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Longitudinal Instabilities of Unbunched and Bunched Proton Beams due to Resonators with High Q-Factors

by

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Introduction

The longitudinal motion of unbunched (coasting) and bunched beams can be unstable if the coupling impedance of the equipment of the ring exceeds critical values. In this context one wants to know to what extent resonators with high Q factors can be retained in electron machines when protons are stacked or accelerated.

Scheme of deriving stability criteria

Since there is no radiation damping for proton beams, particles move according to Liouville's theorem and the phase space density-function satisfies Liouville's equation

$$\left(\frac{\partial}{\partial t} + \zeta \frac{\partial}{\partial \zeta} + \dot{U} \frac{\partial}{\partial U}\right) \rho \; (\zeta, U, t) = 0 \qquad (1)$$

where U, ζ are canoncial variables

$\rho \; (\zeta, U, t)$	particle density in phase space
ζ	azimuth
U	P - P o
h	harmonic number
P o	average particle momentum
P	particle momentum

From equation (1) follows that the total derivative of the density vanishes along a particle trajectory. We want to cast (1) into a form in which the interaction terms explicitely appear. We are interested in the stability of some equilibrium distribution ρ against small perturbations $\tilde{\rho}$ (t, ζ, U). Changing to variables $\overline{\zeta}$, $\overline{U}$ of interaction-free motion we obtain from (1)

$$\left(\frac{\partial}{\partial t} + \overline{\zeta} \frac{\partial}{\partial \overline{\zeta}} + \dot{\overline{U}} \frac{\partial}{\partial \overline{U}}\right) \tilde{\rho} \; (t, \; \zeta, \; U) = \dot{\overline{U}}_I \cdot \frac{\partial \rho}{\partial \overline{U}} \qquad (2)$$

In this equation $\dot{\overline{U}}_I$ denotes the change of $\overline{U}$ due to the perturbation $\tilde{\rho}$ and the interaction

$$\dot{\overline{U}} \sim F\left(\tilde{\rho}, \overline{U}, \overline{\zeta}, t\right)$$

Keeping first order terms only, we substitute

$$\dot{\overline{U}}_I \frac{\partial \rho}{\partial U} \rightarrow \dot{\overline{U}}_I \frac{\partial \rho_o}{\partial \overline{U}} \tag{3}$$

on the right hand side of (2) neglecting $\dot{\overline{U}}_I \frac{\partial \tilde{\rho}}{\partial \overline{U}}$.

This leads to a linear equation for $\tilde{\rho}$ if only linear terms of $F\left(\tilde{\rho}, \overline{U}, \overline{\zeta}, t\right)$ are retained.

$$F\left(\tilde{\rho}, \ldots\right) \sim \tilde{\rho} \tag{4}$$

Equation (2) together with (3) and (4) form the basic tool for stability investigations.

The advantange of this procedure is:

1.) solutions can be found at least approximately
2.) stability criteria at threshold can be given .

The disadvantage of this procedure is:

1.) No stability limit of the form $|\tilde{\rho}| < L$, which may restrict the perturbations if nonlinearties are present, can be given.
2.) Equations (2), (3), (4) don't describe the environment of the beam.

Starting from the linear system for pertubations $\tilde{\rho}$ of equilibrium distribution ρ_o one can derive stability criteria at threshold for unbunched and bunched proton beams.

Unbunched beam

Since the stability of coasting beams according to equations (2), (3), (4) is treated extensively in literature (1,2), only a summary is given here.

Looking for solutions of $\tilde{\rho}\ (\bar{\zeta}, \bar{U}, t)$ in terms of $\tilde{\rho}_t\ (\bar{\zeta}, \bar{U})\ e^{i\omega t}$ one obtains the following dispersion relation for ω

$$1 = \frac{-ieI_o |Z| (X+iY)}{h \frac{\partial \omega_o}{\partial W}} \int dW \frac{d\rho_o}{dW} \frac{1}{W-W_1} \qquad (5)$$

h	=	harmonic number
I_o	=	beam current
$\lvert Z \rvert$	=	absolute value of shunt impedance at harmonic number h
$X + iY$	=	phase factor of shunt impedance
$\omega_o/2\pi$	=	revolution frequency
W	=	angular momentum
W_1	=	$\dfrac{\omega - n\omega_o}{h \frac{\partial \omega_o}{\partial W}}$

Equation (5) has been solved [1,2,9] taking $\rho_o(P)$ from realistic beam shapes. The result is an inequality relation for the coupling impedance at the harmonic number h

$$\frac{|Z|}{h} \leq \frac{P \cdot c\ \beta}{eI_o} |\eta| \frac{\Delta P}{P} \cdot \frac{\Delta P_{se}}{P} \cdot \Lambda \qquad (6)$$

β	particle velocity (in units of c)
η	$1/\gamma^2_{tr} - 1/\gamma^2$ (γ, γ_{tr} are particle and transition energy in restmass units, respectively.
Λ	numerical factor (Λ is assumed to be near one in this note)

The relative spreads $\frac{\Delta P}{P}$ and $\frac{\Delta P_{se}}{P}$ are defined in fig.1.

ΔP total spread at half height

$\frac{\Delta P_s \text{low}}{2}$ width of the low energy tail at half height

$\frac{\Delta P_s \text{high}}{2}$ width of the high energy tail at half height

$$\frac{\Delta P_{se}}{P} = \begin{cases} \frac{\Delta P_{s\ low}}{P} & \text{if } X > 0 \\ \frac{\Delta P_{s\ high}}{P} & \text{if } X < 0 \end{cases}$$

For Gaussian-like distribution functions $\frac{\Delta P_{se}}{P} \cdot \frac{\Delta P}{P}$ can be replaced by $\left(\frac{\Delta P}{P}\right)^2$, but this does not apply for distributions with flat top.

If the coupling impedance is below the critical value of (6) the instability is cured by Landau damping. If the impedance exceeds the critical value an initial perturbation of the equilibrium distribution increases.

Although the further motion cannot be described by (3), (4), (5), (6), the environment of the beam can be sketched.

The increase of an initial perturbation leads to perfect self bunching at the harmonic number h , and the beam loses energy according to

$$P = I_o^2 \text{ Re } Z \qquad (7)$$

If there is no Landau damping (rectangular distribution for example), the instability can occur for any Z. The complex frequency shift is then given by

$$\Delta\omega = f_o \sqrt{\frac{2\pi\eta I_o}{Pc}} \quad \sqrt{iZn}$$

The growth time follows from

$$\frac{1}{\tau} = \text{Im } \Delta\omega \tag{8}$$

In the worst case one obtains

$$\frac{1}{\tau} = f_o \sqrt{\frac{2\pi |\eta| I_o e}{Pc}} \quad \sqrt{|Z| n} \tag{9}$$

Using (8) and the critical impedance from (6) we get the growth rate that is "compensated" by Landau damping

$$\frac{1}{\tau_L} = n\ f_o\ |\eta| \left(2\pi\beta\ \frac{\Delta P}{P} \cdot \frac{\Delta P_{se}}{P} \cdot \Lambda\right)^{1/2} \tag{10}$$

In order to avoid energy loss one has to restrict the coupling impedance to (6) in storage rings with coasting proton beams.

In proton accelerators the spilt particles induce high voltages at the harmonic number h

$$U_s = I_o\ Z$$

after self bunching, which in turn pertubates the synchrotron motion of the protons being accelerated.

Bunched Beams

Classification of oscillation modes

In the case of bunched beams particles can perform phase oscillations around the equilibrium state. The oscillation modes of a bunched beam are classified by two mode numbers

$$< \mu, m > \qquad (11)$$

The first number denotes the mode of single bunch motion, the second number describes the relative motion of single bunches within the beam.

For $\mu = 1,2 \ldots\ldots \infty$ we have dipole-, quadrupole-, sextupole-, octupole-, etc. modes respectively. In the dipole mode a single bunch oscillates as a whole, whereas all other modes characterize bunch-shape oscillations. Fig. 2 gives an illustration of bunch shape oscillations up to octupole oscillations.

For different m = 0, 1, 2, ..., M-1 (M number of bunches) the phase difference of oscillations between two adjacent bunches is $2\pi m/M$.

In general the states $< \mu, m >$ are not states of the system (2),(3), i.e. the modes $< \mu, m >$ are coupled. If, however, all bunches have:

i. the same equilibrium shape
ii. the same particle number
iii. the same synchrotron frequency
iiii. the same distance from each other

the states (11) are approximately uncoupled and eq. (2) can be solved (3,4,5,6,7). The dispersion relation for the complex synchrotron frequencies $\omega_{\mu,m}$ in $\tilde{\rho} \sim e^{i\omega_{\mu,m}t}$ reads

$$1 = \frac{\Delta\omega_{\mu,m}}{W_\mu} \int_0^\infty dr \, \frac{\frac{d\rho_o}{dr} \cdot r^{2\mu}}{\omega_{\mu,m} - \mu\omega_s(r)} \quad ; \quad W_\mu = \int_0^\infty dr \, \frac{\partial\rho_o}{\partial r} \, r^{2\mu} \qquad (12)$$

$\rho_o(r)$ equilibrium distribution of phase oscillation amplitude

$\omega_s(r)$ undisturbed synchrotron frequency as a function of amplitude

The complex shifts $\Delta\omega\mu,m$ are given by the following expression

$$\Delta\omega\mu,m = f_s(0)\frac{\eta}{|\eta|} \frac{I_o}{U_c \cos \zeta_o} \frac{M}{B} \sum i \{Z^+_{\nu M+m} \Gamma_{\mu,\nu M+m} - Z^-_{\nu M-m} \Gamma_{\mu,\nu M-m}\} \quad (13)$$

$f_s(0) = \omega_s(0) / 2\pi$

U_c = main peak voltage

ζ_o = phase angle (with respect to U_c)

$Z^{\pm}_{\nu\ M\pm m}$ = coupling impedance at $(\nu\ M\pm m)\ \omega_o \pm \omega_s(0)$

The coefficients follow from tabulated functions

$$\Gamma\mu,\nu M\pm m = F\mu\ (\chi)$$

with

$$\chi = \pi \cdot \lambda \cdot \left(\nu\frac{M}{h} \pm \frac{m}{h}\right) \quad (\lambda \text{ bunch length in radians}) \quad (14)$$

The modes μ are essentially excited if $\chi \approx \mu\ \pi$. At these values the functions $F\mu\ (\chi)$ become

$$F\mu\ (\chi) \sim \frac{1}{\sqrt{\mu}} \qquad \text{for} \quad \chi \sim \mu\ \pi \quad (15)$$

For $\chi \to 0$ one has

$$\frac{F\mu(\chi)}{\chi} \to 0 \qquad \text{for} \quad \mu > 1 \quad ,$$

whereas

$$\frac{F_1(\chi)}{\chi} \to \frac{1}{2} \quad (16)$$

Thus, for vanishing bunch length, i.e. $B \to 0$, $\chi \to 0$, only the dipole mode can be excited, which is plausible since bunch-shape oscillations can occur only if the driving voltage varies within the bunch.

If there is no Landau-damping one obtains from 7

$$\omega_{M,m} - \mu\omega_s(0) = \Delta\omega_{\mu,m} \qquad (17)$$

i.e. the oscillation frequency of mode μ is near $\mu\,\omega_s(0)$.

In the case of a small bunch length (13) and (17) reproduce previous results concerning the stability of bunched beams(3,4,6).
From (13) we learn that the stability is only governed by the real part of the shunt impedances. For the growth or damping rates, respectively, we can write

$$\frac{1}{\tau_{\mu,m}} \sim \frac{-\eta}{|\eta|} \sum \left(X^{+}_{\nu M+m}\, \Gamma_{\mu,\nu M+m} - X^{-}_{\nu M-m}\, \Gamma_{\mu,\nu M-m} \right) \qquad (18)$$

Fig. 3 demonstrates the influence of resonators at different frequencies on mode m.

From (18) follows that for $\eta < 0$ (above transition energy) a resonator at $(\nu M+m)\omega_o$ excites (+) mode m while a resonator at $(\nu + 1)$ M-m damps (-) the same mode. For even M the mode $\bar{m} = \frac{M}{2}$ is always stable if the resonator has its maximum impedance at $\omega \leq (\nu+\frac{1}{2})M\omega_o$. The same is true for $m = 0$. Let us call $m^{*} = M-m$ the mirror mode of m. Then we find in the case of a single resonator the following simple duality (see fig. 3, fig.4).

If m is stable, m^{*} is unstable.
If m is unstable, m^{*} is stable.

Below transition energy ($\eta > 0$) the roles of $X^{+}_{\nu M+m}$ and $X^{-}_{\nu M-m}$ are interchanged. Thus, if a resonator is moved through all multiples of the revolution frequency we have a sequence of damping and antidamping for each mode. The same is true in an accelerator where the revolution frequency is changed during acceleration and a resonator is present at a high harmonic number. If the acceleration time is sufficiently small, the increase of phase oscillation amplitude during acceleration is a second order effect.

It should be emphasized, however, that (18) is only valid if $|\Delta\omega| << \omega(0)$, i.e. for "weak interaction". What happens in the case of "strong interaction" can be demonstrated for $\mu = 1$, $m = 0$, $M = h$. Instead of (13) and (17) one has to solve[6)]

$$\omega_s^2-\omega^2 = -\frac{\eta}{|\eta|}\frac{\omega_s^2\, I_b}{U_c \cos\psi}\left\{i\left(X^+-X^-\right) - \left(Y^++Y^-\right)\right\} \tag{19}$$

Writing

$$\Delta X = X^+-X^- \approx \frac{\partial X}{\partial\omega}\,\omega_s$$

$$Y^++Y^- \sim 2Y$$

the solution of (19) is

$$\omega = i\,\omega_s \frac{\frac{\eta}{|\eta|}\Delta X}{2\,U_c \cos\psi} \pm \sqrt{\omega_s^2-2\frac{\eta Y I_b}{|\eta| U_c \cos\psi}} - O\,(\Delta X^2) \tag{20}$$

For weak interaction $\left(\frac{2|X|I_b}{U_c \cos\psi} << 1\right)$ the second term in (20) is real giving a real frequency shift, while the first term governs the stability behaviour according to (18). For strong interaction $\left(\frac{2|X|I_b}{U_c \cos\psi} > 1\right)$ the second term can become imaginary, and the stability behaviour is no longer determined by the real part of the shunt impedance only.

For stability one must have (above transition) :

$$\Delta X < 0 \quad (10) \tag{21}$$

From $\Delta X < 0$ follows $Y > 0$

and therefore

$$\frac{2\,|X|\,I_b}{U_c \cos\psi} < 1 \quad (10) \tag{22}$$

To study the effect of Landau damping we must go back to eq. (12) Using realistic distribution functions $\rho_o(r)$ the dispersion relation can be solved numerically [(7)]. The stability of the system can be roughly characterized by [(7)]

$$|\delta\omega_{sL}| > \frac{4}{\sqrt{\mu}}\,|\Delta\omega_{\mu,m}|\,, \tag{23}$$

where $\delta\,\omega_{sL}$ is the full frequency spread within the bunch due to non-linearity of the phase focussing force.

The Landau-spread $\delta\,\omega_{sL}$ follows from the theory of non-linear oscillations[8]. If the equation of phase oscillation is expanded up to the cubic term in the phase

$$\ddot{\zeta} + \omega_s^2\,\zeta + \alpha\,\zeta^2 + \beta\,\zeta^3 = 0 \qquad (24)$$

a relation for the spread is obtained in the form

$$\delta\,\omega_{sL} = \left(\frac{3\beta}{8\omega_\sigma} - \frac{5}{12}\,\frac{\alpha^2}{\omega_s^3}\right)\frac{\lambda^2}{4} \qquad (25)$$

The nonlinear equation for phase oscillations including an additional Landau-cavity at the n-th harmonic with a driving voltage U_L reads

$$\ddot{\zeta} + P^2\left\{(\sin\zeta - \sin\zeta_o) + \frac{U_L}{U_c}\,(\sin n\,\zeta - \sin\zeta_{oL})\right\} = 0 \qquad (26)$$

ζ_o phase angle with respect to main voltage U_c

ζ_{oL} phase angle with respect to U_L

Expanding (26) according to (24) and using (25) one arrives at

$$\left|\frac{\delta\omega_{sL}}{\omega_s}\right| = \frac{\lambda^2}{4}\left|\frac{1}{16}\left(1+\frac{U_L}{U_c}\,\frac{\cos\,\zeta_{oL}}{\cos\,\zeta_s}\,n^3\right)+\frac{5}{48}\left(\tan^2\zeta_o+\frac{U_L^2}{U_c^2}\,\frac{\sin^2\zeta_{oL}}{\sin^2\zeta_o}\,n^4\right)\right| \qquad (27)$$

In order to find the dependence of the critical shunt impedance on the bunch shape mode we insert (13) into (23). Because of (15) we have

$$|z| \sim \mu \qquad (27)$$

So the dependence of the critical impedance on the bunch-shape mode for bunched beams is the same as the dependence of critical impedance on the self-bunching mode for coasting beams.

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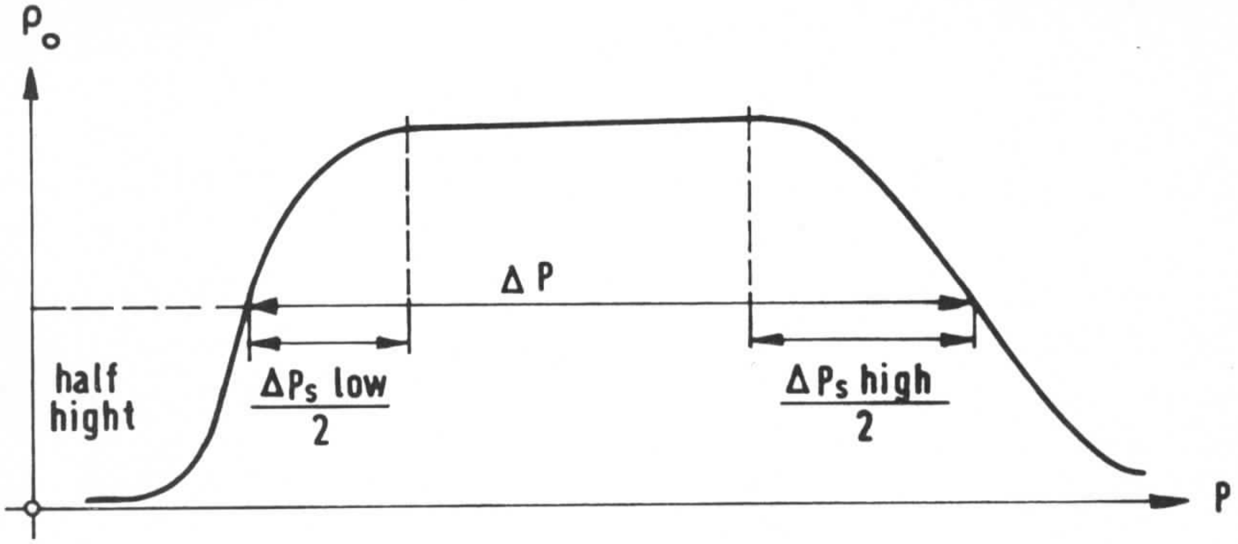
Figure captures

Fig.1 Defination of relative spreads $\frac{\Delta P}{P}$, $\frac{\Delta P_{se}}{P}$

Fig.2 Illustration of bunch shape oscillations

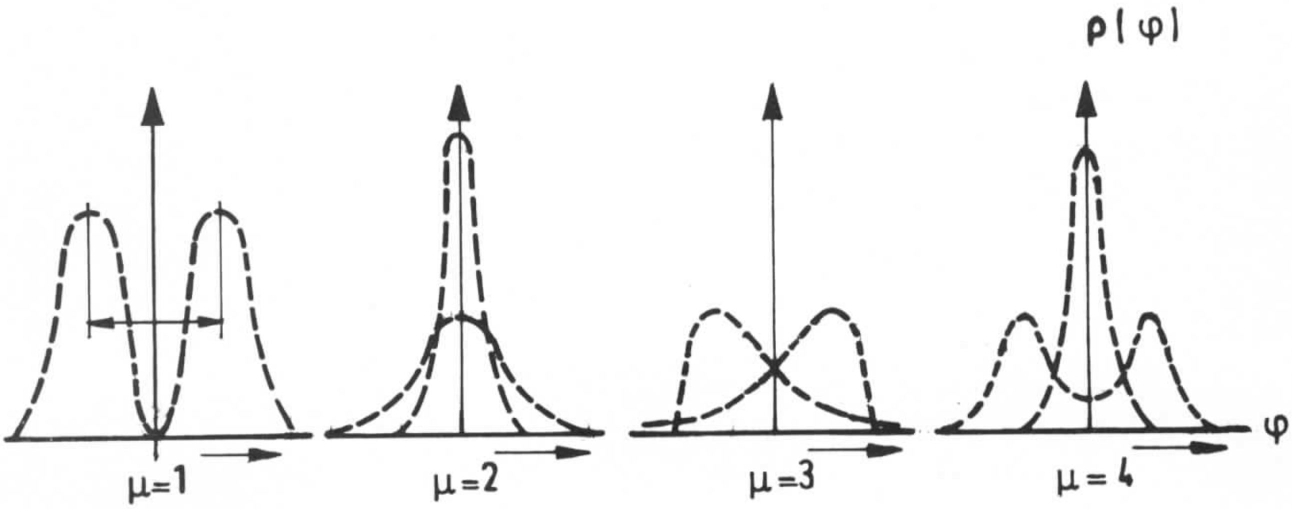
Fig.3 Mode-resonator correspondence

Fig.4 Mode-mirror mode relation



ρ_o
ΔP
half
hight
$\frac{\Delta P_s \text{ low}}{2}$
$\frac{\Delta P_s \text{ high}}{2}$
P

Fig. 1



$\rho(\varphi)$
φ
$\mu=1$
$\mu=2$
$\mu=3$
$\mu=4$

Fig. 2

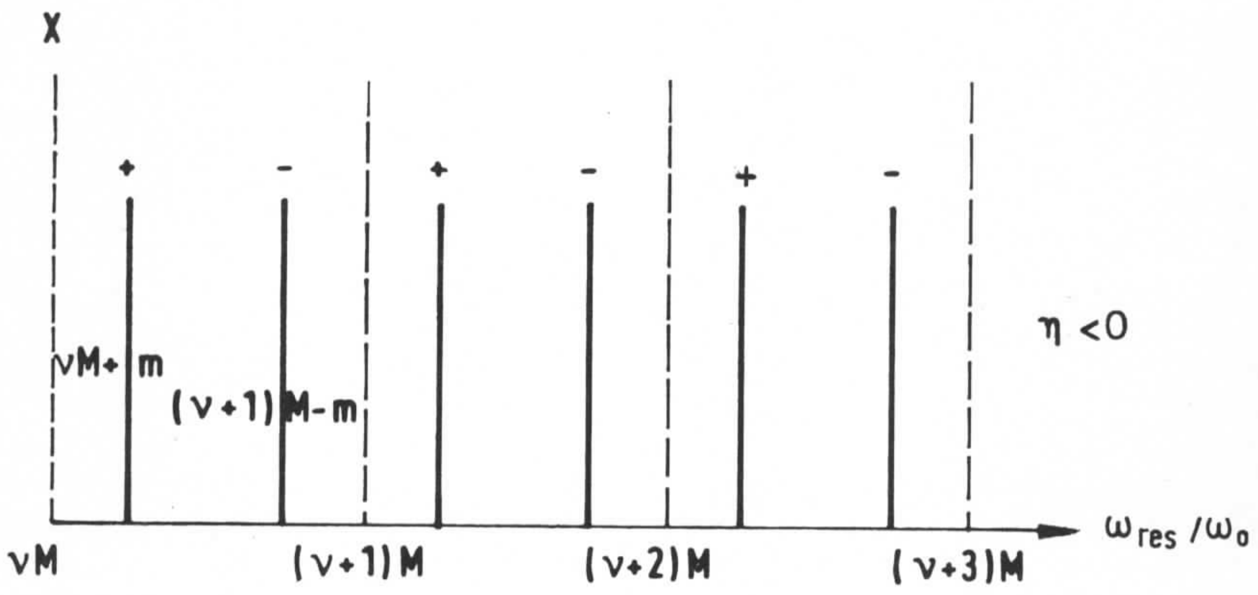


Fig. 3

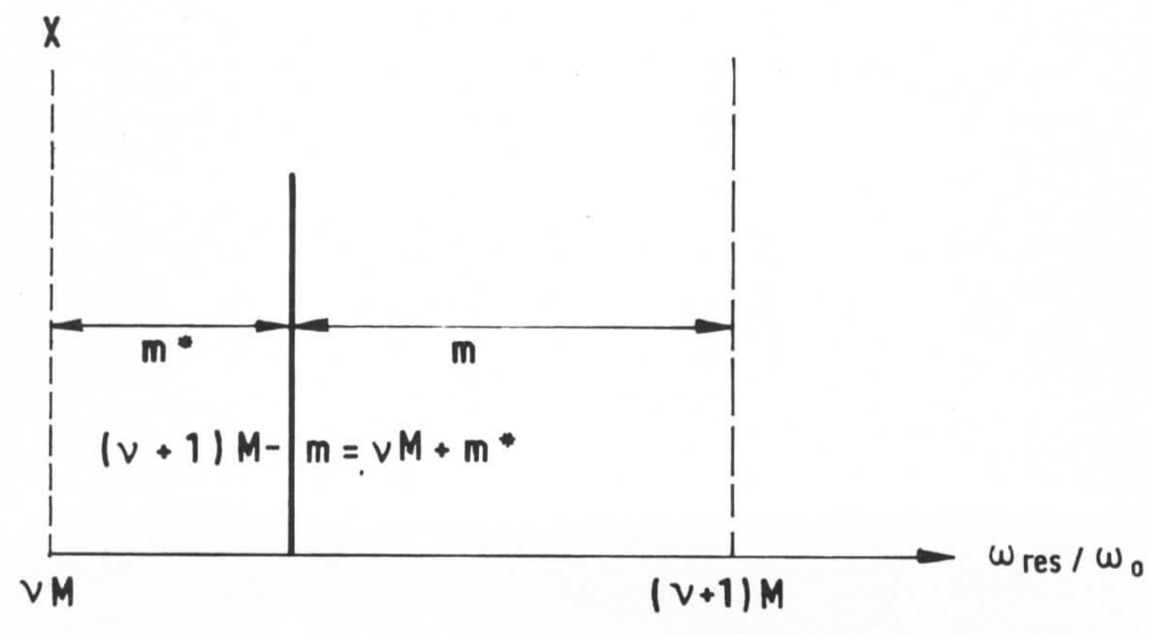


Fig. 4