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Is a Quarter Resonance Responsible for the Space Charge Limit in Storage Rings ?

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I. Introduction

Two stored beams of electrons and positrons colliding in a storage ring influence each other via electromagnetic forces. The effect is a growth of transverse oscillation amplitudes of the particles and finally their losses. This space-charge limit was first described by Amman and Ritson[1].

Operating storage rings, however, have shown that this limit is much lower than theoretically expected. Robinson suggested the nonlinearities to be responsible for this discrepancy. With the aid of monte carlo camputations Courant[2] could show that the space charge limit is in fact lower including the nonlinearities.

In the meantime several theoretical approaches have been made to calculate the space charge limit analytically using a two- or threedimensional particle distribution [3) 4) 5)] in the bunches.

In these theories distributions of dQ-shifts are derived, the maximum of which should correspond to the measured dQ-shift. All these theories, however, give no quantitive results on the maximum dQ-shift.

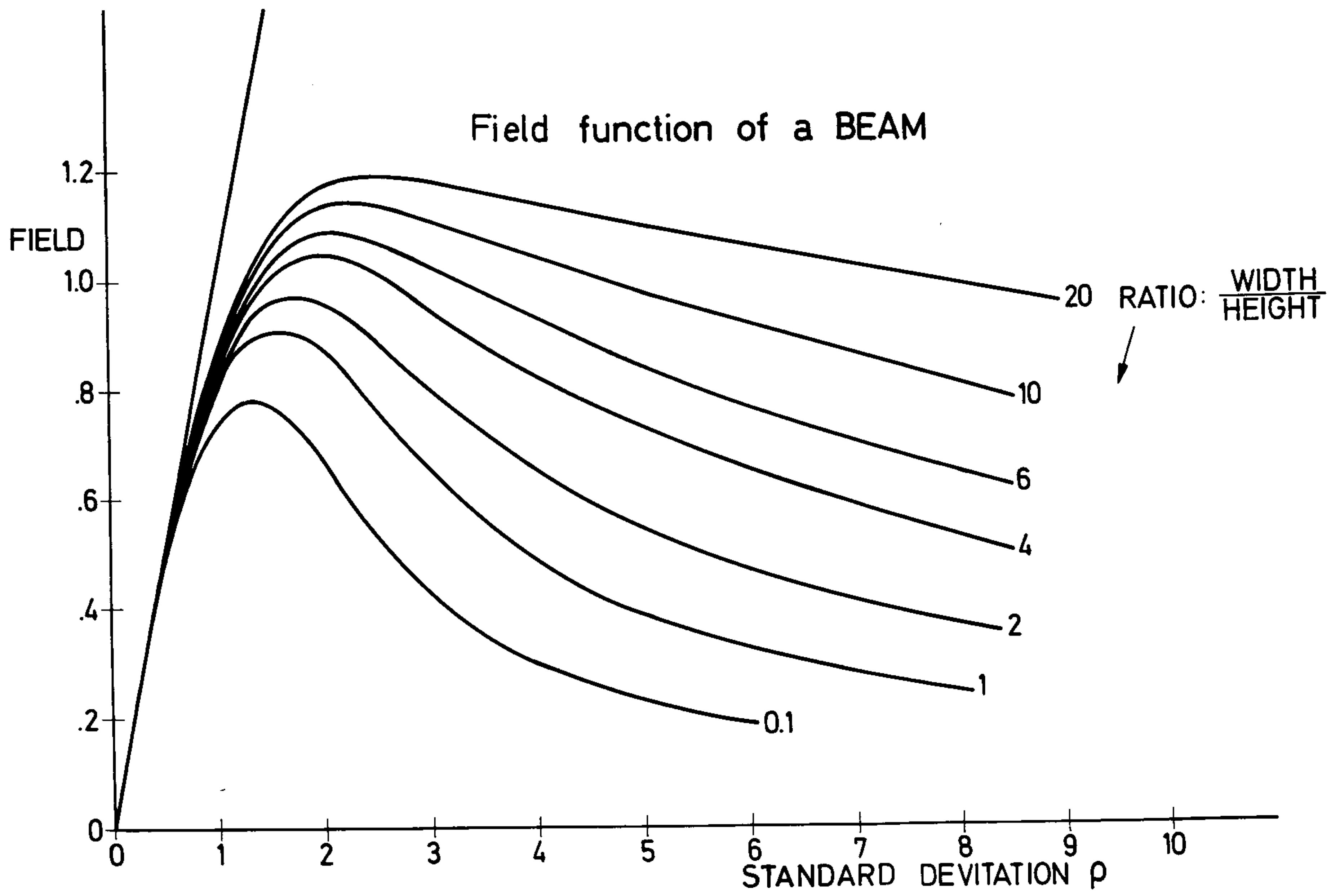
Another approach will be made in this paper which gives quantitative results with the aid of an assumption which seems not unreasonable, but could not yet be proved.

In this model, the space charge effect will be understood as an effect of a nonlinear lens which gives rise to higher order optical resonances. The following discussion, however, is restricted to the strongest nonlinearity producing a fourth order resonance.

II. Electromagnetic Field of the Beam.

For a gaussian charge distribution, the electromagnetic field was analyzed by P.M. Lapostalle et.al.[6] and J.E.Augustin[3].

Fig. 1 shows this field in the z-direction for various ratios σ_x/σ_z - width to height - of the beam. In fig. 2 the same field without the linear part is shown for $\sigma_x/\sigma_z = 1$ and $\sigma_x/\sigma_z = 20$. The linear part, that is the quadrupol-term, will be regarded as part of the linear storage ring optic and is



Field function of a BEAM
FIELD
1.2
1.0
.8
.6
.4
.2
0
0
1
2
3
4
5
6
7
8
9
10
STANDARD DEVITATION ρ
20
10
6
4
2
1
0.1
RATIO: WIDTH / HEIGHT

Fig. 1

- 1/2 -

therefore of no interest for the following discussions. The linear effect can always be compensated by beam optical corrections.

The field, being antisymmetric, involves only odd powers in the transverse coordinate.[6)3)]

(II.1) $$F \sim s - a_3 s^3 + a_5 s^5 - a_7 s^7 + \ldots\ldots\ldots$$

(s being the coordinate in units of standard deviations ($s = {}^z/\sigma_z$))

We mainly concern ourselves with the octupolar field of strength a_3. The approximation of the field shown in fig. 2 with only the octupole term given by (II.1) ($a_3 = \frac{1}{4}$ to $a_3 = \frac{1}{6}$) is rather poor. A better approximation is given for $a_3 \gtrsim \frac{1}{9}$ to $\frac{1}{12}$. That means that for nearly all particles up to some 3 standard deviations a pure octupolar field with $a_3 \approx \frac{1}{9}$ to $\frac{1}{12}$ approximates the real field function rather well.

III. 1/4 - Resonance

To analyze the effect of an octupole, the method of Kobayashi [7) 8)] seems to be very appropriate.

For the particle motion in phase space this method results in:

(III.1) $$-\frac{1}{2}\varepsilon\,(X^2+Y^2) + F(X^2+Y^2)^2 - 2EX^2Y^2 = \text{const}$$

with: $$\varepsilon = 8\,\pi\,\Delta\,Q$$

$\Delta\,Q$ being the deviation of the working-point Q from the next quarter resonance

$$2\,\pi\,Q = (2\pi k+\varepsilon)/4 \qquad k \text{ integer value}$$

$$F = B + \frac{1}{4}(E-C)$$

$$B = \frac{1}{2}\sum_i h_i\,\left(1-\frac{1}{2}\sin^2 2\psi_i\right)$$

$$C = E\cos 4\phi = \frac{1}{2}\sum_i h_i \cos 4\psi_i$$

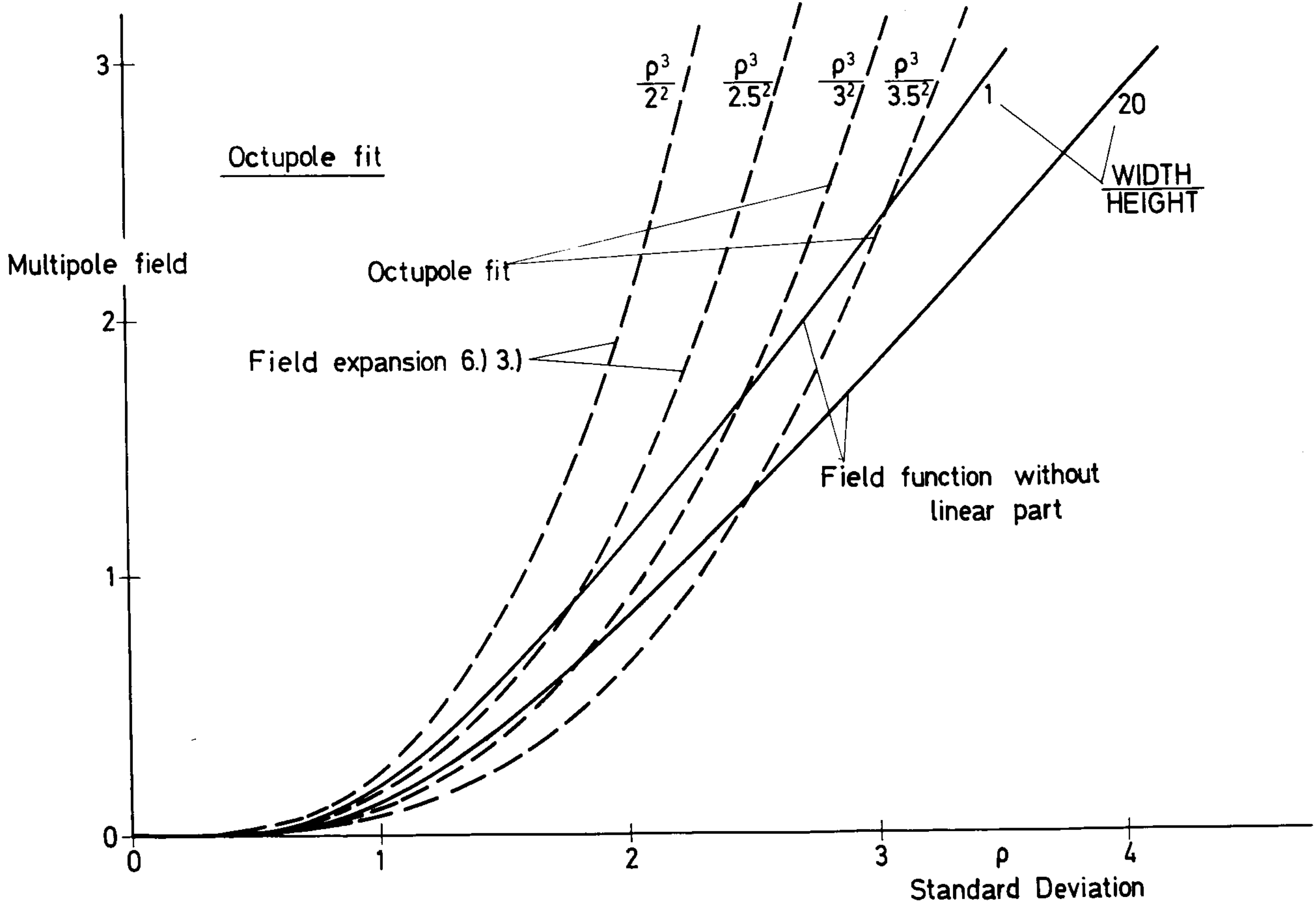

Octupole fit
Multipole field
$\frac{\rho^3}{2^2}$
$\frac{\rho^3}{2.5^2}$
$\frac{\rho^3}{3^2}$
$\frac{\rho^3}{3.5^2}$
1
20
WIDTH / HEIGHT
Octupole fit
Field expansion 6.) 3.)
Field function without linear part
3
2
1
0
0
1
2
3
ρ
4
Standard Deviation

Fig. 2

(III.2) $$S = -E \sin 4\phi = \frac{1}{2} \sum_i h_i \sin 4\psi_i$$

h_i octupole at point i

$$h_i = Q_i \beta_i^2 L_{oct} \qquad Q_i = \frac{1}{6} \frac{e}{pc} g_i''$$

β_i : amplitude function

ψ_i : phase function $\psi = \int \frac{ds}{\beta}$

(III.3) $$\eta = X \cos \phi + Y \sin \phi$$
$$\mathring{\eta} = -X \sin \phi + Y \cos \phi$$

$$\eta = \frac{x}{\sqrt{\beta}} \qquad x: \text{ amplitude}$$

With the working point Q, with p the number of interaction points and with the assumption that all interaction points are optically identical (h_i=const=h) we get:

$$\psi_i = 2\pi \frac{Q}{p} i \qquad i = 0, 1, 2, \ldots . p-1$$

$$C = \frac{1}{2} h \sum_{i=o}^{p-1} \cos \left(8\pi \frac{Q}{p} i\right) = \frac{1}{2} h \frac{\sin (4\pi Q) \cos \left(4\pi Q \frac{p-1}{p}\right)}{\sin \left(4\pi \frac{Q}{p}\right)}$$

(III.4) $$S = \frac{1}{2} h \frac{\sin(4\pi Q) \sin\left(4\pi Q \frac{p-1}{p}\right)}{\sin \left(4\pi \frac{Q}{p}\right)}$$

$$E = \frac{1}{2} h \frac{\sin 4\pi Q}{\sin 4\pi \frac{Q}{p}}$$

$$F = \frac{3}{8} hp + \frac{1}{8} h \frac{\sin 4\pi Q}{\sin 4\pi Q/p}$$

Eq.(III.1.) will be simplified with two transformations:

$$(X,Y) \Rightarrow (V,W) \Rightarrow (\rho,\chi)$$

$$V = \frac{X}{\sigma}\sqrt{\beta} \quad ; \quad W = \frac{Y}{\sigma}\sqrt{\beta}$$

and

$$V = \rho \cos\chi \; ; \; W = \rho \sin\chi$$

Eq.(III.1) and (III.4) thus give:

$$\text{(III.5)} \qquad -\frac{1}{2}\varepsilon\rho^2 + \frac{3}{8} hp \frac{\sigma^2}{\beta}\rho^4 + \frac{1}{4} E \frac{\sigma^2}{\beta}\rho^4 \cos 4\chi = \text{const}$$

Now we introduce the octupole:

$$a_3 = \frac{1}{n^2} \approx (\text{with } n^2 \;\; 9 \text{ to } 12)$$

It follows from (II.1) that for $\rho = n$ the quadrupole field and the octupole field just cancel:

$$k\, n\, \sigma = Q\, (n\sigma)^3$$

From general beam optics, the quadrupole strength k is given by:[9)]

$$\text{(III.6)} \qquad \delta Q = -\frac{1}{4\pi}\int_0^L k\beta ds \approx -\frac{1}{4\pi} k\,\beta\, L$$

With (III.6) the octupolar field is related to δQ. This is convenient for comparison because all measured space charge limits are given in terms of δQ:

$$\text{(III.7)} \qquad h = Q\beta^2 L = \frac{4\pi\delta Q}{\beta L (n\sigma)^2}\beta^2 \cdot L = 4\pi\delta Q \frac{1}{n^2}\frac{\beta}{\sigma^2}$$

Eq.(III.7) with (III.5) gives:

$$-\frac{1}{2}\varepsilon\rho^2 - \frac{3}{2}\pi\delta Q p \frac{\rho^4}{n^2} - \frac{1}{2}\pi\delta Q \frac{\rho^4}{n^2} \frac{\sin 4\pi Q}{\sin 4\pi \frac{Q}{p}} \cdot \cos 4\chi = \text{const}$$

For all electron-positron storage rings $\delta Q > o$, because the space-charge at the interaction point acts like a focusing lens. The value ε is negative for $q < Q < q + \frac{1}{4}$. For $q - \frac{1}{4} < Q < q$ there is $\varepsilon > 0$ but in the formula (III.1) we have to replace ε by $-\varepsilon$. So for

$$q - \frac{1}{4} < Q < q + \frac{1}{4} \qquad\qquad q \text{ integer value}$$

we have:

(III.8) $$U = |\varepsilon|\ \rho^2 - 3\pi\delta Q \frac{\rho^4}{n^2} - \pi\delta Q \frac{\rho^4}{n^2} \frac{\sin 4\pi Q}{\sin 4\pi\frac{Q}{p}} \cos(4\chi) = \text{const.}$$

In this range of working points we find all storage rings except VEPP-2 in the horizontal plane.

From (III.8) we get the phase diagram of fig.3. The fixpoints of this phase diagram result from :

$$\frac{\partial U}{\partial \rho} = 0 \quad \text{and} \quad \frac{\partial U}{\partial \chi} = 0$$

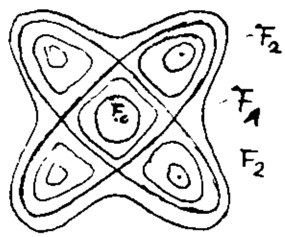


fig. 3

So:

$$0 = 2\rho \left\{ |\varepsilon| - 6\pi\delta Qp \frac{\rho^2}{n^2} - 2\pi\delta Q \frac{\rho^2}{n^2} \frac{\sin 4\pi Q}{\sin 4\pi\frac{Q}{p}} \cos 4\chi \right\}$$

$$0 = \sin 4\chi \quad \Rightarrow \quad \chi = m\frac{\pi}{4} \qquad\qquad m = 0, 1, 2 \ldots\ldots$$

fixpoints:

F_o : $\rho = 0$;

F_1 : $\left(\chi_1 = \frac{\pi}{4}(2m)\ ;\ \rho_1\right)$ and $\cos 4\chi = +1$

$$F_2 : \left(\chi_2 = \frac{\pi}{4}\ (2m+1);\ \rho_2\right) \quad \text{and } \cos 4\chi = -1$$

From that:

$$\frac{\rho_{1,2}}{n^2} = \frac{|\varepsilon|}{2\pi\delta Q \left(3p \pm \frac{\sin 4\pi Q}{\sin 4\pi \frac{Q}{p}}\right)}$$

To come back to normal beam dimensions $s = \frac{x}{\sigma}$, the tilt of eq (III.3) has to be reversed . So we get for the inner fixpoints F_1:

(III.9)
$$\frac{s_1^2}{n^2} = \frac{|\varepsilon| \cdot M}{2\pi\delta Q \left(3p + \frac{\sin 4\pi Q}{\sin 4\pi \frac{Q}{p}}\right)}$$

with

$$M = \max\ (\cos^2\phi,\ \ \sin^2\phi) \qquad \phi = \pi Q\ \frac{p-1}{p}$$

We distinguish now in the phase diagram an inner part within the separatrices and the inner fixpoints and the part outside of these limits. In the inner part we have a beat factor $\beta = \frac{\rho(\pi/4)}{\rho(o)} < 1$ for all particles, whereas outside of this region all particle trajectories have a $\beta > 1$.

At this point we need the assumption mentioned in the introduction, which, as said, could not yet be proved:

Up to the space charge limit we assume that most of the particles - up to about 2,5 to 3,5 standard deviations - are within the inner part of the separatrices. So we assume the space charge limit to be given by:

(III.10)
$$s_1 < n$$

This assumption corresponds with observations: Up to the space charge limit no growth in beam dimensions - because of $\beta < 1$ - is observed.

Outside the inner fixpoints we have to consider a general multipole field instead of a pure octupole. We assume that in this part of the phase diagram, i.e. at large amplitudes, the stable trajectories change to instable ones due to higher order resonances.

If the beam current is increased over the limit (III.10) a rapidly increasing percentage of the beam intensity moves outside the separatrix, causes a beam enlargement because of $\beta > 1$ and will soon be brought to large amplitudes and instable trajectories due to quantum fluctuations.

In accordance with observations only one beam, the weaker one, will be lost. The stronger beam produces the multipole and is itself not influenced by the weaker beam because the space charge limit is not yet reached for the stronger beam.

Assuming that the assumptions made are in some agreement with reality, we get from (III.10) and (III.9) the following limit for the maximum current that can be stored:

$$\delta Q < \frac{M\cdot|\varepsilon|}{2\pi\left(3p+\frac{\sin 4\pi Q}{\sin 4\pi\frac{Q}{p}}\right)} \tag{III.11}$$

with the well known dependence of δQ on the beam current.

IV. Comparison with measurements

Equation (III.11) involves only the number p of interaction points and the workingpoint of the storage ring. These values as well as the measured and computed values for δQ are shown in table 1 for the various storage rings.

In addition to the computed values $\delta Q_{(III.11)}$, there are also so called corrected values δQ_{corr}. In the derivation of (III.1) there was made the approximation $\sin\frac{\varepsilon}{4}\approx\frac{\varepsilon}{4}$ which was for some cases very poor. So we set:

$$\delta Q_{corr} \approx \delta Q_{(III.11)} \; \frac{\sin \varepsilon/4}{\varepsilon/4}$$

Table 1

Storage Ring	Q	p	$\delta Q_{(III.11)}$	δQ_{korr}	δQ_M
1.) Stanford (500 MeV)	0,88	1	0,13	0,11	0,1
2.) A C O	0,845	2	0,050	0,045	0,04
A C O	0,845	4	0,032	0,028	0,027
3.) ADONE	3,15	2	0,053	0,049	0,04
ADONE	3,07	6	0,030	0,024	0,026
4.) VEPP-2	0,8356	2	0,045	0,043	} 0,03 ?
VEPP-2	0,8356	4	0,029	0,027	
5.) C E A	6,81	1	0,060	0,059	0,06
6.) Stanford (500 MeV)	0,88	ϕ=1,6 mrad	0,013	0,012	0,01

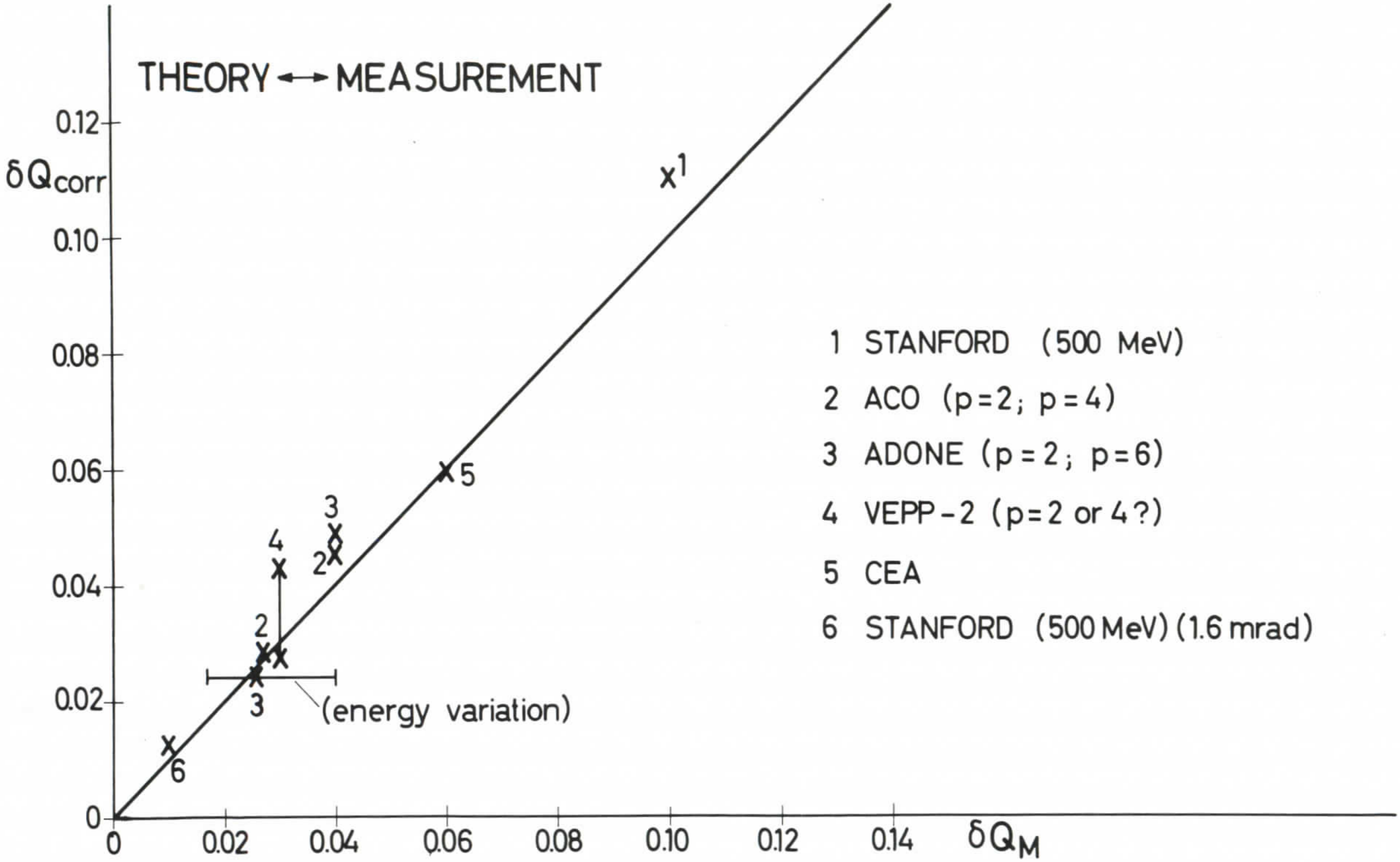

THEORY ↔ MEASUREMENT
δQcorr
0.12
0.10
0.08
0.06
0.04
0.02
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
δQM
(energy variation)
1 STANFORD (500 MeV)
2 ACO (p=2; p=4)
3 ADONE (p = 2; p=6)
4 VEPP-2 (p=2 or 4?)
5 CEA
6 STANFORD (500 MeV) (1.6 mrad)

Fig. 4

This procedure, is mathematically not correct, but it gives the direction of the correction.

In fig. 4 the measured values δQ_M are shown against the values δQ_{corr}. Not for all points it was easy to get a complete set of measured values $(Q, p, \delta Q_M)$; we thus have to make some remarks on the different points.

To 1.)

All three values Q,p and δQ_M are known from HEPL-366 (1965).[11)]

To 2.).

Most measurements with ACO are made with $Q_z = 0.845$. The value $\delta Q_M = 0,04$ for p = 2 got known by private communication (M.Sands) in February 1969. In an early paper[12)] the current ratio for p = 2 and p = 4 was reported to be:

$$\frac{I\ (p=2)}{I\ (p=4)} = \frac{1,5}{1}$$

This gives for p = 4 the value $\delta Q_M = \frac{0,04}{1,5} = 0,027$

To 3.)

For p = 2 a value $\delta Q_M = 0.04$ was given in LNF - 69/31 [13)]. The working point at that time with Adone was Q =3.15. During the informal meeting on storage rings in Frascati (1970) for p = 6 and Q = 3,07 a value $\delta Q = 0,026$ was presented.

This year in Chicago the Frascati people showed an energy-dependence of the Q-shift. Also,only a weak dependence on p was observed. Those measurements cannot be described with the model (pure octupole approximation)discussed in this paper.

To 4.)

In Erevan a value $\delta Q = 0.03$ was reported. From other publications on VEPP-2 the working point seems to be $Q_z = 0,8356$. It is, however, not known whether p = 2 or 4.

To 5.)

Measurements with different crossing angles of the two beams were described in Cambridge [14]. For these cases, eq.(III.11) has to be modified by a well known factor containing the crossing angle 2ϕ.

For vertical crossing we get:

$$\delta Q_{(III.11)} \Rightarrow \quad \delta Q = \delta Q_{(III.11)} \cdot \frac{\sigma_z\,(\sigma_x+\sigma_z)}{\sqrt{\sigma_z^2+\sigma_s^2 tg^2\phi}\,\left(\sigma_x+\sqrt{\sigma_z^2+\sigma_s^2 tg^2\phi}\right)}$$

As a whole, we get a rather good agreement of calculated and measured values except for the most recent observations with Adone.

V. Conclusion

In spite of the relative good agreement of theoretical model and measurements there is a series of questions open:

1.) The octupole model described here does, by itself, not produce instable trajectories. Looking for such trajectories, we should know the real multipole field. Assuming a gaussian intensity distribution, the phase diagram was computed for the general multipole field. The result was only an increased beating of amplitudes but no real instability. If however one assumes an intensity distribution over 6 standard deviations which is somewhat narrower than the gaussian, one gets resonances. So it would be very interesting to have measurements of the beam intensity over at least 4 to 6 standard deviations.

2.) The p-dependence of the value δQ is given correct for the ACO case, but not for the recent measurement at Adone. So one should know the exact beam parameters as bunchform, intensity distribution, luminosity values at the different interaction points and the measured working point for different values of p.

3.) The observed energy dependence at Adone principally cannot be described by optical resonances assuming a constant multipole strength. On the other hand, the multipole field depends on the intensity distribution over the bunch volume, which could give an energy dependence.

4.) To test this model, the dependence of the value δQ on the working point at different values p should be measured. In accordance with the results of Bassetti[15] and observations at Adone one should get an increasing space charge limit for electron positron storage rings going from higher values nearmost to an integer working point. For proton-proton or electron-electron storage rings a working point just below an integer value seems to be the optimum.

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