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The Amplitude Distributions in Lengthened Bunches

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1. Introduction

Bunch lengthening due to RF cavities in electron storage rings and some of the effects associated with it have been considered in Ref. 1. In particular, results were given for the bunch shape, the potential well, the focusing electric field, the variation of the synchrotron frequency ν_s, and the Fourier spectrum of the synchrotron oscillation.

This work is now extended to include the distribution function for the amplitude of the synchrotron oscillations.

2. Outline of the procedure for finding the amplitude distribution function

In synchrotron phase space with the phase τ and the suitably scaled energy error ε as coordinate axes the particle density is described by a distribution function $\psi(\varepsilon,\tau)$. The integral of this function over ε is known, it is the bunch shape $I(\tau)$:

$$I(\tau) = \int_{+\infty}^{+\infty} \psi(\varepsilon,\tau)\, d\varepsilon \qquad (1)$$

The trajectories for particles which cross the τ axis a time $\tau_o - \tau_s$ before the synchronous particle are also known. Since the system has an invariant Hamiltonian (which does not depend on time explicitly), the trajectories divide phase space into "rings". Particles in one ring stay in it forever when slow stochastic instability is neglected.

From the calculation of the synchrotron frequency ν_s it is also known where a particle is on its trajectory n time steps Δt after it started on the τ axis. The trajectories and the lines of equal time divide the phase space into small areas, each bordered by 2 trajectories and 2 time lines as shown in Fig.1.

Since the Hamiltonian is conservative, the areas enclosed between tow trajectories are all equal, and the particle density along a trajectory is conserved. Hence, all the areas in a given ring contain the same number of particles. To obtain a complete knowledge of the distribution function $\psi(\varepsilon,\tau)$ it is therefore sufficient to know the trajectories and to calculate the density along them. The amplitude a is defined by the time difference $(\tau_o - \tau_s)$ between the crossing print τ_o of the τ-axis and the synchronous time τ_s . τ_o is ahead of τ_s.

From the calculation of the bunch shape $I(\tau)$ it is known how many particles are inside a strip of synchrotron phase space which covers the interval $\tau_1 \leq \tau \leq \tau_2$ and all the energies.

Such a strip is also shown in Fig.1. It is clear that only rings which are at least as far away from τ_s as the strip can contribute to the current within that strip. The contribution of a particular ring is proportional to the product of the number of the areas from that ring which fall within the strip, and the number of particles within each area.

For infinitesimal distances $\Delta\tau$ between the trajectories and infinitesimal time steps Δt, the areas become polygons (quadrilaterals and triangles), and tje first factor above is obtained by a simple geometrical calculation, in which the number of areas which are completely inside a strip is counted, and the contribution of those areas which are only partly inside the strip is obtained by assuming that the paricle density inside an area is constant. The outcome of this calculation is a matrix M. Its elements M_{ik} are the number of areas from the k-th ring which fall into the i-th strip. The elements M_{ik} vanish for $i < k$, i.e. M is an upper triangular matrix.

Let the population of an area in the k-th ring be A_k. The total current in the i-th strip, I_i, is then given by

$$I_i = \sum_{k=i}^{\infty} M_{ik} A_k \qquad (2)$$

There is one such equation for each strip. If the number of strips and rings is made equal a system of linear equations is obtained which can be solved for the populations A_k. Multiplying these numbers A_k by the number of areas in a given ring, given by the synchrotron frequency $\nu(a)$ which will be plotted later on.

Since all the information necessary for this calculation is only known in the form of tables inside the computer program, the infinitesimal areas are necessarily finite, and all the assemptions made (polygons, constant density, etc.) approximately true. The stepsizes which were used to obtain the results given below have been determined by trial and error.

3. Results

The result of the calculation of the amplitude distribution $\rho(a)$ is shown in Fig.2 and 3. The parameter on the curves is q, which determines the bunch lengthening.

$$q = \frac{QZ_o}{2\pi_o^2 \dot{V}_{rf}} \qquad (3)$$

Here, Q is the bunch charge, Z_o is the impedance of free space ($Z_o = 1{,}20\pi\ \Omega$), σ_o is the unperturbed bunch length in time units, and $\dot{V}_{rf}$ is the derivative of the RF voltage applied. The curves were obtained for $d = g/c\sigma_o = 10$. Here g is the length of the cavity and c is the velocity of light.

Fig.2 and 3 differ by the width of the histogram bins chosen, all other parameters are the same. At the smaller bin width a numerical instability exists at large values of q and small amplitudes, while for higher amplitudes there is excellent agreement between the two distribution functions. When the bunch charge is increased the peak of the distribution function moves towards higher amplitudes, and the density at small amplitudes is reduced. The bunches become hollow. At small bunch charges there is excellent agreement between the computed density distribution and the expected Maxwell distribution. This is demonstrated in Fig. 4 for $q = 10^{-4}$ and $d = 10$.

Reference:

1. E. Keil, PEP-126 (1975)

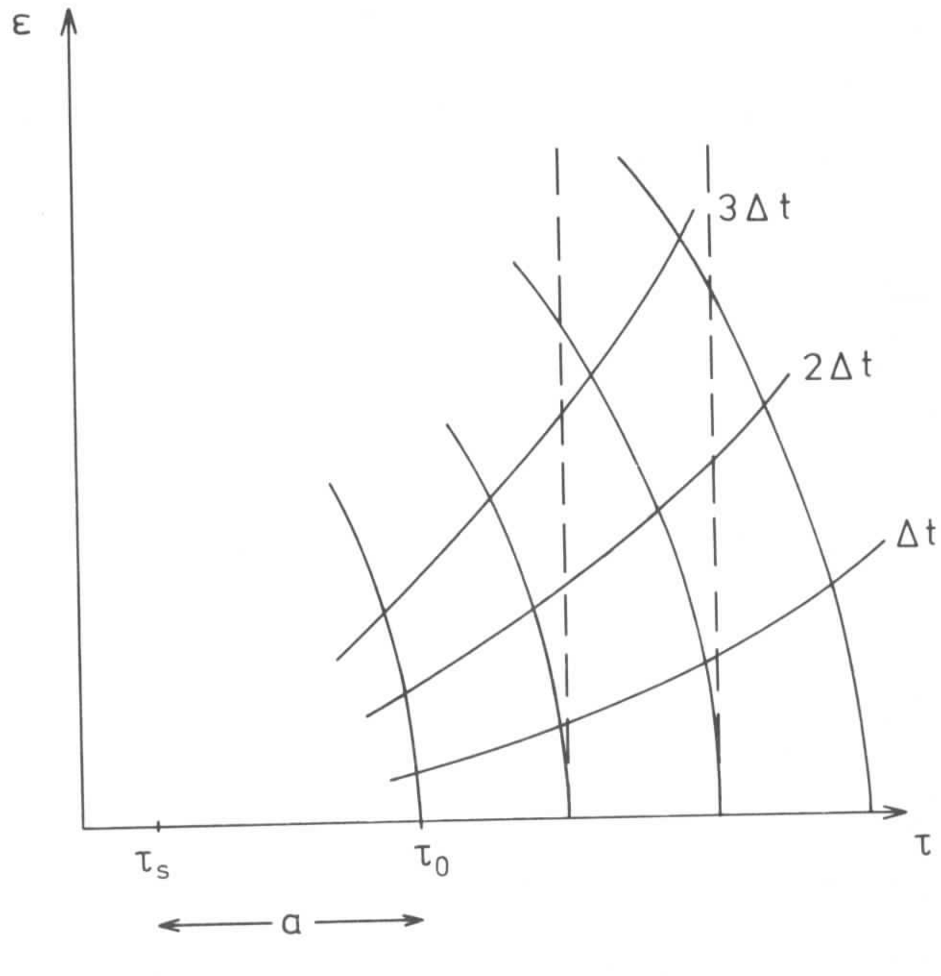


Fig. 1

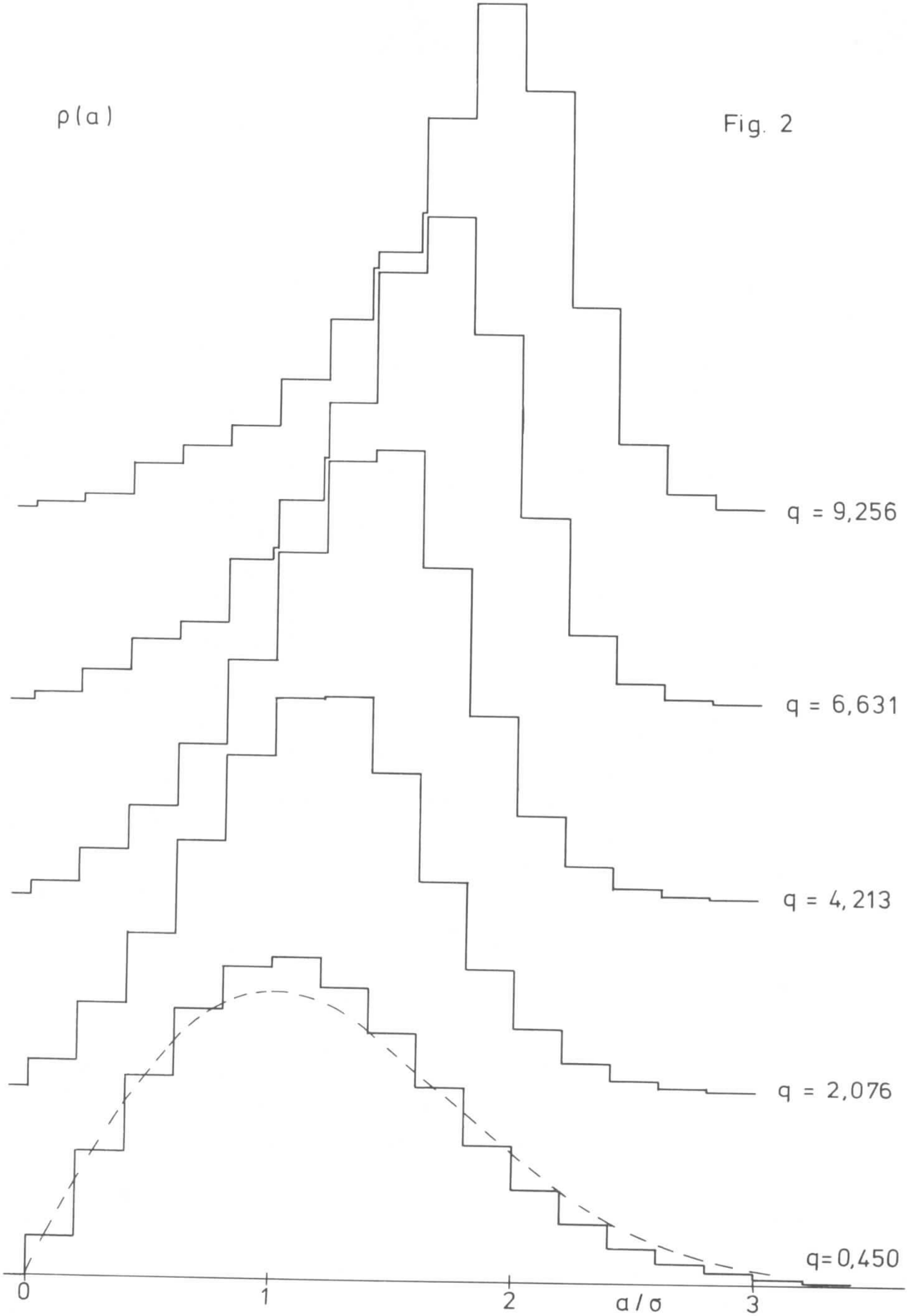


Fig. 2

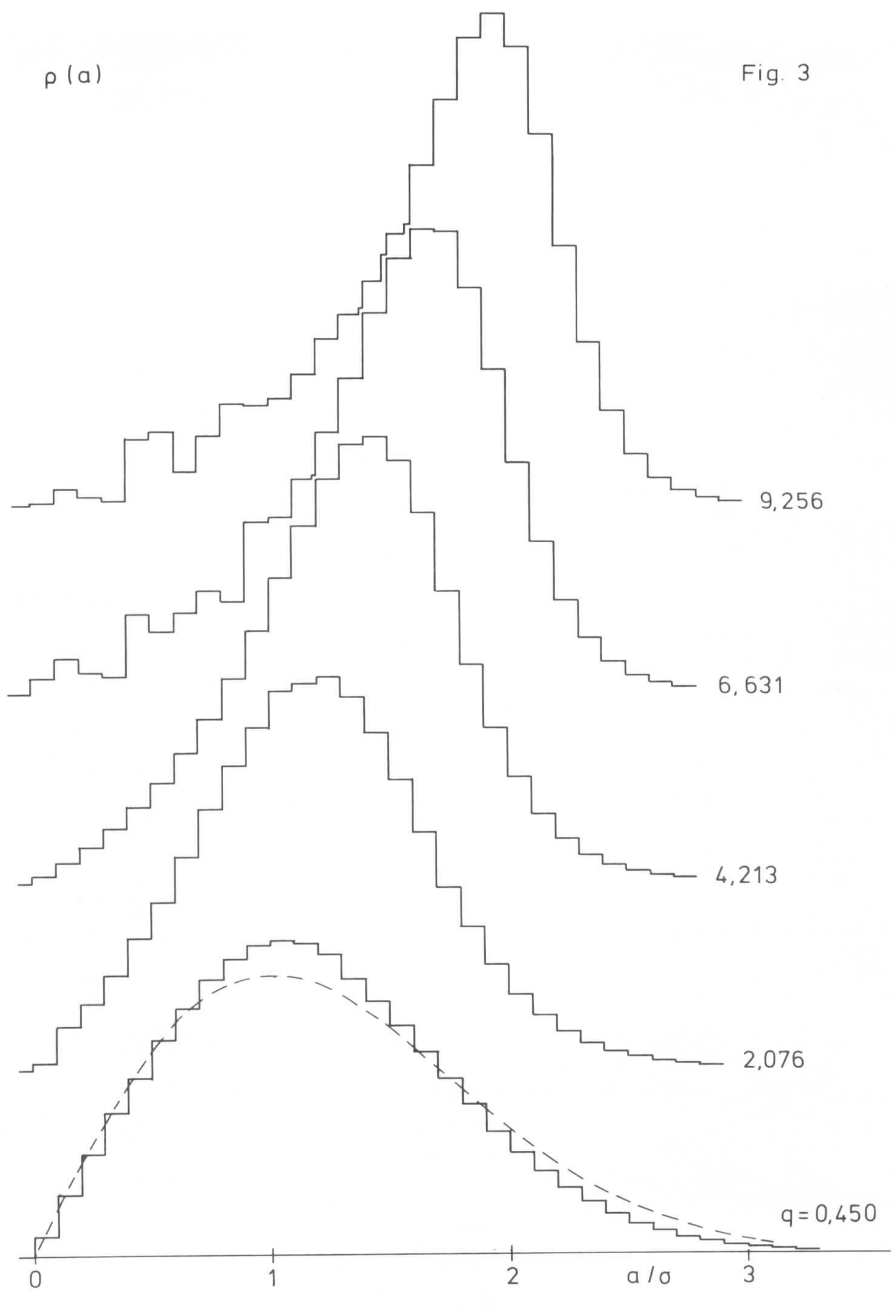
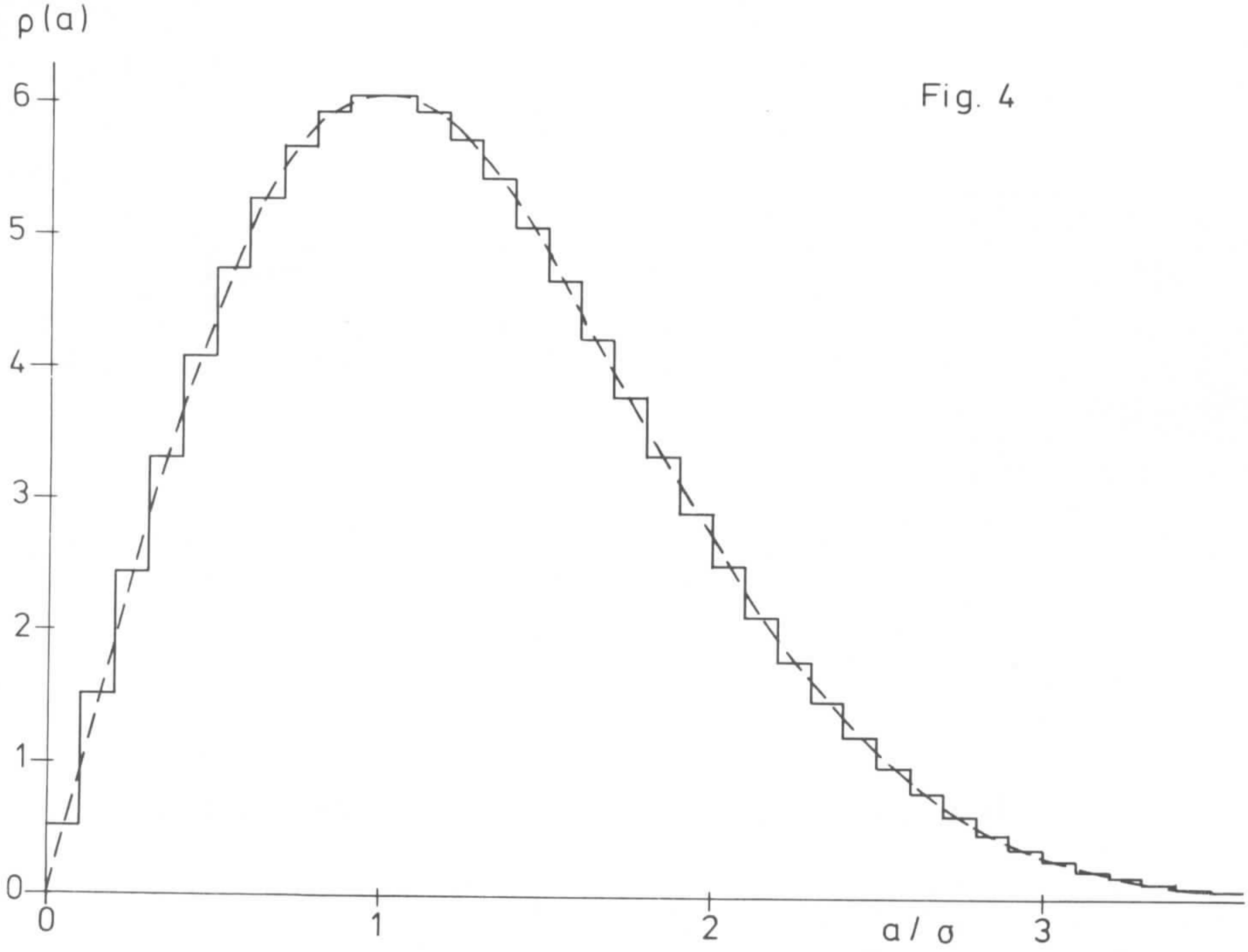


Fig. 3



ρ(a)
6
5
4
3
2
1
0
0
1
2
3
a / σ
Fig. 4