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Betatron Frequency Shifts for PETRA

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1) Formulae for protons

The protons pass several electron bunches in one interaction region. The corresponding linear tune shifts produced by one bunch are given by[1)]

$$\Delta Q_{xp} = \frac{N_{be} r_p \beta_{xp}}{2\pi\gamma_p \sigma_x (\sigma_x + \sigma_t)} \sqrt{\frac{\pi}{2u}}\, e^{-u} \sum_{n=o}^{\infty} (2n+1) \left(\frac{\sigma_x - \sigma_t}{\sigma_x + \sigma_t}\right)^n I_{n+1/2}(u) \qquad (1)$$

$$\Delta Q_{zp} = \frac{N_{be} r_p \beta_{zp}}{2\pi\gamma_p \sigma_t (\sigma_t + \sigma_x)} \sqrt{2\pi u}\, e^{-u} \sum_{n=o}^{\infty} \left(\frac{\sigma_x - \sigma_t}{\sigma_x + \sigma_t}\right)^n \left(I'_{n+1/2}(u) - I_{n+1/2}(u) \right) \qquad (2)$$

with $u = \left(\frac{s\Phi}{2\sigma_t}\right)^2$, $\sigma_t^2 = \sigma_z^2 + \sigma_s^2 \tan^2\Phi$

N_{be} = number of electrons per bunch, r_p = classical proton radius

γ_p = proton energy in units of rest energy

$\beta_{x,z\,p}$ = amplitude function for the protons in the point, where the protons pass the electron bunch

$\sigma_{s,z,z}$ = standard deviations for longitudinal, horizontal and vertical dimensions of the bunch, 2Φ = angle between the beam directions, s = distance from the center of the interaction region.

The modified Bessel functions $I_{n+1/2}$ of order $n + 1/2$ can be represented by exponential functions.

The formulae are derived with $\gamma_{p,e}^2 \gg 1$ and under the condition that the variation of the amplitude function is small within one bunch length, i.e. $\sigma_s \ll \beta_{x,\,z\,o}$. The variation from one bunch to the next may be arbitrary. For s = 0 the formulae simplify to

$$\Delta Q_{xop} = \frac{N_{be} r_p \beta_{xop}}{2\pi\gamma_p \sigma_x (\sigma_x + \sigma_t)} \qquad (3)$$

$$\Delta Q_{zop} = \frac{N_{be} r_p \beta_{zop}}{2\pi\gamma_p \sigma_t (\sigma_t + \sigma_x)} \qquad (4)$$

For s $\neq 0$ and an approximately round electron beam the tune shifts due to the long range forces are

$$\Delta Q_{xp} = \frac{N_{be} r_p \beta_{xp}}{2\pi\gamma_p \sigma_x (\sigma_x + \sigma_t)} \frac{1}{2u} \left\{ 1 - e^{-2u} + 3 \frac{\sigma_x - \sigma_t}{\sigma_x + \sigma_t} \left(1 - \frac{1}{u} + \left(1 + \frac{1}{u}\right) e^{-2u}\right) + 0\left(\left(\frac{\sigma_x - \sigma_t}{\sigma_x + \sigma_t}\right)^2\right)\right\} \tag{5}$$

$$\Delta Q_{zp} = \frac{N_{be} r_p \beta_{zp}}{2\pi\gamma_p \sigma_t (\sigma_x + \sigma_t)} \frac{1}{2u} \left\{ (4u+1) e^{-2u} - 1 + \frac{\sigma_x - \sigma_t}{\sigma_x + \sigma_t} \left(\frac{3}{u} - 1 - e^{-2u}\left(4u + 5 + \frac{3}{u}\right)\right) + \right.$$

$$\left. + 0 \left(\left(\frac{\sigma_x - \sigma_t}{\sigma_x + \sigma_\tau}\right)^2\right)\right\} \tag{6}$$

2) Formulae for electrons

For the tune shifts produced by an unbunched beam formulae are known only for the case where the beam cross section is round[2]. Introducing different amplitude functions for the electrons and protons one obtains:

$$\Delta Q_x = \frac{N_p r_e}{2\pi\gamma_e \Phi^2 C} F_x\left(\frac{\ell}{2\beta_{op}}, \frac{2\Phi\beta_{op}}{\sigma_o}\right) \tag{7}$$

$$\Delta Q_z = \frac{N_p r_e}{2\pi\gamma_e \Phi^2 C} F_y\left(\frac{\ell}{2\beta_{op}}, \frac{2\Phi\beta_{op}}{\sigma_o}\right) \tag{8}$$

with

$$F_x(a,b) = \int_{-a}^{a} \left(\frac{\beta_{op}}{\beta_{xoe}} + \frac{\beta_{xoe}}{\beta_{op}} \frac{1}{s^2}\right)\left(1 - \exp\left\{-\frac{b^2 s^2}{2(1+s^2)}\right\}\right) ds$$

$$F_z(a,b) = \int_{-a}^{a} \left(\frac{\beta_{zoe}}{\beta_{op}} + \frac{\beta_{op}}{\beta_{zoe}} s^2\right)\left(\left(\frac{1}{s^2} + \frac{b^2}{1+s^2}\right) \exp\left\{-\frac{b^2 s^2}{2(1+s^2)}\right\} - \frac{1}{s^2}\right) ds$$

N_p = number of protons of the beam

C = circumference

r_e = classical electron radius

ℓ = lenght of the interaction region

β_{op}, β_{xoe}, β_{zoe} = amplitude functions for protons and electrons in the center of the interaction region

3) Numerical values

If one assumes the following beam parameters

$$N_p = 6{,}2\cdot10^{14} \quad (\div 12A),$$

$$N_{be} = 1{,}4\cdot10^{9} \quad (\div 115 \text{ mA} \div 3 \text{ MW})$$

$$\beta_{xop} = \beta_{zop} = 200 \text{ cm}$$

$$\beta_{xoe} = \beta_{zoe} = 50 \text{ cm}$$

$$\gamma_p = 120 \qquad \gamma_e = 30\,000$$

$$\sigma_{xop} = \sigma_{zop} = \sigma_{xoe} = \sigma_{zoe} = 0{,}016 \text{ cm}$$

$$\sigma_s = 1{,}5 \text{ cm} \quad , \qquad \ell = 21{,}5 \text{ m}$$

$$s = n\cdot 30 \text{ cm}, \qquad n = 0, \pm 1, \pm 2, \ldots \pm 36$$

one obtains for the tune shifts

$$\Delta Q_{xp} = 3{,}9\cdot10^{-4} \qquad \Delta Q_{zp} = -4{,}2\cdot10^{-5}$$

$$\Delta Q_{xe} = 0{,}11 \qquad \Delta Q_{ze} = -0{,}079$$

With these values the luminosity is about $3\cdot10^{31}\text{cm}^{-2}\text{sec}^{-1}$.

If these tune shifts should be to large the transverse deflecting field at the interaction point (ℓ = 2 m, R = 200 m, B(15 GeV) = 2,5 kG) can be applied. In this case the interaction length is reduced by a factor of 10 which reduces the Q-shifts by an order of magnitude.

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References

1) A.Piwinski; Internal Report DESY H1/1 (1969)

2) E.Keil, C.Pellegrini, A.M.Sessler; CRISP 72-34 (ISABELLE PROJECT) BNL 17017 (1972) .