal is just called a $*$-ideal.

The homomorphisms that arise between $*$-algebras, called $*$-*homomorphisms*, are those that preserve in addition to the multiplicative also the involutive structure, i.e., a map $\alpha : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphisms if $\alpha(x^*) = \alpha(x)^*$ for all $x \in \mathcal{A}$. If the $*$-algebras are unital, we also demand that $*$-homomorphisms be unit-preserving.

Often one needs a $*$-algebra with more structure than just an involution. Specifically, one wants an abstract measure of length, i.e., a norm. In the case of $*$-algebras, one requires the norm to satisfy an additional property: A norm $\|\cdot\| : \mathcal{A} \to \mathbb{R}$ is said to be a C^*-*norm* if

$$\|x^* x\| = \|x\|^2, \qquad \forall x \in \mathcal{A}.$$

Definition 1.2. A C^*-*algebra*[2] is a $*$-algebra $\mathcal{A}$ equipped with a C^*-norm $\|\cdot\|$ such that $\mathcal{A}$ is complete with respect to the norm, i.e., $(\mathcal{A}, \|\cdot\|)$ is a Banach space.

1.2.2 The GNS Construction

The algebra of observables already tells us a lot about the structure of the theory at hand. Nevertheless, to give the observables (i.e., the elements of the algebra) any operational meaning we need a notion of states upon which the observables act.

Definition 1.3. A (algebraic) *state* $\omega : \mathcal{A} \to \mathbb{C}$ is a continuous positive linear functional on $\mathcal{A}$ of norm 1, i.e., $\omega(\mathbf{1}) = 1$ and $\omega(x^* x) \geq 0$ for all $x \in \mathcal{A}$.

The algebraic approach is closely related to the familiar Hilbert space formulation. To make this precise, we invoke the *Gel'fand–Naimark–Segal construction*, usually abbreviated as *GNS construction*. Via this powerful tool we will be able to manufacture bounded or unbounded $*$-representations of a $*$-algebra on a Hilbert space. Hence, we first define $*$-representations.

Definition 1.4. A $*$-*representation* π of a unital $*$-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a homomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{D})$ into the linear operators on a dense subspace $\mathcal{D} \subset \mathcal{H}$ such that $\pi(\mathbf{1}) = \text{id}$ and $\pi(x^*) = \pi(x)^*$ for all $x \in \mathcal{A}$.

[2] Pronounced 'sea star'-algebra, it has, however, nothing to do with the star-shaped animals of the class Asteroidea.

Given the definitions above, we are equipped to state and prove the GNS construction for $*$-algebras.

Theorem 1.1 (GNS construction)**.** *Let ω be a state on $\mathcal{A}$. Then there exists a $*$-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{D})$ into the linear operators on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ with inner product $(\cdot\,,\cdot)$ and a unit vector $\Omega \in \mathcal{D}$ such that*

(i) $\omega(x^* y) = (\pi(x)\Omega, \pi(y)\Omega)$ *for all $x, y \in \mathcal{A}$,*

(ii) $\mathcal{D} = \{\pi(x)\Omega \mid x \in \mathcal{A}\}$.

Furthermore, the set $(\mathcal{D}, \pi, \Omega)$, called the GNS triple *of ω, is unique up to unitary equivalence.*

Proof. Let $x, y \in \mathcal{A}$. By a proof analogous to the Cauchy-Schwarz inequality we see that

$$|\omega(x^* y)|^2 \le \omega(x^* x)\,\omega(y^* y).$$

Hence, the set $\mathcal{I} = \{x \in \mathcal{A} \mid \omega(x^* x) = 0\}$ is a left ideal of $\mathcal{A}$, and we have obtained a pre-Hilbert space $\mathcal{D} = \mathcal{A}/\mathcal{I}$ with a positive definite inner product

$$([x],[y]) = \omega(x^* y)$$

for all $[x],[y] \in \mathcal{D}$. We denote by $\mathcal{H}$ the completion of $\mathcal{D}$. Making the identification $\Omega = [\mathbf{1}]$, we define the $*$-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{D})$ by

$$\pi(x)\Omega = [x\mathbf{1}] = [x],$$

i.e., Ω is a *cyclic vector*, and thus

$$\omega(x^* y) = (\pi(x)\Omega, \pi(y)\Omega).$$

Let $(\mathcal{D}', \pi', \Omega')$ be another GNS triple to the state ω, and introduce the operator $U : \mathcal{D} \to \mathcal{D}'$, $U\pi(x)\Omega = \pi'(x)\Omega'$. Then, U is an isometry and can be extended to a unitary operator of $\mathcal{H}$ onto $\mathcal{H}'$. Hence, the two representations π and π' are unitarily equivalent. □

Working with general $*$-algebras, we have not excluded the case of $*$-representations onto unbounded operators. For that reason we cannot uniquely extend the representation to the whole Hilbert space, and hence self-adjoint elements of the algebra may not be represented by self-adjoint operators on the Hilbert space. These problems could be remedied, however, if we restricted ourselves to C^*-algebras, i.e., algebras of bounded operators.

1.3 Differential Geometry

The powerful language of differential geometry, specifically that of differentiable manifolds, has found manifold applications in physics, most prominently the formulation of General Relativity. Since we are concerned with QFT on curved spacetimes, we will also need many results from differential geometry and therefore introduce some basic notions of differential geometry in Sects. 1.3.1 and 1.3.2 and in particular differential forms in Sect. 1.3.3, conformal transformations in Sect. 1.3.5 and the causal structure of Lorentzian manifolds in Sect. 1.3.4. Our conventions are consistent with those of Abraham, Marsden and Ratiu [31]. The sections on Pseudo-Riemannian Geometry and Lorentzian Geometry are partly based on the books by O'Neill [32], and Bär, Ginoux and Pfäffle [33] respectively. Good references for the topics discussed here are also the books by Besse [34] and Wald [35].

Within this section M will always mean a smooth, i.e., C^∞, Hausdorff and paracompact, n-dimensional manifold possibly with additional properties as specified below.

1.3.1 Fundamentals

Let us start by introducing the notions of vector bundles and sections of vector bundles. A (smooth, finite-dimensional) *($\mathbb{K}$-)*[3]*vector bundle*

$$\pi : E \to M$$

is a triple (E, π, M), where E is a manifold, called the *total space*, and π is a smooth, surjective map, called the *bundle projection* such that to every point $x \in M$ there is associated a $\mathbb{K}$-vector space $E_x = \pi^{-1}(x)$, called the *fiber* of E at x. Moreover, there exists to every such x an open neighbourhood $\mathcal{O} \subset M$ with a *local trivialization*, i.e., a diffeomorphism $\phi : \pi^{-1}(\mathcal{O}) \to \mathcal{O} \times E_x$ such that its projection to the first factor gives the bundle projection: $\mathrm{pr}_1 \circ \phi = \pi$.

To each vector bundle E we can associate the dual bundle $\pi^* : E^* \to M$ which has as its fibers E^*_x the dual spaces of the fibers E_x. An inner product on E, i.e., a non-degenerate symmetric bilinear form

$$(\cdot, \cdot) : E \times E \to \mathbb{K}$$

which is not necessarily positive definite, yields an isomorphism between E and E^*.

A *vector bundle homomorphism* from a vector bundle $\pi : E \to M$ to a vector bundle $\pi' : F \to N$ is a smooth map $\psi : E \to F$ which is fiber respecting and a linear map between fibers, i.e., ψ induces a smooth map $\psi' : M \to N$ between the base spaces such that $\psi \restriction_{E_x} : E_x \to F_{\psi'(x)}$ is linear.

We introduced vector bundles so that we may define functions which have at each point of the manifold a value in the corresponding fiber of the vector bundle. Such functions are called *sections*. Given a vector bundle $E \to M$, a smooth section on $\mathcal{O} \subset M$ is a smooth function $s : \mathcal{O} \to E$ such that $\pi \circ s = \mathrm{id}_{\mathcal{O}}$. The space of smooth section of a vector bundle E is denoted by $\Gamma(E)$.

The most prominent and important examples of vector bundles are the tangent and the cotangent bundle. We denote by TM the *tangent bundle* and by T^*M its dual, the *cotangent bundle*. The fibers of these bundles at a point $x \in M$ are the *tangent space* T_xM and the *cotangent space* T^*_xM respectively. The tangent space at a point x may be defined as the space of derivations of smooth functions at x. Hence, we see that the *differential* df of a function $f \in C^\infty(M)$ is an element of the cotangent bundle.

Using the (co-)tangent space, we can introduce some of the basic objects studied in differential geometry: *vectors*, *covectors* and *tensors*. Elements of T_xM and T^*_xM

[3]Throughout this work $\mathbb{K}$ shall denote the field of real or complex numbers, i.e., $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

are called vectors and covectors respectively. A tensor of type (p,q) is an element of the vector space $T^p_q(T_xM)$ constructed by taking the tensor product of p copies of T_xM and q copies of T^*_xM, i.e.,

$$T^p_q(T_xM) \doteq T_xM^{\otimes p} \otimes T^*_xM^{\otimes q}.$$

A *vector field* on M is a section of the tangent bundle TM. The set of all smooth vector fields on M is denoted $\mathfrak{X}(M)$. Sections of the dual T^*M are called *one-forms*. More generally, sections of the p-th exterior power[4] $\Lambda^p T^*M$ of the cotangent bundle are called *p-forms* or *differential forms*. The set of all smooth p-forms on M is denoted $\Omega^p(M)$. Those of compact support are denoted by $\Omega^p_0(M)$. Similar to tensors, *tensor fields* of type (p,q) are defined as sections of $T^p_q(M)$, where $T^p_q(M)$ is the bundle which has at $x \in M$ the fiber $T^p_q(T_xM)$. Following common practice, we will use the term 'tensor' as a shorthand for 'tensor field'.

Given a real vector bundle we can *complexify* it by tensorizing each fiber with $\mathbb{C}$. The complexified cotangent bundle for example is given by

$$T^*_{\mathbb{C}}M \doteq \bigcup_{x\in M} \{x\} \times T^*_xM \otimes \mathbb{C}.$$

The sections of this bundle are the *complex-valued differential forms* which will be denoted by $\Omega^p(M,\mathbb{C})$. Analogously, we may define the complex-valued differential forms by $\Omega^p(M,\mathbb{C}) \doteq \Omega^p(M) \otimes \mathbb{C}$. In general, we can directly complexify the sections of a real vector bundle instead of complexifying the fibers first.

To identify fibers of a vector bundle at different points of the manifold one introduces the notion of parallel transport using connections. A *connection D* on a m-dimensional vector bundle $E \to M$ is a $\mathbb{K}$-linear map

$$D : \Gamma(E) \to \Gamma(E \otimes T^*M)$$

such that the Leibniz rule $D(fs) = f Ds + s \otimes df$ holds for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

In a coordinate neighbourhood of M and a local trivialization of E a connection D applied to a section s can be written as

$$Ds = D\sum_{a=1}^m e_a s^a = \sum_{a=1}^m e_a \otimes ds^a + \sum_{a,b=1}^m e_b \otimes \omega^b_{\ a} s^a,$$

where $\omega \in \Gamma(\operatorname{End} E \otimes T^*M)$ is a 1-form valued matrix called the *connection form* and $(e_a)_a$ is the m-dimensional basis of the fiber in the local trivialization. Thus, we see that two different connections on a vector bundle differ by a connection form.

A connection defines a derivative along a vector field $X \in \mathfrak{X}(M)$. Namely, given a vector bundle $E \to M$ with connection D, the *covariant derivative* along X is $D_X(s) \doteq (Ds)(X)$ for all $s \in \Gamma(E)$. This leads to the notion of *parallel transport*

[4]The *exterior algebra* $\Lambda(V)$ over a vector space V is the antisymmetric quotient of the tensor algebra $T(V)$, i.e., it is the quotient algebra $\Lambda(V) \doteq T(V)/I$ where I is the ideal generated by all $x \otimes x$, $x \in V$. The *exterior* or *wedge product* is the product on this algebra: $\omega \wedge \eta = \omega \otimes \eta \mod I$, $\omega, \eta \in \Lambda(V)$. The *p-th exterior power* of a vector space V is the subspace $\Lambda^p(V) \subset \Lambda(V)$ spanned by the p-fold wedge product of elements in V, i.e., $\Lambda^p(V) \doteq V^{\otimes p}/J$, $J = I \cap V^{\otimes p}$.

along a smooth curve $\gamma : I \to M$, $I \subset \mathbb{R}$: A section $s \in \Gamma(E)$ is called *parallel* to γ if $D_{\dot\gamma} s = 0$.

Let us also introduce a special type of connections: Given an inner product $(\cdot\,,\cdot)$ on E, we say that D is a *metric connection* if $d(s,t)(X) = (D_X s, t) + (s, D_X t)$ holds for all $s, t \in \Gamma(E)$. An important example of such a connection is the Levi-Civita connection which we will define in the next subsection.

We will often make use of smooth mappings $\psi : M \to M'$ between smooth manifolds M and M'. Such a mapping ψ is called a *diffeomorphism* if its inverse ψ^{-1} is also smooth. Using such a smooth map ψ, we can define the *pull-back* of a vector bundle $E \to M'$: The *pull-back bundle* $\psi^* E$ is a vector bundle over M whose fibers are given by $(\psi^* E)_x \doteq E_{\psi(x)}$ for each $x \in M$. This also gives us the pull-back of sections: Given a section $s \in \Gamma(M)$, the *pull-back section* is defined as $\psi^* s \doteq s \circ \psi$ and thus a section of $\psi^* E$.

To define the pull-back of (0, q) tensor fields, we notice that the differential $d\psi$ is a vector bundle homomorphism from TM to $\psi^* TM'$. Hence, the pull-back of a tensor $t \in T^0_q(M)$ is defined as

$$(\psi^* t)(x) \doteq t(\psi(x)) \circ d\psi(x)^{\otimes q}$$

for each $x \in M$.

If ψ is even a diffeomorphism, the construction above can be repeated for the inverse ψ^{-1} to yield the *push-forward* $\psi_* = (\psi^{-1})^*$ (leading to the notions of *push-forward bundle* and *push-forward section*). In the case where ψ is not surjective, we can restrict to its range and will speak of ψ^{-1} and the push-forward ψ_* as if ψ were a diffeomorphism.

1.3.2 Pseudo-Riemannian Geometry

For our purposes having just a manifold is not enough. We need to equip the manifold with additional structure and therefore introduce what is called a *metric tensor*, i.e., a smooth, non-degenerate symmetric $(0,2)$ tensor.[5] The metric tensor gives rise to a metric and hence an inner product $g_x(\cdot\,,\cdot) \doteq g(x)(\cdot\,,\cdot)$ on $T_x M$: For every $x \in M$, i.e., $g_x(\cdot\,,\cdot) : T_x M \times T_x M \to \mathbb{R}$ assigns to each pair (v, w) a real number $g_x(v, w)$.

The maximal dimension of any subspace $\mathcal{O} \subset T_x M$ on which g_x is negative definite is called the *index* of g_x. Since g is continuous and non-degenerate, the index of g_x is the same for all x and henceforth called the index of g and denoted $\mathrm{Ind}(g)$.

The pair (M, g) is called a *pseudo-Riemannian manifold*. We will distinguish two special cases: If $\mathrm{Ind}(g) = 0$, (M, g) is called a *Riemannian manifold*, and if $\mathrm{Ind}(g) = 1$ and $n \geq 2$, we call (M, g) a *Lorentzian manifold*.

At a point $x \in M$ the metric tensor g induces two *canonical isomorphisms* between $T_x M$ and $T^*_x M$. Namely, $\flat : T_x M \to T^*_x M$, $v \mapsto v^\flat$ (*flat*) and its inverse

[5] At some points we will have to relax this definition and also take into account non-smooth but still continuous or degenerate metric tensors.

$\sharp : T_x^*M \to T_xM,\ \omega \mapsto \omega^\sharp$ (*sharp*), where

$$v^\flat(w) \doteq g_x(v,w),$$
$$\omega^\sharp(\eta) \doteq g_x^{-1}(\omega,\eta)$$

for all $w \in T_xM$ and $\eta \in T_x^*M$ and the *inverse metric tensor* g^{-1} is defined by $g^{-1}(v^\flat, w^\flat) \doteq g(v,w)$ for all $v,w \in T_xM$.

The canonical isomorphisms $\sharp$ and $\flat$ extend to the respective bundles TM and T^*M and more generally to tensor field. Hence, via the isomorphism $T^p_q(M) \to T^q_p(M)$ we get an inner product on $T^p_q(M)$ which we still denote $g_x(\cdot,\cdot)$ or merely $g(\cdot,\cdot)$.

A type of mappings between pseudo-Riemannian manifolds stands out: Given two pseudo-Riemannian manifolds (M,g) and (M',g'), an *isometry* is a diffeomorphism $\psi : M \to M'$ such that $\psi^*g' = g$, i.e., it preserves the metric tensor or equivalently the inner product induced by the metric tensor. More generally, if we relax ‘diffeomorphism’ to ‘smooth injective map’ such that the metric tensor is preserved within the range $\psi(M)$ we call ψ an *isometric embedding*.

Isometries preserve *symmetries* of a pseudo-Riemannian manifold. If $\psi_t : M \to M$ is a one-parameter group of isometries, then the vector field $X \in \mathfrak{X}(M)$ which generates the flow, i.e., $\dot{\psi}_t = X$, is called a *Killing vector field*. Equivalently, a Killing field is a vector field which satisfies the *Killing equation*

$$\mathcal{L}_X g = 0.$$

$\mathcal{L}_X$ is the *Lie derivative* along X and it is defined as the derivation[6] such that

$$\mathcal{L}_X f = df(X) \quad \text{and} \quad \mathcal{L}_X Y = [X,Y]$$

for functions $f \in C^\infty(M)$ and vector fields $X,Y \in \mathfrak{X}(M)$.

Before we go on to study differential forms, we will introduce some of the most prominent objects studied in pseudo-Riemannian geometry, namely the Levi-Civita connection and three tensors describing the curvature: the Riemann tensor, the Ricci tensor and the Ricci scalar.

On the tangent bundle of M there exists a connection $\nabla : \mathfrak{X}(M) \to \mathfrak{X}(M) \otimes \Omega^1(M)$ called the *Levi-Civita connection* which is singled out by the metric g. It is the unique connection such that

$$dg(Y,Z)(X) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \qquad \text{(metric connection)}$$
$$[X,Y] = \nabla_X Y - \nabla_Y X \qquad \text{(torsion-free)}$$

holds for all $X,Y,Z \in \mathfrak{X}(M)$. We extend ∇ to one-forms ω by applying the Leibniz rule to $\omega(X)$. Taking this result and using $\nabla_X(s \otimes t) = (\nabla_X s) \otimes t + s \otimes (\nabla_X t)$ for arbitrary tensor fields s,t, one has defined the Levi-Civita connection for tensors too.

The Levi-Civita connection facilitates the definition of a *geodesic*, which is a smooth curve $\gamma : I \to M$, $I \subset \mathbb{R}$ with associated tangent vector field $\dot{\gamma}$ that satisfies the *geodesic equation*

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0, \tag{1.1}$$

[6]Remember that as a derivation $\mathcal{L}_X$ satisfies the Leibniz rule $\mathcal{L}_X(s \otimes t) = \mathcal{L}_X(s) \otimes t + s \otimes \mathcal{L}_X(t)$ for tenors s,t. Thus, the definition can be extended to tensors.

i.e., geodesics are parallel to their tangent vector field – they are auto-parallel. A geodesic which satisfies the above equation is said to be *affinely parametrized*. Instead of (1.1) one may also use the equation $\nabla_{\dot{\gamma}}\dot{\gamma} = \alpha\dot{\gamma}$ with some function α along the curve. However, it can be shown that every geodesic can be reparametrized to yield (1.1).

Given the Levi-Civita connection ∇, the $(1,3)$ tensor $R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$,

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

is called the *Riemann curature* tensor of M. A pseudo-Riemannian manifold is said to be *flat* if R vanishes. Derived from the Riemann curature there exist several more curvature tensors. The *Ricci curvature* tensor is defined as the symmetric $(0,2)$ tensor

$$Ric(X,Y) = \mathrm{tr}(Z \to R(X,Z)Y),$$

where tr denotes the trace of the linear map. Pseudo-Riemannian manifolds with vanishing Ricci curvature are called *Ricci-flat*. Ricci-flat Lorentzian manifolds have physical importance as the vacuum solutions of Einstein's equations. Contracting the tensor Ric, we obtain the *Ricci scalar*:

$$S = \mathrm{tr}_g\, Ric.$$

Here the trace depends on the metric since Ric is a $(0,2)$ tensor while one requires a $(1,1)$ tensor to take the trace.

1.3.3 Differential Forms

Since we will be concerned mainly with differential forms, we will now review some operations on forms and important results regarding these. The results of the first part of this subsection are irrespective of the metric structure of M.

The *exterior derivative* $\boldsymbol{d}^p : \Omega^p(M) \to \Omega^{p+1}(M)$ uniquely extends the notion of a differential of a (smooth) function to p-forms. Usually, $\boldsymbol{d}$ is written for arbitrary p instead of $\boldsymbol{d}^p$. Also, remember the following properties of $\boldsymbol{d}$: It is a $\wedge$-antiderivative, $\boldsymbol{d} \circ \boldsymbol{d} = 0$ and $\boldsymbol{d}$ commutes with pull-backs.

A form ω is called *closed* if $\boldsymbol{d}\omega = 0$ and *exact* if $\omega = \boldsymbol{d}\eta$ for some form η. Obviously, all exact forms are closed. The converse, however, is in general false.

Definition 1.5. The p-th *de Rham cohomology group* $\mathrm{H}^p(M)$ of a manifold M is a device to measures the extent to which closed p-forms are not exact, i.e.,

$$\mathrm{H}^p(M) \doteq \frac{\{\omega \in \Omega^p(M) \mid \omega \text{ closed}\}}{\{\omega \in \Omega^p(M) \mid \omega \text{ exact}\}} = \frac{\ker \boldsymbol{d}^p}{\mathrm{range}\, \boldsymbol{d}^{p-1}}.$$

Analogously, the p-th *de Rham cohomology group with compact support* $\mathrm{H}^p_0(M)$ is given by

$$\mathrm{H}^p_0(M) \doteq \frac{\{\omega \in \Omega^p_0(M) \mid \omega \text{ closed}\}}{\{\omega \in \Omega^p_0(M) \mid \omega \text{ exact}\}}$$

Note the important fact that the de Rham cohomology is a homotopy invariant, and consequently we arrive at the following important result:

Theorem 1.2 (Poincaré lemma)**.** *Let $\omega \in \Omega^p(M)$ be a closed form. For every $x \in M$ there exists a* contractible *neighbourhood $\mathcal{O} \subset M$, i.e., $\mathcal{O}$ is homotopy equivalent to a point, and a form $\eta \in \Omega^{p-1}(\mathcal{O})$ such that*

$$\iota^* \omega = \boldsymbol{d}\eta,$$

where $\iota : \mathcal{O} \hookrightarrow M$ is the inclusion map. In particular, if M is contractible, then η exists globally since we can choose $\mathcal{O} = M$.

Proof. $\mathcal{O}$ is homotopy equivalent to a point, and thus the de Rham cohomology group $\mathrm{H}^p(\mathcal{O})$ is isomorphic to that of a point, i.e., $\mathrm{H}^p(\mathcal{O}) = \{0\}$. □

For a meaningful theory of integration on manifolds, we also need the notion of a *volume form* and the *orientation* of a manifold. A smooth n-form $\mu \in \Omega^n(M)$ is called a volume form if $\mu(x) \neq 0$ for all $x \in M$. If there exists a volume form μ on M, M is called *orientable*. From now on, we shall assume (M, g) to be an orientable, smooth n-dimensional manifold with a volume form μ. μ assigns an orientation to M: A basis $v_1, \ldots, v_n \in T_x M$ is called *positively oriented* if $\mu(v_1, \ldots, v_n) > 0$.

This leads to a standard theorem regarding integration of differential forms, namely *Stokes' theorem*, a proof of which can be found e.g. in Chap. 7.2 of [31].

Theorem 1.3 (Stokes' theorem)**.** *Let $\mathcal{O} \subset M$ be a relatively compact open subset (i.e., its completion $\overline{\mathcal{O}}$ is compact) and suppose that its boundary $\partial\mathcal{O}$ is C^1. Further, let $\partial\mathcal{O}$ (and $\mathcal{O}$) have the induced orientation from M, and let $\omega \in \Omega^{n-1}(M)$. Then*

$$\int_{\mathcal{O}} \boldsymbol{d}\omega = \int_{\partial\mathcal{O}} \iota^* \omega,$$

where $\iota : \partial\mathcal{O} \hookrightarrow M$ denotes the inclusion map.

For the second part, we will assume M to be an oriented smooth manifold equipped with a metric tensor g. Under these assumptions, the ambiguity in the choice of the volume form is eliminated. On (M, g) there exists a unique volume form μ_g, called *g-volume*, such that

$$\mu_g(x)(v_1, \ldots, v_n) = \sqrt{|\det[g_x(v_i, v_j)]|}$$

for a positively oriented basis $v_1, \ldots, v_n \in T_x M$.

Integrating an inner product $(\cdot, \cdot)$ on a vector bundle E over M using the g-volume, yields a natural inner product on the sections of E:

Definition 1.6. Given $s, t \in \Gamma(E)$, we define

$$\langle s, t \rangle_{(M,g)} \doteq \int_M (s, t)\mu_g, \tag{1.2}$$

whenever the integral exists and omit the index (M, g) if there is no ambiguity on which pseudo-Riemannian manifold the inner product is to be taken.

In particular, the integral exists whenever s, t are square-integrable and real-valued, i.e., $s, t \in L^2(E, \mu_g)$ so that $\|s\|_2$ and $\|t\|_2$ are finite, where

$$\|\cdot\|_2 \doteq \Big(\int_M |\cdot|^2 \mu_g \Big)^{1/2}$$

and $|\cdot|$ is a norm on E.

We also apply the g-volume μ_g to introduce the *Hodge star operator* $*$ on (M, g). The Hodge star is the unique map $* : \Omega^p(M) \to \Omega^{n-p}(M)$ such that

$$\omega \wedge *\eta = g(\omega, \eta)\mu_g \tag{1.3}$$

holds pointwise for all $\omega, \eta \in \Omega^p(M)$. It has the properties

$$*1 = \mu_g, \; *\mu_g = (-1)^{\mathrm{Ind}(g)},$$
$$**\omega = (-1)^{\mathrm{Ind}(g)+p(n-p)}\omega,$$

where $\omega \in \Omega^p(M)$.

Using the Hodge operator, we can write the inner product (1.2) for every pair $\omega, \eta \in \Omega^p(M)$ of differential forms as

$$\langle \omega, \eta \rangle = \int_M \omega \wedge *\eta. \tag{1.4}$$

The Hodge operator facilitates the definition of another product: the *interior product* (also called *contraction*). We define the interior product of a vector field $X \in \mathfrak{X}(M)$ or a one-form $\xi \in \Omega^1(M)$ with a p-form $\omega \in \Omega^p(M)$ as

$$\boldsymbol{i}_X \omega \doteq (-1)^{\mathrm{Ind}(g)} * (X^\flat \wedge *\omega),$$
$$\boldsymbol{i}^\xi \omega \doteq (-1)^{\mathrm{Ind}(g)} * (\xi \wedge *\omega).$$

Using the properties of the Hodge operator, one can see that this definition coincides with the usual one as found e.g. in Chap. 5.1 of [31].

Together with the exterior derivative the Hodge star enables us to define the differential operator $\boldsymbol{\delta}$, called the *codifferential*. The codifferential $\boldsymbol{\delta} : \Omega^{p+1}(M) \to \Omega^p(M)$ acting on a $p+1$-form ω is defined by

$$\boldsymbol{\delta}\omega \doteq (-1)^{np+1+\mathrm{Ind}(g)} * \boldsymbol{d} * \omega.$$

This definition makes the codifferential $\boldsymbol{\delta}$ the *formal adjoint* of the exterior derivative $\boldsymbol{d}$, i.e., $\langle \boldsymbol{d}\omega, \eta \rangle = \langle \omega, \boldsymbol{\delta}\eta \rangle$ for all $\omega \in \Omega^p(M), \eta \in \Omega^{p+1}(M)$ such that $\operatorname{supp}\omega \cap \operatorname{supp}\eta$ is compact.

Note that the codifferential is not a derivation and hence does not satisfy the Leibniz law, e.g. for $f \in C^\infty(M)$ and $\omega \in \Omega^p(M)$ one calculates

$$\boldsymbol{\delta}(f\omega) = f\boldsymbol{\delta}\omega + (-1)^{np+1}\boldsymbol{i}^{\boldsymbol{d}f}\omega. \tag{1.5}$$

Furthermore, one can show that for 1-forms η the codifferential satisfies $\boldsymbol{\delta}\eta = -\operatorname{tr}(\nabla\eta^\sharp)$, i.e., it is equal to minus the divergence.

Analogously to the case of the exterior derivative, a form ω is called *coclosed* if $\boldsymbol{\delta}\omega = 0$ and *coexact* if $\omega = \boldsymbol{\delta}\eta$ for some form η. Using these notions, the bijectivity of the Hodge star yields

$$\mathrm{H}^{n-p}(M) \cong \frac{\{\omega \in \Omega^p(M) \mid \omega \text{ coclosed}\}}{\{\omega \in \Omega^p(M) \mid \omega \text{ coexact}\}},$$
$$\mathrm{H}_0^{n-p}(M) \cong \frac{\{\omega \in \Omega_0^p(M) \mid \omega \text{ coclosed}\}}{\{\omega \in \Omega_0^p(M) \mid \omega \text{ coexact}\}}$$

by applying it to Definition 1.5. We can apply this to obtain a relation between $\mathrm{H}^p(M)$ and $\mathrm{H}_0^{n-p}(M)$:[7]

Proposition 1.1. *$\mathrm{H}^p(M) \cong \mathrm{H}_0^{n-p}(M)$ if $\mathrm{H}^p(M)$ is finite-dimensional.*

Proof. If $\mathrm{H}^p(M)$ is finite-dimensional, then it is isomorphic to its dual $\mathrm{H}^p(M)^*$. One can show that $\langle\cdot,\cdot\rangle$ gives a well-defined non-degenerate pairing of $\mathrm{H}^p(M)$ and $\mathrm{H}_0^{n-p}(M)$ (one says that $\langle\cdot,\cdot\rangle$ is cohomological). Thus, $\mathrm{H}_0^{n-p}(M)$ is isomorphic to $\mathrm{H}^p(M)^*$ and hence also to $\mathrm{H}^p(M)$. □

For future purposes we will also need a linear, second order differential operator called the *Laplace-de Rham operator* which is defined as

$$\Box \doteq \boldsymbol{d} \circ \boldsymbol{\delta} + \boldsymbol{\delta} \circ \boldsymbol{d} = (\boldsymbol{d} + \boldsymbol{\delta}) \circ (\boldsymbol{d} + \boldsymbol{\delta}). \tag{1.6}$$

This operator generalizes in a suitable way the usual d'Alembert operator to p-forms on pseudo-Riemannian manifolds. In a coordinate neighbourhood $\mathcal{O} \subset M$ the Laplace-de Rham operator acting on a p-form $\omega \in \Omega^p(M)$ is given in components by (cf. Eq. (10.2) of [37] by Lichnerowicz)

$$(\Box\omega)_{\mu_1\cdots\mu_p} \doteq -\nabla^\nu\nabla_\nu\omega_{\mu_1\cdots\mu_p} + \sum_k Ric_{\mu_k\nu}\,\omega_{\mu_1\cdots}{}^{\nu}{}_{\cdots\mu_p} + \sum_{k\neq l} R_{\mu_k\nu\mu_l\lambda}\,\omega_{\mu_1\cdots}{}^{\nu\cdots\lambda}{}_{\cdots\mu_p},$$

where Einstein summation convention is implied for repeated greek indices. This is a specific case of the Weitzenböck formula (1.15) which we will present later.

We conclude this subsection with a useful formula for the Laplace-de Rham operator which follows from Stokes' theorem:

Corollary 1.1 (Green's identity for $\Box$)**.** *Let $\omega, \eta \in \Omega^p(M)$ and $\mathcal{O} \subset M$ relatively compact with boundary $\partial\mathcal{O}$. Then,*

$$\begin{aligned}\langle\Box\omega,\eta\rangle_{(\mathcal{O},g)} - \langle\omega,\Box\eta\rangle_{(\mathcal{O},g)} = \int_{\partial\mathcal{O}} \iota^*\big(&\omega\wedge *\boldsymbol{d}\eta - \eta\wedge *\boldsymbol{d}\omega \\ &+ \boldsymbol{\delta}\omega\wedge *\eta - \boldsymbol{\delta}\eta\wedge *\omega\big).\end{aligned} \tag{1.7}$$

Applying this identity, we see that $\Box$ is *formally self-adjoint*, i.e., $\langle\Box\omega,\eta\rangle = \langle\omega,\Box\eta\rangle$ for all $\omega,\eta \in \Omega^p(M)$ such that $\operatorname{supp}\omega \cap \operatorname{supp}\eta$ is compact.

[7] This also follows from the more general Poincaré duality, cf. [36].

1.3.4 Lorentzian Geometry

By a *spacetime* (M, g) we shall mean a connected, oriented and time-oriented, four-dimensional Lorentzian manifold (M, g). For brevity, we will often write $\boldsymbol{M}$ for (M, g).

We will now study the *causal structure* of this spacetime. First, the inner product induced by g enabled us to distinguish at each point $x \in M$ three different regions in the tangent space: A tangent vector $v \in T_x M$ is[8]

$$\begin{aligned} \textit{spacelike} \quad &\text{if} \quad g(v,v) > 0, \\ \textit{lightlike} \text{ or } \textit{null} \quad &\text{if} \quad g(v,v) = 0, \\ \textit{timelike} \quad &\text{if} \quad g(v,v) < 0, \\ \textit{causal} \quad &\text{if} \quad v \text{ is non-spacelike.} \end{aligned}$$

This definition can be extended to piecewise C^1-curves and to vector fields, e.g. a C^1-curve in M is called spacelike if its tangent vectors are spacelike and a vector field is called timelike if it is timelike at every point. Replacing vectors with covectors and vector fields with 1-forms we obtain the same structure on the cotangent bundle.

Next, the *time orientation* of $\boldsymbol{M}$ enables us to distinguish between *future* and *past*: Given a time orientation τ, i.e., a smooth timelike vector field $\tau : M \mapsto TM$, a causal vector $v \in T_x M$ is said to be *future directed* (*past directed*) if $g(\tau(x), v) < 0$ $(g(\tau(x), v) > 0)$. The notions 'future/past directed' can be extended to curves and vector fields as before. Further, we say that a covector ω is future respectively past directed if the associated vector $v = \omega^\sharp$ is and write $\omega \rhd 0$ respectively $\omega \lhd 0$.

Finally, combining these two notions, the *causal future* $J^+_{\boldsymbol{M}}(p)$ (*causal past* $J^-_{\boldsymbol{M}}(x)$) of a point $x \in M$ is defined to be the set of points in M which can be reached from x by future (past) directed causal curves. More generally, the causal future/past of a subset $\mathcal{O} \subset M$ is $J^\pm_{\boldsymbol{M}}(\mathcal{O}) \doteq \cup_{x \in \mathcal{O}} J^\pm_{\boldsymbol{M}}(x)$. Defining $J_{\boldsymbol{M}}(\mathcal{O}) \doteq J^+_{\boldsymbol{M}}(\mathcal{O}) \cup J^-_{\boldsymbol{M}}(\mathcal{O})$, we say that two subsets $\mathcal{O}, \mathcal{O}' \subset M$ are *causally separated* if $\mathcal{O}' \subset M \setminus J_{\boldsymbol{M}}(\mathcal{O})$. A subset $\mathcal{O} \subset M$ is called future/past compact if $\mathcal{O} \cap J^\pm_{\boldsymbol{M}}(x)$ is compact for all $x \in M$.

To solve differential equations in $\boldsymbol{M}$, we need a suitable surface on which to specify initial values. This role is played by Cauchy surfaces.

Definition 1.7. A *Cauchy surface* is a subset $\Sigma \subset M$ which is intersected exactly once by every inextendible[9] timelike curve.

Consequently, a Cauchy surface is met by every inextendible causal curve. A necessary condition for a spacetime to allow for a Cauchy problem to be posed is the existence of a Cauchy surface. Thus, we define:

Definition 1.8. A spacetime $\boldsymbol{M}$ is called *globally hyperbolic* if it admits a Cauchy surface.

Equivalently, $\boldsymbol{M}$ is globally hyperbolic if it satisfies the *strong causality condition*, i.e., there are no almost closed causal curves, and if for all $p, q \in M$ the set

[8] Recall that we defined the signature of a Lorentzian metric tensor g such that $\mathrm{Ind}(g) = 1$.

[9] A piecewise C^1-curve is *extendible* provided it has a continuous extension to any of the endpoints of the curves parameter interval. A curve which is not extendible is called *inextendible*.

$J^+_{\boldsymbol{M}}(p) \cap J^-_{\boldsymbol{M}}(q)$ is compact. It was shown by Bernal and Sanchez [38] that every globally hyperbolic spacetime manifold M is diffeomorphic to $\mathbb{R} \times \Sigma$ and that it is foliated by smooth Cauchy surfaces.

1.3.5 Conformal Geometry

Let (M, g) and $(M, \widetilde{g})$ be pseudo-Riemannian manifolds. The two metric tensors g and $\widetilde{g}$ (and thus the two pseudo-Riemannian manifolds) are said to be *conformally equivalent* if there exists a smooth, strictly positive function Ω – the *conformal factor* – such that $\widetilde{g} = \Omega^2 g$. The conformal rescaling from g to $\widetilde{g}$ is called a *conformal transformation*. This invites a generalization of the notions of isometry and isometric embedding:

Definition 1.9. Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two pseudo-Riemannian manifolds. A smooth injective map $\psi : M \hookrightarrow \widetilde{M}$ such that

$$\psi^* \widetilde{g} = \Omega^2 g$$

is called a *conformal embedding*. If ψ is even a diffeomorphism, i.e., M and $\widetilde{M}$ are diffeomorphic, we call it a *conformal isometry*.

Given a conformal isometry $\psi : M \to \widetilde{M}$, the inner product structure (i.e., the causal structure if g is Lorentzian) induced by the metric tensor in the two pseudo-Riemannian manifolds is identical due to the positivity of the conformal factor, e.g. for all $v \in \mathfrak{X}(M)$ with $g(v, v) > 0$ the relation $\widetilde{g}(v, v) > 0$ holds too.

Let the automorphism $\psi : M \to M$ be a conformal isometry between $\boldsymbol{M} = (M, g)$ and $\widetilde{\boldsymbol{M}} = (M, \widetilde{g})$ with $\psi^* \widetilde{g} = \Omega^2 g$. Also, it is convenient to define $\Upsilon \doteq \Omega^{-1} \boldsymbol{d}\Omega$. Operators with respect to the rescaled metric tensor $\widetilde{g}$ are denoted by a tilde ($\sim$) symbol.

Many physical fields transform nicely under conformal transformations: They are *conformally invariant*. We call a section $s \in \Gamma(E)$ of a vector bundle $E \to M$ conformally invariant if it transforms as

$$\psi^* s = \Omega^w s.$$

for some $w \in \mathbb{R}$ called the *conformal weight* of s. That is, s can be considered as a function $s = s(x, g)$ depending also on the metric tensor, and it has the homogeneity property $s(x, \Omega^2 g) = \Omega^w s(x, g)$. Sections of a specific conformal weight are denoted by a superscript w giving the weight, e.g. $\Gamma^w(E)$ denotes the space of sections of conformal weight w of E, and, similarily, the space of differential p-forms of conformal weight w is denoted by $\Omega^{p,w}(M)$.

In the following, we review the behavior of the differential operators $\boldsymbol{d}$, $\boldsymbol{\delta}$, $\Box$, cf. Theorem 1.159 of [34], and the geodesic equation subject to conformal transformations, cf. Appendix D of [35].

Proposition 1.2. *Let M be of even dimension and $\phi \in \Omega^{p,0}(M)$. Then,*

$$\psi^* \widetilde{\boldsymbol{\delta}} \phi = \Omega^{-2}(\boldsymbol{\delta}\phi - (4 - 2p)\, \boldsymbol{i}^{\Upsilon} \phi), \tag{1.8}$$

$$\begin{aligned} \psi^* \widetilde{\Box} \phi = \Omega^{-2}(&\Box\phi - (4 - 2p)\, \boldsymbol{d}\boldsymbol{i}^{\Upsilon} \phi + (8 - 4p)\, \Upsilon \wedge \boldsymbol{i}^{\Upsilon} \phi \\ &- 2\, \Upsilon \wedge \boldsymbol{\delta}\phi - (2 - 2p)\, \boldsymbol{i}^{\Upsilon} \boldsymbol{d}\phi). \end{aligned} \tag{1.9}$$

Proof. Recall that the exterior derivative $\boldsymbol{d}$ is defined independently from the metric tensor, i.e., $\psi^*\widetilde{\boldsymbol{d}} = \boldsymbol{d}$. Hence, to compute the codifferential $\widetilde{\boldsymbol{\delta}}$ and the Laplace-de Rham operator $\widetilde{\square}$ on $\widetilde{\boldsymbol{M}}$, we just need to calculate the transformed Hodge star operator $\widetilde{*}$. It follows from (1.3) that

$$\psi^* \widetilde{*} \phi = \Omega^{4-2p} * \phi. \qquad \square$$

Of particular interest will be the operator $\boldsymbol{\delta d}$ (often termed *Maxwell operator*) and the codifferential $\boldsymbol{\delta}$ on 1-forms of conformal weight 0 and -2. So we state the corresponding transformations here explicitly:

Corollary 1.2. *Let M be of even dimension, $\phi \in \Omega^{1,0}(M)$ and $\varphi \in \Omega^{1,-2}(M)$. Then,*

$$\psi^* \widetilde{\boldsymbol{\delta}} \boldsymbol{d} \phi = \Omega^{-2} \boldsymbol{\delta d} \phi, \tag{1.10}$$

$$\psi^* \widetilde{\boldsymbol{\delta}} \phi = \Omega^{-2}(\boldsymbol{\delta}\phi - 2\boldsymbol{i}^{\Upsilon}\phi) = \Omega^{-2}(\boldsymbol{\delta}\phi - 2g(\Upsilon, \phi)), \tag{1.11}$$

$$\psi^* \widetilde{\boldsymbol{\delta}} \varphi = \Omega^{-4} \boldsymbol{\delta} \varphi, \tag{1.12}$$

i.e., the differential operators $\boldsymbol{\delta d}$ and $\boldsymbol{\delta}$ are conformally invariant on the 1-forms ϕ and φ respectively.

Proof. While (1.10) and (1.11) are a direct consequence of (1.8), let us work out the proof of (1.12) in a little more detail. By (1.11), we have

$$\psi^* \widetilde{\boldsymbol{\delta}} \varphi = \Omega^{-2}(\boldsymbol{\delta} \psi^* \varphi - 2\boldsymbol{i}^{\Upsilon} \psi^* \varphi) = \Omega^{-2}(\boldsymbol{\delta} \Omega^{-2} \varphi - 2\boldsymbol{i}^{\Upsilon} \Omega^{-2} \varphi).$$

Using (1.5), we obtain

$$\psi^* \widetilde{\boldsymbol{\delta}} \varphi = \Omega^{-4}(\boldsymbol{\delta}\varphi - 2\boldsymbol{i}^{\Upsilon}\varphi) - \Omega^{-2} \boldsymbol{i}^{d\Omega^{-2}} \varphi,$$

which yields the result after applying the chain rule. $\square$

The geodesic equation is also not conformally invariant. That is, a geodesic γ with respect to the metric tensor g is, in general, not a geodesic with respect to the transformed metric tensor $\widetilde{g}$. A short calculation [35] yields

$$\psi^* \widetilde{\nabla}_{\dot{\gamma}} \dot{\gamma} = 2\dot{\gamma}\, \boldsymbol{i}_{\dot{\gamma}} \Upsilon - g(\dot{\gamma}, \dot{\gamma}) \Upsilon^{\sharp},$$

where $\widetilde{\nabla}$ is the Levi-Civita connection with respect to $\widetilde{g}$ and $\dot{\gamma}$ is the tangent vector field to γ. Nevertheless, in the case that γ is a null geodesic the second term vanishes, and one sees that, after a reparametrization, γ satisfies the geodesic equation according to the arguments after (1.1). Hence, null geodesics are conformally invariant.

So far we have only considered the conformal transformation behaviour of sections and operators under conformal isometries $\psi : M \to M$. The generalization to conformal embeddings $\psi : M \to \widetilde{M}$ with $\psi^* \widetilde{g} = \Omega^2 g$ is slightly more complicated. Let $\widetilde{E} \to \widetilde{M}$ be a vector bundle and $E = \psi^* \widetilde{E}$ the pull-back bundle over M. Further, let $\widetilde{s} \in \Gamma^w(\widetilde{E})$ be a section conformal weight w in the sense defined above. Then, we can *define* a section $s \in \Gamma^w(E)$ such that

$$\psi^* \widetilde{s} = \Omega^w s$$

holds. Since this case will occur frequently in the following, let us introduce a *conformal pull-back* and a *conformal push-forward*:

$$s = \psi^*_w \widetilde{s} \doteq \Omega^{-w} \psi^* \widetilde{s},$$
$$\widetilde{s} = \psi^w_* s \doteq \psi_* \Omega^w s.$$

If the conformal factor Ω is defined on $\widetilde{M}$ instead of M, i.e., $\psi^* \Omega^{-2} \widetilde{g} = g$, we have instead $\psi^*_w \widetilde{s} = \psi^* \Omega^{-w} \widetilde{s}$ and $\psi^w_* s \doteq \Omega^w \psi_* s$.

1.4 Asymptotically Flat Spacetimes

The study of isolated systems is a valueable tool to which one can attribute much of the progress that has been made in the history of physics. Only by understanding isolated system can we model the universe by treating various subsystems individually. Examples are the investigation of isolated mass and charge distributions in Newtonian gravity and electromagnetism, respectively.

As one could have anticipated, doing the same in general relativity is complicated by the absence of a fixed background with respect to which one would specify the asymptotic behaviour of the system. Nevertheless, one was able to circumvent these difficulties by the definition of asymptotically flat spacetimes by Bondi and subsequent work of Sachs, Newman, Penrose and others.

Our treatment of asymptotically flat spacetimes is based on Chapter 11 of [35], and we refer to this account and the references listed therein for more information on this matter. Let us start directly with the (rather involved) definition and omit any further motivation like the example of the conformal compactification of Minkowski spacetime which may be found in the literature. The definition which we introduce here can be found in [21] and is a modification of that in [35] and [39].

Definition 1.10. A spacetime $\boldsymbol{M} = (M, g)$ is called an *asymptotically flat spacetime at past null and time infinity* if there exists a spacetime $\widetilde{\boldsymbol{M}} = (\widetilde{M}, \widetilde{g})$, which might fail to be smooth at the preferred point i^- (*past time infinity*), and a conformal embedding $\psi : M \hookrightarrow \widetilde{M}$ with a conformal factor $\Omega \in C^\infty(\widetilde{M})$ (i.e., $\Omega \restriction_{\psi(M)} > 0$ and $\psi^* \Omega^{-2} \widetilde{g} = g$) such that the following conditions are satisfied:

(1) $J^+_{\widetilde{\boldsymbol{M}}}(i^-)$ is closed and $\psi(M) = J^+_{\widetilde{\boldsymbol{M}}}(i^-) \setminus \partial J^+_{\widetilde{\boldsymbol{M}}}(i^-)$. We call $\mathscr{I}^- \doteq \partial J^+_{\widetilde{\boldsymbol{M}}}(i^-) \setminus \{i^-\}$ *past null infinity*.

(2) There exists a strongly causal neighbourhood $\mathcal{O}$ of $\partial J^+_{\widetilde{\boldsymbol{M}}}(i^-)$ such that g satisfies the vacuum equations $Ric = 0$ within the preimage of $\mathcal{O} \cap \psi(M)$.

(3) On $\mathscr{I}^-$ we have $\Omega = 0$ and $\boldsymbol{d}\Omega \neq 0$, whereas on i^- we have $\Omega = 0 = \boldsymbol{d}\Omega$ and a non-degenerate Hessian $\lim_{i^-} \widetilde{\nabla} \boldsymbol{d}\Omega = 2\widetilde{g}$.

(4) The map of null directions at i^- into the integral curves $n \doteq \boldsymbol{d}\Omega^{\widetilde{\sharp}} \restriction_{\mathscr{I}^-}$ is a diffeomorphism. Furthermore, given a function $\omega \in C^\infty(\widetilde{M} \setminus \{i^-\})$ which is strictly positive on $\psi(M) \cup \mathscr{I}^-$ and satisfies $\widetilde{\boldsymbol{\delta}} \omega^4 \boldsymbol{d}\Omega = 0$ on $\mathscr{I}^-$, the vector field $\omega^{-1} n$ is complete[10] on $\mathscr{I}^-$.

[10] A vector field $X \in \mathfrak{X}(M)$ is called *complete* if for each $x \in M$ there exists a curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = x$ and $X \restriction_\gamma = \dot{\gamma}$.

Here $\widetilde{\nabla}$, $\widetilde{\boldsymbol{\delta}}$, $\widetilde{\sharp}$ are the Levi-Civita connection, the codifferential and the canonical isomorphism sharp with respect to the unphysical metric tensor $\widetilde{g}$. Note that condition (2) can be weakened significantly by requiring only that the stress-energy tensor $\widetilde{T}$ in the unphysical spacetime is such that $\Omega^{-2}\widetilde{T}$ is smooth on $\mathscr{I}^-$ (cf. Chapter 11 of [35]).

A spacetime which satisfies the above definition will be called *asymptotically flat* for brevity in all that follows. The spacetime $\boldsymbol{M}$ is termed the *physical spacetime* while $\widetilde{\boldsymbol{M}}$ is termed the *unphysical spacetime*.

The main difference of this definition with that in [35] is our use of i^- as the distinguished locus instead of spatial inifnity i^0. In [21] future null infinity i^+ was chosen instead of i^- but all results found in this thesis can be easily translated to the other case. Moreover, equivalently to [21], we only have a compactification into the past such that we only have past null infinity and not future null infinity too.

As the definition of asymptotically flat spacetimes is rather opaque, let us give two examples to obtain some intuition: Whereas spatially flat Friedmann-Lemaître-Robertson-Walker spacetimes are asymptotically flat, Schwarzschild or Kerr spacetimes are *not* asymptotically flat in the sense of this definition because they give rise to a unphysical metric tensor which is singular at i^-. Note, however, that the Schwarzschild spacetime is asymptotically flat in the sense of [35, 39].

One notices that the definition allows for a great freedom in choosing the unphysical spacetime: A conformal rescaling of the unphysical metric $\widetilde{g}$ by a factor ω, i.e.,

$$\widetilde{g} \to \widetilde{g}' = \omega^2 \widetilde{g}, \quad n \to n' = \omega^{-1} n$$

is always possible. This is the *conformal gauge freedom* in the choice of the unphysical metric. Now, we can choose $\omega_B = \omega$ such that the null vector field $n_B = \omega_B^{-1} n$ satisfies the geodesic equation $\widetilde{\nabla}_{n_B} n_B = 0$ on $\mathscr{I}^-$ and n_B is complete. Thus, n_B is called the *null geodesic generator* of $\mathscr{I}^-$.

Since $\mathscr{I}^-$ is the future null-cone of i^-, we have that it is diffeomorphic to $\mathbb{R} \times \mathrm{S}^2$. Then, having fixed ω_B and n_B, we introduce coordinates in a neighbourhood of $\mathscr{I}^-$. First, we can use Ω itself as a coordinate since $\boldsymbol{d}\Omega \neq 0$ on $\mathscr{I}^-$. Secondly, we can introduce the coordinate u as the affine parameter along the null geodesics generator n_B scaled such that $\boldsymbol{d}u(n_B) = 1$ (i.e., $n_B = \partial_u$). Last, we use the standard spherical coordinates (θ, ϕ) on the spherical cross section of $\mathscr{I}^-$. This yields the *Bondi frame* $(\Omega, u, \theta, \phi)$ in a neighbourhood of $\mathscr{I}^-$ in the unphysical spacetime with the *Bondi metric tensor*

$$\widetilde{g}_B \restriction_{\mathscr{I}^-} = -\boldsymbol{d}\Omega \otimes \boldsymbol{d}u - \boldsymbol{d}u \otimes \boldsymbol{d}\Omega + \boldsymbol{d}\theta \otimes \boldsymbol{d}\theta + \sin^2\theta \, \boldsymbol{d}\phi \otimes \boldsymbol{d}\phi .$$

at the locus $\mathscr{I}^-$. Because S^2 and $\mathbb{C}_\infty \doteq \mathbb{C} \cup \{\infty\}$ are diffeomorphic, we may just as well work with complex stereographic coodinates $\zeta = e^{i\phi} \cot\theta/2$. These give the metric tensor the following form:

$$\widetilde{g}_B \restriction_{\mathscr{I}^-} = -\boldsymbol{d}\Omega \otimes \boldsymbol{d}u - \boldsymbol{d}u \otimes \boldsymbol{d}\Omega + 2\,\frac{\boldsymbol{d}\zeta \otimes \boldsymbol{d}\overline{\zeta} + \boldsymbol{d}\overline{\zeta} \otimes \boldsymbol{d}\zeta}{(1+\zeta\overline{\zeta})^2}.$$

As a consequence of the existence of a Bondi frame we have that the structure at null infinity is *universal* for all asymptotically spacetimes. This will be of major importance in our construction of a field theory at $\mathscr{I}^-$.

Thus, let $\mathscr{I}_1^-$ and $\mathscr{I}_2^-$ be null infinity with null geodesic generators n_1, n_2 associated to two arbitrary asymptotically flat spacetimes. Owing to the existence of a Bondi frame, there exists a conformal isometry $\varphi : \mathscr{I}_1^- \to \mathscr{I}_2^-$ with respect to the induced (degenerate) metrics such that $\varphi^* n_2 = \omega^{-1} n_1$.[11] The set of all such diffeomorphisms forms a group called the *Bondi-Metzner-Sachs group* (abbreviated *BMS group*) and denoted BMS.

Let us analyze the structure of the BMS group (cf. [40, 41] by McCarthy). The conformal automorphisms of the sphere are the Möbius transformations

$$\zeta \to \zeta' = \frac{a\zeta + b}{c\zeta + d} \quad \text{with} \quad ad - bc = 1$$

and $a, b, c, d \in \mathbb{C}$. Such a transformation amounts to a conformal transformation with conformal factor

$$K(\zeta)^{-1} \doteq \frac{(a\zeta + b)(\overline{a}\overline{\zeta} + \overline{b}) + (c\zeta + d)(\overline{c}\overline{\zeta} + \overline{d})}{1 + \zeta\overline{\zeta}}.$$

This implies that the affine parameter u along $\mathscr{I}^-$ can only transform as

$$u \to u' = K(\zeta)(u + \alpha(\zeta))$$

with $\alpha \in C^\infty(\mathbb{C}_\infty)$. Accordingly, the BMS group is the infinite-dimensional Lie group $BMS \cong SO^+(1,3) \ltimes C^\infty(\mathrm{S}^2)$, i.e., it is isomorphic to the semidirect product of the proper, orthochronous Lorentz group (which is isomorphic to the Möbius group) with the infinite-dimensional Abelian normal subgroup $C^\infty(\mathrm{S}^2)$. In particular, BMS contains the *supertranslations*

$$\zeta \to \zeta' = \zeta, \qquad u \to u' = u + \alpha(\zeta),$$

which are the asymptotic translations of all asymptotically flat spacetime.

This last statement can be made more precise. Namely, given a Killing field X in the physical spacetime, we obtain a conformal Killing field $\widetilde{X} = \psi_* X$ in $\psi(M)$, i.e.,

$$\mathcal{L}_{\widetilde{X}} \widetilde{g} = 2\Omega^{-1} \widetilde{g} \mathcal{L}_{\widetilde{X}} \Omega + \Omega^2 \mathcal{L}_{\widetilde{X}}(\psi_* g) = 2\Omega^{-1} \widetilde{g} \mathcal{L}_{\widetilde{X}} \Omega,$$

which has a smooth extension, also denoted $\widetilde{X}$, to $\widetilde{M}$, as proven by Geroch in [42]. Since $\Omega = 0$ at $\mathscr{I}^-$, $\mathcal{L}_{\widetilde{X}} \widetilde{g} = 2\Omega^{-1} \widetilde{g} \mathcal{L}_{\widetilde{X}} \Omega$ at $\mathscr{I}^-$, too. From the smoothness of $\Omega^{-1} \mathcal{L}_{\widetilde{X}} \Omega$ it then follows that $\boldsymbol{d}\Omega(\widetilde{X}) \restriction_{\mathscr{I}^-} = 0$, i.e., $\widetilde{X}$ satisfies the Killing equation at $\mathscr{I}^-$. Therefore, $\widetilde{X} \restriction_{\mathscr{I}^-}$ can be defined intrinsically on $\mathscr{I}^-$ where it gives an infinitesimal symmetry. And, as BMS is the symmetry group of $\mathscr{I}^-$, every such $\widetilde{X}$ generates a one-parameter subgroup of BMS. An example are the vector fields $\alpha(\zeta) n$ which lead to the supertranslations defined above.

Nevertheless, to construct the full BMS group, these considerations do not suffice. We also have to take into account so called *asymptotic Killing fields* (cf. [35, 21]):

[11] The degenerate metric on the submanifold $\mathscr{I}^- \hookrightarrow M'$ induced by the Bondi metric tensor is $0 \cdot \boldsymbol{d}u \otimes \boldsymbol{d}u + 2(1 + \zeta\overline{\zeta})^{-2}(\boldsymbol{d}\zeta \otimes \boldsymbol{d}\overline{\zeta} + \boldsymbol{d}\overline{\zeta} \otimes \boldsymbol{d}\zeta)$. The null geodesic generator $n = \partial_u$ can be defined intrinsically on $\mathscr{I}^-$.

Definition 1.11. A vector fields $X \in \mathfrak{X}(M)$ is called an asymptotic Killing field if it has a smooth extension $\widetilde{X}$ to the unphysical spacetime ($\widetilde{X} \restriction_{\psi(M)} = \psi_* X$) such that $\widetilde{X}$ is tangent to $\mathscr{I}^-$ and $\Omega^2 \mathcal{L}_{\widetilde{X}}(\psi_* g)$ is smooth and vanishes at $\mathscr{I}^-$.

Therefore, $\widetilde{X}$ satisfies the Killing at $\mathscr{I}^-$ and gives rise to an element of BMS.[12]

We close the discussion on asymptotically flat spacetimes and the BMS group by mentioning that, since the BMS group is a subset of the conformal gauge transformations at $\mathscr{I}^-$, a tensor of conformal weight w transforms under a BMS transformations as it would under a conformal transformation.

1.5 Distributions on Manifolds

Distributions occur distributed throughout various subjects in physics and mathematics. In particular, they appear as propagators and fundamental solutions in QFT and the theory of partial differential equations. Here, we will need a generalization of the usual notion of distributions to *distributions on manifolds*, cf. Chap. 1.1 of [33], and Chap. 2.8 and Appendix of Friedlander [43].

After familiarizing ourselves with the basic notions of distributions on manifolds in Sect. 1.5.1, we will review the concept of the *wavefront set* and its generalization to distributional sections in Sect. 1.5.2.

For more information on distributions in general and the wavefront set in particular we refer to the book by Hörmander [44].

1.5.1 Fundamentals

Let (M, g) be an n-dimensional Riemannian manifold[13] and consider the $\mathbb{K}$-vector bundle $E \to M$ with some metric. This gives us a norm $|\cdot|$ on $E \otimes T^*M^{\otimes p}$ induced by the metric tensor g and the metric of E. We denote by $\Gamma_0(E)$ the space of compactly supported smooth sections of E and call its elements *test sections*. Further, denote by D a connection on E and T^*M and thus also on $E \otimes T^*M^{\otimes p}$.

Before we define distributions, we need to define a family of seminorms and thereby a topology on $\Gamma_0(E)$. For every $k \in \mathbb{N}$ and every compact subset $K \subset M$

$$\|f\|_{k,K} \doteq \sum_{j \leq k} \sup_{x \in K} |D^j f(x)|$$

is a norm on $\Gamma_0(E)$. Hence, we say that a sequence $(f_m)_{m \in \mathbb{N}}$ converges to f in $\Gamma_0(E)$ if there exists a $K \subset M$ such that $\operatorname{supp} f_m \subset K$ for all m and $\|f - f_m\|_{k,K} \to 0$ for each k.

Definition 1.12. Given some Fréchet space V,[14] a continuous linear map $u : \Gamma_0(E) \to V$ is called a *V-valued distributional section of E*. The space of V-valued distributions in E is denoted by $\Gamma_0(E, V)'$ and if $V = \mathbb{K}$, we write $\Gamma_0(E)'$. We use

[12] Here one sees – from a slightly different perspective – why α cannot depend on u: Killing fields are constant in the direction of the tangent vector fields of geodesics.

[13] Using a Riemannian manifold here amounts to no restriction on the metric we use later. The reason for having a Riemannian metric here is the need for a norm on T^*M.

[14] Note that, if the fibers of E are complex, V has to be complex and vice versa.

equivalent notation for compactly supported distributions, which we denote in general by $\Gamma(E,V)'$.[15]

If we are dealing with with compactly supported real- or complex-valued differential forms as test sections, we write $\Omega_0^p(M)'$ or $\Omega_0^p(M,\mathbb{C})' = \Omega_0^p(M)' \otimes \mathbb{C}$ for the distributions.

Note that, since the vector bundle E is by definition finite-dimensional, we have that $\Gamma_0(E)' \cong \Gamma_0(M)' \otimes E^*$. Therefore, given an inner product $(\cdot,\cdot)$ on E, we can associate to each $u \in \Gamma(E)$ a distributional section of $\Gamma_0(E)'$, which we denote by the same symbol, via

$$u(f) \doteq \langle u, f\rangle = \int_M (u,f)\,\mu,$$

where $f \in \Gamma_0(E)$ and μ is a volume form on M. Sometimes we will also write $\langle u, f\rangle$ if u does not correspond to a smooth section, and then the definition can be understood in the reverse direction.

Now we can formulate a simple, yet powerful relation between integral maps and bidistributions (cf. Theorem 5.2.1 of [44]):

Theorem 1.4 (Schwartz kernel theorem)**.** *Every bidistribution $k \in (\Gamma_0(E)\otimes\Gamma_0(E))'$ defines a linear map $K : \Gamma_0(E) \to \Gamma_0(E)'$ and vice versa:*

$$k(f \otimes h) = \langle Kf, h\rangle$$

for all $f,h \in \Gamma_0(E)$. k is called the (Schwartz) kernel *of K.*

Below we will often use the same symbol for the kernel and the map as is customary. Some of the linear maps K considered below arise as *convolutions* with a distribution. For two smooth functions $f,h \in C_0^\infty(\mathbb{R})$ the convolution $f * h$ is defined as

$$(f * h)(x) \doteq \langle f(y), h(x-y)\rangle.$$

This definition extends directly to distributions $u \in (C_0^\infty(\mathbb{R}))'$. $u * f \in C^\infty(\mathbb{R})$ is defined by $(u(y) * f(y))(x) \doteq u(f(x-y))$.

1.5.2 The Wavefront Set

To specify where a distribution is smooth and where it is singular, one introduces the *singular support*: $\operatorname{sing\,supp} u$ of a distribution u is the complement of the union of all open subsets $\mathcal{O} \subset M$ on which u is smooth.

The wavefront has been introduced by Hörmander as a refinement of the notion of singular support. It is used to characterize the singularity structure of distributions by studying its Fourier transform at each point. Thereby it not only gives us information about the location of singularities but also about the directions in which the singularities occur. In Sect. 2.3 we will use the wavefront set to introduce the microlocal spectrum condition. Since $\Gamma_0(E)' \cong \Gamma_0(M)' \otimes E^*$, we can first discuss the wavefront set for scalar distributions $\Gamma_0(M)'$ and generalize to $\Gamma_0(E)'$ later.

[15] We remark that in our notation $\Gamma_0(M)' = \mathscr{D}'(M)$ and $\Gamma(M)' = \mathscr{E}'(M)$.

Let $(\mathcal{O},\kappa)$ be a coordinate chart of M. We define the Fourier transform $\widehat{f}$ of a square-integrable function $f \in L^2(\mathcal{O}, d^n x)$ by

$$\widehat{f}(\xi) \doteq \int_{\mathbb{R}^n} f(\kappa^{-1}(x))\, e^{-i x \cdot \xi}\, d^n x.$$

With this choice of normalization Parseval's identity and the convolution theorem are as follows:

$$\langle f, \overline{h} \rangle = (2\pi)^{-n} \langle \widehat{f}, \overline{\widehat{h}} \rangle, \quad \widehat{f * h} = \widehat{f}\,\widehat{h} \quad \text{and} \quad \widehat{f\,h} = (2\pi)^{-n} \widehat{f} * \widehat{h}$$

for all $f, h \in L^2(\mathbb{R}, dx)$ and the first equation applies also to $f, h \in L^2(\mathcal{O}, d^n x)$.

The Fourier transform can be extended to distributions: For compactly supported distributions $u \in \Gamma(\mathcal{O})'$ it is defined by $\widehat{u}(\xi) \doteq u_\kappa(e^{-i x \cdot \xi})$, where $u_\kappa \in \Gamma_0(\mathbb{R}^n)'$ is given by $u_\kappa(f \circ \kappa^{-1}) = u(f)$ for all $f \in \Gamma_0(\mathcal{O})$.

We can now employ the Fourier transform to give a condition on the smoothness of a compactly supported distribution $u \in \Gamma(\mathcal{O})'$: u is smooth if and only if for each $N \in \mathbb{N}_0$ there exists a constant C_N such that

$$|\widehat{u}(\xi)| \le C_N (1 + |\xi|)^{-N}.$$

We use this bound to identify the directions in which the Fourier transform $\widehat{u}$ of some distribution u is not of rapid decay and hence lead to its singularities.

Definition 1.13. We say that $\xi \in \mathbb{R}^n \setminus \{0\}$ is a *singular direction* of a compactly supported distribution $u \in \Gamma(\mathcal{O})'$ if there exists a conical[16] neighbourhood V of ξ and a $N \in \mathbb{N}_0$ such that $(1 + |\eta|)^N |\widehat{u}(\eta)|$ has no bound for $\eta \in V$. The set of all singular directions of u is denoted by $\Sigma(u)$.

Localizing this, we obtain the singular directions of a distribution $u \in \Gamma_0(M)'$ at a point $x \in M$

$$\Sigma_x(u) \doteq \bigcap_{\chi} \Sigma(\chi u),$$

where $\chi \in \Gamma_0(\mathcal{O})$ for some coordinate neighbourhood $\mathcal{O}$ of x such that $\chi(x) \neq 0$. This leads to the definition of the wavefront set as the set of the singular directions at all points:

Definition 1.14. The *wavefront set* $\mathrm{WF}(u)$ of a distribution $u \in \Gamma_0(M)'$ is defined as

$$\mathrm{WF}(u) \doteq \big\{(x, \xi) \in T^*M \setminus 0 \mid \xi \in \Sigma_x(u)\big\},$$

where $T^*M \setminus 0$ is the cotangent bundle with the zero section removed. Note that $\mathrm{WF}(u)$ is independent of the choice of local coordinates, cf. Chap. 8.2 of [44].

We will now give as an example the wavefront sets of two simple but important distributions (cf. [45] by Strohmaier): The Dirac δ-distribution and the distributions $\delta^\pm$ defined by

$$\delta^\pm(f) \doteq \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \frac{f(x)}{x \pm i\epsilon}\, dx.$$

[16]A *cone* is a subset V such that $\lambda V \subset V$ for all $\lambda > 0$.

Example 1.1 (Dirac δ-distribution)**.** For any $f \in L^2(\mathbb{R}, dx)$ we have

$$\widehat{f\,\delta} = f(0).$$

Hence, δ has support $\{0\}$ and does not decay in any direction. We obtain

$$\mathrm{WF}(\delta) = \{0\} \times (\mathbb{R} \setminus \{0\}).$$

Example 1.2 ($\delta^{\pm}$-distribution)**.** First we calculate the Fourier transform of $1/(x + i\epsilon)$ for $\epsilon > 0$ using the residue theorem[17]

$$\int_{\mathbb{R}} \frac{e^{-i\xi x}}{x + i\epsilon}\, dx = -2\pi i\, \Theta(\xi)\, e^{-\xi\epsilon}.$$

Taking the limit $\epsilon \downarrow 0$, this gives $\widehat{\delta^+}(\xi) = -2\pi i\, \Theta(\xi)$. Then, applying the convolution theorem, we obtain the Fourier transform of $\delta^+(f)$ for all $f \in L^2(\mathbb{R}, dx)$ as

$$\widehat{f\,\delta^+} = \frac{1}{2\pi}(\widehat{f} * \widehat{\delta^+}) = -i \int_{-\infty}^{\xi} \widehat{f}(k)\, dk.$$

Since this decays rapidly as $\xi \to -\infty$ and does not decay as $\xi \to \infty$ (tends to $-2\pi i\, f(0)$), we get

$$\mathrm{WF}(\delta^+) = \{0\} \times \mathbb{R}_+$$

and, by an analogous computation, $\mathrm{WF}(\delta^-) = \{0\} \times \mathbb{R}_-$. Moreover, it follows that

$$\delta^+ - \delta^- = -2\pi i\, \delta, \tag{1.13}$$

i.e., the δ-distribution may be split into two parts.

The wavefront set can also be generalized to distributions in vector bundles. Suppose we have a distribution $u \in \Gamma_0(E)' \cong \Gamma_0(M)' \otimes E^*$ in a vector bundle $E \to M$ with m-dimensional fibers. Locally u can be written in components $(u_1, \ldots, u_m)$, $u_i \in \Gamma_0(M)'$ via a local trivilization. Accordingly, we have

$$\mathrm{WF}(u) = \bigcup_i \mathrm{WF}(u_i)$$

independent of the particular trivilization.

Here one notices a possible refinement of the wavefront set to study the individual compontent, e.g. to identify those components which are singular. This leads to the so called *polarization set* introduced by Dencker [46]. Nevertheless, the information contained in the wavefront set is sufficient for the problems considered here.

Using the wavefront set of the bidistribution k associated to every linear map $K : \Gamma_0(E) \to \Gamma_0(E)'$ via the Schwartz kernel theorem, we can extend the definition of K to certain distributions $u \in \Gamma(E)'$ (cf. Theorem 8.2.13 of [44]):

[17] Θ denotes the Heaviside step-function.

Proposition 1.3. *The map K is defined for every $u \in \Gamma(E)'$ with*

$$\mathrm{WF}(u) \cap \big\{(y,\eta) \mid (x,y,0,-\eta) \in \mathrm{WF}(k) \text{ for some } y \in M\big\} = \emptyset,$$

and we have

$$\mathrm{WF}(Ku) \subset \big\{(x,\xi) \mid (x,y,\xi,-\eta) \in \mathrm{WF}(k) \text{ for some } (y,\eta) \in \mathrm{WF}(u) \cup (M \times \{0\})\big\}.$$

Moreover, formulating the comment following Theorem 8.2.14 in [44] as a proposition, we get

Proposition 1.4. *Let $u \in (C_0^\infty(\mathbb{R}))'$ be a distribution on $\mathbb{R}$, then the wavefront set of the Schwartz kernel k associated to the operator $u*$, i.e., $k(f \otimes h) = \langle u * f, h\rangle$ for all $f, h \in C_0^\infty(\mathbb{R})$, is given by*

$$\mathrm{WF}(k) = \{(x,y,\xi,-\xi) \mid (x-y,\xi) \in \mathrm{WF}(u)\}.$$

Let us now give two examples of Schwartz kernels and their wavefront sets that arise as an extension of the two examples considered above.

Example 1.3 (Diagonal distribution δ^Δ)**.** We define the *diagonal distribution*

$$\delta^\Delta(f \otimes h) \doteq \langle f, h\rangle$$

for all $f, h \in L^2(E,\mu)$. The wavefront set can be directly calculated as

$$\mathrm{WF}(\delta^\Delta) = \{(x,x,\xi,-\xi) \in T^*(M \times M)\backslash 0\}.$$

If $f, h \in L^2(\mathbb{R}, dx)$, we have $f = \delta * f$ so that δ^Δ can be seen as the Schwartz kernel associated to the linear operator $\delta *$.

Example 1.4 ($\delta^{\pm\Delta}$-distribution)**.** Similar to $\delta^\pm$, we can define $\delta^{\pm\Delta}$ as the positive and negative frequency part of δ^Δ:

$$\delta^{\pm\Delta}(f \otimes h) \doteq \langle \delta^\pm * f, h\rangle = \lim_{\epsilon \downarrow 0} - \int_{\mathbb{R}^2} \frac{f(x)h(y)}{x - y \mp i\epsilon}\, dx\, dy$$

for all $f, h \in L^2(\mathbb{R}, dx)$. Applying Proposition 1.4 to the results of Example 1.2, we obtain

$$\mathrm{WF}(\delta^{\pm\Delta}) = \{(x,x,\xi,-\xi) \in \mathbb{R}^4 \mid \pm\xi > 0\}.$$

To determine the range of test functions that $\delta^{\pm\Delta}$ can be applied to, we calculate the Fourier transform using Parseval's identity, the convolution theorem and Example 1.2:

$$\delta^{\pm\Delta}(f \otimes h) = \langle \delta^\pm * f, h\rangle = -i\,\langle \Theta(\xi)\widehat{f}(\xi), \widehat{h}(-\xi)\rangle. \tag{1.14}$$

Thus, $\delta^{\pm\Delta}$ can be defined as a distribution on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ by Plancherel's theorem and the Schwartz inequality.

1.6 Wave Equations

In the discussion of the quantization of a specific quantum field we will always start from a dynamical equation, the *equation of motion*, which describes the behavior of some classical field in space and time. Therefore, this section is attributed to the study of a specific yet very important type of equation of motion: the wave equation. We highly recommend the books [33] and [43] on which most of this section is based and which cover this subject in great depth.

For the duration of this section let $\boldsymbol{M} = (M, g)$ be a globally hyperbolic spacetime with volume form μ_g and $\pi : E \to M$ a vector bundle equipped a with an inner product $(\cdot, \cdot)$.

A *partial differential operator* $P : \Gamma(E) \to \Gamma(E)$ of order k acting on sections of the vector bundle E is given locally in each coordinate neighbourhood $\mathcal{O} \subset M$ with coordinates $(x_1, \ldots, x_4)$ as a polynomial

$$P = p(x, \partial) = \sum_{|\alpha| \leq k} A_\alpha \partial^\alpha,$$

where $A_\alpha \in \Gamma(\operatorname{End} E)$ and $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_4^{\alpha_4}$ for some multi-index α. The leading term of the polynomial $p(x, \xi)$ is called the *principal symbol* σ_P of P:

$$\sigma_P(x, \xi) \doteq \sum_{|\alpha| = k} A_\alpha \xi^\alpha,$$

where $\xi = \sum_\mu \xi_\mu dx^\mu \in T^*M$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. Furthermore, the *characteristic set* $\operatorname{Char} P$ of P is defined as $\operatorname{Char} P \doteq \ker \sigma_P \subset T^*M$, the set of zeros of σ_P.

We may also act with partial differential operators on distributional sections $\Gamma_0(E)'$ by introducing the formal adjoint P^* of P. Given $u \in \Gamma_0(E)'$ and $f \in \Gamma_0(E)$,

$$\langle Pu, f \rangle = \langle u, P^* f \rangle \quad \text{with} \quad P^* = \sum_{|\alpha| \leq k} (-\partial)^\alpha A_\alpha.$$

In the case that $P^* = P$ we call P formally self-adjoint.

1.6.1 Normally Hyperbolic Operators

We already defined a partial differential operator in (1.6) – the Laplace-de Rham operator $\Box$. It is an example of a *normally hyperbolic*[18] differential operator or *wave operator* because its principal symbol is given by the metric tensor, i.e., a partial differential operator P acting on sections of $E \to M$ is normally hyperbolic if and only if

$$\sigma_P(x, \xi) = -g_x(\xi, \xi)\, \mathrm{id}_{E_x},$$

i.e., $\operatorname{Char} P = \{(x, \xi) \in T^*M \mid g_x(\xi, \xi) = 0\}$. In a coordinate neighbourhood of M with coordinates $(x_1, \ldots, x_4)$ a normally hyperbolic operator P may be written as

$$P = -g^{\mu\nu} \frac{\partial^2}{\partial x_\mu \partial x_\nu} + A^\mu \frac{\partial}{\partial x_\mu} + B,$$

[18]Normally hyperbolic operators are a subset of the hyperbolic differential operators.

where $g^{\mu\nu}$ denotes the matrix of the inverse metric tensor, $A \in \Gamma(TM \otimes \operatorname{End} E)$, $B \in \Gamma(\operatorname{End} E)$ and Einstein summation convention is assumed.

All normally hyperbolic operators on a vector bundle can be associated to a connection on this bundle. We make this precise by the following proposition which can be found e.g. in [47] by Baum and Kath:

Proposition 1.5. *Let $P : \Gamma(E) \to \Gamma(E)$ be a normally hyperbolic operator on E. There exists a uniquely determined connection $D : \Gamma(E) \to \Gamma(E \otimes T^*M)$ and an endomorphism $B \in \Gamma(\operatorname{End} E)$ on E such that the* Weitzenböck formula

$$P = D^*D + B \tag{1.15}$$

holds. $D^* : \Gamma(E \otimes T^*M) \to \Gamma(E)$ is the formal adjoint of D with respect to the pairing on $E \otimes T^*M$ induced by the inner product on E and the inverse metric tensor.

Using this proposition, we can formulate a Green's identity for each normally hyperbolic operator that is associated to a metric connection.

Proposition 1.6 (Green's identity)**.** *Let D be a metric connection on E and $P = D^*D + B$ a normally hyperbolic operator. If $\mathcal{O} \subset M$ is a relatively compact subset with C^1 boundary $\iota : \partial\mathcal{O} \hookrightarrow \mathcal{O}$, then*

$$\langle Ps, t\rangle_{(\mathcal{O},g)} - \langle s, P^*t\rangle_{(\mathcal{O},g)} = \int_{\partial\mathcal{O}} \iota^* * \big((Ds,t) - (s,Dt)\big) \tag{1.16}$$

for all $s, t \in \Gamma(E)$ with $\operatorname{supp} s \cap \operatorname{supp} t$ *compact.*[19]

Proof. Integrating the condition for metric connections, $g(\xi, \boldsymbol{d}(s,t)) = (Ds, t \otimes \xi) + (s \otimes \xi, Dt)$, yields

$$\langle D^*(s \otimes \xi), t\rangle_{(\mathcal{O},g)} = -\langle Ds, t \otimes \xi\rangle_{(\mathcal{O},g)} + \int_{\mathcal{O}} \xi \wedge * \boldsymbol{d}(s,t)$$

for all $\xi \in \Omega^1(M)$. Therefore, we have

$$\langle D^*Ds, t\rangle_{(\mathcal{O},g)} = -\langle Ds, Dt\rangle_{(\mathcal{O},g)} + \int_{\mathcal{O}} \boldsymbol{d} * \omega$$

with $\omega \in \Omega^1(M)$ such that $\omega(X) = (D_X s, t)$ for all $X \in \mathfrak{X}(M)$. Further, we have by Stokes' theorem

$$\int_{\mathcal{O}} \boldsymbol{d} * \omega = \int_{\partial\mathcal{O}} \iota^* * \omega.$$

With these results and $P^* = D^*D + B^*$, we obtain

$$\langle Ps, t\rangle_{(\mathcal{O},g)} - \langle s, P^*t\rangle_{(\mathcal{O},g)} = \int_{\partial\mathcal{O}} \iota^* * (\omega - \eta)$$

with ξ as before and η defined by $\eta(X) = (s, D_X t)$ for $X \in \mathfrak{X}(M)$. □

[19] (Ds,t) and (s,Dt) are symbolic for the 1-forms defined by $(D_X s, t)$ and $(s, D_X t)$ for $X \in \mathfrak{X}(M)$ as described in the proof and $*$ is the Hodge star operator.

1.6.2 Green's Operators

Normally hyperbolic operators play a distinguished role in physics since they give rise to *wave equations*. It is known that the Cauchy problem for wave equations in a globally hyperbolic spacetime is well-posed (cf. Chap. 3 of [33] and Chap. 5 of [43]), and thus we have:

Theorem 1.5 (Green's operators)**.** *Let P be a formally self-adjoint*[20] *normally hyperbolic operator and $\boldsymbol{M}$ globally hyperbolic. Then there exist unique* advanced $(+)$ *and* retarded $(-)$ Green's operators $G^{\pm}$ *satisfying*

(1) $G^{\pm}(Pu) = u$,

(2) $P(G^{\pm}f) = f$,

(3) $\operatorname{supp}(G^{\pm}f) \subset J^{\pm}_{\boldsymbol{M}}(\operatorname{supp} f)$

for all $u, f \in \Gamma_0(E)'$ which are past $(+)$ respectively future $(-)$ compact.

Above $G^{\pm}$ acting on a distributional section has to be understood in the sense of Proposition 1.3, i.e., in the smeared sense:

$$\langle G^{\pm}f, \phi\rangle = \langle f, G^{\mp}\phi\rangle = \langle u, \phi\rangle,$$

where $\phi \in \Gamma(E)$ future respectively past compact and $\operatorname{supp} f \cap \operatorname{supp}\phi$ compact. Further, note that $G^{\pm}$ commutes with any partial differential operator Q that commutes with P since $PG^{\pm}Qf = Qf = QPG^{\pm}f = PQG^{\pm}f$.

Having found advanced and retarded Green's operators, let us also introduce the *causal propagator*[21]

$$Gf \doteq (G^{+} - G^{-})f$$

for all past and future compact $f \in \Gamma_0(E)'$. That is, to each such f there exists a (weak) solution $u = Gf \in \Gamma_0(E)'$, $\operatorname{supp} u = J_{\boldsymbol{M}}(\operatorname{supp} f)$, of the homogeneous equation $Pu = 0$. The opposite direction holds too:

Proposition 1.7. *Let $u \in \Gamma_0(E)'$ be such that $Pu = 0$. Then, there exists past and future compact $f \in \Gamma_0(E)'$ such that $u = Gf$.*

Proof. Let $\chi \in C^{\infty}(M)$ and $1-\chi$ be past and future compact respectively. Therefore, $u = \chi u + (1-\chi)u$ and $P\chi u = P(\chi - 1)u = f$ for some $f \in \Gamma_0(E)'$. We can apply G^{+} to $P\chi u$ and G^{-} to $P(\chi-1)u$ to see that both $P\chi u$ and $P(\chi-1)u$ are past and future compact due to them being equal. Thus, we obtain

$$Gf = G^{+}P\chi u - G^{-}P(\chi - 1)u = \chi u + (1-\chi)u = u. \qquad \square$$

Another property of the causal propagator which will be used a lot is that it is not a bijective map and that its kernel is given by

$$\ker G = \{Pu \mid u \in \Gamma_0(E)' \text{ past and future compact}\} \tag{1.17}$$

[20] Many results below have an appropriate generalization to P which are not formally self-adjoint. Nevertheless, for simplicitly we will only consider the case of formally self-adjoint wave operators.

[21] In the following we will sometimes add a subscript $\boldsymbol{M}$ to Green's operators and the causal propagator to explicitly indicate the spacetime under investigation.

as a direct consequence of the definition of Green's operators and the causal propagator. Moreover, we are often only interested in smooth solutions which are compactly supported on every Cauchy surface. Such solutions arise from G acting on elements of $\Gamma_0(E)$.

This observation allows us to construct the phase space of a field theory whose equation of motion is given by the homogenous wave equation $Pu = 0$, $u \in \Gamma(E)$ with formally self-adjoint P. Following Crnković and Witten [48], we identify the phase space with the space of solutions which is isomorphic to

$$\mathscr{P} \doteq \Gamma_0(E)/\ker G. \tag{1.18}$$

There is a canonical way to equip this space with a symplectic structure without making any preferred choice of initial data (cf. Dimock [49]):

Proposition 1.8. *Let* $(\cdot,\cdot)$ *be a inner product on* E *and* $[f],[h] \in \mathscr{P}$ *with representatives* $f,h \in \Gamma_0(E)$*. Then*

$$\sigma([f],[h]) \doteq G(f \otimes h) \doteq \langle Gf, h\rangle \tag{1.19}$$

is a symplectic form on the vector space $\mathscr{P}$.

Proof. The bilinear form $\sigma(\cdot,\cdot)$ is by definition of $\mathscr{P}$ well-defined. It is weakly non-degenerate because $(\cdot,\cdot)$ is non-degenerate and $Gf \neq 0$ for all $[0] \neq [f] \in \mathcal{S}$. It is anti-symmetric because of the formal adjointness of G^+ and G^-. □

The (bi-)distribution $G(\cdot \otimes \cdot)$ is called the *commutator distribution* and it is also defined for past and future compact $f \in \Gamma_0(E)'$. Among some other properties, the commutator distribution G is a bisolution of the homogenous wave equation, i.e., it solves the wave equation in both factors: $(P \otimes \mathrm{id}) \circ G = 0 = (\mathrm{id} \otimes P) \circ G$.

We also need some general results on conformally invariant wave operators and their Greens operators. Let P be a normally hyperbolic wave operator acting on sections $\Gamma^w(E)$ of conformal weight w. If P is conformally invariant, it transforms as

$$\psi^* \circ \widetilde{P} = \Omega^{w-2} P \circ \psi^*_w = \Omega^{w-2} P\, \Omega^{-w} \circ \psi^* \tag{1.20}$$

for all conformal isometries $\psi : M \to \widetilde{M}$ with conformal factor Ω and $\widetilde{P}$ is the corresponding wave operator acting on $\psi_* \Gamma^w(E)$.

To establish the conformal transformation behavior of the advanced and retarded Green's operators and hence also of the causal propagator, we make use of their definition: Let $\widetilde{G^\pm}$ be the Green's operators associated to $\widetilde{P}$. From $\psi^* \circ \widetilde{G^\pm}\widetilde{P} = \psi^* = G^\pm P \circ \psi^*$ it follows that

$$\psi^* \circ \widetilde{G^\pm} = \Omega^w\, G^\pm \circ \psi^*_{w-2} = \Omega^w\, G^\pm\, \Omega^{2-w} \circ \psi^*, \tag{1.21}$$

i.e., $G^\pm$ acts on sections of conformal weight $w - 2$. The same transformation behaviour holds true for the causal propagator.

1.6.3 Propagation of Singularities

We will not give a full account of the propagation of singularities here nor state the theorem in all its generality as done by Duistermaat and Hörmander [50] and Dencker [46]. Instead we will derive a propagation of singularities theorem for wave operators from the wavefront set of Green's operators in a coordinate neighbourhood.

Let P be a formally self-adjoint normally hyperbolic operator acting on distributional sections $\Gamma_0(E)'$ of a vector bundle $\pi : E \to M$ with inner product $(\cdot,\cdot)$. We define *Green's distributions* $G^\pm \in (\Gamma_0(E)\otimes\Gamma_0(E))'$ as the Schwartz kernels of Green's operators

$$G^\pm(f\otimes h) \doteq \langle G^\pm f, h\rangle.$$

Note that they are also defined for past and future compact $f \in \Gamma_0(E)'$. Furthermore, they are bisolutions of P, namely,

$$(P\otimes \mathrm{id})G^\pm = \delta^\Delta = (\mathrm{id}\otimes P)G^\pm.$$

We define *local Green's distributions* $G^\pm_{\mathcal{O}}$ on each globally hyperbolic coordinate neighbourhood $\mathcal{O}\subset M$ as the restriction of $G^\pm$ to $(\Gamma_0(\pi^{-1}\mathcal{O})\otimes\Gamma_0(\pi^{-1}\mathcal{O}))'$. The local Green's distributions can be expressed via a sum over Riesz distributions. Analyzing this sum as in Chap. 4.5 of [45] one obtains the wavefront set of $G^\pm_{\mathcal{O}}$.

Proposition 1.9. *The wavefront sets of the local Green's distributions are*

$$\begin{aligned}\mathrm{WF}(G^\pm_{\mathcal{O}}) = &\{(x,x,\xi,-\xi)\in T^*(\mathcal{O}\times\mathcal{O})\backslash 0\}\\ &\cup\{(x,y,\xi,-\eta)\in T^*(\mathcal{O}\times\mathcal{O})\backslash 0 \mid (x,\xi)\sim(y,\eta),\ y\in J^\pm_M(x)\},\end{aligned} \tag{1.22}$$

where $(x,\xi)\sim(y,\eta)$ means that there exists a null geodesic γ connecting x and y such that ξ is cotangent to γ and η is the parallel transport of ξ from x to y along γ.

From this we also see that the wavefront sets of $G^\pm_{\mathcal{O}}$ and $G_{\mathcal{O}}$ contain no elements of the form $(x,y,\xi,0)$ since this would already imply that $(x,y,0,0)$ is an element.

Let us now prove the following two important results which are localized generalizations to curved spaces of the corresponding statements in Chap. 11.5 of [51] by Friedlander and Joshi:

Proposition 1.10. *If $u\in\Gamma_0(E)'$, then*

$$\mathrm{WF}(u)\subset \mathrm{WF}(Pu)\cup \mathrm{Char}\,P.$$

Proof. Suppose $(x,\xi)\in \mathrm{WF}(u)\setminus\mathrm{WF}(Pu)$. Choose a bump function $\chi\in C_0^\infty(\mathcal{O})$ in a globally hyperbolic coordinate neighbourhood $\mathcal{O}\subset M$ such that $\chi=1$ in a small neighbourhood of x. Hence, we even have $(x,\xi)\in\mathrm{WF}(\chi u)\setminus\mathrm{WF}(P\chi u)$. Since χu is compactly supported, we obtain

$$\chi u = G^+_{\mathcal{O}} P\chi u.$$

Applying Proposition 1.3, we see from (1.22) that the wavefront at (x,ξ) must come from the second part of $\mathrm{WF}(G^+_{\mathcal{O}})$, and thus $(x,\xi)\in\mathrm{Char}\,P$. □

We are now ready to state a special case of the propagation of singularities theorem:

Theorem 1.6. *(Propagation of singularities) Let $u \in \Gamma_0(E)'$. The wavefront set* $\mathrm{WF}(u)$ *is invariant under the null geodesic flow in the complement of* $\mathrm{WF}(Pu)$*. That is, if $(x,\xi) \in \mathrm{WF}(u) \setminus \mathrm{WF}(Pu)$ and $(x,\xi) \sim (y,\eta)$ with $y \in \operatorname{supp} u$, then also $(y,\eta) \in \mathrm{WF}(u)$.*

Proof. It suffices to prove this theorem locally.

Assume that $(x,\xi) \in \mathrm{WF}(u) \setminus \mathrm{WF}(Pu) \subset \operatorname{Char} P$. Taking a χ as in the proof of the last proposition which in addition satisfies $\operatorname{supp} \chi u \cap \operatorname{sing\,supp} Pu = \emptyset$. Again we have $\chi u = G^{\pm}_{\mathcal{O}} P \chi u$ from which we see that the wavefront at (x,ξ) must come from the second part of $\mathrm{WF}(G^{\pm}_{\mathcal{O}})$ interacting with $\mathrm{WF}(P\chi u)$ along the geodesic defined by (x,ξ). Hence, there exists a $(y,\eta) \in \mathrm{WF}(P\chi u)$ with $(y,\eta) \sim (x,\xi)$ for some $y \in \operatorname{supp} \boldsymbol{d}\chi$.

To prove the other direction, let $x, y \in \mathcal{O} \cap \operatorname{supp} u$ be two points lying on a null geodesic through a globally hyperbolic coordinate neighbourhood $\mathcal{O} \subset M$, i.e., we have $(x,\xi) \sim (y,\eta)$ for null covectors ξ and η cotangent to the geodesic at x and y respectively, such that $(x,\xi) \notin \mathrm{WF}(Pu)$. Choose a bump function $\chi \in C^\infty_0(\mathcal{O})$ with $\chi = 1$ in a small neighbourhood of x and $y \in \operatorname{supp} \chi u$. Now, assume that $(y,\eta) \in \mathrm{WF}(P\chi u)$. Then, by reversing argument of the previous case, we also have $(x,\xi) \in \mathrm{WF}(\chi u)$ □

With the propagation of singularities theorem at hand we immediately obtain the global version of Proposition 1.9:

Proposition 1.11. *Let $G^{\pm}$ and G be Green's distributions and the commutator distribution defined above. Then,*

$$\begin{aligned}
\mathrm{WF}(G^{\pm}) &= \left\{(x,x,\xi,-\xi) \in T^*(M \times M)\backslash 0\right\} \\
&\qquad \cup \left\{(x,y,\xi,-\eta) \in T^*(M \times M)\backslash 0 \mid (x,\xi) \sim (y,\eta),\ y \in J^{\pm}_{M}(x)\right\}, \\
\mathrm{WF}(G) &= \left\{(x,y,\xi,-\eta) \in T^*(M \times M)\backslash 0 \mid (x,\xi) \sim (y,\eta)\right\}.
\end{aligned}$$

2

Quantum Field Theory

In this chapter we will give an introduction to the theory of quantum fields on curved spacetimes and present some results in the framework of general local covariance. In the first section we introduce the notion of locally covariant quantum field theories and study the field algebra corresponding to the classical theory with equations of motion given by a wave operator. The second section then defines the notion of a locally covariant conformal quantum field theory and adopts the results developed in the first section to this picture. In the last section we proof the bulk to boundary correspondence in asymptotically flat spacetimes for an abstractly defined conformal quantum field theory.

Hence, this chapter will lay down the foundation for the discussion of the vector potential in the following chapter. Moreover, as we show many results in a very general context, we will be able to devote the next chapter more to the distinct properties of the vector potential rather than some general results.

2.1 Locally Covariant Quantization

In their seminal paper [24] Brunetti, Fredenhagen and Verch, inspired by work of Hollands and Wald [15, 16], introduced the principle of general local covariance in local QFT using the language of category theory. Within that framework one can not only rigorously compare quantum field theories on different background spacetimes but also understand quantum fields as natural transformations between certain functors.

2.1.1 Fundamentals

Since this framework is based on categories, we start by defining the two essential categories:

Definition 2.1. Using the unital $*$-algebras (over $\mathbb{R}$ or $\mathbb{C}$) as objects and the injective $*$-homomorphisms as morphisms, we construct a category which we shall denote $*$-Alg. As usual, composition of morphisms is the composition of maps and the unit element is the identity map.

Definition 2.2. We denote by $\mathsf{GlobHyp}$ the category which has as its objects globally hyperbolic spacetimes $\boldsymbol{M} = (M, g)$. The morphisms between any two objects $\boldsymbol{M}, \boldsymbol{M'}$ are the isometric embeddings $\psi : M \hookrightarrow M'$ such that

(i) orientation and time-orientation are preserved and

(ii) the image $\psi(M)$ is an open globally hyperbolic subset of M'.[1]

[1] This is equivalent to demanding that every causal curve in $\boldsymbol{M'}$ whose endpoints lie in $\psi(M)$ is entirely contained in $\psi(M)$.

The composition of morphisms is the composition of maps and the unit element is given by the identity map.

Now we can state the main definitions to state the concept of local covariance and the notion of a quantum field.

Definition 2.3. A *locally covariant quantum field theory* (abbreviated *LCQFT*) is a covariant functor $\mathscr{A} : \mathsf{GlobHyp} \to *\text{-}\mathsf{Alg}$, i.e., we have the diagram

$$\begin{array}{ccc} \boldsymbol{M} & \xrightarrow{\psi} & \boldsymbol{M'} \\ {\scriptstyle \mathscr{A}}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{A}} \\ \mathscr{A}(\boldsymbol{M}) & \xrightarrow[\mathscr{A}(\psi)]{} & \mathscr{A}(\boldsymbol{M'}) \end{array}$$

for all $\boldsymbol{M}, \boldsymbol{M'} \in \mathrm{Obj}(\mathsf{GlobHyp})$ and all arrows $\psi : \boldsymbol{M} \to \boldsymbol{M'}$ in $\mathsf{GlobHyp}$.

The functor $\mathscr{A}$ is called *causal* (i.e., it satisfies Einstein causality) if for any two morphisms $\psi_i \in \mathrm{Hom}_{\mathsf{GlobHyp}}(\boldsymbol{M}_i, \boldsymbol{M})$, $i = 1, 2$ with causally separated images of ψ_i in $\boldsymbol{M}$ one has

$$\big[\mathscr{A}(\psi_1)(\mathscr{A}(\boldsymbol{M}_1)), \mathscr{A}(\psi_2)(\mathscr{A}(\boldsymbol{M}_2))\big] = \{0\},$$

where $[\mathcal{A}, \mathcal{B}] = \{xy - yx \mid x \in \mathcal{A}, y \in \mathcal{B}\}$ for subalgebras $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$. Further, $\mathscr{A}$ is said to obey the *time-slice axiom* if

$$\mathscr{A}(\psi)(\mathscr{A}(\boldsymbol{M})) = \mathscr{A}(\boldsymbol{M'})$$

holds for all $\psi \in \mathrm{Hom}_{\mathsf{GlobHyp}}(\boldsymbol{M}, \boldsymbol{M'})$ such that $\psi(M)$ contains a Cauchy surface of $\boldsymbol{M'}$, i.e., $\psi(M)$ is an open globally hyperbolic neighbourhood of the Cauchy surface.

Within this formalism quantum fields can be seen as natural transformations between the functor $\mathscr{A}$ that generates the $*$-algebra of the quantum field theory and a functor $\mathscr{D} : \mathsf{GlobHyp} \to \mathsf{Top}$ that assigns to each spacetime a topological vector space of 'test functions', e.g. a set of compactly supported distributional sections of a vector bundle $E_{\boldsymbol{M}} \to M$. Accordingly,

$$\mathscr{D}(\boldsymbol{M}) \subset \Gamma(E_{\boldsymbol{M}})' \quad \text{and} \quad \mathscr{D}(\psi) \doteq \psi_*$$

for all $\psi \in \mathrm{Hom}_{\mathsf{GlobHyp}}(\boldsymbol{M}, \boldsymbol{M'})$.

Definition 2.4. A *quantum field* is a natural transformation Φ in Top between the functors $\mathscr{A}$ and $\mathscr{D}$. That is, given two objects $\boldsymbol{M}, \boldsymbol{M'} \in \mathrm{Obj}(\mathsf{GlobHyp})$, the morphisms $\Phi_{\boldsymbol{M}}$ and $\Phi_{\boldsymbol{M'}}$ make the diagram

$$\begin{array}{ccc} \mathscr{D}(\boldsymbol{M}) & \xrightarrow{\Phi_{\boldsymbol{M}}} & \mathscr{A}(\boldsymbol{M}) \\ {\scriptstyle \psi_*}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{A}(\psi)} \\ \mathscr{D}(\boldsymbol{M'}) & \xrightarrow[\Phi_{\boldsymbol{M'}}]{} & \mathscr{A}(\boldsymbol{M'}) \end{array}$$

commute for all $\psi \in \mathrm{Hom}_{\mathsf{GlobHyp}}(\boldsymbol{M}, \boldsymbol{M'})$.

In many cases a quantum field $\Phi_{\boldsymbol{M}}$ may be seen as a $*$-algebra-valued distribution, i.e., an element of $\mathscr{D}(\boldsymbol{M})' \otimes \mathscr{A}(\boldsymbol{M})$. In general, however, $\Phi_{\boldsymbol{M}}$ may not even be linear and thus not a distribution.

2.1.2 The Field Algebra

An example of a LCQFT is the functor $\mathscr{F}^0 : \mathsf{GlobHyp} \to *\text{-}\mathsf{Alg}$ generating the Borchers-Uhlmann algebra $\mathfrak{F}^0_{\boldsymbol{M}}$ for each spacetime $\boldsymbol{M}$ if $\mathscr{D}(\boldsymbol{M})$ is a complex topological $\mathbb{C}$-vector space.

Definition 2.5. The *Borchers-Uhlmann algebra* $\mathfrak{F}^0_{\boldsymbol{M}}$ is the free unital $*$-algebra generated by $\mathscr{D}(\boldsymbol{M})$, i.e.,

$$\mathfrak{F}^0_{\boldsymbol{M}} \doteq \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathscr{D}(\boldsymbol{M})^{\otimes n}$$

with only a finite number of terms non-zero. The $*$-operation is defined by

$$\boldsymbol{f}^* = \bigoplus_{n=1}^{\infty} (f_{n,1} \otimes f_{n,2} \otimes \cdots \otimes f_{n,n-1} \otimes f_{n,n})^* = \bigoplus_{n=1}^{\infty} (\overline{f_{n,n}} \otimes \overline{f_{n,n-1}} \otimes \cdots \otimes \overline{f_{n,2}} \otimes \overline{f_{n,1}}),$$

where $f_{i,j} \in \mathscr{D}(\boldsymbol{M})$.[2]

Proposition 2.1. $\mathscr{F}^0 : \mathsf{GlobHyp} \to *\text{-}\mathsf{Alg}$ *defined by*

$$\mathscr{F}^0(\boldsymbol{M}) \doteq \mathfrak{F}^0_{\boldsymbol{M}} \quad \textit{and} \quad \mathscr{F}^0(\psi) \doteq \psi_*$$

for each $\boldsymbol{M}, \boldsymbol{M'} \in \mathsf{Obj}(\mathsf{GlobHyp})$ *and any arrow* $\psi \in \mathrm{Hom}_{\mathsf{GlobHyp}}(\boldsymbol{M}, \boldsymbol{M'})$ *is a functor and therefore a LCQFT. By abuse of notation, the $*$-algebra homomorphism ψ_* applied to $\mathfrak{F}^0_{\boldsymbol{M}}$ shall mean that the push-forward ψ_* has to be applied to each factor of each summand of all elements of $\mathfrak{F}^0_{\boldsymbol{M}}$.*

Proof. Given another arrow $\psi' \in \mathrm{Hom}_{\mathsf{GlobHyp}}(\boldsymbol{M'}, \boldsymbol{M''})$, compatibility with composition follows from $\psi_* \circ \psi'_* = (\psi \circ \psi')_*$. Moreover, $\mathscr{U}$ preserves the unit since the unit element of $\mathfrak{F}^0(\boldsymbol{M})$ is $\mathbf{1} = 1 \oplus 0 \oplus \cdots$ regardless of the spacetime $\boldsymbol{M}$. □

Working with Borcher-Uhlmann algebra has the disadvantage of neglecting the dynamical content of the theory. In particular, it yields a LCQFT which is neither causal nor does it satisfy the time-slice axiom. To obtain a functor which has these properties we will equip the Borchers-Uhlmann algebra with two relations containing the dynamics: A field equation and *canonical commutation relations* (abbreviated *CCRs*).

Let $\psi \in \mathrm{Hom}_{\mathsf{GlobHyp}}(\boldsymbol{M}, \boldsymbol{M'})$ be an isometry between two arbitrary spacetimes $\boldsymbol{M}, \boldsymbol{M'} \in \mathsf{Obj}(\mathsf{GlobHyp})$. First, we assign to every spacetime $\boldsymbol{M}$ a vector bundle $E_{\boldsymbol{M}} \to M$ such that $\psi_* E_{\boldsymbol{M}} = E_{\boldsymbol{M'}}$. Secondly, we need an inner product $(\cdot,\cdot)_{\boldsymbol{M}}$ on each $E_{\boldsymbol{M}}$ such that $\psi^*(\cdot,\cdot)_{\boldsymbol{M'}} = (\psi^*(\cdot), \psi^*(\cdot))_{\boldsymbol{M}}$. Last, we set $\mathscr{D}(\boldsymbol{M}) = \Gamma_0(E_{\boldsymbol{M}})$. Then, for every spacetime $\boldsymbol{M} \in \mathsf{Obj}(\mathsf{GlobHyp})$ the field equations are given by a formally self-adjoint wave operator $P_{\boldsymbol{M}}$ acting on a $\mathscr{D}(\boldsymbol{M})$ and the CCRs are given by the commutator distribution $G_{\boldsymbol{M}}$ which is obtained from the causal propagator associated to $P_{\boldsymbol{M}}$ (cf. Eq. (1.19)). To have a well-defined locally covariant theory, we require the wave operator P to be locally covariant, i.e., $\psi_* \circ P_{\boldsymbol{M}} = P_{\boldsymbol{M'}} \circ \psi_*$ must hold for all $\psi \in \mathrm{Hom}_{\mathsf{GlobHyp}}(\boldsymbol{M}, \boldsymbol{M'})$.

[2] We use boldface characters for elements of the algebra to distinguish them from their individual summands. Moreover, we will sometimes omit trivial summands, e.g. we identify $c \in \mathbb{C}$ with $c \oplus 0 \oplus \cdots \in \mathfrak{F}^0(\boldsymbol{M})$ and $f \in \mathscr{D}(\boldsymbol{M})$ with $0 \oplus f \oplus 0 \oplus \cdots \in \mathfrak{F}^0(\boldsymbol{M})$.

Definition 2.6. The *field algebra* $\mathcal{F}_M$ is given by the quotient

$$\mathcal{F}_M \doteq \mathcal{F}^0_M / \mathcal{I},$$

where $\mathcal{I}$ is the closed $*$-ideal generated by

$$\ker G_M \cap \mathscr{D}(\boldsymbol{M}) \quad \text{and} \quad -iG_M(f \otimes h) \oplus f \otimes h - h \otimes f$$

for all $f, h \in \mathscr{D}(\boldsymbol{M})$.

Note that quotienting by the $*$-ideal generated from the field equations by

$$\ker G_M \cap \mathscr{D}(\boldsymbol{M}) = \{P_M u \mid u \in \mathscr{D}(\boldsymbol{M})\}$$

we have that two different test functions to which the causal propagator assigns the same classical field correspond to the same algebra element (cf. Proposition 1.8 and the discussion which preceeds it).

Proposition 2.2. *Let $\boldsymbol{M}, \boldsymbol{M}' \in \mathrm{Obj}(\mathsf{GlobHyp})$ and $\psi \in \mathrm{Hom}_{\mathsf{GlobHyp}}(\boldsymbol{M}, \boldsymbol{M}')$. Denote by q_M the projection map $\mathcal{F}^0_M \to \mathcal{F}_M$. Then $\mathscr{F} : \mathsf{GlobHyp} \to *\text{-}\mathsf{Alg}$ defined by*

$$\mathscr{F}(\boldsymbol{M}) \doteq \mathcal{F}_M \quad \textit{and} \quad \mathscr{F}(\psi) \doteq q_{M'} \circ \psi_*$$

is a functor and therefore a LCQFT.

Proof. As a consequence of the local covariance of P and the uniqueness of the causal propagator we have $\psi_* \circ G_M = G_{M'} \circ \psi_*$ which then yields

$$\psi_* G_M(f \otimes h) = G_{M'}(\psi_* f \otimes \psi_* h)$$

for all $f, h \in \mathscr{D}(\boldsymbol{M})$. Thus, if we denote by $\mathcal{I}$ and $\mathcal{I}_\psi$ the $*$-ideals such that $\mathcal{F}_M = \mathcal{F}^0_M / \mathcal{I}$ and $\mathcal{F}_{M'} \supset \mathcal{F}_{\psi(M)} = \mathcal{F}^0_{\psi(M)} / \mathcal{I}_\psi$, we have $\psi_* \mathcal{I} = \mathcal{I}_\psi$ and consequently $q_{M'} \circ \psi_*$ is a well-defined $*$-algebra homomorphism. Then, the covariance of the functor $\mathscr{F}$ follows from Proposition 2.1 and the injectivity of the projection map. □

Having included these abstract relations containing the dynamics of the field in the definition of $\mathscr{F}$, we can show their physical implications:

Proposition 2.3. *The LCQFT $\mathscr{F}$ is causal and satisfies the time-slice axiom.*

Proof. Let $\psi_i \in \mathrm{Hom}_{\mathsf{GlobHyp}}(\boldsymbol{M}_i, \boldsymbol{M})$, $i = 1, 2$ with causally separated images of ψ_i in $\boldsymbol{M}$. Then, using the CCR relation in the field algebra,

$$[\boldsymbol{f}, \boldsymbol{h}] = f \otimes h - h \otimes f = iG_M(f \otimes h) = 0$$

for all $0 \oplus f \oplus 0 \oplus \cdots = \boldsymbol{f} \in \mathscr{F}(\psi_1)(\mathcal{F}_{M_1})$ and $0 \oplus h \oplus 0 \oplus \cdots = \boldsymbol{h} \in \mathscr{F}(\psi_2)(\mathcal{F}_{M_2})$ because $J_M(\operatorname{supp} f) \cap \operatorname{supp} h = \emptyset$. This proves causality.

To see that the time-slice axiom is fulfilled we use a proof similiar to one used by Sanders [52] for the scalar field (which was adapted from [49]). Consider $\psi \in \mathrm{Hom}_{\mathsf{GlobHyp}}(\boldsymbol{M}', \boldsymbol{M})$ such that $\mathcal{O} = \psi(M')$ contains a Cauchy surface of $\boldsymbol{M}$. Given any $f \in \mathscr{D}(\boldsymbol{M})$, we construct

$$h = (1 - \chi) G^+_M f + \chi G^-_M f,$$

where $\chi \in C^\infty(M)$ with $\chi \restriction_{J^+_M(\mathcal{O})\setminus\mathcal{O}} = 1$ and $\chi \restriction_{J^-_M(\mathcal{O})\setminus\mathcal{O}} = 0$. Then,

$$f - P_M h = P_M(\chi G_M f) \quad \text{with} \quad \operatorname{supp}(P_M(\chi G_M f)) \subset \mathcal{O}, \tag{2.1}$$

and the result follows by the field equation relation in the field algebra $\mathscr{F}(\boldsymbol{M})$. □

We can now find a quantum field Φ in the sense of Definition 2.4 as a natural transformation between $\mathscr{D}$ and $\mathscr{F}$:

Proposition 2.4. *There is a quantum field $\Phi : \mathscr{D} \dot{\to} \mathscr{F}$ given by*

$$\Phi_M(f) \doteq q_M(\boldsymbol{f}) \quad \textit{with} \quad \boldsymbol{f} = 0 \oplus f \oplus 0 \oplus \cdots$$

for each $\boldsymbol{M} \in \mathrm{Obj}(\mathsf{GlobHyp})$ and any $f \in \mathscr{D}(\boldsymbol{M})$.

Proof. Φ has the properties of a natural transformation since

$$\mathscr{F}(\psi)(\Phi_M(f)) = q_{M'}(\psi_* q_M(\boldsymbol{f})) = q_{M'}(q_{\psi(M)}(\psi_* \boldsymbol{f})) = q_{M'}(\psi_* \boldsymbol{f}) = \Phi_{M'}(\psi_* f)$$

for all arrows $\psi \in \mathrm{Hom}_{\mathsf{GlobHyp}}(\boldsymbol{M}, \boldsymbol{M}')$. □

2.2 Locally Covariant Conformal Quantization

In Sect. 2.1 we introduced the notion of locally covariant quantum field theory. When working with conformally invariant fields like the massless scalar, Dirac and vector fields, one can extend this concept: Instead of the category $\mathsf{GlobHyp}$ we may also consider a category with a bigger class of morphisms including conformal embeddings as done by Pinamonti [25]. Note, however, that the definitions found here differ slightly from those of [25].[3]

2.2.1 Fundamentals

Definition 2.7. Denote by $\mathsf{CGlobHyp}$ the extension of the category $\mathsf{GlobHyp}$ (cf. Definition 2.2) which has as its morphism *conformal* embeddings $\psi : M \hookrightarrow \widetilde{M}$, such that

(i) ψ preserves orientation and time-orientation and

(ii) the image $\psi(M)$ is an open globally hyperbolic subset of $\widetilde{M}$.

That is, $\mathsf{GlobHyp}$ is the subcategory of $\mathsf{CGlobHyp}$ where all non-isometric arrows have been removed. A conformal quantum field theory is then defined analogously to a LCQFT (cf. Definition 2.3):

Definition 2.8. A *locally covariant conformal quantum field theory* (abbreviated *LCCQFT*) is a covariant functor $\mathscr{C}\!\mathscr{A} : \mathsf{CGlobHyp} \to *\text{-}\mathsf{Alg}$, i.e., we have

$$\begin{array}{ccc} \boldsymbol{M} & \xrightarrow{\quad\psi\quad} & \widetilde{\boldsymbol{M}} \\ {\scriptstyle \mathscr{C}\mathscr{A}}\downarrow & & \downarrow{\scriptstyle \mathscr{C}\mathscr{A}} \\ \mathscr{C}\mathscr{A}(\boldsymbol{M}) & \xrightarrow[\mathscr{C}\mathscr{A}(\psi)]{} & \mathscr{C}\mathscr{A}(\widetilde{\boldsymbol{M}}) \end{array}$$

for all $\boldsymbol{M}, \widetilde{\boldsymbol{M}} \in \mathrm{Obj}(\mathsf{CGlobHyp})$ and all arrows $\psi \in \mathrm{Hom}_{\mathsf{CGlobHyp}}(\boldsymbol{M}, \widetilde{\boldsymbol{M}})$.

[3] In [25] Pinamonti doesn't consider a possible conformal weight of the inner product on the vector bundle.

The introduction of conformal quantum fields needs somewhat more care. First we define the functor $\mathscr{CD} : \mathsf{CGlobHyp} \to \mathsf{Top}$ similar to $\mathscr{D}$ such that it assigns to each object in CGlobHyp a topological vector space of test functions of a specific conformal weight. Typically, $\mathscr{CD}$ will have as its objects spaces of compactly supported distributional sections of a specific conformal weight $w-2$ and the morphisms that $\mathscr{CD}$ induces are conformal push-forwards.

Therefore, let $E_{\boldsymbol{M}} \to M$ be a vector bundle associated to every $\boldsymbol{M}$ such that $\psi_* E_{\boldsymbol{M}} = E_{\widetilde{\boldsymbol{M}}}$ for all $\psi \in \mathrm{Hom}_{\mathsf{CGlobHyp}}(\boldsymbol{M}, \widetilde{\boldsymbol{M}})$. Equip each of these vector bundles $E_{\boldsymbol{M}}$ with an inner product $(\cdot,\cdot)_{\boldsymbol{M}}$ such that

$$\psi^*(\cdot,\cdot)_{\widetilde{\boldsymbol{M}}} = \Omega^{\nu}\,(\psi^*(\cdot), \psi^*(\cdot))_{\boldsymbol{M}}.$$

We call ν the conformal weight of the inner product. Morveover, we set $\mathscr{CD}(\boldsymbol{M}) \subset \Gamma^{w}(E_{\boldsymbol{M}})'$. A conformal quantum field is then defined as follows:

Definition 2.9. A *conformal quantum field* Φ of conformal weight[4] $-\nu - w - 4$ is a linear natural transformation between the functor $\mathscr{CD}$ and a LCCQFT $\mathscr{CA}$.

2.2.2 The Field Algebra

Having the notion of a LCCQFT at our disposal, we can now define the field algebra for a conformal quantum field theory. The resulting definition is very similar to Definition 2.6.

Let $\mathscr{CD}(\boldsymbol{M}) = \Gamma_0^{w-2}(E_{\boldsymbol{M}})$. Equations (1.20) and (1.21) show that, if we want to have a connection between the classical and quantum theory, namely, that classical and quantum field have equal conformal weight w and that we have a commutator given by the commutator distribution, $(\cdot,\cdot)_{\boldsymbol{M}}$ has to be of conformal weight $-2-2w$. If that is the case, the commutator distribution is conformally invariant, i.e., $G_{\boldsymbol{M}}(f \otimes h) = G_{\widetilde{\boldsymbol{M}}}(\psi_*^{w-2} f \otimes \psi_*^{w-2} h)$ for $f, h \in \Gamma_0^{w-2}(E_{\boldsymbol{M}})$ and $\psi \in \mathrm{Hom}_{\mathsf{CGlobHyp}}(\boldsymbol{M}, \widetilde{\boldsymbol{M}})$.

Definition 2.10. The *conformal field algebra* $\mathcal{CF}_{\boldsymbol{M}}$ is given by the quotient

$$\mathcal{CF}_{\boldsymbol{M}} \doteq \mathcal{CF}_{\boldsymbol{M}}^0/\mathcal{I} \quad \text{with} \quad \mathcal{CF}_{\boldsymbol{M}}^0 \doteq \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathscr{CD}(\boldsymbol{M})^{\otimes n},$$

where $\mathcal{I}$ is the closed $*$-ideal generated by the elements

$$\ker G_{\boldsymbol{M}} \cap \mathscr{CD}(\boldsymbol{M}) \quad \text{and} \quad -iG_{\boldsymbol{M}}(f \otimes h) \oplus f \otimes h - h \otimes f$$

for all $f, h \in \mathscr{CD}(\boldsymbol{M})$. Moreover, the $*$-operation is given by complex conjugation and the reversal of order in the tensor products as in Definition 2.5.

Proposition 2.5. *The field algebra* $\mathcal{CF}_{\boldsymbol{M}}$ *extends to a LCCQFT* $\mathscr{CF} : \mathsf{CGlobHyp} \to *\text{-}\mathsf{Alg}$ *via the assignments*

$$\mathscr{CF}(\boldsymbol{M}) \doteq \mathcal{CF}_{\boldsymbol{M}} \quad \textit{and} \quad \mathscr{CF}(\psi) \doteq q_{\widetilde{\boldsymbol{M}}} \circ \psi_*^{w-2}$$

[4]The conformal weight is choosen such that the conformal weights of the test sections, the inner product and the measure are cancelled.

for all $\psi \in \mathrm{Hom}_{\mathsf{CGlobHyp}}(\boldsymbol{M}, \widetilde{\boldsymbol{M}})$, *where we denote by* $q_{\boldsymbol{M}}$ *the projection map* $q_{\boldsymbol{M}} : \mathfrak{CF}^0_{\boldsymbol{M}} \to \mathfrak{CF}_{\boldsymbol{M}}$ *and, by abuse of notation,* ψ_*^{w-2} *applied to an element of* $\mathfrak{CF}_{\boldsymbol{M}}$ *means that* ψ_*^{w-2} *has to be applied to each factor of each sum in* $\mathfrak{CF}_{\boldsymbol{M}}$. *Furthermore,* $\mathscr{CF}$ *is causal and satisfies the time-slice axiom with respect to conformal embeddings.*

Proof. Analogous to Proposition 2.2 combined with Proposition 2.3. □

Proposition 2.6. *There is a conformal quantum field* $\Phi : \mathscr{CD} \dot{\to} \mathscr{CF}$ *of conformal weight* w *given by*

$$\Phi_{\boldsymbol{M}}(f) \doteq q_{\boldsymbol{M}}(\boldsymbol{f}) \quad \text{with} \quad \boldsymbol{f} = 0 \oplus f \oplus 0 \oplus \cdots$$

for each $\boldsymbol{M} \in \mathrm{Obj}(\mathsf{CGlobHyp})$ *and any* $f \in \mathscr{CD}(\boldsymbol{M})$.

Proof. Analogous to Proposition 2.4 □

2.3 Hadamard States

Our motivation for studying the wave front set of distributions on manifolds in Sect. 1.5.2 was that, as shown by Radzikowski [11] for the scalar field and in more generality by Sahlmann and Verch [12], a condition on the wave front set of two-point distributions can be related to the Hadamard condition (cf. [8] by Wald). The Hadamard condition is believed to be a necessary condition for physical states of quantum fields on globally hyperbolic spacetimes because Hadamard states mimic the UV-behaviour of the Minkowski vacuum. Moreover, applying an appropriate renormalization prescription, Hadamard states can be seen to be states with finite energy density.

First, we will specify how to associate a two-point distributions to a state and give some of its properties. Afterwards, we introduce the microlocal spectrum condition to single out the two-point distributions of Hadamard form.

Let $\boldsymbol{M}$ be a globally hyperbolic spacetime and $\mathscr{A}$ a locally covariant (conformal) quantum field theory with test functions $\mathscr{D}$ and (conformal) quantum field Φ. We can associate to a state ω on $\mathscr{A}(\boldsymbol{M})$ a hierarchy of *n-point distributions* $\{\omega_n\}_{n\in\mathbb{N}}$, $\omega_n : \mathscr{D}(\boldsymbol{M})^{\otimes n} \to \mathbb{C}$ via

$$\omega_n(f_1 \otimes \ldots \otimes f_n) \doteq \omega(\Phi_{\boldsymbol{M}}(f_1) \otimes \cdots \otimes \Phi_{\boldsymbol{M}}(f_n)).$$

Definition 2.11. We say that the state ω is *quasi-free* if all its odd n-point distributions vanish while the even n-point distributions can be expressed using only the two-point distribution ω_2:

$$\omega_n(f_1 \otimes \ldots \otimes f_n) = \sum_\sigma \omega_2(f_{\sigma(1)} \otimes f_{\sigma(2)}) \cdots \omega_2(f_{\sigma(n-1)} \otimes f_{\sigma(n)}),$$

where the sum is over all ordered pairings, that is, over all permutations σ of $\{1,\ldots,n\}$ such that $\sigma(1) < \sigma(3) < \cdots < \sigma(n-1)$ and $\sigma(1) < \sigma(2), \ldots, \sigma(n-1) < \sigma(n)$. Hence, a quasi-free state is determined entirely by its two-point distribution.

Equivalently, a quasi-free state can be defined as a state ω on $\mathscr{A}(\boldsymbol{M})$ with quantum field $\Phi_{\boldsymbol{M}}$ which satisfies

$$\omega(e^{i\Phi_M(f)}) = e^{-\frac{1}{2}\omega(f\otimes f)} \tag{2.2}$$

with $f \in \mathscr{D}(\boldsymbol{M})$ in the sense that all the identities (the n-point distributions) which can be obtained from (2.2) by functional differentiation must be satisfied. Hence, the right-hand side may be understood as the generating functional of the n-point distributions.

The two-point distribution ω_2 to a state ω on a (conformal) field algebra $\mathscr{F}(\boldsymbol{M})$ with wave operator $P_{\boldsymbol{M}}$ and commutator distribution $G_{\boldsymbol{M}}$ has the following properties:

(1) Positivity, $\omega_2(f \otimes \overline{f}) \geq 0$;

(2) Field equations, $\omega_2(P_{\boldsymbol{M}} f \otimes h) = 0 = \omega_2(f \otimes P_{\boldsymbol{M}} h)$;

(3) CCRs, $\omega_2(f \otimes h) - \omega_2(h \otimes f) = i\, G_{\boldsymbol{M}}(f \otimes h)$.

for any $f, h \in \mathscr{D}(\boldsymbol{M})$. Property (1) follows from the positivity of the state ω whereas properties (2) and (3) follow from the relations imposed on the field algebra defined in Definition 2.6 (respectively Definition 2.10).

We are interested in states that satisfy an appropriate generalization of the spectrum condition from axiomatic QFT on Minkowski spacetime.[5] It is not possible to formulate such a condition globally on arbitrary spacetimes due to the absence of a timelike Killing field. Nevertheless, a microlocal approach can be taken:

Definition 2.12. A two-point distribution ω_2 satisfies the *microlocal spectrum condition* if

$$\mathrm{WF}(\omega_2) = \{(x, y, \xi, -\eta) \in T^*(M \times M)\backslash 0 \mid (x, \xi) \sim (y, \eta),\ \xi \rhd 0\},$$

where $\sim$ has the meaning described in Proposition 1.9 above.

Using the microlocal spectrum condition, the following can be shown (cf. [11] and [12]):

Theorem 2.1. *Let ω be a quasi-free state on a field algebra. If the associated two-point distribution ω_2 satisfies the microlocal spectrum condition, ω is a* Hadamard state *and we say that ω_2 is of* Hadamard form.

2.4 Bulk to Boundary Correspondence

In QFT on Minkowski spacetime we can use the Poincaré group to select a distinguished state: The vacuum state is the unique state which is invariant under the action of the Poincaré group [1]. In QFT on curved spacetimes we will have, in general, not enough symmetries to have the standard notion of a vacuum state. Nevertheless, if the spacetime under consideration is asymptotically flat, one can do slightly better due to the presence of asymptotic symmetries which give rise to the BMS group. Quantum field theory on asymptotically flat spacetimes was already considered in [54, 55] by Frovlov and Ashtekar respectively. In this section, however, we will describe a construction, introduced by Dappiaggi, Moretti and Pinamonti in [19] and elaborated by Moretti in [20, 21], which yields asymptotic vacuum states, i.e., states which are invariant under the action of supertranslations. In [19, 20, 21] only the scalar field was considered, and in [22] by Dappiaggi,

[5]The spectrum condition requires that the eigenvalues of the energy-momentum operator are future-directed and causal, cf. Streater and Wightman [53].

Hack and Pinamonti and the PhD thesis of Hack [23] the bulk to boundary correspondence was extended to the Dirac field. Here we develop the bulk to boundary correspondence in a more abstract setting already including the scalar field. This will allow us to extend the bulk to boundary construction to the vector potential with only little extra effort in Sect. 3.3.

Let us recall some facts about asymptotically flat spacetimes from Sect. 1.4: All asymptotically flat spacetimes are, by definition, equipped with a universal conformal boundary $\mathscr{I}^-$ called past null infinity. By a choice of Bondi coordinates (u,ζ), $\mathscr{I}^-$ can be treated as an intrinsic manifold with degenerate metric tensor

$$g_{\mathscr{I}^-} \doteq 2(1+\zeta\overline{\zeta})^{-2}(\boldsymbol{d}\zeta\otimes\boldsymbol{d}\overline{\zeta}+\boldsymbol{d}\overline{\zeta}\otimes\boldsymbol{d}\zeta) \tag{2.3}$$

and induced volume form

$$\mu_{\mathscr{I}^-}(u,\zeta) \doteq \boldsymbol{d}u\wedge\mu_{S^2}(\zeta) = -2i\,\boldsymbol{d}u\wedge\frac{\boldsymbol{d}\zeta\wedge\boldsymbol{d}\overline{\zeta}}{(1+\zeta\overline{\zeta})^2} = \sin^2\theta\,\boldsymbol{d}u\wedge\boldsymbol{d}\theta\wedge\boldsymbol{d}\phi,$$

where $\zeta = e^{i\phi}\cot\theta/2$. Note that when we integrate over $\mathscr{I}^- \cong \mathbb{R}\times S^2 \cong \mathbb{R}\times\mathbb{C}_\infty$ with respect to the volume form $\mu_{\mathscr{I}^-}(u,\zeta)$, i.e., in a coordinate chart with the point ∞ of $\mathbb{C}_\infty$ removed, then the integral is still understood as a Lebesgue integral with the integration over $\mathbb{C}\cong\mathbb{R}^2$ performed as an integration over the plane, treating the real and complex part as independet real variables.

Under a *BMS* transformation, i.e., a transformation from a Bondi frame (u,ζ) into another one with coordinates (u',ζ'), where

$$\begin{aligned}
\zeta \to \zeta' &= \frac{a\zeta+b}{c\zeta+d} && \text{with} \quad ad-bc=1,\\
u \to u' &= K(\zeta)(u+\alpha(\zeta)) && \text{with} \quad \alpha\in C^\infty(\mathbb{C}_\infty),\\
K(\zeta)^{-1} &= \frac{|a\zeta+b|^2+|c\zeta+d|^2}{1+|\zeta|^2},
\end{aligned}$$

we have $\mu_{\mathscr{I}^-}(u',\zeta') = K(\zeta)^3\mu_{\mathscr{I}^-}(u,\zeta)$ for the volume form.

2.4.1 The Boundary Algebra

The universality of past null infinity $\mathscr{I}^-$ allows us to study quantities at null infinity which correspond to all asymptotically flat spacetimes. Here, we are going to define a boundary algebra defined intrinsically on $\mathscr{I}^-$ which contains all field algebras on asymptotically flat spacetimes of a conformal quantum field.

Therefore, let $\mathscr{CF}$ be the locally covariant conformal quantum field theory defined in Proposition 2.5 which maps each spacetime to the conformal field algebra $\mathfrak{CF}_M$ constructed from the $\mathbb{C}$-vector bundle $E_M \to M$ on which we have an inner product $(\cdot,\cdot)_M$ of conformal weight $-2-2w$, compactly-supported sections $\Gamma_0^{w-2}(E_M)$ of conformal weight $w-2$, a connection D_M such that $P_M = D_M^* D_M + B_M$ with $B_M \in \Gamma(\operatorname{End} E_M)$ is a formally self-adjoint and conformally invariant wave operator acting on $\Gamma^w(E)$ and a causal propagator G_M for P_M (cf. Definition 2.10). Further, denote by $\Phi_M : \Gamma_0^{w-2}(E_M) \to \mathfrak{CF}_M$ the conformal quantum field constructed as in Proposition 2.6.

For the remainder of this section we also make the following choices: We denote by $\boldsymbol{M} \in \mathrm{Obj}(\mathsf{CGlobHyp})$ an arbitrary asymptotically flat spacetime conformally embedded into the unphysical spacetime $(\widetilde{M}, \widetilde{g}) = \widetilde{\boldsymbol{M}} \in \mathrm{Obj}(\mathsf{CGlobHyp})$ via $\psi : M \hookrightarrow \widetilde{M}$ with conformal factor Ω defined on $\widetilde{M}$. We choose Bondi coordinates (u, ζ) on $\mathscr{I}^-$ and define the conformal embedding $\iota : \mathscr{I}^- \hookrightarrow \widetilde{M}$ with conformal factor ω_B.

Before we construct the algebra on $\mathscr{I}^-$, let us develop a 'classical' theory on the boundary, i.e., we want to find a natural phase space with a sympletic product. The playground is set by the following:

Proposition 2.7. *The pull-back bundle $E_{\mathscr{I}^-} \doteq \iota^* E_{\widetilde{\boldsymbol{M}}}$ is universal for all asymptotically flat spacetimes. It is equipped with the induced inner product $(\cdot, \cdot)_{\mathscr{I}^-}$ defined by*

$$\iota^*(\cdot, \cdot)_{\widetilde{\boldsymbol{M}}} \doteq \omega_B^{-2-2w} \left(\iota^*(\cdot), \iota^*(\cdot)\right)_{\mathscr{I}^-}$$

Proof. Let $\widetilde{\boldsymbol{M}'}$ be another unphysical spacetime for an arbitrary globally hyperbolic asymptotically flat spacetime. Then, we have another conformal embedding $\iota' : \mathscr{I}^- \hookrightarrow \widetilde{M}'$. Composing ι and ι', we obtain the conformal isometry $\iota^{-1} \circ \iota' : \widetilde{M}' \restriction_{\mathscr{I}^-} \to \widetilde{M} \restriction_{\mathscr{I}^-}$. This proves the universality of $E_{\mathscr{I}^-}$ and $(\cdot, \cdot)_{\mathscr{I}^-}$. □

With the inner product just defined, we can introduce yet another pairing:

$$\langle f, h \rangle_{\mathscr{I}^-} \doteq \int_{\mathscr{I}^-} (f, h)_{\mathscr{I}^-} \, \mu_{\mathscr{I}^-}$$

for all $f, h \in \Gamma(E_{\mathscr{I}^-})$ such that the integral is well-defined. Using this, we can define a phase space on the boundary:

Proposition 2.8. *The bilinear form*

$$\varsigma(f, h) \doteq 2 \, \langle f, \partial_u h \rangle_{\mathscr{I}^-}$$

defines a BMS-invariant symplectic product on the boundary phase space

$$\mathscr{B} \doteq \{ f \in \Gamma^w(E_{\mathscr{I}^-}) \mid f, \partial_u f \in L^2(E_{\mathscr{I}^-}, \mu_{\mathscr{I}^-}) \}$$

of smooth sections of conformal weight w which are square-integrable and have square-integrable derivative in u-direction.

Proof. Since the smoothness in combination with the square-integrability implies $\lim_{u \to \pm\infty} f = 0$, we can use integration by parts to see that $\varsigma(\cdot, \cdot)$ is antisymmetric. Moreover, this bilinear form is weakly non-degenerate because $(\cdot, \cdot)_{\mathscr{I}^-}$ is non-degenerate and $\ker \partial_u \cap \mathscr{B} = \{0\}$.

Consider a BMS transformation $(u, \zeta) \mapsto (u', \zeta')$ with conformal factor $K(\zeta)^{-1}$ and supertranslation $\alpha(\zeta)$. This yields the transformation

$$\langle \partial_u f, h \rangle_{\mathscr{I}^-} \mapsto \langle \partial_{u'} f', h' \rangle_{\mathscr{I}^-} = \int_{\mathscr{I}^-} K^{-2-2w} (K^w f, K^{-1} \partial_u K^w h)_{\mathscr{I}^-} K^3 \mu_{\mathscr{I}^-} = \langle f, \partial_u h \rangle_{\mathscr{I}^-},$$

where $f = f(u, \zeta)$, $h = h(u, \zeta)$, $\mu_{\mathscr{I}^-} = \mu_{\mathscr{I}^-}(u, \zeta)$ and $f' = f(u', \zeta')$, $h' = h(u', \zeta')$, because the integral is translation invariant. It follows that the symplectic form is invariant under the action of the BMS group. □

If we make additional assumptions on the normally hyperbolic operator respectively the metric connection D_M, we can show that the phase space $(\mathscr{B}, \varsigma(\cdot,\cdot))$ is intimately related to the phase space $\mathscr{P}(M) = \Gamma_0^{w-2}(E_M)/\ker G_M$ in the bulk spacetime, as defined in (1.18), equipped with the symplectic product $\sigma(\cdot,\cdot)$ given by the commutator distribution G_M.

Proposition 2.9. *The* bulk to boundary map

$$b \doteq \iota_w^* \circ G^-_{\widetilde{M}} \circ \psi_*^{w-2}$$

is a symplectomorphism from the bulk phase space into the boundary phase space if

$$\iota^*(D_{\widetilde{M}} G^-_{\widetilde{M}} \psi_*^{w-2} f)(\partial_u) = \partial_u \omega_B^w b(f)$$

is satisfied for all $f \in \mathscr{P}(M)$.

Proof. Let $[f],[h] \in \mathscr{P}(M)$ with representatives $f,h \in \Gamma_0^{w-2}(E_M)$. Further, set $\widetilde{f} = \psi_*^{w-2} f$ and $\widetilde{h} = \psi_*^{w-2} h$. The bulk to boundary map is independent of the chosen representative because

$$\psi_*^{w-2} \ker G_M \subset \ker G_{\widetilde{M}} \quad \text{and} \quad \iota^* \circ G_{\widetilde{M}} \circ \psi_*^{w-2} = \iota^* \circ G^-_{\widetilde{M}} \circ \psi_*^{w-2}.$$

As $J^-_{\widetilde{M}}(\operatorname{supp} \psi_* f) \cap J^+_{\widetilde{M}}(i^-)$ is compact, we see that $\operatorname{supp} G^-_{\widetilde{M}} \psi_* f \cap (\mathscr{I}^- \cup \{i^-\})$ is compact too. Hence, we must have that $\omega_B b(f)$ and $\omega_B^{-1} \partial_u b(f)$ are square integrable with respect to the volume form $\mu_{\mathscr{I}^-}$ and, because ω_B is bounded and non-zero even that $b(f), \partial_u b(f) \in L^2(E_{\mathscr{I}^-}, \mu_{\mathscr{I}^-})$. It follows that $b(f) \in \mathscr{B}$.

Since the commutator distribution is conformally invariant, we obtain

$$G_M(f \otimes h) = G_{\widetilde{M}}(\widetilde{f} \otimes \widetilde{h}) = G_{\widetilde{M}}(\widetilde{f} \otimes P_{\widetilde{M}} G^-_{\widetilde{M}} \widetilde{h}).$$

Applying Green's identity (1.16), gives[6]

$$\begin{aligned} G_M(f \otimes h) &= \int_{\mathscr{I}^-} \iota^* \widetilde{*}\big((G^-_{\widetilde{M}} \widetilde{f}, D_{\widetilde{M}} G^-_{\widetilde{M}} \widetilde{h})_{\widetilde{M}} - (D_{\widetilde{M}} G^-_{\widetilde{M}} \widetilde{f}, G^-_{\widetilde{M}} \widetilde{h})_{\widetilde{M}}\big) \\ &= \int_{\mathscr{I}^-} \omega_B^{-2-2w} \big((\omega_B^w b(f), \omega_B^{-1} \partial_u \omega_B^w b(h))_{\mathscr{I}^-} \\ &\qquad - (\omega_B^{-1} \partial_u \omega_B^w b(f), \omega_B^w b(h))_{\mathscr{I}^-}\big)\, \omega_B^3 \mu_{\mathscr{I}^-}, \end{aligned}$$

where $\widetilde{*}$ is the Hodge operator with respect to the unphysical metric $\widetilde{g}$. Then, the derivative of ω_B^w cancels due to the asymmetry, and an integration by parts gives the equality of the symplectic products, i.e.,

$$G_M(f \otimes h) = \langle b(f), \partial_u b(h) \rangle_{\mathscr{I}^-} - \langle \partial_u b(f), b(h) \rangle_{\mathscr{I}^-} = \varsigma(b(f), b(h)). \qquad \square$$

Henceforth, we always assume that the connection D is such that it satisfies the condition in Proposition 2.9. Having constructed the classical theory on the boundary, we are now ready to define the *boundary algebra* which is an algebra that is universal for all asymptotically flat spacetimes.

[6] Note that the locus i^- has measure zero.

Definition 2.13. The *boundary algebra* $\mathcal{B}$ is the quotient of the free unital $*$-algebra $\mathcal{B}^0$ by the $*$-ideal $\mathcal{I}$ imposing the commutation relations:

$$\mathcal{B} \doteq \mathcal{B}^0/\mathcal{I} \quad \text{with} \quad \mathcal{B}^0 \doteq \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathscr{B}^{\otimes n},$$

where $\mathcal{I}$ is generated by elements of the form

$$-i\varsigma(f,h) \oplus f \otimes h - h \otimes f \quad \text{with} \quad f,h \in \mathscr{B}.$$

The $*$-operation is given by complex conjugation and the reversal of order in the tensor products as in Definition 2.5.

Just as the bulk to boundary map gives a map between the bulk and the boundary phase space, there exists an injective $*$-algebra homomorphism between the field algebra in the bulk spacetime and the boundary algebra.

Proposition 2.10. *The bulk to boundary map b induces an injective $*$-algebra homomorphism $b : \mathcal{CF}_M \to \mathcal{B}$ defined by*

$$b(\Phi_M(f)) = \Phi_M(b(f))$$

for all $f \in \Gamma_0^w(E_M)$.

Proof. Since the bulk to boundary map is a symplectomorphism, this is an immediate consequence of Definition 2.13. □

2.4.2 The Boundary State

Having constructed a boundary algebra, we can consider states on this algebra. To obtain a positive state using the methods we do below, we have to make another assumption: From now on we shall assume that the inner product $(\cdot,\cdot)_{\mathscr{I}^-}$ is positive definite for all real-valued $f \in \mathscr{B}$.

Proposition 2.11. *The two-point distribution*

$$\beta_2(f \otimes h) \doteq \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int\limits_{\mathbb{R}^2 \times \mathbb{C}} \frac{(f(u,\zeta), h(u',\zeta))_{\mathscr{I}^-}}{(u-u'-i\epsilon)^2} \, du \, du' \, \mu_{S^2}(\zeta) \tag{2.4}$$

with $f,h \in \mathscr{B}$ uniquely determines a quasi-free state β on the boundary algebra $\mathcal{B}$.

Proof. The two-point distribution β_2 is well-defined as can be seen from Example 1.4, because $f(u,\zeta)$ and $\partial_{u'} h(u',\zeta)$ are square-integrable with respect to $\mu_{\mathscr{I}^-}$. Furthermore, the state is uniquely defined via (2.2).

To see that positivity is satisfied, we employ a (component-wise) Fourier transformation in u-direction to obtain (cf. Eq. (1.14))

$$\beta_2(f \otimes h) = \frac{1}{\pi} \int\limits_{\mathbb{R} \times \mathbb{C}} k \, \Theta(k) (\widehat{f}(k,\zeta), \widehat{h}(-k,\zeta))_{\mathscr{I}^-} \, dk \, \mu_{S^2}(\zeta). \tag{2.5}$$

Thus, we have

$$\beta_2(f \otimes \overline{f}) = \frac{1}{\pi} \int\limits_{\mathbb{R}\times\mathbb{C}} k\,\Theta(k)(\widehat{f}(k,\zeta), \widehat{\overline{f}}(k,\zeta))_{\mathscr{I}^-}\, dk\, \mu_{\mathbb{S}^2}(\zeta),$$

which proves the positiviy of β_2 by the properties of the inner product $(\cdot,\cdot)_{\mathscr{I}^-}$.

To see that the state also satisfies the commutation relations, we calculate

$$\begin{aligned}\beta_2(h \otimes f) &= \lim_{\epsilon\downarrow 0} \frac{1}{\pi} \int\limits_{\mathbb{R}^2\times\mathbb{C}} \frac{(f(u,\zeta), h(u',\zeta))_{\mathscr{I}^-}}{(u'-u-i\epsilon)^2}\, du\, du'\, \mu_{\mathbb{S}^2}(\zeta) \\ &= \lim_{\epsilon\downarrow 0} \frac{1}{\pi} \int\limits_{\mathbb{R}^2\times\mathbb{C}} \frac{(f(u,\zeta), h(u',\zeta))_{\mathscr{I}^-}}{(u-u'+i\epsilon)^2}\, du\, du'\, \mu_{\mathbb{S}^2}(\zeta).\end{aligned}$$

Doing an integration by parts, we get

$$\beta_2(f \otimes h) = \lim_{\epsilon\downarrow 0} -\frac{1}{\pi} \int\limits_{\mathbb{R}^2\times\mathbb{C}} \frac{(f(u,\zeta), \partial_{u'} h(u',\zeta))_{\mathscr{I}^-}}{u-u'-i\epsilon}\, du\, du'\, \mu_{\mathbb{S}^2}(\zeta),$$

and thus it follows from (1.13) combined with Example 1.4 that $\beta_2(f \otimes h) - \beta_2(h \otimes f) = i\varsigma(f,h)$. □

The proof of the last proposition shows the construction of the boundary state in reverse: One starts with the symplectic form (2.8) and rewrites it to take the form

$$i\varsigma(f,h) = 2i\,\delta^{\Delta}(f \otimes \partial_u h),$$

where ∂_u is here symbolically for the derivative along the $\mathbb{R}$-coordinate. Then, one uses the splitting of the diagonal distribution into $\delta^{\pm\Delta}$ (cf. (1.13) and Example 1.4) in the 'u-direction' and calls the $\delta^{+\Delta}$-term $\beta_2(f \otimes h)$. This construction is equivalent to explicitly selecting the part of $i\varsigma(f,h)$ which has positive frequency in u-direction by employing a Fourier transform to obtain (2.5).

Since the symplectic form is invariant under BMS transformations, we immediately have a similar statement for the state β. $\mathcal{B}$ being the free $*$-algebra generated by $\mathscr{B}$ (up to the quotient by the commutation relations), we obtain a natural representation ρ of the BMS group by $*$-automorphisms on $\mathcal{B}$: Any $a \in BMS$ induces per definition a conformal isometry $\alpha : \mathscr{I}^- \to \mathscr{I}^-$. Therefore, the representation ρ is given by α_* acting on the generating space $\mathscr{B}$, i.e., by α_* acting on each factor in each component of the direct sum.

Proposition 2.12. *The boundary state β is invariant under the action of the BMS group, i.e., $\beta \circ \rho = \beta$.*

Proof. We only have to show that the two-point distribution is invariant under BMS transformations. Starting in the Bondi frame (u,ζ), take an arbitrary BMS transformation $(u,\zeta) \mapsto (u',\zeta')$ with supertranslation $\alpha(\zeta)$ and conformal factor

$K(\zeta)^{-1}$. Suppressing the coordinate dependence of K, the transformation gives

$$\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int\limits_{\mathbb{R}^2\times\mathbb{C}}\frac{(f(u,\zeta),h(u',\zeta))_{\mathscr{I}^-}}{(u-u'-i\epsilon)^2}\,du\,du'\,\mu_{S^2}(\zeta)$$
$$\mapsto \lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int\limits_{\mathbb{R}^2\times\mathbb{C}}\frac{K^{-2-2w}(K^w f(u,\zeta),K^w h(u',\zeta))_{\mathscr{I}^-}}{(Ku-Ku'-i\epsilon)^2}K^4\,du\,du'\,\mu_{S^2}(\zeta)$$
$$=\lim_{\epsilon'\downarrow 0}\frac{1}{\pi}\int\limits_{\mathbb{R}^2\times\mathbb{C}}\frac{(f(u,\zeta),h(u',\zeta))_{\mathscr{I}^-}}{(u-u'-i\epsilon')^2}\,du\,du'\,\mu_{S^2}(\zeta)$$

because the integral is translation invariant. □

2.4.3 The Bulk State

We can now use the state β constructed on the boundary and the injective $*$-algebra homomorphism b to define a state on the field algebra $\mathcal{CF}_{\boldsymbol{M}}$ in the bulk spacetime $\boldsymbol{M}$. In this subsection we are going to prove the following important result:

Theorem 2.2. *The pulled-back state*

$$\omega \doteq \beta \circ b$$

of the conformal field algebra $\mathcal{CF}_{\boldsymbol{M}}$ satisfies the Hadamard condition.

ω is by definition of the $*$-homomorphism b a state on the conformal field algebra $\mathcal{CF}_{\boldsymbol{M}}$ and its two-point distribution is given by

$$\omega_2(f\otimes h)=\beta_2(b(f)\otimes b(h))$$

for $f,h\in\Gamma_0^{w-2}(E_M)$. We will prove that ω is a Hadamard state by showing that ω_2 satisfies the microlocal spectrum condition. The proof will utilize the propagation of singularities to show that the microlocal spectrum condition for ω_2 is equivalent to showing that β_2 has the following wavefront set:

Lemma 2.1. *The two-point distribution β_2 has the wavefront set*

$$\mathrm{WF}(\beta_2)=\{(x,x,\xi,-\xi)\in T^*(\mathscr{I}^-\times\mathscr{I}^-)\backslash 0 \mid \xi(\partial_u)>0\}.$$

Proof. Reviewing Example 1.2, it is clear that $\delta^\pm$ and $\partial_x\delta^\pm$ have the same wave front set.[7] Then, the result is a consequence of the wavefront sets of δ^Δ and $\delta^{+\Delta}$ determined in Example 1.3 and Example 1.4 respectively. □

According to the propagation of singularities theorem, the singularities of a solution to a wave operator propagate along null geodesics. Hence, we will also need a result on null geodesics propagating to $\mathscr{I}^-$.

Lemma 2.2. *There is no null geodesic joining any $x\in\psi(M)$ with i^- (cf. Lemma 4.3 of [21]).*

[7] This is also a consequence of the microlocal elliptic regularity theorem found e.g. in Theorem 8.3.1 of [44].

Proof. Past null infinity $\mathscr{I}^-$ is given by $\mathscr{I}^- = \partial J^+_{\widetilde{M}}(i^-)\setminus\{i^-\}$. Hence, if there was a geodesic γ connecting x to i^-, then there would exist a point $y \in \mathscr{I}^-$ such that γ is contained in $\mathscr{I}^-$ in the past of y and contained in $\psi(M)$ in the future of y. In the past of y, however, we have that γ is cotangent to $\boldsymbol{d}\Omega$ and, by the uniqueness of solutions to the geodesic equation, it follows that γ is one of the integral curve generating $\mathscr{I}^-$. Since all the null geodesics generating $\mathscr{I}^-$ are complete, such a point y cannot exist. □

From this lemma it follows that a null geodesics γ crossing $\mathscr{I}^-$ intersects $\mathscr{I}^-$ at only one point $x \in \mathscr{I}^-$ and that the cotangent vector of γ at x is proportional to $\boldsymbol{d}u$.

Proof of Theorem 2.2. Instead of working with ω_2 we will work with $\widetilde{\omega}_2$ defined by

$$\widetilde{\omega}_2(f\otimes h) \doteq \beta_2(\iota_w^* G^-_{\widetilde{M}} f \otimes \iota_w^* G^-_{\widetilde{M}} h)$$

for $f,h \in \Gamma_0^{w-2}(\widetilde{M})$ with $\{i^-\}\notin \operatorname{supp} f \cup \operatorname{supp} h$. Since $\left(J^-_{\widetilde{M}}(\operatorname{supp} f)\cup J^-_{\widetilde{M}}(\operatorname{supp} h)\right)\cap J^+_{\widetilde{M}}(i^-)$ is compact, this is indeed well-defined. Moreover, from the support of the retarded propagator $G^-_{\widetilde{M}}$ we see that $\operatorname{supp}\widetilde{\omega}_2 = J^+_{\widetilde{M}}(i^-)\setminus\{i^-\}$.

Before we can apply the propagation of singularities theorem we calculate

$$\widetilde{\omega}_2(P_{\widetilde{M}} f\otimes P_{\widetilde{M}} h) = \beta_2(\iota_w^* f \otimes \iota_w^* h).$$

Therefore, the wavefront set of $(P_{\widetilde{M}}\otimes P_{\widetilde{M}})\widetilde{\omega}_2$ is given by the wavefront set of β_2, and we have

$$\mathrm{WF}((P_{\widetilde{M}}\otimes P_{\widetilde{M}})\,\widetilde{\omega}_2) = \{(x,x,\xi,-\xi)\in T^*(\widetilde{M}\times\widetilde{M})\backslash 0 \mid x\in\mathscr{I}^-,\ \xi(\partial_u)>0\}.$$

Moreover, it holds true that

$$\mathrm{WF}((P_{\widetilde{M}}\otimes \mathrm{id})\,\widetilde{\omega}_2) \supset \mathrm{WF}((P_{\widetilde{M}}\otimes P_{\widetilde{M}})\,\widetilde{\omega}_2) \subset \mathrm{WF}((\mathrm{id}\otimes P_{\widetilde{M}})\,\widetilde{\omega}_2).$$

Hence, applying the propagation of singularities theorem, we obtain

$$\begin{aligned}\mathrm{WF}(\widetilde{\omega}_2) = \{(x,y,\xi,-\eta)\in T^*(\widetilde{M}\times\widetilde{M})\backslash 0 \mid \exists z\in\mathscr{I}^-,\ \zeta\in T^*_z\widetilde{M} \text{ s.t.}\\ x,y\in J^+_{\widetilde{M}}(i^-)\setminus\{i^-\},\ (x,\xi)\sim(y,\eta)\sim(z,\zeta),\ \zeta(\partial_u)>0\}.\end{aligned}$$

Restricting again to test functions with support contained in $\psi(M)$ and using the conformal invariance of null geodesics, we get

$$\mathrm{WF}(\omega_2) = \{(x,y,\xi,-\eta)\in T^*(M\times M)\backslash 0 \mid (x,\xi)\sim(y,\eta),\ \xi \rhd 0\}$$

by Lemma 2.2 and the fact that $\zeta(\partial_u)>0$ implies that ζ is future directed. □

As the boundary state β is BMS invariant and the boundary symmetries correspond to asymptotic symmetries of the bulk, we can prove that the bulk state ω is invariant under all isometries of the bulk spacetime (cf. Proposition 3.4 of [21]).

Proposition 2.13. *Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be the one-parameter group of asymptotic symmetries $\varphi_t : M \to M$ generated by the flow given by a complete asymptotic Killing field. Then,*

$$b\circ(\varphi_t)_* = (\alpha_t)_*\circ b$$

for a one-parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ of conformal isometries $\alpha_t : \mathscr{I}^-\to\mathscr{I}^-$ induced by a one-parameter subgroup $\{a_t\}_{t\in\mathbb{R}}\subset BMS$.

Proof. This is an immediate consequence of the discussion at the end of Sect. 1.4, i.e., the asymptotic Killing field in $\boldsymbol{M}$ yields a Killing field at (the locus) $\mathscr{I}^-$ and hence induces a one-parameter subgroup of BMS transformation on (the manifold) $\mathscr{I}^-$ in a specific Bondi frame. □

Corollary 2.1. *The bulk state ω is invariant under each one-parameter group of $*$-automorphism $\{\rho_t\}_{t\in\mathbb{R}}$, $\rho_t : \mathcal{CF}_{\boldsymbol{M}} \to \mathcal{CF}_{\boldsymbol{M}}$ induced by a one-parameter group of asymptotic symmetries $\{\varphi_t\}_{t\in\mathbb{R}}$, $\varphi_t : M \to M$ via push-forwards, i.e., $\rho_t \circ \Phi_{\boldsymbol{M}} = \Phi_{\boldsymbol{M}} \circ (\varphi_t)_*$.*

3

The Electromagnetic Field

Classical electromagnetism is described by Maxwell's well known field equations. In non-relativistic physics one usually finds Maxwell's equations written in terms of the *electric field* and the *magnetic field* together with the *charge density* and the *current density*. But in relativistic physics, we see, e.g. by applying Lorentz transformations, that the electric and magnetic field and are deeply intertwined. This motivates an arguably more fundamental formulation of Maxwell's equations in terms of a two-form F, the electromagnetic *field strength tensor*, which combines the electric and magnetic field. Thus, we write Maxwell's equations as

$$\boldsymbol{d}F = 0 \quad \text{and} \quad \boldsymbol{\delta}F = -j, \tag{3.1}$$

where j, the *source one-form*, combines charge and current density.

The electric and magnetic field, as measured by an observer travelling with a four-velocity v, can be recovered from F as the spatial compontents of the respective one-forms

$$E \doteq \boldsymbol{i}_v F \quad \text{and} \quad B \doteq \boldsymbol{i}_v * F,$$

cf. Proposition 8.3.1 of [31] and Chapter 4.2 of [35].

The form of Maxwell's equations as given in (3.1) generalizes to arbitrary spacetimes without modification. Therefore, we will take an Occam's razor approach and conjecture that Maxwell's equations as stated here remain a valid description of physics in non-flat and non-trivial spacetimes under the approximation that the background metric remains non-dynamical.

Maxwell's equations have an even simpler formulation in terms of a potential. If the second de Rham cohomology $\mathrm{H}^2(M)$ is trivial, the equation $\boldsymbol{d}F = 0$ implies that there exists globally a one-form A, traditionally called the *vector potential* or also *electromagnetic potential*, such that

$$F = \boldsymbol{d}A. \tag{3.2}$$

In terms of the vector potential A, Maxwell's equations (3.1) thus reduce to

$$\boldsymbol{\delta}\boldsymbol{d}A = -j. \tag{3.3}$$

However, the potential A is not uniquely defined by F. We see from (3.2) that A is only determined up to a *gauge*, i.e., A can be replaced by $A' = A + \Lambda$ with Λ closed. Two such vector potentials A and A' are called *gauge equivalent*, in symbols $A \sim A'$, and hence, to each A corresponds a *gauge equivalence class* $[A]$ of vector potentials differing by closed one-forms.

It is common to remove the gauge freedom at least partially by *gauge fixing*, i.e., by implementing a so called *gauge condition* on A. We introduce only one such condition here: The *Lorenz gauge condition* $\boldsymbol{\delta}A = 0$.

In all that follows, we will be concerned solely with the homogeneous Maxwell equations, i.e., $j = 0$, and use instead of (3.1) or (3.3)

$$\boldsymbol{\delta d} A = 0, \tag{3.4}$$

which we will call “Maxwell’s equation”. This equation is conformally invariant on 1-forms of conformal weight 0 because of (1.10).

Although we acknowledge that, starting from (3.1), the restriction $\mathrm{H}^2(M) = \{0\}$ has to be made to arrive at (3.4), we will in the next sections consider the vector potential described by this dynamical equation in its own right. We justify this by the following observation: Seeing F as a representative of $\mathrm{H}^2(M)$, every solution F of (3.1) can be written as

$$F = f + dA,$$

where $f \in \Omega^2(M)$ is not exact (or zero) and $A \in \Omega^1(M)$. Now, f has purely topological origin and may be understood via Stokes’ theorem as the effect of a magnetic topological charge (cf. [56] by Misner and Wheeler). A discussion of the quantization of the electromagnetic field which rests on the field strength tensor instead can be found in the diploma thesis of Lang [57] and his joint work with Dappiaggi [58].

3.1 Classical Phase Space

Within this section we will construct a phase space for the vector potential on a globally hyperbolic spacetime $\boldsymbol{M}$. To achieve this goal, we will need a suitable vector space with a symplectic form on it. The approach presented here is based on results by Dimock [26] and, more recently, Pfenning [27].

The naïve choice for a phase space is the space of solutions to Maxwell’s equation which are compactly supported on every Cauchy surface. This space would, however, exclude non-compactly supported gauge transformations and make no reference to the possible gauge equivalence of different solutions. Before we can make a more educated choice we need some additional results. From now on, $G^\pm$ and G will always denote Green’s operators and the causal propagator of the Laplace-de Rham operator $\square$.

Definition 3.1. The space of (co-)closed p-forms will be denoted by $\Omega^p_{\boldsymbol{d}}(M)$ and $\Omega^p_{\boldsymbol{\delta}}(M)$, i.e.,

$$\Omega^p_{\boldsymbol{d}}(M) \doteq \{f \in \Omega^p(M) \mid \boldsymbol{d} f = 0\},$$
$$\Omega^p_{\boldsymbol{\delta}}(M) \doteq \{f \in \Omega^p(M) \mid \boldsymbol{\delta} f = 0\}.$$

Lemma 3.1. *Let a past* (+) *or future* (−) *compact* $A \in \Omega^1(M)$ *satisfy* $\boldsymbol{\delta d} A = f$ *for some* $f \in \Omega^1_{\boldsymbol{\delta}}(M)$*. Then*

$$A \sim G^\pm f.$$

Proof. There exists $\lambda \in \Omega^0(M)$, given by $\lambda = G^\pm \boldsymbol{\delta} A$, such that $\square\lambda = \boldsymbol{\delta d}\lambda = \boldsymbol{\delta} A$. Thus, $A = G^\pm f + \boldsymbol{d}\lambda \sim G^\pm f$. □

This enables us to show that each solution of Maxwell’s equations is gauge related to a *Lorenz solution*, i.e., a solution A of $\square A = 0$ which satisfies the Lorenz condition $\boldsymbol{\delta} A = 0$ and thus also solves Maxwell’s equation $\boldsymbol{\delta d} A = 0$.

Proposition 3.1. *$\delta dA = 0$ for $A \in \Omega^1(M)$ if and only if*

$$A \sim Gf$$

for some $f \in \Omega^1_\delta(M)$ which is past and future compact.

Proof. Let $f \in \Omega^1_\delta(M)$ past and future compact. Then $A = Gf$ solves $\Box A = 0$ and furthermore $\delta A = \delta Gf = G\delta f = 0$.

Conversely, let $\delta dA = 0$. Choose $\chi_+ \in C^\infty(M)$ past and $\chi_- \in C^\infty(M)$ future compact with $\chi_+ + \chi_- = 1$ and set $A_\pm = \chi_\pm A$. Applying Lemma 3.1 to A_+ and A_- with $\delta dA_+ = -\delta dA_- = f \in \Omega^1_\delta(M)$ past and future compact, we obtain $A_\pm \sim \pm G^\pm f$ and hence $A \sim Gf$. □

We finish the investigation of Lorenz solutions with the following proposition (cf. the similar Prop. A.3 of Fewster and Pfenning [18]) which shows the relation imposed on the test forms of two gauge equivalent Lorenz solutions.

Proposition 3.2. *Let $f, f' \in \Omega^1_\delta(M)$ be past and future compact. $Gf \sim Gf'$ if and only if*

$$f - f' = \delta\lambda \tag{3.5}$$

for some $\lambda \in \Omega^2_d(M)$.

Proof. Assume that $Gf \sim Gf'$. Then $f - f' = h$ for some past and future compact $h \in \Omega^1_\delta(M)$ with $dGh = Gdh = 0$. It follows, that there exists $\lambda \in \Omega^2(M)$ such that

$$dh = \Box\lambda.$$

Taking coderivatives, we obtain $\Box(h - \delta\lambda) = 0$, which gives $h = \delta\lambda$ by applying G^+. Hence, $f - f' = \delta\lambda$. On the other hand, taking the exterior derivate yields $\Box d\lambda = 0$, which implies $d\lambda = 0$ by the same argument as before.

Conversely, let $f, f' \in \Omega^1_\delta(M)$ be past and future compact and $\lambda \in \Omega^2_d(M)$ such that $f - f' = \delta\lambda$. This implies $d(f - f') = \Box\lambda$. Applying G, we see that $Gf \sim Gf'$. □

Corollary 3.1. *Let $f \in \Omega^1_\delta(M)$ be past and future compact. Any past and future compact $f' \in \Omega^1_\delta(M)$ such that $Gf \sim Gf'$ is cohomologous to f, i.e., f and f' are in the same equivalence class of $*\mathrm{H}^3(M)$.*

If $\mathrm{H}^1(M) = \{0\}$ or $\mathrm{H}^2(M) = \{0\}$, the results above have the following special form:

Proposition 3.3. *Let $f, f' \in \Omega^1_\delta(M)$ be past and future compact and $\mathrm{H}^1(M) = \{0\}$ or $\mathrm{H}^2(M) = \{0\}$. Then, $Gf \sim Gf'$ implies*

$$f - f' = \delta d\lambda \quad \text{and} \quad Gf - Gf' = d\Lambda \tag{3.6}$$

for some $\lambda \in \Omega^1(M)$ which is past and future compact and $\Lambda \in \Omega^0(M)$.

Proof. If $\mathrm{H}^1(M) = \{0\}$, then we have to every closed 1-form an exact 0-form so that

$$G(f - f') = d\Lambda.$$

Since $\boldsymbol{\delta d}\Lambda = 0$, we have $\Lambda = G\eta$ for some past and future compact $\eta \in \Omega^0(M)$. This yields

$$f - f' = \boldsymbol{d}\eta - \Box\lambda = \boldsymbol{\delta d}\lambda,$$

because $\boldsymbol{\delta}(f - f') = 0$ implies $\eta = \boldsymbol{\delta}\lambda$ as can be seen by applying G^+ to the right-hand side.

If, on the other hand, $\mathrm{H}^2(M) = \{0\}$, there exists to every closed 2-form an exact 1-form so that the first statement follows immediately from (3.5). Applying G and subtracting $\Box\lambda \in \ker G$, we get

$$G(f - f') = G(\boldsymbol{\delta d}\lambda - \Box\lambda) = G\boldsymbol{d\delta}\lambda.$$

Setting $\Lambda = G\boldsymbol{\delta}\lambda$ we have shown the second statement too. □

Thus, regarding the residual gauge freedom of Lorenz solutions, the cases $\mathrm{H}^1(M) = \{0\}$ and $\mathrm{H}^2(M) = \{0\}$ behave alike. These results allow us to write down a phase space $\mathscr{M}(\boldsymbol{M})$ for the vector potential on a globally hyperbolic spacetime $\boldsymbol{M}$:

$$\mathscr{M}(\boldsymbol{M}) \doteq \Omega^1_{0,\delta}(M)/\boldsymbol{\delta}\Omega^2_{0,d}(M)$$

and in the case that $\mathrm{H}^1(M) = \{0\}$ or $\mathrm{H}^2(M) = \{0\}$

$$\mathscr{M}(\boldsymbol{M}) = \Omega^1_{0,\delta}(M)/\boldsymbol{\delta d}\Omega^1_0(M).$$

This space is isomorphic to the space of gauge equivalence classes of solutions to Maxwell's equation which are gauge related to a Lorenz solution with compact support on every Cauchy surface. Also note that the elements of $\mathscr{M}(\boldsymbol{M})$ are of conformal weight -2 due to (1.8) and that $\Omega^{1,-2}_{0,\delta}(M)$, i.e., the space of compactly supported coclosed 1-forms of conformal weight -2, is a conformally covariant space by (1.12).

Now we need to find a symplectic form on $\mathscr{M}(\boldsymbol{M})$. This task is accomplished using Proposition 1.8:

Proposition 3.4. *Let* $[f],[h] \in \mathscr{M}(\boldsymbol{M})$ *with representatives* $f,h \in \Omega^1_{0,\delta}(M)$*. Then,*

$$\sigma([f],[h]) \doteq G(f \otimes h) \doteq \langle Gf, h\rangle \tag{3.7}$$

is a non-degenerate symplectic form on the vector space $\mathscr{M}(\boldsymbol{M})$ *if* $\mathrm{H}^1(M) = \{0\}$ *or* $\mathrm{H}^2(M) = \{0\}$.

Proof. All we have to show is that two elements from the same equivalence class have vanishing symplectic product. Suppose that $[f] = [h]$, i.e., $h = f + \boldsymbol{\delta d}\lambda$, then

$$\langle Gf, h\rangle = \langle Gf, \boldsymbol{\delta d}\lambda\rangle = \langle \boldsymbol{\delta d}Gf, \lambda\rangle = 0$$

by the properties of the causal propagator. □

It comes to no surprise, thanks to (1.21), that the symplectic form (3.7) is conformally invariant due to the conformal invariance of Maxwell's equation.

What we did not consider is the general case where neither $\mathrm{H}^1(M) = \{0\}$ nor $\mathrm{H}^2(M) = \{0\}$. To show that the symplectic form is well-defined we used the results of Proposition 3.3. In particular, to have a well-defined symplectic structure on $\mathscr{M}(\boldsymbol{M})$ given by (3.7), we need the conditions given in (3.6) to be satisfied. Judging from the proof made in Proposition 3.3 it seems highly doubtful that these conditions hold in the general case.

We will now use Green's identity to write down the symplectic form on a fixed but arbitrary Cauchy surface:

Corollary 3.2. *Choosing a Cauchy surface $\Sigma \subset M$ with inclusion map $\iota : \Sigma \hookrightarrow M$, we can write (3.7) as*

$$\sigma([f],[h]) = \int_{\Sigma} \iota^*(Gh \wedge *\boldsymbol{d}Gf - Gf \wedge *\boldsymbol{d}Gh). \tag{3.8}$$

Proof. To see the equivalence of (3.7) and (3.8), define $\Sigma_{\pm} \doteq J^{\pm}_{\boldsymbol{M}}(\Sigma) \setminus \Sigma$. Then,

$$\begin{aligned}\langle Gf, h\rangle_{(\Sigma_{\mp},g)} &= \langle Gf, \boldsymbol{\delta d}G^{\pm}h\rangle_{(\Sigma_{\mp},g)} \\ &= \pm \int_{\Sigma} \iota^*(G^{\pm}h \wedge *\boldsymbol{d}Gf - Gf \wedge *\boldsymbol{d}G^{\pm}h),\end{aligned}$$

where we used the properties of Green's operators (cf. Theorem 1.5) in the first and Green's identity for $\Box$ in the second equality (the sign difference is a consequence of the orientation of M). Adding the two results, we obtain (3.8). ☐

Taking two solutions A and A' of Maxwell's equation which are compactly supported on Σ, (3.8) takes a form which might be more familiar to some readers, namely,

$$\int_{\Sigma} \iota^*(A' \wedge *F - A \wedge *F'),$$

where $F = \boldsymbol{d}A$ and $F' = \boldsymbol{d}A'$.

3.2 Quantization

In this section we will discuss the quantization of the vector potential in the framework of locally covariant conformal quantization introduced in Sect. 2.2. Moreover, we will discuss the relation of this quantization scheme with the indefinite metric approach by Gupta [59] and Bleuler [60] known from quantum field theory on Minkowski spacetime.

3.2.1 Locally Covariant Conformal Quantization

As we want to construct the field algebra for the vector potential using the symplectic form (3.7), we have no choice but to restrict the space of admissible spacetimes.

Definition 3.2. Denote by $\mathsf{CGlobHyp}'$ the subcategory of $\mathsf{CGlobHyp}$ which contains only the objects $\boldsymbol{M} \in \mathsf{CGlobHyp}$ such that either $\mathrm{H}^1(M) = \{0\}$ or $\mathrm{H}^2(M) = \{0\}$.

Having made this restriction, we can formulate a field algebra for the vector potential. The resulting definition is very similar to that of the conformal field algebra in Definition 2.10.

Definition 3.3. The *field algebra $\mathcal{A}_{\boldsymbol{M}}$ of the vector potential* in the spacetime $\boldsymbol{M} \in$ Obj(CGlobHyp′) is given by the quotient

$$\mathcal{A}_{\boldsymbol{M}} \doteq \mathcal{A}^0_{\boldsymbol{M}} / \mathcal{I} \quad \text{with} \quad \mathcal{A}^0_{\boldsymbol{M}} \doteq \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \Omega^{1,-2}_{0,\delta}(M, \mathbb{C})^{\otimes n},$$

where $\mathcal{I}$ is the closed $*$-ideal generated by the elements

$$\boldsymbol{\delta d}\Omega^{1,0}_0(M,\mathbb{C}) \quad \text{and} \quad -iG_{\boldsymbol{M}}(f \otimes h) \oplus f \otimes h - h \otimes f$$

for all $f, h \in \Omega^{1,-2}_{0,\delta}(M,\mathbb{C})$. As in Definition 2.5, the $*$-operation is given by complex conjugation.

Note the difference to the formulation of Definition 2.6 and Definition 2.10: Instead of quotienting only by $G_{\boldsymbol{M}} \cap \Omega^{1,-2}_{0,\delta}(M,\mathbb{C})$ we also quotient by $\boldsymbol{\delta d}\Omega^{1,0}_0(M,\mathbb{C})$ to take care of the residual gauge freedom of Lorenz solutions (cf. Proposition 3.2 and the discussion thereafter). Since the latter set already includes the former, we need only to quotient by the latter.

Then, a proof analogous to that of Proposition 2.2 combined with Proposition 2.3 shows the following:

Corollary 3.3. *The field algebra $\mathcal{A}_{\boldsymbol{M}}$ defines a* locally covariant conformal quantum field theory of the vector potential $\mathscr{A}$: CGlobHyp′ $\to *$-Alg *via the assignments*

$$\mathscr{A}(\boldsymbol{M}) \doteq \mathcal{A}_{\boldsymbol{M}} \quad \textit{and} \quad \mathscr{A}(\psi) \doteq q_{\widetilde{\boldsymbol{M}}} \circ \psi^{-2}_*$$

for all $\psi \in \mathrm{Hom}_{\mathsf{CGlobHyp}'}(\boldsymbol{M}, \widetilde{\boldsymbol{M}})$ and $q_{\boldsymbol{M}}$ is the projection map $\mathcal{A}^0_{\boldsymbol{M}} \to \mathcal{A}_{\boldsymbol{M}}$. Furthermore, $\mathscr{A}$ is causal and satisfies the time-slice axiom with respect to conformal embeddings.

Thus, we can introduce the quantum field of the eletromagnetic potential as a special case of Proposition 2.6.

Corollary 3.4. *The* conformal quantum field *(of conformal weight 0)* of the vector potential *is the natural transformation A given by*

$$A_{\boldsymbol{M}}(f) \doteq q_{\boldsymbol{M}}(\boldsymbol{f}) \quad \textit{with} \quad \boldsymbol{f} = 0 \oplus f \oplus 0 \oplus \cdots$$

for each $\boldsymbol{M} \in$ Obj(CGlobHyp′) and any $f \in \Omega^{1,-2}_{0,\delta}(M,\mathbb{C})$.

Equivalently, one may define the conformal quantum field A as the natural transformation such that $A_{\boldsymbol{M}} : \mathscr{M}(\boldsymbol{M}) \otimes \mathbb{C} \to \mathcal{A}_{\boldsymbol{M}}$ with $A_{\boldsymbol{M}}([f]) = 0 \oplus [f] \oplus 0 \oplus \cdots$ and $[f] \in \mathscr{M}(\boldsymbol{M}) \otimes \mathbb{C}$ if we consider $\mathscr{M}_{\mathbb{C}}$ as the functor of test functions taking $\boldsymbol{M} \in$ Obj(CGlobHyp′) to $\mathscr{M}(\boldsymbol{M}) \otimes \mathbb{C}$ with conformal push-forwards (of weight -2) as morphisms.

3.2.2 The Gupta-Bleuler Formalism

The Gupta-Bleuler formalism [59, 60] is discussed in several introductory books on quantum field theory and is usually one of the first subjects to be teached when introducing quantum electrodynamics. Problems with the naïve quantization of the vector potential arise already at a fundamental level as shown by Strocchi in [61, 62]. Namely, Maxwell's equations cannot be satisfied as operator identities if one treats the vector potential as a local quantum field. These problems can be (partially) solved in the Gupta-Bleuler approach.

Since these early papers, the Gupta-Bleuler formalism was also treated at an algebraic level as local constraints on a C^*-algebra in [63] by Grundling and Lledó and applied to the vector potential on ultrastatic spacetimes by Furlani in [64]. Going over to arbitrary globally hyperbolic spacetimes, we can, however, not straightforwardly apply the Gupta-Bleuler formalism as it is bound to a notion of positive frequency and to the representation of the algebra of observables of the 1-form wave equation on a *complete indefinite inner product space* (a so called Krein space). A few comments on this can also be found in Chap. IV D of [18]. Let us therefore reformulate the Gupta-Bleuler procedure in language more appropriate for our needs.

In the Gupta-Bleuler formalism one starts by adding a gauge breaking term to the Lagrangian or directly to the equations of motions. That is, instead of Maxwell's equation (3.4) we consider the equation

$$\Box\phi - \lambda \boldsymbol{d}\boldsymbol{\delta}\phi = 0,$$

where $\phi \in \Omega^1(M)$ and $\lambda \in \mathbb{R}$ plays the role of a gauge parameter. Choosing the Feynman gauge $\lambda = 0$, we obtain the 1-form wave equation $\Box\phi = 0$.

We then define the field algebra of the 1-form field ϕ in accordance with Definition 2.6 as the quotient algebra

$$\mathcal{F}_{\boldsymbol{M}} \doteq \mathcal{F}^0_{\boldsymbol{M}}/\mathcal{I} \quad \text{with} \quad \mathcal{F}^0_{\boldsymbol{M}} \doteq \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \Omega^1_0(M,\mathbb{C})^{\otimes n},$$

where $\mathcal{I}$ is the closed $*$-ideal generated by the elements

$$\Box\Omega^1_0(M,\mathbb{C}) \quad \text{and} \quad -iG_{\boldsymbol{M}}(f\otimes h) \oplus f\otimes h - h\otimes f$$

for all $f,h \in \Omega^1_0(M,\mathbb{C})$ and $\boldsymbol{M} \in \mathrm{Obj}(\mathsf{CGlobHyp}')$

Now, consider a quasi-free *indefinite* state ω on $\mathcal{F}$, i.e., a continuous linear functional on $\mathcal{F}$ such that $\omega(\mathbf{1}) = 1$ which is completely determined by the 2-point distribution $\omega_2 \in (\Omega^1_0(M,\mathbb{C}) \otimes \Omega^1_0(M,\mathbb{C}))'$. Using this indefinite state, we could, in principle, perform a procedure analogous to the GNS construction to obtain a Krein space.

Here, we will, however, stick to the algebraic picture. Hence, we define the quantum field

$$\Phi : \Omega^1_0(M,\mathbb{C}) \to \mathcal{F}_{\boldsymbol{M}}$$

as described in Proposition 2.4. Furthermore, suppose that ω is a Hadamard state. This enables us to write the Gupta-Bleuler condition as

$$\omega(\boldsymbol{a} \otimes \Phi(\boldsymbol{d}h) \otimes \boldsymbol{b}) = 0,$$

where $\boldsymbol{a},\boldsymbol{b} \in \mathcal{F}_{\boldsymbol{M}}$ and $h \in C^\infty_0(M,\mathbb{C})$. The connection to the usual condition, i.e., the Lorentz gauge condition on the annihilation operator, becomes clear if we write (formally) $\Phi(\boldsymbol{d}h) = \boldsymbol{\delta}\Phi(h)$. Since ω is quasi-free, it suffices to consider the condition

$$\omega_2(f \otimes \boldsymbol{d}h) = 0 \tag{3.9}$$

for $f \in \Omega_0^1(M, \mathbb{C})$. That is, to obtain the physical space of observables respectively the physical space of test forms, we need to find the subspace of $\Omega_0^1(M, \mathbb{C})$ such that (3.9) is satisfied.

Using the Schwartz kernel theorem, condition (3.9) yields

$$\omega_2(f \otimes \boldsymbol{d}h) = \langle \omega_2 f, \boldsymbol{d}h \rangle_{\boldsymbol{M}} = \langle \boldsymbol{\delta}\omega_2 f, h \rangle_{\boldsymbol{M}} = 0.$$

Therefore, we recover Dimock's condition $\boldsymbol{\delta} f = 0$ on the space of test forms and thus also the field algebra defined in Definition 3.3 if the codifferential $\boldsymbol{\delta}$ commutes with the linear operator ω_2. This is the case for the Minkowski vacuum and also for the state that we will construct below. Nevertheless, the Gupta-Bleuler formalism cannot be applied for the bulk to boundary construction because $\Box$ is not a conformally invariant operator.

In the usual Gupta-Bleuler procedure for the quantization of the vector potential on Minkowski spacetime we find that the inner product on subset of the original Krein space which satisfies the Gupta-Bleuler condition is positive semi-definite. There is no obvious way to proof an analogous statement in this algebraic approach if no concrete state is chosen.

3.3 Bulk to Boundary Correspondence

We already laid down the foundation for the bulk to boundary correspondence in asymptotically flat spacetimes in Sect. 2.4. Several aspects of the vector potential, however, call for a special treatment. We will see in this section that some of the assumptions that we made for the general case do not seem to be satisfied for the vector potential.

Let $(M, g) = \boldsymbol{M} \in \mathrm{Obj}(\mathsf{CGlobHyp}')$ be an asymptotically flat spacetime embedded into the unphysical spacetime $(\widetilde{M}, \widetilde{g}) = \widetilde{\boldsymbol{M}} \in \mathrm{Obj}(\mathsf{CGlobHyp}')$ via the conformal embedding $\psi : M \hookrightarrow \widetilde{M}$ with conformal factor Ω. Denote by $\iota : \mathscr{I}^- \hookrightarrow \widetilde{M}$ the conformal embedding of past null infinity into the unpyhsical spacetime in a Bondi frame (u, ζ).

If we followed the general course of Sect. 2.4, we would first define a vector bundle with an inner product on $\mathscr{I}^-$. The induced inner product on the pull-back bundle $\iota^* T^* \widetilde{M}$ is, however, indefinite because it is given by the pull-back of the inverse unphysical metric tensor. To deal with this problem, we will study the vector potential at past null infinity and find some properties that allow us to circumnavigate these problems.

Proposition 3.5. *Let* $f \in \Omega_{0,\boldsymbol{\delta}}^{1,-2}(M)$. *Then,*

$$\widetilde{g}(\boldsymbol{d}\Omega, G^{\pm}_{\widetilde{\boldsymbol{M}}} \psi_*^{-2} f) = 0. \tag{3.10}$$

Proof. As f is coclosed, so are $\psi_*^{-2} f$ and $G^{\pm}_{\widetilde{\boldsymbol{M}}} \psi_*^{-2} f$ by (1.12). Furthermore, by the conformal invariance of the vector potential we have $\psi^* G^{\pm}_{\widetilde{\boldsymbol{M}}} \psi_*^{-2} f = G^{\pm}_{\boldsymbol{M}} f$ and, applying the codifferential,

$$0 = \tilde{\boldsymbol{\delta}} G^{\pm}_{\widetilde{M}} \psi_*^{-2} f = -2\Omega^{-3} \psi_* g(\psi^* \boldsymbol{d}\Omega, \psi^* G^{\pm}_{\widetilde{M}} \psi_*^{-2} f) = -2\Omega^{-5} \tilde{g}(\boldsymbol{d}\Omega, G^{\pm}_{\widetilde{M}} \psi_*^{-2} f)$$

by (1.11) and $\tilde{\boldsymbol{\delta}}$ denotes the codifferential with respect to the unphysical metric $\tilde{g}$. Since Ω extends smoothly to the whole unphysical spacetime, we obtain (3.10). □

Hence, for a Lorenz solution at most the 3 components conormal to $\boldsymbol{d}\Omega$ are non-zero. This has grave consequences for the remaining gauge freedom at $\mathscr{I}^-$: There is none! Let us define the map

$$b' = \iota^* \circ G^-_{\widetilde{M}} \circ \psi_*^{-2}$$

as a preliminary bulk to boundary map so as to simplify notation.

Proposition 3.6. *If $f, f' \in \Omega^{1,-2}_{0,\boldsymbol{\delta}}(M)$ such that $[f] = [f']$, then*

$$b'f = b'f'.$$

Proof. $[f] = [f']$ implies that $G^-_{\widetilde{M}} \psi_*^{-2}(f - f') = G^-_{\widetilde{M}} \psi_*^{-2} \boldsymbol{\delta}\boldsymbol{d}\lambda = \boldsymbol{d}\Lambda$ for some $\lambda \in \Omega^{1,0}_0(\widetilde{M})$ according to Proposition 3.3. For $\tilde{g}(\boldsymbol{d}\Omega, \boldsymbol{d}\Lambda) = 0$ to be satisfied this requires that $\iota^*\Lambda$ is independent of u and thus constant along the null geodesic generator along $\mathscr{I}^-$. This is only possible if $\Lambda = 0$ because $J^-_{\widetilde{M}}(\operatorname{supp} \psi_*\lambda) \cap J^+_{\widetilde{M}}(i^-)$ and hence $\operatorname{supp} \boldsymbol{d}\Lambda \cap (\mathscr{I}^- \cup \{i^-\})$ are compact. □

Proposition 3.5 also simplifies the symplectic form of the vector potential written on $\mathscr{I}^-$ via Green's identity considerably.

Proposition 3.7. *Let $[f], [h] \in \mathscr{M}(\boldsymbol{M})$ with representatives $f, h \in \Omega^{1,-2}_{0,\boldsymbol{\delta}}(M)$. On the null surface $\mathscr{I}^-$ we can write the symplectic product (3.7) of the vector potential as*

$$\sigma([f],[h]) = 2 \int_{\mathscr{I}^-} g_{\mathscr{I}^-}(b'f, \partial_u b'h)\, \mu_{\mathscr{I}^-}, \tag{3.11}$$

where $g_{\mathscr{I}^-}(\cdot,\cdot)$ denotes the contraction of the angular components with respect to the metric (2.3).

Proof. Using Green's identity for □ and the conformal invariance of $G^{\pm}$, we have

$$\begin{aligned}
\langle G_M f, h\rangle_M &= \langle G_{\widetilde{M}} \psi_*^{-2} f, \tilde{\boldsymbol{\delta}} \boldsymbol{d} G^-_{\widetilde{M}} \psi_*^{-2} h\rangle_{\widetilde{M}} \\
&= \int_{\mathscr{I}^-} \iota^* \big(G^-_{\widetilde{M}} \psi_*^{-2} h \wedge * \boldsymbol{d} G^-_{\widetilde{M}} \psi_*^{-2} f - G^-_{\widetilde{M}} \psi_*^{-2} f \wedge * \boldsymbol{d} G^-_{\widetilde{M}} \psi_*^{-2} h\big) \\
&= \int_{\mathscr{I}^-} \iota^* \tilde{*} \big(\tilde{g}(G^-_{\widetilde{M}} \psi_*^{-2} f, \tilde{\nabla} G^-_{\widetilde{M}} \psi_*^{-2} h) - \tilde{g}(\tilde{\nabla} G^-_{\widetilde{M}} \psi_*^{-2} f, G^-_{\widetilde{M}} \psi_*^{-2} h)\big),
\end{aligned}$$

where $\tilde{\boldsymbol{\delta}}$, $\tilde{\nabla}$ and $\tilde{*}$ denote the codifferential, the Levi-Civita connection and the Hodge operator with respect to $\tilde{g}$. Since $\tilde{g}(\boldsymbol{d}\Omega, G^-_{\widetilde{M}} \psi_*^{-2} f) = 0 = \tilde{g}(\boldsymbol{d}\Omega, G^-_{\widetilde{M}} \psi_*^{-2} h)$, this gives (3.11) after an integration by parts. □

Using the non-degenerateness of the symplectic form, we can now study the remaining non-zero component that is conormal to the S^2-part (i.e., conormal to $\boldsymbol{d}\zeta$ and $\boldsymbol{d}\overline{\zeta}$) of a Lorenz solution at $\mathscr{I}^-$.

Corollary 3.5. *If $f,h \in \Omega^{1,-2}_{0,\delta}(M)$ such that*

$$g_{\mathscr{I}^-}(\boldsymbol{d}\zeta, b'f) = g_{\mathscr{I}^-}(\boldsymbol{d}\zeta, b'h) \quad \text{and} \quad g_{\mathscr{I}^-}(\boldsymbol{d}\overline{\zeta}, b'f) = g_{\mathscr{I}^-}(\boldsymbol{d}\overline{\zeta}, b'h),$$

then $f = h$. That is, $\iota^*\widetilde{g}(\boldsymbol{d}u, G^-_{\widetilde{M}}\psi^{-2}_* f)$ is completely determined by $g_{\mathscr{I}^-}(\boldsymbol{d}\zeta, b'f)$ and $g_{\mathscr{I}^-}(\boldsymbol{d}\overline{\zeta}, b'f)$.

Proof. Assume that $f \neq h$ satisfy the condition above. Then, $\sigma([f-h],\cdot) = 0$. This leads to a contradiction as the symplectic product is non-degenerate. □

Putting all these results together, we obtain at $\mathscr{I}^-$ *effectively* the gauge conditions

$$\iota^*\widetilde{g}(\boldsymbol{d}\Omega, G^-_{\widetilde{M}}\psi^{-2}_* f) = 0 \quad \text{and} \quad \iota^*\widetilde{g}(\boldsymbol{d}u, G^-_{\widetilde{M}}\psi^{-2}_* f) = 0,$$

which were enforced in [54, 55] explicitly.[1] With only two independent degrees of freedom remaining, we (heuristically) recover the two polarization states of the photon.

We can now define a vector bundle equipped with a positive definite inner product intrinsically on $\mathscr{I}^-$.

Definition 3.4. We define on $\mathscr{I}^-$ the vector bundle

$$E \doteq \bigcup_{(u,\theta,\phi)\in\mathscr{I}^-} \{(u,\theta,\phi)\} \times T^*_{(\theta,\phi)}S^2 \otimes \mathbb{C},$$

and equip it with the positive definite (on real-valued elements) inner product $g_{\mathscr{I}^-}(\cdot,\cdot)$.[2] Moreover, denote by $\kappa : \iota^* T^*_{\mathbb{C}}\widetilde{M} \to E$ the canonical vector bundle epimorphism from the pull-back of the complexified unphysical cotangent bundle to $\mathscr{I}^-$ into E.

Accordingly, we have a symplectic product

$$\varsigma(f,h) \doteq 2\int_{\mathscr{I}^-} g_{\mathscr{I}^-}(f, \partial_u h)\,\mu_{\mathscr{I}^-},$$

cf. Proposition 2.8, on the *boundary phase space*

$$\mathscr{B} \doteq \{f \in \Gamma^0(E) \mid f, \partial_u f \in L^2(E, \mu_{\mathscr{I}^-})\}.$$

Therefore, we can now establish a result analogous to Proposition 2.9, i.e., we find a map from the bulk phase space into the boundary phase space:

[1]While the second equality is satisfied in [54, 55], here we only have that the component tangent to $\boldsymbol{d}u$ carries no physical information.

[2]Note that $g_{\mathscr{I}^-}$ is also a section in $E^* \otimes E^*$.

Proposition 3.8. *The* bulk to boundary map

$$b \doteq \kappa \circ b' = \kappa \circ \iota^* \circ G^-_{\widetilde{M}} \circ \psi_*$$

is a symplectomorphism from the complexified bulk phase space $(\mathscr{M}(\boldsymbol{M}) \otimes \mathbb{C}, \sigma(\cdot,\cdot))$ into the boundary phase space $(\mathscr{B}, \varsigma(\cdot,\cdot))$.

Proof. By Proposition 3.6, Corollary 3.5 and Proposition 2.9 we know that b is well-defined and injective. Furthermore, since $g_{\mathscr{I}^-}(\kappa(\cdot),\kappa(\cdot)) = g_{\mathscr{I}^-}(\cdot,\cdot)$, we have by Proposition 3.7 that b is a symplectomorphism. □

This finishes the discussion of the classical theory on the boundary. The boundary algebra is the one defined in Definition 2.13. As in the general case, we can then define a $*$-algebra monomorphism from the bulk field algebra $\mathcal{A}_{\boldsymbol{M}}$ into the boundary algebra $\mathcal{B}$. Similar to Proposition 2.10 we obtain:

Corollary 3.6. *The bulk to boundary map b induces an injective $*$-algebra homomorphism $b : \mathcal{A}_{\boldsymbol{M}} \to \mathcal{B}$ defined by*

$$b(A_{\boldsymbol{M}}(f)) = A_{\boldsymbol{M}}(b(f))$$

for all $f \in \Omega_0^{1,-2}(M)$.

Defining the quasifree state β on $\mathcal{B}$ according to Proposition 2.11 via its two-point distribution

$$\beta_2(f \otimes h) \doteq \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int\limits_{\mathbb{R}^2 \times \mathbb{C}} \frac{g_{\mathscr{I}^-}(f(u,\zeta), h(u',\zeta))}{(u - u' - i\epsilon)^2} \, du \, du' \, \mu_{S^2}(\zeta)$$

with $f, h \in \mathscr{B}$, it can be pulled back to $\mathcal{A}_{\boldsymbol{M}}$ using the $*$-algebra homomorphism b to yield the quasifree Hadmard state $\omega = \beta \circ b$. Let us put this result, proven for the general case in Theorem 2.2, into the following form:

Corollary 3.7. *The pulled-back state*

$$\omega \doteq \beta \circ b$$

of the field algebra $\mathcal{A}_{\boldsymbol{M}}$ satisfies the Hadamard condition and it is uniquely determined by its two-point distribution

$$\omega_2(f \otimes h) \doteq \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int\limits_{\mathbb{R}^2 \times \mathbb{C}} \frac{g_{\mathscr{I}^-}(b(f)(u,\zeta), b(h)(u',\zeta))}{(u - u' - i\epsilon)^2} \, du \, du' \, \mu_{S^2}(\zeta).$$

That is, we found for all asymptotically flat spacetimes a quasi-free Hadamard state ω on the (conformal) field algebra $\mathcal{A}_{\boldsymbol{M}}$ of the vector potential defined in Definition 3.3 which is induced by a quasi-free state β defined on the boundary algebra $\mathcal{B}$. Furthermore, this state is invariant under all asymptotic symmetries (generated by a complete asymptotic Killing field) including all isometries of the physical spacetimes as shown in Proposition 2.13 and Corollary 2.1. Therefore, ω can be understood as an asymptotic vacuum state of the vector potential.

Conclusions

In Chap. 1 we laid down the mathematical foundation of this thesis. It includes no novel results but might offer a partly different perspective on some of the mentioned definitions and propositions.

Chap. 2 discusses several aspects of quantum field theory on curved spacetimes in the algebraic approach. In Sect. 2.1 we introduced the notion of general local covariance and studied the field algebra of bosonic quantum fields as a locally covariant quantum field theory. This departs from the existing literature only in as much that the field algebra and its properties are often only presented for the scalar field. We extended these results in Sect. 2.1 to conformal embeddings instead of isometric embeddings which yielded the notion of general local coformal covariance and investigated the field algebra as a locally conformally covariant quantum field theory. Similar results had already been obtained for the conformally coupled massless scalar field. After a short introduction of Hadamard states in Sect. 2.3, we presented in Sect. 2.4 one of the main results of this thesis: the bulk to boundary correspondence in asymptotically flat spacetimes for an abstractly defined bosonic quantum field. In particular, we showed the 'holographic' construction of a Hadamard state which can be interpreted as an asymptotic vacuum state employing the correspondence of the quantum field theory in the bulk and a quantum field theory on the boundary of the asymptotically flat spacetime. The proof for the Hadamard property found in this thesis, while relying on the same properties of $\mathscr{I}^-$, differs from that in the literature and is arguably more intuitive and generic.

In Chap. 3 we addressed the main goal of this thesis, namely, the quantization of the vector potential in asymptotically flat spacetimes. To that end, we first investigated the classical phase space of the vector potential in globally hyperbolic spacetimes in Sect. 3.1, slightly improving on the results already present in the literature. In particular, we found a well-defined symplectic product for the vector potential whenever the first or the second de Rham cohomology group is trivial. Using this symplectic product, we quantized the vector potential in Sect. 3.2 as a locally conformally covariant quantum field by restricting the category of admissible spacetimes to those satisfying either $\mathrm{H}^1(M) = \{0\}$ or $\mathrm{H}^2(M) = \{0\}$. Moreover, we discussed the relation of this quantization with the Gupta-Bleuler approach. In Sect. 3.3 we finally studied the vector potential in asymptotically flat spacetimes. A straightforward application of the bulk to boundary correspondence was, however, not possible since the induced symmetric bilinear form on null infinity given by the Bondi metric is not positive definite. Nevertheless, using some special features of the vector potential at past null infinity, we were able to amend the method developed in Sect. 2.4 appropriately to obtain a Hadmard state for the vector potential.

There are many problems which were not covered in this thesis. It is possible, for example, to construct KMS states on the boundary $\mathscr{I}^-$ which can then be pulled-back to the bulk to yield Hadamard states there. These states have a natural interpretation as asymptotic equilibrium states [65]. In the context of the vector potential these states could be useful for the discussion of the cosmic microwave background. Furthermore, we did not study Wick polynomials and renormalization. For example, one could calculate the expectation value of the renormalized stress-energy tensor and then consider cosmological applications via the semiclassical Einstein equation [23]. Another point requiring further investigation are the problems that arise for both the field strength tensor and the vector potential in spacetimes of non-trivial topology [57, 58].

Concerning the general results of this thesis obtained in Sect. 2.4, it seems desirable to generalize these even further to horizons different from $\mathscr{I}^-$ e.g. the Schwarzschild horizon of a spacetime with a black hole [66]. In particular, an extension of the categorical formulation of general local covariance might allow for a fundamental discussion of the relation of bulk and boundary quantum field theories.

We hope to come back to these matters in the near future.

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So long, and thanks for all the fish. — Douglas Adams

Erklärung

Ich versichere, diese Arbeit selbständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel verfasst zu haben. Ich gestatte die Veröffentlichung dieser Arbeit.

Hamburg, den 07.06.2011

Daniel Siemssen