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THEORY OF COUPLED SYNCHRO-BETATRON OSCILLATIONS (I)

by

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1. Introduction and formulation of the problem

Investigating the coupled synchro- betatron oscillations in a circular accelerator or storage ring one usually assumes, that the transverse oscillation amplitude of the circulating particle, measured with respect to a given closed reference trajectory can be separated into two parts (y=x, z)

$$y(s) = \eta(s) \ D_y(s) + y_\beta(s) \qquad (1.1)$$
$$\eta \equiv \frac{\Delta p}{p}$$

The first part takes into account the dispersion of the machine and the second part describes the free betatron oscillations about the instantaneous orbit $\eta \cdot D_y(s)$. This decomposition is well justified if $\eta(s)$ is a weakly varying function of s compared with $D_y(s)$ and $y_\beta(s)$. Even in the case that synchrotron and betatron oscillations are coupled, this separation allows one to write down analytical expressions for the complete revolution matrix of the transverse and longitudinal oscillations. Knowing this revolution matrix one can study problems like the stability of the particle motion or synchro- betatron resonances. ((1), (2), (3))

But this decomposition, although appropriate for the investigation of many questions, has the disadvantage that the symplectic structure of the transfer matrices for the transverse and longitudinal oscillation amplitudes is lost, if these two degrees of freedom are coupled. On the other hand this symplectic structure of the transfer matrix allows one in a straightforward manner to extend both the linear theory of Courant-Snyder (4) and the theoretical treatment of radiative processes (influence of the radiation on the particle motion) to the general case of multidimensional coupled systems. The problem of the coupled betatron oscillations has been treated already in this way ((5), (6), (7)).

A. Chao has proposed a method allowing for a simultaneous investigation of the coupled synchro- betatron oscillations without decomposing the oscillation amplitude according to (1.1). Using this method and a suitable set of variables for the description of the particle motion, one can show that even in the general (coupled) case the symplectic structure of the transfer matrices is preserved in the thin-lens approximation (8). In this approximation A. Chao has investigated the influence of the synchrotron radiation on the particle motion for the general case of a coupling between the synchrotron and betatron oscillations.

The purpose of the present report will be to review the linear theory of synchro- betatron oscillations in connection with the work of Chao, and especially

1) to extend the theory of Courant and Snyder to the multidimensional case of coupled synchro- betatron oscillations;

2) to demonstrate how to extend the investigations of A. Chao to thick lenses in a systematic way.

All results will be deduced from the general equations of motion (9).
The results obtained in this way can be used

a) to extend the computing code PETROS (10) to the six-dimensional phase space of the coupled synchro- betatron oscillations;

b) to develop a tracking program for the coupled synchro- betatron oscillations, which enables one to simulate and study for example the linear (and by taking into account nonlinear cavity fields also nonlinear) synchro- betatron resonances.

2. Derivation of the equation of motion for the coupled synchro- betatron oscillations

2.1 Particle motion in an electromagnetic field

We start our investigation of the coupled synchro- betatron oscillation with the derivation of the corresponding equations of motion.
In the presence of an electric field $\vec{\varepsilon}$ and a magnetic field $\vec{B}$ the relativistic equation of motion for a charged particle with rest-mass m_o reads

$$e\,\vec{\varepsilon} + \frac{e}{c}\,\dot{\vec{r}} \times \vec{B} + \vec{R} = \frac{d}{dt}\left(\frac{E}{c^2}\,\dot{\vec{r}}\right) \equiv \frac{1}{c^2}\left(\dot{E}\,\dot{\vec{r}} + E\,\ddot{\vec{r}}\right), \qquad (2.1)$$

where the energy of the particle E is given by

$$E = \frac{m_o c^2}{\sqrt{1 - \frac{1}{c^2}(\dot{\vec{r}})^2}} \qquad (2.1a)$$

and where $\vec{R}$ is the radiation reaction force consisting of two parts

$$\vec{R} = \vec{R}^D + \delta\vec{R}, \qquad (2.1b)$$

a continuous part $\vec{R}^D$ (radiation damping) given by ((9), (11)):

$$\vec{R}^D = -\frac{2}{3}\,\frac{e^2}{c^5}\,\gamma^4\,\dot{\vec{r}}\cdot\left\{(\ddot{\vec{r}})^2 + \frac{\gamma^2}{c^2}\,(\dot{\vec{r}}\cdot\ddot{\vec{r}})^2\right\} \qquad (2.1c)$$

and a stochastic part $\delta\vec{R}$ caused by the quantum fluctuations of the radiation field. $\delta\vec{R}$ will be specified in detail in chapter 8 where we shall study the influence of the synchrotron radiation on the particle motion.
Taking, instead of the time variable t, the arc length ℓ of the particle trajectory as independent variable (12):

$$\frac{d}{dt} = \frac{d\ell}{dt}\cdot\frac{d}{d\ell} = v\cdot\frac{d}{d\ell} \qquad (2.2a)$$

$$\dot{E} = v\,\frac{dE}{d\ell} \qquad (2.2b)$$

$$\dot{\vec{r}} = v\,\frac{d\vec{r}}{d\ell} \qquad (2.2c)$$

$$\ddot{\vec{r}} = v\,\frac{d}{d\ell}\left(v\,\frac{d\vec{r}}{d\ell}\right) = v\,\frac{dv}{d\ell}\,\frac{d\vec{r}}{d\ell} + v^2\,\frac{d^2\vec{r}}{d\ell^2}\,. \qquad (2.2d)$$

and taking into account

$$v\,\frac{d}{d\ell}\left(\frac{E}{c^2}\cdot v\right) = v^2\cdot\frac{d}{d\ell}\,\frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m_o v}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\frac{dv}{d\ell}$$

$$= m_o v^2\cdot\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\cdot\frac{v}{c^2}\,\frac{dv}{d\ell} + m_o v\,\frac{dv}{d\ell}\cdot\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$= m_o\cdot\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\cdot v\,\frac{dv}{d\ell}$$

$$= \frac{d}{d\ell}\,\frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \frac{dE}{d\ell} \qquad (2.3)$$

eq. (2.1) now reads

$$e\cdot\vec{\varepsilon} + e\cdot\frac{v}{c}\,\frac{d\vec{r}}{d\ell}\times\vec{B} + \vec{R} = \frac{1}{c^2}\cdot\left\{v^2\,\frac{dE}{d\ell} + Ev\cdot\frac{dv}{d\ell}\right\}\cdot\frac{d\vec{r}}{d\ell} + E\,\frac{v^2}{c^2}\,\frac{d^2\vec{r}}{d\ell^2}$$

$$= \frac{1}{c^2}\cdot v\,\frac{d}{d\ell}(Ev)\cdot\frac{d\vec{r}}{d\ell} + E\,\frac{v^2}{c^2}\,\frac{d^2\vec{r}}{d\ell^2}$$

$$= \frac{dE}{d\ell}\cdot\frac{d\vec{r}}{d\ell} + \frac{v^2}{c^2}\,E\cdot\frac{d^2\vec{r}}{d\ell^2}\,. \qquad (2.4)$$

Making use of

$$\left(\frac{d\vec{r}}{d\ell}\right)^2 = 1 \tag{2.5a}$$

$$\Longrightarrow \frac{d\vec{r}}{d\ell}\cdot\frac{d^2\vec{r}}{d\ell^2} = 0 \tag{2.5b}$$

and multiplying eq. (2.4) with $\frac{d\vec{r}}{d\ell}$ (scalar product) we get for the change of energy of the particle:

$$\frac{dE}{d\ell} = (e \cdot \vec{\varepsilon} + \vec{R})\cdot\frac{d\vec{r}}{d\ell} \tag{2.6}$$

Putting (2.6) into (2.4) we finally obtain the equations of motion for a particle in an electromagnetic field ($v \approx c$, $\frac{v}{c} \approx 1$)

$$[(e\cdot\vec{\varepsilon} + \vec{R})\ \frac{d\vec{r}}{d\ell}]\cdot\frac{d\vec{r}}{d\ell} + E\cdot\frac{d^2\vec{r}}{d\ell^2} = e\cdot\vec{\varepsilon} + \vec{R} + e\,\frac{d\vec{r}}{d\ell}\times\vec{B}\,. \tag{2.7}$$

2.2 Reference trajectory and coordinate system

For our further investigations we shall assume that an ideal closed orbit exists for a particle with fixed energy E_0, if we neglect the variation of energy caused by the radiation losses and the accelerating fields. This orbit is also called reference trajectory or equilibrium orbit. We further assume that this ideal closed orbit consists of piece-wise plane curves either in the horizontal or vertical plane, so that there is no torsion.

Vectors lying on the reference trajectory will be called $\vec{r}_0(s)$, where s is a parameter which is chosen as the arc length of the equilibrium orbit. An arbitrary particle trajectory is then described by its displacement $\delta\vec{r}$ from the ideal trajectory $\vec{r}_0(s)$:

$$\vec{r} = \vec{r}_0(s) + \delta\vec{r}(s)$$

In order to calculate this displacement vector $\delta\vec{r}$ we define in the well-known manner ((13), (14), (15), (16)) a system of mutually orthogonal unit vectors moving along the equilibrium orbit with the considered particle and consisting of

the normal unit vector $\vec{n}(s)$,
the tangent unit vector $\vec{t}(s)$ and
the binormal unit vector $\vec{b}(s) = \vec{t}(s) \times \vec{n}(s)$.

We require the vector $\vec{n}$ to be directed outwards if the motion takes place in the horizontal plane and upwards if the motion takes place in the vertical plane (6). Choosing the direction of $\vec{n}$ in this way implies, that the curvature K(s) appearing in the Frenet formulae

$$\vec{t}(s) = \vec{r}_0'(s) = \frac{d\vec{r}_0}{ds}; \tag{2.8}$$

$$\begin{cases} \frac{d\vec{t}}{ds} = -K(s)\cdot\vec{n}(s)\ ; \\ \frac{d\vec{n}}{ds} = +K(s)\cdot\vec{t}(s)\ ; \\ \frac{d\vec{b}}{ds} = 0 \end{cases} \tag{2.9}$$

is always positive in the horizontal plane and it is negative in the vertical plane iff the centre of curvature lies above the reference trajectory.

Using this coordinate system we can describe $\delta\vec{r}$ in terms of the so-called natural coordinates, and it is obvious that only two components are needed, namely the projections of $\delta\vec{r}$ onto the direction of the normal and of the binormal

$$\delta\vec{r}(s) = (\delta\vec{r}\cdot\vec{n})\cdot\vec{n} + (\delta\vec{r}\cdot\vec{b})\cdot\vec{b}.$$

But this representation of $\delta\vec{r}$ has the disadvantage that the direction of $\vec{n}$ changes discontinuously if the particle trajectory is going over from the vertical plane to the horizontal plane and vice versa. Therefore it is advantageous to introduce new unit vectors $\vec{e}_x(s)$ and $\vec{e}_z(s)$ which change their direction continuously by putting

$$\vec{e}_x(s) = \begin{cases} \vec{n}(s), \text{ if the orbit lies in the horizontal plane;} \\ \vec{b}(s), \text{ " " " " " vertical " } ; \end{cases}$$

$$\vec{e}_z(s) = \begin{cases} \vec{b}(s), \text{ if the orbit lies in the horizontal plane;} \\ \vec{n}(s), \text{ " " " " " vertical " } . \end{cases}$$

Thus we obtain

$$\vec{r}(s,x,z) = \vec{r}_0(s) + x(s)\cdot\vec{e}_x(s) + z(s)\cdot\vec{e}_z(s) \tag{2.10}$$

and the Frenet formulae now read

$$\begin{cases} \frac{d}{ds}\ \vec{e}_x(s) = K_x(s)\cdot\vec{t}(s); \\ \frac{d}{ds}\ \vec{e}_z(s) = K_z(s)\cdot\vec{t}(s); \\ \frac{d}{ds}\ \vec{t}(s) = -K_x(s)\cdot\vec{e}_x(s) - K_z(s)\cdot\vec{e}_z(s) \end{cases} \tag{2.11}$$

with

$$K_x(s)\cdot K_z(s) = 0 \tag{2.12}$$

where $K_x(s)$, $K_z(s)$ designate the curvature in x-direction and z-direction respectively.

2.3 The arc length s of the equilibrium orbit as independent variable

According to eq. (2.10) the vector $\vec{r}$ of the particle trajectory depends on the arc length s of the equilibrium orbit. Therefore we shall rewrite eq. (2.7) in terms of this variable s (12).

From

$$\frac{d}{d\ell} = \frac{1}{\ell'}\frac{d}{ds} \qquad \text{with} \quad \ell' = \frac{d\ell}{ds}$$

it follows

$$\frac{d\vec{r}}{d\ell} = \frac{1}{\ell'}\ \frac{dr}{ds} \equiv \frac{1}{\ell'}\cdot\vec{r}' \tag{2.13a}$$

$$\frac{d^2\vec{r}}{d\ell^2} = \frac{1}{\ell'}\cdot\frac{d}{ds}\ \left(\frac{\vec{r}'}{\ell'}\right)$$

$$= \frac{1}{(\ell')^2}\cdot\vec{r}'' - \frac{1}{\ell'}\cdot\vec{r}'\cdot\frac{\ell''}{(\ell')^2}$$

$$= \frac{1}{(\ell')^2}\cdot\{\vec{r}'' - \frac{1}{2}\ \frac{\vec{r}'}{(\ell')^2}\cdot\frac{d}{ds}(\ell')^2\}\ . \tag{2.13b}$$

Introducing the relative deviation of energy $\eta(s)$

$$\eta(s) = \frac{E - E_o}{E_o} \Longrightarrow E = E_o(1+\eta) \tag{2.14}$$

and taking into account (2.13a, b), eq. (2.7) can be written in the form

$$\frac{1}{E_o\cdot(1+\eta)}[(e\cdot\vec{\varepsilon}+\vec{R})\cdot\vec{r}']\cdot\vec{r}' + \vec{r}'' - \frac{1}{2}\ \frac{\vec{r}'}{(\ell')^2}\ \frac{d}{ds}(\ell')^2 =$$

$$= \frac{(\ell')^2}{E_o\cdot(1+\eta)}\cdot[e\cdot\vec{\varepsilon}+\vec{R}\,] + \frac{e\cdot\ell'}{E_o\cdot(1+\eta)}\cdot\vec{r}'\times\vec{B}. \tag{2.15}$$

Using (2.8), (2.11) and (2.12) the derivatives $\vec{r}'$ and $\vec{r}''$ are given by

$$\begin{aligned} \vec{r}'(s) &= \vec{t}(s) + x'\cdot\vec{e}_x + z'\cdot\vec{e}_z + x\cdot(K_x\cdot\vec{t}) + z\cdot(K_z\cdot\vec{t}) \\ &= (1 + K_x\cdot x + K_z\cdot z)\cdot\vec{t} + x'\cdot\vec{e}_x + z'\cdot\vec{e}_z; \end{aligned} \tag{2.16a}$$

$$\begin{aligned} \vec{r}''(s) &= (K_x\cdot x' + K_x'\cdot x + K_z\cdot z' + K_z'\cdot z)\cdot\vec{t} + \\ &+ x''\cdot\vec{e}_x + z''\cdot\vec{e}_z + \\ &+ (1 + K_x\cdot x + K_z\cdot z)\cdot(-K_x\cdot\vec{e}_x - K_z\cdot\vec{e}_z) + \\ &+ x'\cdot(K_x\cdot\vec{t}) + z'\cdot(K_z\cdot\vec{t}) = \\ &= (K_x'\cdot x + K_z'\cdot z + 2K_x\cdot x' + 2K_z\cdot z')\cdot\vec{t} + \\ &+ [\,x'' - K_x\cdot(1 + K_x\cdot x)]\cdot\vec{e}_x + \\ &+ [z'' - K_z\cdot(1 + K_z\cdot z)]\cdot\vec{e}_z. \end{aligned} \tag{2.16b}$$

Linearizing with respect to x, x', z, z' we obtain because of eq. (2.16a, b)

$$\begin{aligned} (\ell')^2 \quad &\equiv \left(\frac{d\ell}{ds}\right)^2 = \left(\frac{d\vec{r}}{ds}\right)^2 \\ &= (1 + K_x\cdot x + K_z\cdot z)^2 + (x')^2 + (z')^2 \\ &= 1 + 2K_x\cdot x + 2K_z\cdot z + \ldots; \\ \frac{1}{(\ell')^2} \quad &= 1 - 2K_x\cdot x - 2K_z\cdot z\ ; \\ \frac{1}{2}\frac{d}{ds}(\ell')^2 \quad &= (1 + K_x\cdot x + K_z\cdot z)\,(K_x\cdot x' + K_x'\cdot x + K_z\cdot z' + K_z'\cdot z) + \\ &+ x'\cdot x'' + z'\cdot z'' \\ &= (K_x\cdot x' + K_x'\cdot x + K_z\cdot z' + K_z'\cdot z) + \ldots; \end{aligned}$$

$$\frac{1}{2}\frac{\vec{r}'}{(\ell')^2}\ \frac{d}{ds}(\ell')^2 = (K_x\cdot x' + K_x'\cdot x + K_z\cdot z' + K_z'\cdot z)\cdot\vec{t} + \ldots; \tag{2.17}$$

$$\ell' = 1 + K_x\cdot x + K_z\cdot z + \ldots; \tag{2.18}$$

$$\vec{r}' \times \vec{B} = \{(1 + K_x\cdot x + K_z\cdot z)\,\vec{t} + x'\cdot\vec{e}_x + z'\cdot\vec{e}_z\} \times$$

$$\times \{B_t\cdot\vec{t} + B_x\cdot\vec{e}_x + B_z\cdot\vec{e}_z\}$$

$$= (1 + K_x\cdot x + K_z\cdot z)\cdot(B_x\cdot\vec{e}_z - B_z\cdot\vec{e}_x) - x'\cdot B_t\cdot\vec{e}_z +$$

$$+ x'\cdot B_z\cdot\vec{t} + z'\cdot B_t\cdot\vec{e}_x - z'\cdot B_x\cdot\vec{t} =$$

$$= [x'\cdot B_z - z'\cdot B_x]\cdot\vec{t} +$$

$$+ [z'\cdot B_t - (1 + K_x\cdot x + K_z\cdot z)\cdot B_z]\cdot\vec{e}_x -$$

$$- [x'\cdot B_t - (1 + K_x\cdot x + K_z\cdot z)\cdot B_x]\cdot\vec{e}_z\,; \qquad (2.19)$$

$$[(e\cdot\vec{\varepsilon} + \vec{R})\cdot\vec{r}']\cdot\vec{r}' = [(e\cdot\varepsilon_t + R_t)\cdot(1 + K_x\cdot x + K_z\cdot z) +$$

$$+ (e\cdot\varepsilon_x + R_x)\cdot x' + (e\cdot\varepsilon_z + R_z)\cdot z']\cdot$$

$$\cdot[(1 + K_x\cdot x + K_z\cdot z)\cdot\vec{t} + x'\cdot\vec{e}_x + z'\cdot\vec{e}_z] =$$

$$= [(e\cdot\varepsilon_t + R_t)\cdot(1 + 2K_x\cdot x + 2K_z\cdot z) +$$

$$+ (e\cdot\varepsilon_x + R_x)\cdot x' + (e\cdot\varepsilon_z + R_z)\cdot z']\cdot\vec{t} +$$

$$+ (e\cdot\varepsilon_t + R_t)\cdot x'\cdot\vec{e}_x +$$

$$+ (e\cdot\varepsilon_t + R_t)\cdot z'\cdot\vec{e}_z. \qquad (2.20)$$

Using $(1 + \eta)^{-1} \approx 1 - \eta$ and substituting (2.16 - 2.20) into eq. (2.15) we find:

$$\frac{1}{E_o}\cdot(1 - \eta)\cdot\{\vec{t}\cdot[(e\cdot\varepsilon_t + R_t)\cdot(1 + 2K_x\cdot x + 2K_z\cdot z) +$$

$$+ (e\cdot\varepsilon_t + R_t)\cdot x' + (e\cdot\varepsilon_z + R_z)\cdot z'] +$$

$$+ \vec{e}_x\cdot(e\cdot\varepsilon_t + R_t)\cdot x' + \vec{e}_z\cdot(e\cdot\varepsilon_t + R_t)\cdot z'\} +$$

$$+ (K_x'\cdot x + K_z'\cdot z + 2K_x\cdot x' + 2K_z\cdot z')\cdot\vec{t} +$$

$$+ [x'' - K_x(1 + K_x\cdot x)]\cdot\vec{e}_x + [z'' - K_z(1 + K_z\cdot z)]\cdot\vec{e}_z -$$

$$- (K_x\cdot x' + K_x'\cdot x + K_z\cdot z' + K_z'\cdot z)\cdot\vec{t} =$$

$$= \frac{1}{E_o}\cdot(1 - \eta)\cdot(1 + 2K_x\cdot x + 2K_z\cdot z)\cdot\{(e\cdot\varepsilon_t + R_t)\cdot\vec{t} +$$

$$+ (e\cdot\varepsilon_x + R_x)\cdot\vec{e}_x + (e\cdot\varepsilon_z + R_z)\cdot\vec{e}_z\} +$$

$$+ \frac{e}{E_o}\cdot(1 - \eta)\cdot(1 + K_x\cdot x + K_z\cdot z)\cdot\{[x'\cdot B_z - z'\cdot B_x]\cdot\vec{t} +$$

$$+ [z'\cdot B_t - (1 + K_x\cdot x + K_z\cdot z)\cdot B_z]\cdot\vec{e}_x -$$

$$- [x'\cdot B_t - (1 + K_x\cdot x + K_z\cdot z)\cdot B_x]\cdot\vec{e}_z\}. \qquad (2.21)$$

If

$$\vec{\varepsilon} = 0\;;\quad \vec{R} = 0\;;\quad \eta = 0;$$

$$B_x = B_x^{(o)};\quad B_z = B_z^{(o)}$$

we find, that because of the definition of the equilibrium orbit

$$x = x' = 0$$

$$z = z' = 0$$

is a solution of eq. (2.21), which implies

$$-K_x\cdot\vec{e}_x - K_z\cdot\vec{e}_z = -\frac{e}{E_o}\cdot\{B_z^{(o)}\cdot\vec{e}_x - B_x^{(o)}\cdot\vec{e}_z\}\,,$$

or

$$K_x = \frac{e}{E_o}\cdot B_z^{(o)}\;;\; K_z = -\frac{e}{E_o}\cdot B_x^{(o)}\,. \qquad (2.22)$$

Putting

$$B_x(s,x,z) = B_x^{(o)}(s,o,o) + \Delta B_x(s,o,o) + x\cdot\left(\frac{\partial B_x}{\partial x}\right)_{x=z=o} + z\cdot\left(\frac{\partial B_x}{\partial z}\right)_{x=z=o}\;; \qquad (2.23a)$$

$$B_z(s,x,z) = B_z^{(o)}(s,o,o) + \Delta B_z(s,o,o) + x\cdot\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} + z\cdot\left(\frac{\partial B_z}{\partial z}\right)_{x=z=o} \qquad (2.23b)$$

we find in linear approximation for the $\vec{t}$-component of eq. (2.21):

$$\frac{1}{E_o}\ (e\cdot\varepsilon_t + R_t)\cdot(1 + 2K_x\cdot x + 2K_z\cdot z - \eta) +$$

$$+ (K_x'\cdot x + K_z'\cdot z + 2K_x\cdot x' + 2K_z\cdot z') -$$

$$- (K_x\cdot x' + K_x'\cdot x + K_z\cdot z' + K_z'\cdot z) =$$

$$= \frac{1}{E_o}\cdot(1 + 2K_x\cdot x + 2K_z\cdot z - \eta)\cdot(e\cdot\varepsilon_t + R_t) +$$

$$+ \frac{e}{E_o}\cdot[x'\cdot B_z^{(o)} - z'\cdot B_x^{(o)}]$$

or

$$K_x\cdot x' + K_z\cdot z' = \frac{e}{E_o}\cdot[x'\cdot B_z^{(o)} - z'\cdot B_x^{(o)}]$$

(satisfied because of eq. (2.22))

The $\vec{e}_x$-component is given by

$$\frac{1}{E_o}\ (e\cdot\varepsilon_t + R_t)\cdot x' + x'' - K_x(1 + K_x\cdot x) =$$

$$= \frac{1}{E_o}\cdot(1 + 2K_x\cdot x + 2K_z\cdot z - \eta)\cdot(e\cdot\varepsilon_x + R_x) +$$

$$+ \frac{e}{E_o}\cdot\{z'\cdot B_t - (1 + 2K_x\cdot x + 2K_z\cdot z - \eta)\cdot B_z^{(o)} -$$

$$- \Delta B_z - x\cdot\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} - z\cdot\left(\frac{\partial B_z}{\partial z}\right)_{x=z=o}\}$$

or taking into account (2.12) and (2.22) by

$$\frac{1}{E_o}\cdot(e\cdot\varepsilon_t + R_t)\cdot x' + x'' = K_x\cdot\eta - K_x^2\cdot x +$$

$$+ \frac{e}{E_o}\cdot\{z'\cdot B_t - x\cdot\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} - z\cdot\left(\frac{\partial B_z}{\partial z}\right)_{x=z=o} - \Delta B_z\} +$$

$$+ \frac{1}{E_o}\cdot(1 + 2K_x\cdot x + 2K_z\cdot z - \eta)\cdot(e\cdot\varepsilon_x + R_x)\ . \qquad (2.24)$$

The $\vec{e}_z$-component now reads

$$\frac{1}{E_o}\cdot(e\cdot\varepsilon_t + R_t)\cdot z' + z'' - K_z(1 + K_z\cdot z) =$$

$$= \frac{1}{E_o}\cdot(1 + 2K_x\cdot x + 2K_z\cdot z - \eta)\cdot(e\cdot\varepsilon_z + R_z) -$$

$$- \frac{e}{E_o}\cdot\{x'\cdot B_t - (1 + 2K_x\cdot x + 2K_z\cdot z - \eta)\cdot B_x^{(o)} -$$

$$- \Delta B_x - x\cdot\left(\frac{\partial B_x}{\partial x}\right)_{x=z=o} - z\cdot\left(\frac{\partial B_x}{\partial z}\right)_{x=z=o}\}$$

or (using again (2.12) and (2.22))

$$\frac{1}{E_o}\cdot(e\cdot\varepsilon_t + R_t)\cdot z' + z'' = K_z\cdot\eta - K_z^2\cdot z -$$

$$- \frac{e}{E_o}\cdot\{x'\cdot B_t - x\cdot\left(\frac{\partial B_x}{\partial x}\right)_{x=z=o} - z\cdot\left(\frac{\partial B_x}{\partial z}\right)_{x=z=o} - \Delta B_x\} +$$

$$+ \frac{1}{E_o}\cdot(1 + 2K_x\cdot x + 2K_z\cdot z - \eta)\cdot(e\cdot\varepsilon_z + R_z). \qquad (2.25)$$

Because of the Maxwell equations the components of the magnetic field must satisfy the following conditions

$$\left(\frac{\partial B_x}{\partial z}\right)_{x=z=o} = \left(\frac{\partial B_z}{\partial x}\right)_{x=z=o};$$

$$\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z} + \frac{\partial B_t}{\partial s}\right)_{x=z=o} = 0$$

or

$$\left(\frac{\partial B_x}{\partial x}\right)_{x=z=o} = +\frac{1}{2}\left[\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial z}\right) - \frac{\partial B_t}{\partial s}\right]_{x=z=o};$$

$$\left(\frac{\partial B_z}{\partial z}\right)_{x=z=o} = -\frac{1}{2}\left[\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right) + \frac{\partial B_t}{\partial s}\right]_{x=z=o}$$

so that eq's (2.24) and (2.25) can be put into the form

$$\frac{1}{E_o}\cdot(e\cdot\varepsilon_t + R_t)\cdot x' + x'' + G_1\cdot x - 2H\cdot z' - (N + H')\cdot z -$$

$$- K_x\cdot\eta - \frac{1}{E_o}\ (1 + 2K_x\cdot x + 2K_z\cdot z - \eta)\cdot(e\cdot\varepsilon_x + R_x) =$$

$$= -\frac{e}{E_o}\cdot\Delta B_z; \qquad (2.26a)$$

$$\frac{1}{E_o}\cdot(e\cdot\epsilon_t + R_t)\cdot z' + z'' + G_2\cdot z + 2H\cdot x' - (N - H')\cdot x -$$

$$- K_z\cdot\eta - \frac{1}{E_o}(1 + 2K_x\cdot x + 2K_z\cdot z - \eta)\cdot(e\cdot\epsilon_z + R_z) =$$

$$= + \frac{e}{E_o}\cdot\Delta B_x \qquad (2.26b)$$

with the following definitions

$$\begin{cases} G_1 = K_x^2 + \frac{e}{E_o}\cdot\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o}; \\ G_2 = K_z^2 - \frac{e}{E_o}\cdot\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o}; \\ H = \frac{1}{2}\cdot\frac{e}{E_o}B_t\ ;\quad H' = \frac{1}{2}\cdot\frac{e}{E_o}\left(\frac{\partial B_t}{\partial s}\right)_{x=z=o}; \\ N = \frac{1}{2}\cdot\frac{e}{E_o}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=o}. \end{cases} \qquad (2.27)$$

These two equations contain three unknown functions

$x(s)$, $z(s)$ and $\eta(s)$

Therefore we need another relationship between x, z and η, which can be obtained from eq. (2.6) by rewriting this equation in terms of the independent variable s and by introducing the relative deviation of energy according to eq. (2.14). Then, using (2.16a), we find

$$\frac{d\eta}{ds} = \frac{1}{E_o}\cdot\{(e\cdot\epsilon_t + R_t)\cdot(1 + K_x\cdot x + K_z\cdot z) +$$

$$+ (e\cdot\epsilon_x + R_x)\cdot x' + (e\cdot\epsilon_z + R_z)\cdot z'\}. \qquad (2.26c)$$

Our next task will be to calculate the electric field produced by the cavities and the radiation reaction force. This will be done in the next two chapters.

2.4 Description of the electric field

For the sake of simplicity we assume pointlike cavities situated in the straight sections at the positions

$$s = s_\nu \quad (\nu = 1, 2, \ldots, N)$$

so that the electric field can be written in the form

$$\frac{e\cdot\vec{\epsilon}}{E_o} = \vec{t}\cdot\frac{e\hat{V}}{E_o}\cdot\sin[\Phi + \psi(s, t)]\sum_\nu \delta(s - s_\nu)\ ; \qquad (2.28)$$

$$K_x(s_\nu) = K_z(s_\nu) = 0\,. \qquad (2.29)$$

Since a particle, circulating on the reference trajectory, has to traverse the cavity always with the same phase, the phase function $\psi(s, t)$ included in eq. (2.28) has to satisfy the following conditions:

1) $\psi(s, \frac{s}{c}) = \psi(o, o)$ for $s < L$;

2) $\psi(o, \frac{L}{c}) = \psi(o, o) - k\cdot 2\pi$ (k = integer)

(L = circumference of the equilibrium orbit).

Both requirements are satisfied by the "ansatz" (9):

$$\psi(s, t) = k\cdot 2\pi\cdot\frac{1}{L}\ (s - ct) \qquad (2.30)$$

The factor

$$\sigma(s) = (s - ct) \qquad (2.31)$$

is given by

$$\begin{aligned}\sigma(s) &= \int_o^s ds - \int_o^s d\ell \\ &= \int_o^s ds\cdot[1 - \ell'(s)] \\ &= -\int_o^s ds\cdot[K_x(s)\cdot x + K_z(s)\cdot z] \end{aligned} \qquad (2.32)$$

where we have also taken into account (2.18).* For our later investigations of the resonance excitation of the synchro- betatron oscillations it will be convenient to supplement the field (2.28) by an additional perturbing electric field $\Delta\vec{\epsilon}$ in the longitudinal direction $\vec{t}$:

$$\frac{e}{E_o}\cdot\Delta\vec{\epsilon} = \frac{e}{E_o}\cdot\Delta\epsilon\cdot\vec{t} \qquad (2.33)$$

---

* For a particle moving on the reference trajectory with the phase Φ relative to the accelerating field the energy losses are exactly compensated by the electric field (definition of Φ).

By virtue of (2.28), (2.30), (2.31) and (2.33) we get

$$\frac{e\cdot\vec{\varepsilon}}{E_o} = \vec{t}\cdot\frac{e\hat{V}}{E_o}\sin\left[\Phi + k\cdot\frac{2\pi}{L}\cdot\sigma(s)\right]\cdot\sum_\nu\delta(s-s_\nu) +$$

$$+\vec{t}\cdot\frac{e}{E_o}\cdot\Delta\varepsilon$$

or after linearization with respect to σ

$$\frac{e\cdot\varepsilon_t}{E_o} = \frac{e\hat{V}}{E_o}\{\sin\Phi + k\cdot\frac{2\pi}{L}\cdot\sigma(s)\cdot\cos\Phi\}\cdot\sum_\nu\delta(s-s_\nu) + \frac{e}{E_o}\cdot\Delta\varepsilon;$$

$$\frac{e\cdot\varepsilon_x}{E_o} = \frac{e\cdot\varepsilon_z}{E_o} = 0.$$

Taking into account the separation of $\vec{R}$ into the parts $\vec{R}^D$ and $\delta\vec{R}$ according to eq. (2.1b) and using the relationships (9)

$$|\delta R_t| \ll |R_t^D|;$$

$$|\delta R_x| \ll |\delta R_t|; \quad |\delta R_z| \ll |\delta R_t|$$

we obtain the following equations for (2.26a, b, c):

$$x'\cdot\frac{e\cdot\hat{V}}{E_o}\cdot\sin\Phi\cdot\sum_\nu\delta(s-s_\nu) +$$

$$+ x'\cdot\frac{R_t^D}{E_o} + x'' + G_1\cdot x - 2H\cdot z' - (N+H')\cdot z - K_x\cdot\eta -$$

$$-\frac{R_x^D}{E_o}\cdot(1 + 2K_x\cdot x + 2K_z\cdot z - \eta) = -\frac{e}{E_o}\cdot\Delta B_z; \qquad (2.34a)$$

$$z'\cdot\frac{e\cdot\hat{V}}{E_o}\sin\Phi\cdot\sum_\nu\delta(s-s_\nu) +$$

$$+ z'\cdot\frac{R_t^D}{E_o} + z'' + G_2\cdot z + 2H\cdot x' - (N-H')\cdot x - K_z\cdot\eta -$$

$$-\frac{R_z^D}{E_o}\cdot(1 + 2K_x\cdot x + 2K_z\cdot z - \eta) = +\frac{e}{E_o}\cdot\Delta B_x; \qquad (2.34b)$$

$$\eta' = \frac{e\cdot\hat{V}}{E_o}\cdot\{\sin\Phi + \sigma(s)\cdot k\frac{2\pi}{L}\cos\Phi\}\cdot\sum_\nu\delta(s-s_\nu) +$$

$$+\frac{e}{E_o}\Delta\varepsilon + \frac{R_t^D}{E_o}\cdot(1 + K_x\cdot x + K_z\cdot z) + \frac{\delta R_t}{E_o} +$$

$$+\frac{R_x^D}{E_o}\cdot x' + \frac{R_z^D}{E_o}\cdot z'. \qquad (2.34c)$$

Because of the new variable σ we have to supplement this system of differential equations for x, z and η by a fourth equation for σ, which can be obtained from (2.32) by taking the derivative with respect to s:

$$\sigma' = -K_x(s)\cdot x - K_z(s)\cdot z \qquad (2.34d)$$

2.5 Calculation of the radiation reaction force

Now we want to determine the radiation reaction force (9)

$$\vec{R}^D = (R_t, R_x, R_z).$$

For this purpose we have to replace the time derivatives in eq. (2.1a) by the derivatives with respect to the arc length s.
By virtue of (2.5a, b) the following expressions can be found for the quantities $(\ddot{\vec{r}})^2$ and $(\dot{\vec{r}}\,\ddot{\vec{r}})^2$ appearing in (2.1c):

$$(\ddot{\vec{r}})^2 = \left(v\frac{dv}{d\ell}\right)^2 + v^4\left(\frac{d^2\vec{r}}{d\ell^2}\right)^2 \qquad (2.35a)$$

$$(\dot{\vec{r}}\,\ddot{\vec{r}})^2 = v^2\cdot\left(v\frac{dv}{d\ell}\right)^2. \qquad (2.35b)$$

Taking into account (2.1a), (2.14), (2.18) and (2.3) the term $v\frac{dv}{d\ell}$ can be written in the form

$$v\frac{dv}{d\ell} = \frac{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}{m_o}\cdot\frac{dE}{d\ell}$$

$$= \frac{m_o^2c^6}{E^3}\;\frac{1}{\ell'}\;\frac{dE}{ds}$$

$$= \frac{m_o^2c^6}{E_o^2}\;(1 - K_x\cdot x - K_z\cdot z - 3\eta)\cdot\eta'. \qquad (2.36a)$$

Using (2.16b, 17 and 18) it follows from (2.13b)

$$\frac{d^2\vec{r}}{d\ell^2} = (1 - 2K_x\cdot x - 2K_z\cdot z)\cdot\{(K_x\cdot x' + K_z\cdot z')\cdot\vec{t} +$$

$$+ [x'' - K_x(1 + K_x\cdot x)]\cdot\vec{e}_x + [z'' - K_z(1 + K_z\cdot z)]\cdot\vec{e}_z\} =$$

$$= (K_x\cdot x' + K_z\cdot z')\cdot\vec{t} + (x'' + K_x^2\cdot x - K_x)\cdot\vec{e}_x +$$

$$+ (z'' + K_z^2\cdot z - K_z)\cdot\vec{e}_z;$$

$$\left(\frac{d^2r}{d\ell^2}\right)^2 = K_x^2 - 2K_x\cdot(x'' + K_x^2\cdot x) + K_z^2 - 2K_z\cdot(z'' + K_z^2\cdot z) \qquad (2.36b)$$

and from (2.2c) we get by using (2.13a, 18)

$$\dot{\vec{r}} = v\,\frac{d\vec{r}}{d\ell} = v \cdot \frac{1}{\ell'} \cdot \vec{r}\,' = v \cdot (1 - K_x \cdot x - K_z \cdot z) \cdot$$

$$\cdot \ \{(1 + K_x \cdot x + K_z \cdot z) \cdot \vec{t} + x' \cdot \vec{e}_x + z' \cdot \vec{e}_z\}$$

$$= v \cdot \{\vec{t} + x' \cdot \vec{e}_x + z' \cdot \vec{e}_z\} \ . \qquad (2.37)$$

Putting (2.36a, b) into eq. (2.35) and putting (2.35, 37) into (2.1c) we obtain after linearization:

$$\vec{R}^D = -\frac{2}{3}\,\frac{e^2}{c^5}\,\gamma_o^4 \cdot (1 + 4\eta) \cdot v \cdot \{\vec{t} + x' \cdot \vec{e}_x + z' \cdot \vec{e}_z\} \cdot$$

$$\cdot\ v^4 \cdot \{ K_x^2 - 2K_x \cdot (x'' + K_x^2 \cdot x) + K_z^2 - 2K_z \cdot (z'' + K_z^2 \cdot z)\} =$$

$$= -\frac{2}{3}\,e^2\gamma_o^4 \cdot \{(K_x^2 + K_z^2) \cdot (1 + 4\eta) - 2K_x \cdot (x'' + K_x^2 \cdot x) -$$

$$- 2K_z(z'' + K_z^2 \cdot z)\} \cdot \vec{t} - \frac{2}{3}\,e^2\gamma_o^4 \cdot (K_x^2 + K_z^2) \cdot [\,x' \cdot \vec{e}_x + z' \cdot \vec{e}_z] \quad (2.38)$$

$$(v \approx c \ \ , \ \ \gamma = \gamma_o(1 + \eta))$$

The expressions

$$K_x \cdot (x'' + K_x^2 \cdot x) \text{ and } K_z(z'' + K_z^2 \cdot z)$$

which appear in eq. (2.38) can be taken from (2.34a, b) by setting approximately

$$R_t^D = R_x^D = R_z^D = 0$$

in these equations.

Then we find by virtue of (2.27) and (2.29)

$$K_x \cdot (x'' + K_x^2 \cdot x) = K_x \cdot \{-\frac{e}{E_o}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \cdot x + 2H \cdot z' +$$

$$+ (N + H') \cdot z + K_x \cdot \eta - \frac{e}{E_o} \cdot \Delta B_z\} \quad ; \qquad (2.39a)$$

$$K_z \cdot (z'' + K_z^2 \cdot z) = K_z \cdot \{+\frac{e}{E_o}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \cdot z - 2H \cdot x' +$$

$$+ (N - H') \cdot z + K_z \cdot \eta + \frac{e}{E_o} \cdot \Delta B_x\} \qquad (2.39b)$$

and eq. (2.38) can be written in the form

$$R_x^D = -\frac{2}{3}\,e^2\gamma_o^4 \cdot (K_x^2 + K_z^2) \cdot x'; \qquad (2.40a)$$

$$R_z^D = -\frac{2}{3}\,e^2\gamma_o^4 \cdot (K_x^2 + K_z^2) \cdot z'; \qquad (2.40b)$$

$$R_t^D = -\frac{2}{3}\,e^2\gamma_o^4 \cdot \{(K_x^2 + K_z^2) \cdot (1 + 2\eta) - 2K_x \cdot [-\frac{e}{E_o}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \cdot x +$$

$$+ 2H \cdot z' + (N + H') \cdot z - \frac{e}{E_o} \cdot \Delta B_z] -$$

$$- 2K_z \cdot [+\frac{e}{E_o}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \cdot z - 2H \cdot x' + (N - H') \cdot x +$$

$$+ \frac{e}{E_o} \cdot \Delta B_x]\} \ . \qquad (2.40c)$$

2.6 The equations of motion for the coupled synchro- betatron oscillations

Putting (2.40a, b, c) into (2.34) and linearizing with respect to x, z, σ and η we get the linearized equations of motion for the coupled synchro- betatron oscillations in their final and most general form, namely:

$$x' \cdot \frac{e\hat{V}}{E_o} \sin\Phi \cdot \sum_\nu \delta(s - s_\nu) + x'' + G_1 \cdot x - (N + H') \cdot z -$$

$$- 2H \cdot z' - K_x \cdot \eta = -\frac{e}{E_o} \cdot \Delta B_z; \qquad (2.41a)$$

$$z' \cdot \frac{e\hat{V}}{E_o} \sin\Phi \cdot \sum_\nu \delta(s - s_\nu) + z'' + G_2 \cdot z - (N - H') \cdot x + 2H \cdot x' -$$

$$- K_z \cdot \eta = +\frac{e}{E_o} \cdot \Delta B_x; \qquad (2.41b)$$

$$\sigma' + K_x \cdot x + K_z \cdot z = 0; \qquad (2.41c)$$

$$\eta' - \sigma(s) \cdot \frac{e\hat{V}}{E_o} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\phi \cdot \sum_\nu \delta(s - s_\nu) - \frac{e\hat{V}}{E_o} \sin\phi \cdot \sum_\nu \delta(s - s_\nu) +$$

$$+ C_1 \cdot (K_x^2 + K_z^2) - \frac{e}{E_o} \cdot \Delta\varepsilon - 2C_1 \cdot \frac{e}{E_o}(K_z \cdot \Delta B_x - K_x \cdot \Delta B_z) - \frac{\delta R_t}{E_o} =$$

$$= - x\cdot C_1\cdot[K_x(K_x^2 + K_z^2) + 2K_x\cdot \frac{e}{E_o}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} - 2K_z(N - H')] -$$

$$- z\cdot C_1\ [K_z(K_x^2 + K_z^2) - 2K_z\cdot \frac{e}{E_o}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} - 2K_x(N + H')] -$$

$$- x'\cdot 4C_1\cdot K_z\cdot H + z'\cdot 4C_1\cdot K_x\cdot H - 2\eta\cdot C_1(K_x^2 + K_z^2) \qquad (2.41d)$$

where we have defined

$$C_1 = \frac{2}{3}e^2\frac{\gamma_o^4}{E_o} \qquad (2.42)$$

Eq.'s (2.41a) and (2.41b) describe the betatron oscillations in x-direction and z-direction, respectively, while (2.41c, d) describe the synchrotron oscillations. Synchrotron and betatron oscillations are coupled because of the curvature of the reference trajectory. This coupling vanishes for

$$K_x = K_z = 0$$

(which means that in a first approximation the betatron and synchrotron oscillations are decoupled in a linear accelerator). There is also a coupling between the horizontal and vertical betatron oscillations which is caused by the fields of rotated quadrupoles (factor N in eq. (2.41)) or by the fields of solenoids (factor H, H' in eq. (2.41)).
In addition to these coupling mechanisms the differential equations for the betatron oscillations contain a damping term of the form

$$y'\cdot \frac{e\hat{V}}{E_o}\cdot\sin\Phi\cdot\sum_\nu \delta(s - s_\nu)$$

$$(y \equiv x, z)$$

originating from the electric fields of the cavities. One can calculate the influence of the accelerating fields on the betatron motion by integrating eq. (2.41a, b) from $s_\nu - 0$ to $s_\nu + 0$, where s_ν designates the position of the ν-th cavity. Then we get $(y = x, z)$:

$$\frac{e\hat{V}}{E_o}\cdot\sin\Phi\cdot\int_{s_\nu-0}^{s_\nu+0} ds\cdot y'(s)\cdot\delta(s - s_\nu) + \int_{s_\nu-0}^{s_\nu+0} ds\cdot y''(s) = 0\ ;$$

$$\frac{e\hat{V}}{E_o}\cdot\sin\Phi\cdot y'(s_\nu - 0) + [\,y'(s_\nu + 0) - y'(s_\nu - 0)] = 0$$

or

$$y'(s_\nu + 0) = [1 - \frac{e\hat{V}}{E_o}\sin\Phi]\cdot y'(s_\nu - 0) \qquad (2.43)$$

Since the quantity $\frac{e\hat{V}}{E_o}\sin\Phi$ which describes the relative gain of energy of a particle traversing a cavity satisfies the inequality

$$0 < \frac{e\hat{V}}{E_o}\sin\Phi \ll 1$$

the tangent vector of the particle trajectory is rotated by a small amount towards the equilibrium orbit by the presence of the electric fields. This is equivalent to a damping of the oscillation amplitude.
In many cases one is allowed to neglect these damping terms for the calculation of the particle dynamics and to consider among those terms describing the emission of photons only the constant factor

$$C_1\cdot(K_x^2 + K_z^2)$$

which just describes the influence of the radiation on the equilibrium particle $(x = z = \eta = 0)$. Making these simplifications the equations of motion for the "undamped" synchro- betatron oscillations can be obtained from (2.41):

$$x'' + G_1\cdot x - (N + H')\cdot z - 2H\cdot z' - K_x\cdot\eta = -\frac{e}{E_o}\cdot\Delta B_z\ ; \qquad (2.44a)$$

$$z'' + G_2\cdot z - (N - H')\cdot x + 2H\cdot x' - K_z\cdot\eta = +\frac{e}{E_o}\cdot\Delta B_x\ ; \qquad (2.44b)$$

$$\sigma' + K_x\cdot x + K_z\cdot z = 0\ ; \qquad (2.44c)$$

$$\eta' - \sigma(s)\cdot\frac{e\hat{V}}{E_o}\cdot k\cdot\frac{2\pi}{L}\cdot\cos\Phi\cdot\sum_\nu\delta(s - s_\nu) =$$

$$=\{\frac{e\hat{V}}{E_o}\sin\Phi\cdot\sum_\nu\delta(s - s_\nu) - C_1\cdot(K_x^2 + K_z^2)\} +$$

$$+\frac{e}{E_o}\cdot\Delta\varepsilon + 2C_1\cdot\frac{e}{E_o}(K_z\cdot\Delta B_x - K_x\cdot\Delta B_z)\,. \qquad (2.44d)$$

These equations can be written in canonical form by using the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\{G_1\cdot x^2 + G_2\cdot z^2 - 2N\cdot xz + (p_x + Hz)^2 + (p_z - Hx)^2\} -$$

$$-\frac{1}{2}\sigma^2\cdot\frac{e\hat{V}}{E_o}\cdot k\cdot\frac{2\pi}{L}\cdot\cos\Phi\cdot\sum_\nu\delta(s - s_\nu) - (K_x\cdot x + K_z\cdot z)\cdot p_\sigma -$$

$$-\sigma\cdot\{[\frac{e\hat{V}}{E_o}\sin\Phi\cdot\sum_\nu\delta(s - s_\nu) - C_1\cdot(K_x^2 + K_z^2)] + \frac{e}{E_o}\cdot\Delta\varepsilon +$$

$$+ 2C_1\cdot\frac{e}{E_o}[K_x\cdot\Delta B_z - K_z\cdot\Delta B_x]\} + x\cdot\frac{e}{E_o}\cdot\Delta B_z - z\cdot\frac{e}{E_o}\cdot\Delta B_x \qquad (2.45a)$$

with

$$p_\sigma \equiv \eta\ . \qquad (2.45b)$$

Putting $\mathcal{H}$ as defined in (2.45) into the Hamilton equations of motion

$$\begin{aligned} x' &= \frac{\partial \mathcal{H}}{\partial p_x} \; ; \quad p_x' = -\frac{\partial \mathcal{H}}{\partial x} \; ; \\ z' &= \frac{\partial \mathcal{H}}{\partial p_z} \; ; \quad p_z' = -\frac{\partial \mathcal{H}}{\partial z} \; ; \\ \sigma' &= \frac{\partial \mathcal{H}}{\partial p_\sigma} \; ; \quad p_\sigma' = -\frac{\partial \mathcal{H}}{\partial \sigma} \end{aligned} \tag{2.46}$$

we find in matrix notation:

$$\vec{y}\,' = \underline{A}\vec{y} + \vec{c}_o + \vec{c}_1 \tag{2.47}$$

with the following definitions

$$\vec{y}^{\,T} = (x, p_x, z, p_z, \sigma, p_\sigma) \; ; \tag{2.48a}$$

$$\vec{c}_o^{\,T} = \left(0, 0, 0, 0, 0, \frac{e\hat{V}}{E_o} \sin\phi \cdot \sum_\nu \delta(s - s_\nu) - C_1(K_x^2 + K_z^2)\right) ; \tag{2.48b}$$

$$\vec{c}_1^{\,T} = \left(0, -\frac{e}{E_o} \cdot \Delta B_z, 0, \frac{e}{E_o} \cdot \Delta B_x, 0, \frac{e}{E_o} \cdot \Delta\varepsilon - 2C_1 \frac{e}{E_o}[K_x \cdot \Delta B_z - K_z \cdot \Delta B_x]\right) \tag{2.48c}$$

and

$$\underline{A} = \begin{pmatrix} 0 & 1 & H & 0 & 0 & 0 \\ -(G_1 + H^2) & 0 & N & H & 0 & K_x \\ -H & 0 & 0 & 1 & 0 & 0 \\ N & -H & -(G_2 + H^2) & 0 & 0 & K_z \\ -K_x & 0 & -K_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{e\hat{V}}{E_o} \cdot k \cdot \frac{2\pi}{L} \cos\Phi \cdot \sum_\nu \delta(s - s_\nu) & 0 \end{pmatrix} \tag{2.48d}$$

(2.47) reprensents a system of linear differential equations of first order which contain besides the quantities x, z, σ and η the momentum variables

$$p_x = x' - H \cdot z \; ; \tag{2.49a}$$

$$p_z = z' + H \cdot x \; ; \tag{2.49b}$$

(The relationships (2.49a, b) are equivalent to the Hamilton equations

$$x' = \frac{\partial \mathcal{H}}{\partial p_x} \; ; \quad z' = \frac{\partial \mathcal{H}}{\partial p_z}) \; .$$

Eliminating the variables p_x and p_z in eq. (2.47) we indeed rediscover (2.44). On the other hand we can get (2.47) from (2.44) by introducing p_x and p_z as additional variables and by transforming the differential equation of second order (2.44) into a differential equation of first order. In this respect the two equations (2.44) and (2.47) are equivalent. For our further discussions it will be convenient to use the equations of motion (mostly) in the form (2.47). The canonical structure of these equations will play an important role in our later investigations.

Rewriting the general equations of motion for the synchro- betatron oscillations (2.41) in terms of the variables x, p_x, z, p_z, σ, $p_\sigma \equiv \eta$ we have to supplement the matrix $\underline{A}$ by a perturbation matrix $\delta\underline{A}$ and we have to add a third inhomogeneous term $\delta\vec{c}$, so that we get

$$\vec{y}\,' = (\underline{A} + \delta\underline{A})\vec{y} + \vec{c}_o + \vec{c}_1 + \delta\vec{c} \tag{2.50}$$

where $\delta\underline{A}$ is defined by:

$$\begin{aligned} \delta\underline{A} &= ((\delta A_{ik})) \; ; \\ \delta A_{22} &= -\frac{e\hat{V}}{E_o} \sin\Phi \cdot \sum_\nu \delta(s - s_\nu) \; ; \\ \delta A_{44} &= \delta A_{22} \; ; \\ \delta A_{61} &= -C_1 \cdot \Big[(K_x^2 + K_z^2) \cdot K_x + 2K_x \cdot \frac{e}{E_o} \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} + \\ &\quad + 4\,K_x \cdot H^2 - 2K_z \cdot (N - H')\Big] \; ; \\ \delta A_{62} &= 4\,C_1 \cdot K_z \cdot H \; ; \\ \delta A_{63} &= -C_1 \cdot \Big[(K_x^2 + K_z^2) \cdot K_z - 2K_z \cdot \frac{e}{E_o} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} + \\ &\quad + 4K_z \cdot H^2 - 2K_x \cdot (N + H')\Big] \; ; \\ \delta A_{64} &= -4C_1 \cdot K_x \cdot H \; ; \\ \delta A_{66} &= -2C_1 \cdot (K_x^2 + K_z^2) \; ; \\ \delta A_{ik} &= 0 \quad \text{(otherwise)} \end{aligned} \tag{2.51a}$$

and $\delta\vec{c}$ is given by

$$\delta\vec{c}^{\,T} = \left(0, 0, 0, 0, 0, \frac{\delta R_t}{E_o}\right) . \tag{2.51b}$$

Eq.'s (2.47) or (2.50) uniquely define the particle motion in a circular accelerator or storage ring. These equations will form the starting point for our further discussions.

3. Introduction of a new reference orbit.

3.1 Equation of motion for the new reference trajectory

The equations of motion for the coupled synchro-betatron oscillations (2.47) and (2.50) form a system of linear and inhomogeneous differential equations with the inhomogeneous parts $\delta\vec{c}$, $\vec{c}_o$ and $\vec{c}_1$. The inhomogeneous term $\delta\vec{c}$ is due to quantum fluctuations of the radiation field and $\vec{c}_o$ is due to the variation of the energy of the circulating particle because of radiation losses and the presence of accelerating fields while the vector $\vec{c}_1$ originates from fields ΔB_x, ΔB_z and $\Delta\varepsilon$, which can be interpreted as field errors or (time-dependent) external fields.

For our further discussions of the particle motion it is advantageous to eliminate the inhomogeneous part $\vec{c}_o$ of eq. (2.50). This is achieved in the well-known manner by looking for the (only) periodic solution $\vec{y}_o$ of the inhomogeneous equation

$$\vec{y}\,' = (\underline{A} + \delta\underline{A})\vec{y} + \vec{c}_o \tag{3.1}$$

namely

$$\vec{y}_o\,' = (\underline{A} + \delta\underline{A})\vec{y}_o + \vec{c}_o \tag{3.2a}$$

$$\vec{y}_o\,(s_o + L) = \vec{y}_o(s_o) \quad \text{(condition of periodicity)}. \tag{3.2b}$$

Then the general solution of (2.50) can be separated into

$$\vec{y} = \vec{y}_o + \vec{\tilde{y}}$$

where the vector $\vec{\tilde{y}}$ describes the synchro-betatron oscillations about the new closed equilibrium trajectory $\vec{y}_o$, which we shall call "six-dimensional closed orbit" in the following.

3.2 Description of the closed orbit by the enlarged transfer matrix

The closed orbit is uniquely defined by eq. (3.2a, b). For an approximate calculation of this new reference trajectory we are allowed to neglect the perturbation matrix $\delta\underline{A}$ in eq. (3.2), and thus eq. (3.2a) (or (3.1) reduces to the simpler equation

$$\vec{y}\,' = \underline{A}\vec{y} + \vec{c}_o \tag{3.4}$$

of the undamped synchro-betatron oscillations. We further assume, that the functions $G_1(s)$, $G_2(s)$, $N(s)$, $H(s)$, $K_x(s)$ and $K_z(s)$ appearing in the coefficient matrix $\underline{A}(s)$ are piece-wise constant, so that the s-axis can be divided into segments

$$s_\mu < s < s_{\mu+1}$$

where $\underline{A}$ is a constant matrix

$$\underline{A}(s) = \text{const.}, \ \underline{A}'(s) = 0 \tag{3.5a}$$

$$\text{for } s_\mu < s < s_{\mu+1}.$$

And since in this case

$$K_x = \text{const.}$$

$$K_z = \text{const.},$$

eq. (2.48b) implies that

$$\vec{c}_o = \text{const.} \ (s_\mu < s < s_{\mu+1}) \tag{3.5b}$$

so that eq. (3.4) represents a linear differential equation with constant coefficients in the intervals $s_\mu < s < s_{\mu+1}$, except at the positions where the cavities are situated. But assuming pointlike cavities (see eq. (2.28)) the integration of the equations of motion can easily be performed for the accelerating fields because of the presence of the δ-functions in (2.48b) and (2.48c).

The solution of eq. (3.4) can now be written in the form

$$\begin{pmatrix} \vec{y}(s) \\ 1 \end{pmatrix} = \underline{\hat{M}}(s, s_o) \begin{pmatrix} \vec{y}(s_o) \\ 1 \end{pmatrix} \tag{3.6}$$

where the "enlarged" transfer matrix $\underline{\hat{M}}$ is defined by

$$\underline{\hat{M}}(s, s_o) = \begin{pmatrix} \underline{M}(s, s_o) & \vec{m}(s) \\ 0 & 1 \end{pmatrix}. \tag{3.7}$$

$\underline{M}(s, s_o)$ represents the (simple) transfer matrix belonging to the homogeneous equation

$$\vec{y}\,' = \underline{A}\vec{y}$$

and satisfying the following conditions

$$\frac{d}{ds}\underline{M}(s, s_o) = \underline{A}\,\underline{M}(s, s_o) \ ; \tag{3.8a}$$

$$\underline{M}(s_o, s_o) = 1 \tag{3.8b}$$

while the vector $\vec{m}(s)$ in eq. (3.7) is a special solution of eq. (3.4)

$$\frac{d}{ds}\vec{m}(s) = \underline{A}\vec{m}(s) + \vec{c}_o \tag{3.9a}$$

with the initial value

$$\vec{m}(s_o) = 0 . \tag{3.9b}$$

For the special case of an infinitesimal transfer matrix $\hat{\underline{M}}(s + \Delta s, s)$ we get from (3.4)

$$\hat{\underline{M}}(s + \Delta s, s) = \begin{pmatrix} [\underline{1} + \underline{A}(s)\cdot\Delta s] & \vec{c}_o\cdot\Delta s \\ 0 & 1 \end{pmatrix} . \tag{3.10}$$

Making use of (3.6) and (3.7) the condition of periodicity (3.2b) then takes the form

$$\begin{pmatrix} \underline{M}(s_o + L, s_o) & \vec{m}(s_o + L) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \vec{y}_o(s_o) \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{y}_o(s_o) \\ 1 \end{pmatrix}$$

from which we can calculate the "initial vector" of the closed orbit $\vec{y}(s_o)$:

$$\vec{y}_o(s_o) = [\underline{1} - \underline{M}(s_o + L, s_o)]^{-1}\cdot\vec{m}(s_o + L) . \tag{3.11}$$

Using this equation and eq. (3.6) we can determine the closed orbit at each position s, if the transfer matrix $\hat{\underline{M}}(s, s_o)$ is known everywhere around the ring.

3.3 Calculation of the transfer matrices

3.3.1 The transfer matrices for special types of lenses

Because of (3.5a, b) it is possible to calculate the transfer matrices for the most important types of lenses explicitly.

3.3.1.1 Synchrotron magnet

$$N = H = H' = \hat{V} = 0 ; \tag{3.12}$$
$$K_x^2 + K_z^2 = \text{const.} \neq 0 ;$$
$$K_x(s)\cdot K_z(s) = 0 ;$$

$$K_x \neq 0 \quad \text{curvature in x-direction} ;$$
$$K_z \neq 0 \quad \text{curvature in z-direction} ;$$
$$G_1 = \text{const.}\ (\neq 0 \text{ for } K_x \neq 0),\ G_2 = \text{const.}\ (\neq 0 \text{ for } K_z \neq 0).$$

In this case the equations of motion (3.4) and (2.48) read ($p_\sigma \equiv \eta$)

$$\begin{aligned} x' &= p_x; \\ p_x' &= -G_1\cdot x + K_x\cdot\eta ; \\ z' &= p_z; \\ p_z' &= -G_2\cdot z + K_z\cdot\eta ; \\ \sigma' &= -K_x\cdot x - K_z\cdot z; \\ \eta' &= -C_1\cdot(K_x^2 + K_z^2). \end{aligned} \tag{3.13}$$

The elements of the enlarged transfer matrices are given by:

$$\hat{M}_{11} = \cos[\sqrt{G_1}(s - s_o)] \quad ;$$

$$\hat{M}_{12} = \frac{1}{\sqrt{G_1}}\sin[\sqrt{G_1}(s - s_o)] \ ;$$

$$\hat{M}_{16} = \frac{K_x}{G_1}\cdot\{1 - \cos[\sqrt{G_1}(s - s_o)\} \quad ;$$

$$\hat{M}_{17} = -\frac{1}{G_1}\cdot C_1\cdot K_x(K_x^2 + K_z^2)\cdot\{(s - s_o) - \frac{1}{\sqrt{G_1}}\sin[\sqrt{G_1}(s - s_o)]\};$$

$$\hat{M}_{21} = -\sqrt{G_1}\cdot\sin[\sqrt{G_1}(s - s_o)] \ ;$$

$$\hat{M}_{22} = \cos[\sqrt{G_1}(s - s_o)] \quad ;$$

$$\hat{M}_{26} = \frac{K_x}{\sqrt{G_1}}\cdot\sin[\sqrt{G_1}(s - s_o)] \quad ;$$

$$\hat{M}_{27} = -\frac{1}{G_1}\cdot C_1\cdot K_x(K_x^2 + K_z^2)\cdot\{1 - \cos[\sqrt{G_1}(s - s_o)]\} \quad ;$$

$$\hat{M}_{33} = \cos[\sqrt{G_2}(s - s_o)] \quad ;$$

$$\hat{M}_{34} = \frac{1}{\sqrt{G_2}}\sin[\sqrt{G_2}(s - s_o)] \quad ;$$

$$\hat{M}_{36} = \frac{K_z}{G_2}\cdot\{1 - \cos[\sqrt{G_2}(s - s_o)]\} \quad ;$$

$$\hat{M}_{37} = -\frac{1}{G_2}\cdot C_1\cdot K_z(K_x^2 + K_z^2)\cdot\{(s - s_o) - \frac{1}{\sqrt{G_2}}\cdot\sin[\sqrt{G_2}(s - s_o)]\};$$

$$\hat{M}_{43} = -\sqrt{G_2}\cdot\sin[\sqrt{G_2}(s-s_o)] \;;$$

$$\hat{M}_{44} = \cos[\sqrt{G_2}(s-s_o)] \;;$$

$$\hat{M}_{46} = \frac{K_z}{\sqrt{G_2}}\cdot\sin[\sqrt{G_2}(s-s_o)] \;;$$

$$\hat{M}_{47} = -\frac{1}{G_2}\cdot C_1\cdot K_z(K_x^2+K_z^2)\cdot\{1-\cos[\sqrt{G_2}(s-s_o)]\} \;;$$

$$\hat{M}_{51} = -\frac{K_x}{\sqrt{G_1}}\sin[\sqrt{G_1}(s-s_o)] \;;$$

$$\hat{M}_{52} = -\frac{K_x}{G_1}\{1-\cos[\sqrt{G_1}(s-s_o)]\} \;;$$

$$\hat{M}_{53} = -\frac{K_z}{\sqrt{G_2}}\sin[\sqrt{G_2}(s-s_o)] \;;$$

$$\hat{M}_{54} = -\frac{K_z}{G_2}\{1-\cos[\sqrt{G_2}(s-s_o)]\} \;;$$

$$\hat{M}_{55} = 1 \;;$$

$$\hat{M}_{56} = -\frac{K_x^2}{G_1}\{(s-s_o)-\frac{1}{\sqrt{G_1}}\sin[\sqrt{G_1}(s-s_o)]\} - \frac{K_z^2}{G_2}\{(s-s_o)-\frac{1}{\sqrt{G_2}}\sin[\sqrt{G_2}(s-s_o)]\} \;;$$

$$\hat{M}_{57} = \frac{1}{G_1}\cdot C_1\cdot K_x^2(K_x^2+K_z^2)\cdot\{\frac{1}{2}(s-s_o)^2+\frac{1}{G_1}\cos[\sqrt{G_1}(s-s_o)]-\frac{1}{G_1}\} + \frac{1}{G_2}\cdot C_1\cdot K_z^2(K_x^2+K_z^2)\cdot\{\frac{1}{2}(s-s_o)^2+\frac{1}{G_2}\cos[\sqrt{G_2}(s-s_o)]-\frac{1}{G_2}\};$$

$$\hat{M}_{66} = 1 = \hat{M}_{77} \;;$$

$$\hat{M}_{67} = -C_1\cdot(K_x^2+K_z^2)\cdot(s-s_o) \;;$$

$$\hat{M}_{ik} = 0 \qquad \text{otherwise} . \tag{3.14}$$

If the quantities G_1 and G_2 are negative, which may be the case, we can make use of the following relationships

$$\cos[\sqrt{-|G|}(s-s_o)] = \cosh[\sqrt{|G|}(s-s_o)] \;;$$
$$\frac{1}{\sqrt{-|G|}}\sin[\sqrt{-|G|}(s-s_o)] = \frac{1}{\sqrt{|G|}}\sinh[\sqrt{|G|}(s-s_o)] . \tag{3.15}$$

3.3.1.2 Quadrupole

with

$$N = H = H' = K_x = K_z = \hat{V} = 0 \;, \; G_1 = +G \;; \; G_2 = -G$$

$$G = \frac{e}{E_o}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \neq 0 \text{ (see eq. (2.27))}.$$

The equations of motion now read

$$x' = p_x \;;$$
$$p_x' = -G\cdot x \;;$$
$$z' = p_z \;;$$
$$p_z' = +G\cdot z$$
$$\sigma' = 0 \;;$$
$$\eta' = 0 \;;$$

with the transfer matrix

$$\hat{M} = ((\hat{M}_{ik})) \;;$$

$$\hat{M}_{11} = \cos[\sqrt{G}(s-s_o)] \;;$$

$$\hat{M}_{12} = \frac{1}{\sqrt{G}}\sin[\sqrt{G}(s-s_o)] \;;$$

$$\hat{M}_{21} = -\sqrt{G}\sin[\sqrt{G}(s-s_o)] \;;$$

$$\hat{M}_{22} = \cos[\sqrt{G}(s-s_o)] \;;$$

$$\hat{M}_{33} = \cos[\sqrt{-G}(s-s_o)] \;;$$

$$\hat{M}_{34} = \frac{1}{\sqrt{-G}}\sin[\sqrt{-G}(s-s_o)] \;;$$

$$\hat{M}_{43} = -\sqrt{-G}\sin[\sqrt{-G}(s-s_o)] \;;$$

$$\hat{M}_{44} = \cos[\sqrt{-G}(s-s_o)] \;;$$

$$\hat{M}_{55} = 1 \;;$$

$$\hat{M}_{66} = 1 \;;$$

$\hat{M}_{77} = 1$;

$\hat{M}_{ik} = 0$ otherwise (and see eq. (3.15)) . (3.16)

3.3.1.3 Rotated quadrupole

$G_1 = G_2 = H = H' = K_x = K_z = \hat{V} = 0$;

$N \neq 0$.

The equations of motion are given by (see eq. (3.4) and (2.48))

$$\begin{aligned} x' &= p_x \\ p_x' &= N \cdot z \\ z' &= p_z \\ p_z' &= N \cdot x \\ \sigma' &= 0 \\ \eta' &= 0 \end{aligned} \qquad (3.17)$$

which means, that the betatron oscillations in x- and z-direction are coupled:

$$\begin{aligned} x'' &= N \cdot z \\ z'' &= N \cdot x . \end{aligned} \qquad (3.18)$$

It follows from (3.18)

$$(x + z)'' = N(x + z) ;$$

$$(x - z)'' = - N(x - z).$$

In this form the differential equations are decoupled and they can be integrated easily. Thus we obtain the following expressions for the matrix elements of the transfer matrix $\underline{\hat{M}}(s_o, s)$:

$$\hat{M}_{11} = \frac{1}{2}\{\cos[\sqrt{-N}(s - s_o)] + \cos[\sqrt{+N}(s - s_o)]\} ;$$

$$\hat{M}_{12} = \frac{1}{2}\{\frac{1}{\sqrt{-N}}\sin[\sqrt{-N}(s - s_o)] + \frac{1}{\sqrt{+N}}\sin[\sqrt{+N}(s - s_o)]\} ;$$

$$\hat{M}_{13} = \frac{1}{2}\{\cos[\sqrt{-N}(s - s_o)] - \cos[\sqrt{+N}(s - s_o)]\} ;$$

$$\hat{M}_{14} = \frac{1}{2}\{\frac{1}{\sqrt{-N}}\sin[\sqrt{-N}(s - s_o)] - \frac{1}{\sqrt{+N}}\sin[\sqrt{+N}(s - s_o)]\} ;$$

$$\hat{M}_{21} = -\frac{1}{2}\{\sqrt{-N}\sin[\sqrt{-N}(s - s_o)] + \sqrt{+N}\sin[\sqrt{+N}(s - s_o)]\} ;$$

$\hat{M}_{22} = \hat{M}_{11}$;

$$\hat{M}_{23} = -\frac{1}{2}\{\sqrt{-N}\sin[\sqrt{-N}(s - s_o)] - \sqrt{+N}\sin[\sqrt{+N}(s - s_o)]\} ;$$

$\hat{M}_{24} = \hat{M}_{13}$;

$\hat{M}_{31} = \hat{M}_{13}$;

$\hat{M}_{32} = \hat{M}_{14}$;

$\hat{M}_{33} = \hat{M}_{11}$;

$\hat{M}_{34} = \hat{M}_{12}$;

$\hat{M}_{41} = \hat{M}_{23}$;

$\hat{M}_{42} = \hat{M}_{21}$;

$\hat{M}_{43} = \hat{M}_{13}$;

$\hat{M}_{44} = \hat{M}_{11}$;

$\hat{M}_{55} = 1$;

$\hat{M}_{66} = 1$;

$\hat{M}_{77} = 1$;

$\hat{M}_{ik} = 0$ otherwise. (3.19)

3.3.1.4 Solenoid

$$G_1 = G_2 = N = K_x = K_z = \hat{V} = 0 \; ;$$
$$H = \text{const.} \neq 0 .$$

The equations of motion are written in the form

$$\vec{y}\,' = \underline{B}\,\vec{y} \tag{3.20a}$$

where the matrix $\underline{B}$ is given by (compare with eq. (2.48))

$$\underline{B} = \begin{pmatrix} 0 & 1 & H & 0 & 0 & 0 \\ -H^2 & 0 & 0 & H & 0 & 0 \\ -H & 0 & 0 & 1 & 0 & 0 \\ 0 & -H & -H^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{3.20b}$$

Making the "ansatz"

$$\vec{y} = \underline{D}(s) \cdot \vec{x}(s) \tag{3.21}$$

eq. (3.20a) is transformed into

$$\vec{x}\,' = \underline{D}^{-1} \cdot (\underline{B}\,\underline{D} - \underline{D}')\vec{x} . \tag{3.22}$$

Choosing the following matrix for $\underline{D}(s)$

$$\underline{D} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 & 0 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.23a}$$

where we have put

$$\theta(s) = H(s - s_o) \tag{3.23b}$$

we obtain from (3.22) ($\vec{x}^T \equiv (x_1, x_2, \ldots, x_6)$)

$$\left.\begin{aligned} x_1' &= x_2 \\ x_2' &= -H^2 \cdot x_1 \end{aligned}\right\} \Rightarrow x_1'' = -H^2 \cdot x_1 \; ;$$

$$\left.\begin{aligned} x_3' &= x_4 \\ x_4' &= -H^2 \cdot x_3 \end{aligned}\right\} \Rightarrow x_3'' = -H^2 \cdot x_3 ;$$

$$x_5' = 0 \; ;$$

$$x_6' = 0 . \tag{3.24}$$

The solution of eq. (3.24) can be written as

$$\vec{x}(s) = \underline{C}(s, s_o)\,\vec{x}(s_o) \tag{3.25a}$$

with

$$\underline{C}(s,s_o) = \begin{pmatrix} \cos\theta & \frac{1}{H}\sin\theta & 0 & 0 & 0 & 0 \\ -H\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & \frac{1}{H}\sin\theta & 0 & 0 \\ 0 & 0 & -H\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.25b}$$

Putting (3.25a) into (3.21) we get

$$\begin{aligned} \vec{y}(s) &= \underline{D}(s) \cdot \underline{C}(s,s_o)\,\vec{x}(s_o) \\ &= \underline{D}(s) \cdot \underline{C}(s,s_o) \cdot \underline{D}^{-1}(s_o) \cdot \vec{y}(s_o) \\ &= \underline{D}(s) \cdot \underline{C}(s,s_o) \cdot \vec{y}(s_o) . \end{aligned} \tag{3.26}$$

This equation implies that the (simple) transfer matrix $\underline{M}(s,s_o)$ is given by

$$\underline{M}(s,s_o) = \underline{D}(s) \cdot \underline{C}(s,s_o) \tag{3.27}$$

Taking into account (3.23a) and (3.25b) we can calculate the elements of the enlarged transfer matrix $\underline{\hat{M}}(s,s_o)$

$$\hat{M}_{11} = \frac{1}{2}\cdot(1 + \cos 2\theta);$$

$$\hat{M}_{12} = \frac{1}{2H}\cdot \sin 2\theta \quad ;$$

$$\hat{M}_{13} = \frac{1}{2} \sin 2\theta \quad ;$$

$$\hat{M}_{14} = \frac{1}{2H}\cdot(1 - \cos 2\theta);$$

$$\hat{M}_{21} = - H\cdot\frac{1}{2} \sin 2\theta \quad ;$$

$$\hat{M}_{22} = \hat{M}_{11} \;;$$

$$\hat{M}_{23} = - H\cdot\frac{1}{2}(1 - \cos 2\theta);$$

$$\hat{M}_{24} = \hat{M}_{13} \;;$$

$$\hat{M}_{31} = - \hat{M}_{13} \;;$$

$$\hat{M}_{32} = - \hat{M}_{14} \;;$$

$$\hat{M}_{33} = \hat{M}_{11} \;;$$

$$\hat{M}_{34} = \hat{M}_{12} \;;$$

$$\hat{M}_{41} = - \hat{M}_{23} \;;$$

$$\hat{M}_{42} = - \hat{M}_{13} \;;$$

$$\hat{M}_{43} = \hat{M}_{21} \;;$$

$$\hat{M}_{44} = \hat{M}_{11} \;;$$

$$\hat{M}_{55} = 1 \;;$$

$$\hat{M}_{66} = 1 \;;$$

$$\hat{M}_{77} = 1 \;;$$

$$\hat{M}_{ik} = 0 \quad \text{otherwise} \qquad (3.28)$$

where θ has to be taken from eq. (3.23b) ((17), (18), (19), (20)).

3.3.1.5 Cavity

$$G_1 = G_2 = N = H = K_x = K_z = 0 \;;$$
$$\hat{V} \neq 0 .$$

Because of (2.48b, d) the equations of motion read

$$x' = p_x \;;$$
$$p_x' = 0 \;;$$
$$z' = p_z \;;$$
$$p_z' = 0 \;;$$
$$\sigma' = 0 \;;$$
$$\eta' = \sigma\cdot\frac{e\hat{V}}{E_o}\cdot k\cdot\frac{2\pi}{L}\cdot\cos\Phi\cdot\sum_\nu \delta(s - s_\nu) + \frac{e\hat{V}}{E_o}\sin\Phi\cdot\sum_\nu \delta(s - s_\nu)$$

and the following expressions are found for the matrix elements of $\underline{\hat{M}}(s_\nu + 0, s_\nu - 0)$:

$$\hat{M}_{kk} = 1 \quad \text{for } k = 1, 2, \ldots, 7$$

$$\hat{M}_{65} = \frac{e\hat{V}}{E_o}\cdot k\cdot\frac{2\pi}{L}\cos\Phi$$

$$\hat{M}_{67} = \frac{e\hat{V}}{E_o}\sin\Phi$$

$$M_{ik} = 0 \quad \text{otherwise.} \qquad (3.29)$$

3.3.2 Approximation Shemes

In the foregoing chapters we have derived explicit expressions for the enlarged transfer matrices for the most important types of lenses (see eq's (3.14), (3.16), (3.19), (3.28), (3.29)).
However in more complicated cases one has to apply suitable approximation schemes for calculating the transfer matrices. Now we want to describe two simple schemes of calculation.

3.3.2.1 Series expansion

Because of (3.5a, b) the equations of motion for each lens are given by linear differential equations with constant coefficients.

And therefore we can write down the following expressions for the simple transfer matrix $\underline{M}(s, s_o)$ and the vector $\vec{m}(s)$ defined in (3.7):

$$\hat{\underline{M}}(s, s_o) = e^{\underline{A}(s-s_o)} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \underline{A}^n (s-s_o)^n ; \quad (3.30a)$$

$$\vec{m}(s) = \left(\sum_{n=1}^{\infty} \frac{1}{n!} \underline{A}^{n-1} (s-s_o)^n \right) \vec{c}_o \quad (3.30b)$$

which can easily be verified by putting (3.30a, b) into the equations for $\underline{M}$ (eq. (3.8a. b)) and $\vec{m}$ (eq. (3.9a, b)). Thus we have obtained a series expansion allowing for an approximate calculation of $\underline{M}$, $\vec{m}$ and hence of the enlarged transfer matrix $\hat{\underline{M}}$, if we terminate the expansion after a finite number of terms. The terms taken into account determine the accuracy of the approximation. It is also worthwhile mentioning that the vector $\vec{m}(s)$ can be put in the form

$$\vec{m}(s) = [\underline{M}(s, s_o) - \underline{1}] \, \underline{A}^{-1} \, \vec{c}_o$$

if $\quad \det(\underline{A}) \neq 0 \quad$ (existence of $\underline{A}^{-1}$).

In this case we only need the matrix $\underline{M}$ for a calculation of the vector $\vec{m}$.

3.3.2.2 Decomposition of a magnet into thin lenses

If the conditions (3.5a, b) do not hold, for example if we take into account the perturbation matrix $\delta\underline{A}$ in eq. (3.1), we can divide the given lens into small (infinitesimal) segments and according to (3.10) we can calculate the infinitesimal transfer matrix $\hat{\underline{M}}(s+\Delta s, s)$. Multiplying the single infinitesimal transfer matrices we obtain the transfer matrix $\hat{\underline{M}}(s_o+ l, s_o)$ for the whole length l of the magnet element.
This approximation is used in the computing code SLIM (31).

3.4 The free synchro- betatron oscillations

Having set up the transfer matrices for the different types of lenses we are able to determine the six-dimensional closed orbit $\vec{y}_o(s)$, which then means that we know the first component of the oscillation amplitude $\vec{y}(s)$ which has been decomposed according to eq. (3.3). Therefore we can restrict our further discussions to the investigation of the second part $\vec{\tilde{y}}$ in eq. (3.3).

Putting eq. (3.3) into (2.50) and taking into account (3.2a) we obtain the following equation for $\vec{\tilde{y}}$

$$\vec{\tilde{y}}' = (\underline{A} + \delta\underline{A}) \, \vec{\tilde{y}} + \vec{c}_1 + \delta\vec{c} \quad (3.31)$$

where the inhomogeneous part $\vec{c}_o$ has indeed disappeared as required, and where only the vectors $\vec{c}_1$ (due to the fields ΔB_x, ΔB_z, $\Delta\varepsilon$) and $\delta\vec{c}$ (due to the quantum fluctuations) are left over.
Eq. (3.31) can be used to study the influence of small external fields (time-dependent or independent of time) on the particle motion. We shall consider these problems in chapter 6.1 (presence of ΔB_x, ΔB_z, $\Delta\varepsilon$ because of misalignment errors) and in chapter 7 (ΔB_x, ΔB_z, $\Delta\varepsilon$ time-dependent external driving fields; resonance excitation of the synchro- betatron oscillations (27)).
As a preparation for the more general case

$$(\Delta B_x, \Delta B_z, \Delta\varepsilon) \neq (0, 0, 0)$$

we want to discuss the ideal machine at the moment (no misalignment errors or perturbing external fields, $\Delta B_x = \Delta B_z = \Delta\varepsilon = 0$). In this case eq. (3.31) reduces to

$$\vec{\tilde{y}}' = (\underline{A} + \delta\underline{A}) \, \vec{\tilde{y}} + \delta\vec{c} \, . \quad (3.32)$$

Eq. (3.32) describes the "free" synchro- betatron oscillations around the six-dimensional closed orbit. For a first approximation it is also reasonable to neglect the small perturbing terms appearing in the matrix $\delta\underline{A}$ and the vector $\delta\vec{c}$ and only to consider the equations of motion

$$\vec{\tilde{y}}' = \underline{A} \, \vec{\tilde{y}} \quad (3.33)$$

of the "free and undamped" synchro- betatron oscillations. The influence of $\delta\underline{A}$ and $\delta\vec{c}$ can be calculated in perturbation theory showing that $\delta\underline{A}$ generally is introducing a damping and $\delta\vec{c}$ is introducing a stochastic excitation of the synchro- betatron oscillations (see chapter 8 for more details). The solution of the unperturbed equation of motion (3.33) can be expressed in the form

$$\vec{\tilde{y}}(s) = \underline{M}(s, s_o) \, \vec{\tilde{y}}(s_o) \quad (3.34)$$

where the (simple) transfer matrix $\underline{M}(s, s_o)$ is known already from chapter 3.3: $\underline{M}(s, s_o)$ is a submatrix of the enlarged transfer matrix $\hat{\underline{M}}(s, s_o)$ so that we have

$$M_{ik} = \hat{M}_{ik} \quad (i, k = 1, 2, \ldots, 6) .$$

All characteristic features of the synchro- betatron oscillations are contained in the structure of the matrix $\underline{M}$. Therefore our next task will be to study the properties of this transfer matrix.

4. The free and undamped synchro- betatron oscillations.

4.1 Hamiltonian form of the equations of motion

In order to study the structural properties of the transfer matrices we take into account, that the free and undamped synchro- betatron oscillations are the solution of the homogeneous part of the complete equations of motion (2.47). We have already shown in chapter 2.6 that the inhomogeneous differential equations (2.47) can be written in canonical form with the Halmiltonian defined in (2.45a). But this implies, that eq. (3.33), as a special case of (2.47), can also be expressed in canonical form. The corresponding Hamiltonian $\mathcal{H}$ is obtained by neglecting in (2.45a) all terms linear in x, z, σ, p_x, p_z and p_σ, which were responsible for the inhomogeneous parts $\vec{c}_0$ and $\vec{c}_1$ of eq. (2.47) and by renaming the components x, p_x, z, p_z, σ, p_σ of the vector $\vec{y}$ which are called now $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ and which are the components of a vector $\vec{\tilde{y}}$:

$$\mathcal{H} = \frac{1}{2}\{G_1\tilde{x}^2 + G_2\tilde{z}^2 - 2N\cdot\tilde{x}\tilde{z} + (\tilde{p}_x + Q\tilde{z})^2 + (\tilde{p}_z - Q\tilde{x})^2\} - \frac{1}{2}\tilde{\sigma}^2\frac{e\hat{V}}{E_0}\cdot k\cdot\frac{2\pi}{L}\cos\Phi\cdot\sum_\nu \delta(s - s_\nu) - (K_x\,\tilde{x} + K_z\,\tilde{z})\,\tilde{p}_\sigma \quad (4.1)$$

so that $\mathcal{H}$ is just a quadratic polynomial of $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$ and $\tilde{p}_\sigma$.

4.2 Symplectic structure of the transfer matrices

All properties of the transfer matrices relevant to the investigation of the stability of the particle motion can now be obtained from the canonical structure of the equations of motion (3.33). One of these properties playing an important role in our further calculations is the socalled symplectic structure of the matrices $\underline{M}(s, s_0)$ stating that

$$\underline{M}^T(s, s_0)\cdot\underline{S}\cdot\underline{M}(s, s_0) = \underline{S} \quad (4.2a)$$

with $\underline{S}$ given by

$$\underline{S} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (4.2b)$$

Then the following theorem holds:

Theorem 1: The transfer matrices of a canonical, linear and homogeneous system of differential equations are symplectic.

In order to prove this theorem we write down the most general Hamiltonian for linear homogeneous differential equations with the variables

$$q_1 \equiv \tilde{x}\ ,\ q_2 \equiv \tilde{z}\ ,\ q_3 \equiv \tilde{\sigma}\ ,\ p_1 \equiv \tilde{p}_x\ ,\ p_2 \equiv \tilde{p}_z\ ,\ p_3 \equiv \tilde{p}_\sigma$$

$$\mathcal{H} = \frac{1}{2}\sum_{i,k=1}^{3} B_{ik}\, q_i q_k + \frac{1}{2}\sum_{i,k=1}^{3} C_{ik}\, p_i p_k + \sum_{i,k}^{3} D_{ik}\, p_i q_k \quad (4.3a)$$

$$\begin{cases} B_{ik} = B_{ki}\ ; \\ C_{ik} = C_{ki}\ . \end{cases} \quad (4.3b)$$

The corresponding canonical equations now read:

$$\begin{aligned} q_i' &= \frac{\partial\mathcal{H}}{\partial p_i} = \sum_{k=1}^{3} C_{ik}p_k + \sum_{k=1}^{3} D_{ik}q_k \\ p_i' &= -\frac{\partial\mathcal{H}}{\partial q_i} = -\sum_{k=1}^{3} B_{ik}q_k - \sum_{k=1}^{3} D_{ki}p_k \end{aligned} \quad (4.4)$$

or in matrix notation

$$\vec{x}' = \underline{K}\vec{x} \quad (4.5a)$$

with

$$\vec{x}^T = (q_1, q_2, q_3, p_1, p_2, p_3)\ ; \quad (4.5b)$$

$$\underline{K} = \begin{pmatrix} \underline{D} & \underline{C} \\ -\underline{B} & -\underline{D}^T \end{pmatrix}\ ; \quad (4.5c)$$

$$\underline{B} = ((B_{ik}))\ ; \quad (4.5d)$$

$$\underline{C} = ((C_{ik}))\ ; \quad (4.5e)$$

$$\underline{D} = ((D_{ik})) \quad (4.5f)$$

and where, because of (4.3b), the following relationships are valid

$$\underline{B}^T = \underline{B}\ ; \qquad \underline{C}^T = \underline{C}\ . \quad (4.6)$$

From eq. (4.5a) we get

$$\vec{x}(s + \Delta s) = [\underline{1} + \underline{K}(s)\cdot\Delta s\,]\,\vec{x}(s)\ . \quad (4.7)$$

Rewriting this equation in terms of the vector

$$\vec{\tilde{y}}^T = (\tilde{x}, \tilde{p}_x, \tilde{z}, \tilde{p}_z, \tilde{\sigma}, \tilde{p}_\sigma)$$

where the components have been arranged in a different way and using

$$\vec{\tilde{y}} = \underline{P}\,\vec{x} \qquad (4.8)$$

with

$$\underline{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4.8b)$$

we get

$$\vec{\tilde{y}}(s + \Delta s) = \underline{P} \cdot [\underline{1} + \underline{K}(s) \cdot \Delta s]\, \underline{P}^{-1} \vec{\tilde{y}}(s)\,. \qquad (4.9)$$

But this equation implies that the infinitesimal transfer matrix $\underline{M}(s + \Delta s, s)$ can be written in the form

$$\underline{M}(s + \Delta s, s) = \underline{P}\,[\underline{1} + \underline{K}(s) \cdot \Delta s]\, \underline{P}^{-1}. \qquad (4.10)$$

Taking into account the following relationships

$$\underline{P}^{-1} = \underline{P}^T \qquad (4.11a)$$

$$\underline{P}^T\, \underline{S}\, \underline{P} = \underline{J} \quad \text{with} \qquad (4.11b)$$

$$\underline{J} = \begin{pmatrix} \underline{0}_3 & -\underline{1}_3 \\ \underline{1}_3 & \underline{0}_3 \end{pmatrix} \quad \text{and}$$

$$\underline{K}^T\, \underline{J} + \underline{J}\, \underline{K} = \underline{0} \quad \text{(eq. 4.5c, 4.6)} \qquad (4.11c)$$

we find, by putting (4.10) into (4.2a), that the symplectic condition is indeed fulfilled for the infinitesimal transfer matrices:

$$\underline{M}^T(s + \Delta s, s) \cdot \underline{S} \cdot \underline{M}(s + \Delta s, s)$$

$$= \underline{P}\,[\underline{1} + \underline{K}(s)\,\Delta s\,]^T\, \underline{P}^T\, \underline{S}\, \underline{P}\,[\underline{1} + \underline{K}(s)\,\Delta s\,]\,\underline{P}^T$$

$$= \underline{P} \cdot \underline{P}^T \cdot \underline{S} \cdot \underline{P} \cdot \underline{P}^T + \Delta s\, \underline{P}\,[\underline{K}^T\, \underline{J} + \underline{J}\, \underline{K}\,] \cdot \underline{P} \qquad \text{(eq. (4.11a, b))}$$

$$= \underline{S} \qquad \text{(eq. (4.11c))}\ . \qquad (4.12)$$

But the group character of the symplectic condition then implies, that "finite" transfer matrices are also symplectic; q.e.d.

One consequence of theorem 1 is, that all transfer matrices calculated in chapter 3.3 satisfy the symplectic condition (4.2).

In order to complete these considerations we want to prove the converse of theorem 1:

Theorem 2: If the transfer matrices of a linear and homogeneous system of differential equations of first order are symplectic one can write the differential equations in canonical form.

Proof: Given a linear and homogeneous system of differential equations

$$\vec{\tilde{y}}' = \underline{A}(s)\, \vec{\tilde{y}} \qquad (4.13a)$$

$$\vec{\tilde{y}}^T \equiv (q_1, p_1, q_2, p_2, q_3, p_3) \qquad (4.13b)$$

and the corresponding transfer matrix $\underline{M}(s + \ell, s)$ satisfying the symplectic condition (4.2a). This implies that the infinitesimal transfer matrix

$$\underline{M}(s + \Delta s, s) = \underline{1} + \Delta s\, \underline{A}(s)$$

is symplectic

$$0 = [\underline{1} + \Delta s\, \underline{A}(s)\,]^T\, \underline{S}\, [\underline{1} + \Delta s\, \underline{A}(s)] - \underline{S}$$

$$= \Delta s\, [\underline{A}^T(s) \cdot \underline{S} + \underline{S} \cdot \underline{A}(s)]$$

so that the matrix of the coefficients $\underline{A}(s)$ must fulfill the condition

$$\underline{A}^T(s) \cdot \underline{S} + \underline{S} \cdot \underline{A}(s) = 0. \qquad (4.14)$$

Transforming the vector $\vec{\tilde{y}}$ to the vector $\vec{x}$

$$\vec{x}^T \equiv (q_1, q_2, q_3, p_1, p_2, p_3) \qquad (4.15)$$

by applying the permutation matrix $\underline{P}$ defined by (4.8b)

$$\vec{\tilde{y}} = \underline{P}\, \vec{x}\ , \quad \vec{x} = \underline{P}^T\, \vec{\tilde{y}} \qquad (4.16)$$

we can write (4.13a) in the form

$$\vec{x}' = \underline{K}\, \vec{x} \qquad (4.17a)$$

with

$$\underline{K} = \underline{P}^T\, \underline{A}\, \underline{P}\,. \qquad (4.17b)$$

Eqs. (4.17) and (4.13a) are equivalent. Therefore it is enough to show, that one can find a Hamiltonian of the form of (4.3a, b) such that the corresponding canonical equations (4.4), also written as

$$\vec{x}' = \begin{pmatrix} \underline{D} & \underline{C} \\ -\underline{B} & -\underline{D}^T \end{pmatrix} \vec{x} \qquad (4.18)$$

are identical with eq. (4.17a).
Therefore we rewrite the relationship (4.14) for the matrix $\underline{A}(s)$ in terms of the matrix $\underline{K}(s)$. We then obtain

$$\underline{K}^T \underline{J} + \underline{J}\,\underline{K} = \underline{0}$$

where $\underline{J}$ is given by (4.11b). By putting

$$\underline{K} = \begin{pmatrix} \underline{K}_1 & \underline{K}_2 \\ \underline{K}_3 & \underline{K}_4 \end{pmatrix}$$

($\underline{K}_\gamma \equiv 3 \times 3$ submatrices)

we get

$$\underline{K}_2^T = \underline{K}_2$$
$$\underline{K}_3^T = \underline{K}_3$$
$$\underline{K}_4^T = -\underline{K}_1$$

which means that eq. (4.17a) is of the form of (4.18) if one makes the following replacements

$$\underline{K}_1 \longrightarrow \underline{D}$$
$$\underline{K}_2 \longrightarrow \underline{C}$$
$$\underline{K}_3 \longrightarrow -\underline{B}$$

and in addition the relationships (4.3) are satisfied ; q.e.d.
The theorems 1 and 2 together state, that the canonical form of the equation of motion and the symplectic structure of the transfer matrices of linear and homogeneous differential equations are equivalent terms.
Proving these two theorems we have restricted ourselves to the sixdimensional case (three spatial coordinates q_1, q_2, q_3 and three momentum variables p_1, p_2, p_3). However it is not difficult to extend these considerations to the general case of an arbitrary number of variables.
In Appendix I we shall derive a necessary and sufficient condition for the canonical structure of general (linear or nonlinear) systems of differential equations of first order, implying the theorems of this chapters as special cases.

4.3 Conclusions from the symplectic structure of the transfer matrices

4.3.1 The eigenvalue spectrum of the revolution matrix $\underline{M}(s_o + L, s_o)$

The symplectic condition (4.2) for the transfer matrix implies that a constant of the motion can be formed of two arbitrary solutions $\vec{\tilde{y}}_1(s)$ and $\vec{\tilde{y}}_2(s)$ of the equations of motion for the free and undamped synchro- betatron oscillations (3.33). This invariant, also called "Lagrange-invariant" is defined by

$$W[\vec{\tilde{y}}_1(s), \vec{\tilde{y}}_2(s)] = \vec{\tilde{y}}_2^T(s) \cdot \underline{S} \cdot \vec{\tilde{y}}_1(s)$$

with

$$\begin{aligned} W[\vec{\tilde{y}}_1(s), \vec{\tilde{y}}_2(s)] &= [\underline{M}(s, s_o)\, \vec{\tilde{y}}_2(s_o)]^T \cdot \underline{S} \cdot [\underline{M}(s, s_o) \cdot \vec{\tilde{y}}_1(s_o)] \\ &= \vec{\tilde{y}}_2^T(s_o) \cdot \underline{M}^T(s, s_o) \cdot \underline{S} \cdot \underline{M}(s, s_o) \cdot \vec{\tilde{y}}_1(s_o) \\ &= \vec{\tilde{y}}_2^T(s_o) \cdot \underline{S} \cdot \vec{\tilde{y}}_1(s_o) \\ &= W[\vec{\tilde{y}}_1(s_o), \vec{\tilde{y}}_2(s_o)] = \text{const.} \end{aligned} \qquad (4.19)$$

Using this invariant we can study the eigenvalue spectrum of the revolution matrix $\underline{M}(s_o + L, s_o)$:

$$\underline{M}(s_o + L, s_o)\, \vec{v}_\mu(s_o) = \lambda_\mu\, \vec{v}_\mu(s_o) \; ; \qquad (4.20)$$
$$(\mu = 1, 2, \ldots, 6)\,.$$

And this spectrum of the eigenvalues λ_μ ($\mu = 1, 2, \ldots, 6$) allows one to investigate the stability of the coupled synchro- betatron oscillations.
We make this investigation in several steps ((4), (5), (7)):
1) We form the Lagrange-invariant with two arbitrary eigenvectors $\vec{v}_\mu(s_o)$ and $\vec{v}_\nu(s_o)$ of $\underline{M} \equiv \underline{M}(s_o + L, s_o)$ and we obtain

$$\begin{aligned} W[\vec{v}_\mu(s_o), \vec{v}_\nu(s_o)] &= W[\underline{M}\, \vec{v}_\mu(s_o), \underline{M}\, \vec{v}_\nu(s_o)] \\ &= W[\lambda_\mu\, \vec{v}_\mu(s_o), \lambda_\nu\, \vec{v}_\nu(s_o)] \\ &= \lambda_\mu\, \lambda_\nu \cdot W[\vec{v}_\mu(s_o), \vec{v}_\nu(s_o)] \end{aligned}$$

which implies

$$\lambda_\mu \cdot \lambda_\nu \neq 1 \Longrightarrow W[\vec{v}_\mu, \vec{v}_\nu] \equiv \vec{v}_\nu^T(s_o)\, \underline{S}\, \vec{v}_\mu(s_o) = 0 \; ;$$
$$\vec{v}_\nu^T(s_o) \cdot \underline{S}\; \vec{v}_\mu(s_o) \neq 0 \Longrightarrow \lambda_\mu \cdot \lambda_\nu = 1 \qquad (4.21)$$

so that the eigenvectors of $\underline{M}$ can be divided into 3 groups

$$(\vec{v}_k\,,\, \vec{v}_{-k}) \qquad k = I, II, III$$

with the properites

$$\underline{M}\, \vec{v}_k = \lambda_k\, \vec{v}_k \; ; \; \underline{M}\, \vec{v}_{-k} = \lambda_{-k}\, \vec{v}_{-k} \; ; \; \lambda_k \cdot \lambda_{-k} = 1 \qquad (4.22a)$$

$$\begin{cases} \vec{v}_{-k}^T \cdot \underline{S} \cdot \vec{v}_k = - \vec{v}_k^T \cdot \underline{S} \cdot \vec{v}_{-k} \neq 0 \; ; \\ \vec{v}_\mu^T \cdot \underline{S} \cdot \vec{v}_\nu = 0 \quad \text{otherwise} \; ; \end{cases} \tag{4.22b}$$

(k = I, II, III).

For our further considerations we put

$$\begin{cases} \lambda_k = e^{-i\,2\pi\,Q_k} \; ; \\ \lambda_{-k} = e^{-i\,2\pi\,Q_{-k}} \; ; \end{cases} \tag{4.23}$$

(k = I, II, III) .

Using (4.22a) we get

$$Q_{-k} = - Q_k \tag{4.24}$$

where the quantity Q_k can be a real or complex number.

2) Eqs. (4.22a) or (4.23) imply that the eigenvalues of the matrix $\underline{M}(s_o + L, s_o)$ always appear in reciprocal pairs

$$(\lambda_k \, , \lambda_{-k} = {}^1/\, \lambda_k)$$

(k = I, II, III)

If λ is an eigenvalue, λ^* is also an eigenvalue because $\underline{M}(s_o + L, s_o)$ is a real matrix.

With these statements we find the following possibilities for the eigenvalue spectrum of the matrix $\underline{M}(s_o + L, s_o)$:

a) all of the six eigenvalues are complex and lie on the unit circle in the complex plane

$$|\lambda_k| = |\lambda_{-k}| = 1 \tag{4.25}$$

and we have

$$Q_k = \text{real} \; ; \tag{4.26a}$$

$$\lambda_{-k} = \lambda_k^* \; ; \tag{4.26b}$$

$$\vec{v}_{-k} = \vec{v}_k^* \; ; \tag{4.26c}$$

b) one reciprocal pair is real, the remaining eigenvalues lie on the unit circle

$$\begin{aligned} &\lambda_I = \lambda_I^* \; ; \; \lambda_{-I} = \lambda_{-I}^* \;\; ; \; \lambda_{-I} = {}^1/\, \lambda_I \;\; ; \\ &\lambda_{-II} = \lambda_{II}^* \;\; ; \;\; |\lambda_{II}| = |\lambda_{-II}| = 1 \; ; \\ &\lambda_{-III} = \lambda_{III}^* \;\; ; \;\; |\lambda_{III}| = |\lambda_{-III}| = 1 \; ; \end{aligned} \tag{4.27}$$

c) two reciprocal pairs are real and the third pair lies on the unit circle

$$\begin{aligned} &\lambda_I = \lambda_I^* \;\; ; \;\; \lambda_{-I} = \lambda_{-I}^* \;\; ; \;\; \lambda_{-I} = {}^1/\, \lambda_I \;\; ; \\ &\lambda_{II} = \lambda_{II}^* \;\; ; \lambda_{-II} = \lambda_{-II}^* \;\; ; \lambda_{-II} = {}^1/\, \lambda_{II} \; ; \\ &\lambda_{-III} = \lambda_{III}^* \;\; ; \;\; |\lambda_{III}| = |\lambda_{-III}| = 1 \; ; \end{aligned} \tag{4.28}$$

d) all reciprocal pairs are real

$$\lambda_k = \lambda_k^* \;\; ; \lambda_{-k} = \lambda_{-k}^* \;\; ; \lambda_{-k} = {}^1/\, \lambda_k \; ; \tag{4.29}$$

(k = I, II, III) ;

e) one eigenvalue, for example λ_I, is complex but does not lie on the unit circle

$$|\lambda_I| \neq 1 \;\; ; \;\; \lambda_I \neq \lambda_I^* \; . \tag{4.30}$$

Then the following condition must be valid

$$\lambda_{-I} = {}^1/\, \lambda_I \tag{4.31}$$

and (with an appropriate choice of the eigenvalues)

$$\begin{aligned} &\lambda_{II} = \lambda_I^* \;\; ; \\ &\lambda_{-II} = {}^1/\, \lambda_I^* \end{aligned} \tag{4.32a}$$

or

$$\begin{aligned} &\lambda_{II} = {}^1/\, \lambda_I^* \; ; \\ &\lambda_{-II} = \lambda_I^* \; . \end{aligned} \tag{4.32b}$$

The third, remaining pair must lie on the unit circle or on the real axis.

Later on we shall show that the particle motion is only stable in case a).

3) If we define

$$\vec{v}_\mu(s) = \underline{M}(s, s_o)\, \vec{v}_\mu(s_o) \;\; ; \tag{4.33}$$

then $\vec{v}_\mu(s)$ is an eigenvector of the revolution matrix

$\underline{M}(s + L, s)$ belonging to the eigenvalue λ_μ

$$\underline{M}(s + L, s)\ \vec{v}_\mu(s) = \lambda_\mu\ \vec{v}_\mu(s)\ . \tag{4.34}$$

Proof:

$$\begin{aligned}\underline{M}(s + L, s)\ \vec{v}_\mu(s) &= \underline{M}(s + L, s)\ \underline{M}(s, s_o)\ \vec{v}_\mu(s_o)\\ &= \underline{M}(s + L, s_o + L)\ \underline{M}(s_o + L, s_o)\ \vec{v}_\mu(s_o)\\ &= \underline{M}(s, s_o)\cdot\underline{M}(s_o + L, s_o)\ \vec{v}_\mu(s_o)\\ &\quad(\text{because } \underline{M}(s + L, s_o + L) = \underline{M}(s, s_o))\\ &= \lambda_\mu\ \underline{M}(s, s_o)\cdot\vec{v}_\mu(s_o) \qquad (\text{eq. } (4.20))\\ &= \lambda_\mu\ \vec{v}_\mu(s) \qquad (\text{eq. } (4.33))\end{aligned}$$

q.e.d.

The eigenvector $\vec{v}_\mu(s)$ has the same eigenvalue as $\vec{v}_\mu(s_o)$ which means that the eigenvalue is independent of s .

4) Defining

$$\vec{v}_\mu(s) = \vec{u}_\mu(s)\ e^{-i\ 2\pi\ Q_\mu \frac{s}{L}} \tag{4.35a}$$

we find

$$\vec{u}_\mu(s + L) = \vec{u}_\mu(s) \tag{4.35b}$$

Proof: Putting (4.35a) into (4.34) and using (4.23) we obtain

$$\vec{u}_\mu(s + L)\ e^{-i\ 2\pi\ Q_\mu \frac{s + L}{L}} = e^{-i\ 2\pi\ Q_\mu}\cdot\vec{u}_\mu(s)\cdot e^{-i\ 2\pi\ Q_\mu \frac{s}{L}}$$

Dividing both sides by

$$e^{-i\ 2\pi\ Q_\mu \frac{s + L}{L}} = e^{-i\ 2\pi\ Q_\mu}\ e^{-i\ 2\pi\ Q_\mu \frac{s}{L}}$$

we get (4.35b). q.e.d.

Eq. (4.35) is called Floquet-theorem. It states: the vectors $\vec{v}_\mu(s)$ are special solutions of the equations of motion (3.33) and they can be written as the product of a periodic function $\vec{u}_\mu(s)$ and a (generally aperiodic) harmonic function

$$e^{-i\ 2\pi\ Q_\mu \frac{s}{L}}$$

5) The general solution of the equations of motion is a linear combination of the special solutions (4.35a) and can be written in the form

$$\vec{y}(s) = \sum_{k=I,II,III} \{A_k\ \vec{u}_k(s)\ e^{-i\ 2\pi\ Q_k \frac{s}{L}} + A_{-k}\ \vec{u}_{-k}(s)\ e^{-i\ 2\pi\ Q_{-k} \frac{s}{L}}\} \tag{4.36}$$

This equation implies that the amplitudes of the synchro- betatron oscillations remain bounded (stable motion) only if the quantities Q_k are real numbers, which also means, that the eigenvalues lie on the unit circle, as we have mentioned already:

$$|\lambda_k| = |\lambda_{-k}| = 1 \tag{4.37}$$

$(k = I, II, III)$ (stability criterion).

If at least one of the exponents Q_k is complex, Q_k or Q_{-k} has a positive imaginary part. In this case the components of $\vec{y}(s)$ grow exponentially and the particle motion becomes unstable.

6) For the following we always assume that the stability criterion (4.37) is satisfied.

Then it follows from (4.26b)

$$\vec{v}_{-k} = \vec{v}_k^{\,*}$$

$(k = I, II, III)$

and eq. (4.22b) reduces to $(\vec{v}^{\,+} \equiv (\vec{v}^{\,*})^T)$

$$\vec{v}_k^{\,+}(s_o)\cdot\underline{S}\ \vec{v}_k(s_o) = -\ \vec{v}_{-k}^{\,+}(s_o)\cdot\underline{S}\cdot\vec{v}_{-k}(s_o) \neq 0 \tag{4.38a}$$

$$\vec{v}_\mu^{\,+}(s_o)\cdot\underline{S}\ \vec{v}_\nu(s_o) = 0 \quad \text{otherwise.} \tag{4.38b}$$

The terms

$$\vec{v}_\mu^{\,+}(s_o)\cdot\underline{S}\ \vec{v}_\mu(s_o)$$

appearing in (4.38a) are purely imaginary:

$$[\vec{v}_\mu^{\,+}(s_o)\cdot\underline{S}\ \vec{v}_\mu(s_o)]^+ = \vec{v}_\mu^{\,+}(s_o)\cdot\underline{S}^+\ \vec{v}_\mu(s_o) = -\ [\vec{v}_\mu^{\,+}(s_o)\cdot\underline{S}\ \vec{v}_\mu(s_o)]$$

because $\underline{S}^+ = -\ \underline{S}$

so that the following normalizing conditions can be used for the vectors $\vec{v}_k(s_o)$ and $\vec{v}_{-k}(s_o)$ $(k = I, II, III)$

$$\vec{v}_k^{\,+}(s_o)\cdot\underline{S}\cdot\vec{v}_k(s_o) = -\ \vec{v}_{-k}^{\,+}(s_o)\cdot\underline{S}\cdot\vec{v}_k(s_o) = i \tag{4.39}$$

$(k = I, II, III)$.

The validity of the symplectic condition (4.2) then implies that the eigenvectors $\vec{v}_k(s)$ and $\vec{v}_{-k}(s)$ $(k = I, II, III)$ at the position s also satisfy the conditions (4.38b) and (4.39):

$$\begin{cases} \vec{v}_k^{+}(s)\cdot\underline{S}\cdot\vec{v}_k(s) = -\vec{v}_{-k}^{+}(s)\cdot\underline{S}\cdot\vec{v}_{-k}(s) = i\,; \\ \vec{v}_\mu^{+}(s)\cdot\underline{S}\cdot\vec{v}_\nu(s) = 0 \quad \text{otherwise}. \end{cases} \tag{4.40}$$

4.3.2 Special case of the completely decoupled machine

4.3.2.1 Properties of the revolution matrix

For the limiting case of a vanishing coupling between the synchro- betatron oscillations we want to calculate the eigenvalues and eigenvectors explicitly. There are two reasons why it is interesting to study this special case:

1) to rediscover the notations and results of the theory of Courant-Snyder;

2) to investigate the influence of the coupling mechanisms on the particle motion.

If there is complete decoupling the coefficient matrix of the equations of motion (3.33) takes the form

$$\underline{A} = \begin{pmatrix} \underline{a}_x & \underline{0}_2 & \underline{0}_2 \\ \underline{0}_2 & \underline{a}_z & \underline{0}_2 \\ \underline{0}_2 & \underline{0}_2 & \underline{a}_\sigma \end{pmatrix} \tag{4.41}$$

so that the equations of motion split into three separate systems $(y = x, z, \sigma)$

$$\frac{d}{ds}\begin{pmatrix} y \\ p_y \end{pmatrix} = \underline{a}_y \begin{pmatrix} y \\ p_y \end{pmatrix}. \tag{4.42}$$

The revolution matrix $\underline{M}(s + L, s)$ can then be written in the form

$$\underline{M}(s + L, s) = \begin{pmatrix} \underline{m}_x(s + L, s) & \underline{0}_2 & \underline{0}_2 \\ \underline{0}_2 & \underline{m}_z(s + L, s) & \underline{0}_2 \\ \underline{0}_2 & \underline{0}_2 & \underline{m}_\sigma(s + L, s) \end{pmatrix} \tag{4.43}$$

where the 2x2 submatrices $\underline{m}_x$, $\underline{m}_z$ and $\underline{m}_\sigma$ are just the transfer matrices of eq. (4.42).

The symplectic condition (4.2) now reads

$$\underline{m}_y^T\cdot\underline{S}_2\cdot\underline{m}_y = \underline{S}_2 \tag{4.44a}$$

with $\underline{S}_2$ defined by

$$\underline{S}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.44b}$$

or

$$\det(\underline{m}_y) = 1. \tag{4.45}$$

Generally the corresponding submatrices

$$\underline{d}_x = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$\underline{d}_z = \begin{pmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{pmatrix}$$

$$\underline{d}_\sigma = \begin{pmatrix} M_{55} & M_{56} \\ M_{65} & M_{66} \end{pmatrix}$$

of the revolution matrix for the coupled synchro- betatron oscillation have determinants differing from 1.

So we can consider the difference

$$\det(\underline{d}_y) - \det(\underline{m}_y) = \det(\underline{d}_y) - 1$$

of the determinant $\det(\underline{d}_y)$ from the value 1 as a measure for the coupling strength of the betatron and synchrotron oscillations at the position s.

Applying (4.44) to the infinitesimal transfer matrix

$$\underline{m}_y(s + \Delta s, s) = 1 + \Delta s\cdot\underline{a}_y$$

we find the following condition for the coefficient matrix $\underline{a}_y(s)$ of eq. (4.42)

$$\underline{a}_y^T\,\underline{S}_2 + \underline{S}_2\,\underline{a}_y = \underline{0}_2 \Leftrightarrow \mathrm{Sp}\,\underline{a}_y = 0$$

which means that $\underline{a}_y$ must be of the form

$$\underline{a}_y = \begin{pmatrix} R & F \\ -G & -R \end{pmatrix}. \tag{4.46}$$

According to Courant-Snyder we can write for the revolution matrix $\underline{m}_y(s + L, s)$ (4)

$$\underline{m}_y(s + L, s) = \begin{pmatrix} \cos 2\pi Q_y + \alpha_y(s)\sin 2\pi Q_y & \beta_y(s)\sin 2\pi Q_y \\ -\gamma_y(s)\sin 2\pi Q_y & \cos 2\pi Q_y - \alpha_y(s)\sin 2\pi Q_y \end{pmatrix} \quad (4.47a)$$

with

$$\beta_y \gamma_y - \alpha_y^2 = 1 \quad (4.47b)$$

where we have used (4.45).

In addition we require

$$\beta_y(s) \geq 0. \quad (4.47c)$$

Knowing the matrix elements of $\underline{m} = \underline{m}_y(s + L, s)$

$$\underline{m} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

we can determine Q, α, β, and γ according to

$$\cos 2\pi Q = \frac{1}{2} \mathrm{Sp}\ \underline{m} \ ;$$

$$\sin 2\pi Q \begin{cases} > 0, \text{ if } m_{12} > 0 \ ; \\ < 0, \text{ if } m_{12} < 0 \ ; \end{cases}$$

$$\alpha = \frac{1}{2\sin 2\pi Q} (m_{11} - m_{22}) \ ;$$

$$\beta = \frac{m_{12}}{\sin 2\pi Q} \ ;$$

$$\gamma = - \frac{m_{21}}{\sin 2\pi Q}$$

(stability considerations do not allow Q to be an integer; see chapter 6).

Using the representation (4.47) of the transfer matrix $\underline{m}_y(s + L, s)$ we can calculate the normalized eigenvectors (see eq. (4.39)) of the revolution matrix (4.43)

$$\begin{aligned} \vec{v}_I^T &= (\vec{w}_x^T, o, o, o, o) \ ; \quad \vec{v}_{-I} = \vec{v}_I^* \\ \vec{v}_{II}^T &= (o, o, \vec{w}_z^T, o, o) \ ; \quad \vec{v}_{-II} = \vec{v}_{II}^* \\ \vec{v}_{III}^T &= (o, o, o, o, \vec{w}_\sigma^T) \ ; \quad \vec{v}_{-III} = \vec{v}_{III}^* \end{aligned} \quad (4.48)$$

with the eigenvalues

$$\begin{aligned} \lambda_I &= e^{-i\, 2\pi\, Q_x} \\ \lambda_{II} &= e^{-i\, 2\pi\, Q_z} \\ \lambda_{III} &= e^{-i\, 2\pi\, Q_\sigma} \end{aligned} \quad (4.49)$$

and where the vector $\vec{w}_y$ (y = x, z, σ) is given by

$$\vec{w}_y(s) = \frac{1}{\sqrt{2\beta_y(s)}} \begin{pmatrix} \beta_y(s) \\ -[\alpha_y(s) + i] \end{pmatrix} e^{-i\Psi_y(s)} \ . \quad (4.50)$$

Comparing (4.49) and (4.23) we can make the following identifications for the decoupled case

$$Q_I \longleftrightarrow Q_x \ ;$$

$$Q_{II} \longleftrightarrow Q_z \ ;$$

$$Q_{III} \longleftrightarrow Q_\sigma \ .$$

The stability condition (4.37) now reads

$$Q_x, Q_z, Q_\sigma \quad \text{real}$$

or using (4.47a)

$$-2 < \mathrm{Sp}\ \underline{m}_y < +2 \ . \quad (4.51)$$

Taking into account (4.33), namely

$$\vec{v}(s + \Delta s) = \underline{M}(s + \Delta s, s)\ \vec{v}(s)$$

or equivalently

$$\begin{aligned} \vec{w}(s + \Delta s) &= \underline{m}(s + \Delta s, s)\ \vec{w}(s) \\ &= (\underline{1} + \Delta s \cdot \underline{a})\ \vec{w}(s) \end{aligned}$$

and

$$\frac{d}{ds}\vec{w}(s) = \underline{a}\cdot\vec{w}(s)$$

the following differential equations can be obtained for α, β and ψ:

$$\psi' = \frac{F}{\beta}\;; \qquad (4.52a)$$

$$\beta' = 2\cdot(R\beta - F\alpha)\;; \qquad (4.52b)$$

$$G\beta^2 - \alpha'\beta - F\alpha^2 = F \qquad (4.52c)$$

where we made use of (4.46).

For the case of decoupled betatron oscillations with

$$\underline{a} = \begin{pmatrix} 0 & 1 \\ -G & 0 \end{pmatrix} \Longrightarrow \quad R = o,\; F = 1$$

(compare with eq. (2.48e))

one finds for example the following relationships

$$\psi' = \frac{1}{\beta}$$

$$\beta' = -2\alpha$$

$$G\beta^2 + \frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 = 1$$

which have been derived already by Courant and Snyder. (Another special case, namely the decoupled synchrotron oscillations, will be investigated in Appendix III).

4.3.2.2 Phase-plane ellipses

Now we want to show that the functions $\alpha(s)$, $\beta(s)$, $\gamma(s)$ and $\psi(s)$ describe the focussing properties of the decoupled machine.

Therefore we define at the position $s = s_o$ an ellipse in the phase-plane which can be written in the form ((7), (16))

$$\begin{pmatrix} y(s_o,\,\delta) \\ p_y(s_o,\,\delta) \end{pmatrix} = \sqrt{\frac{\varepsilon_y}{2}}\{\,\vec{w}_y(s_o)\,e^{i\delta} + \vec{w}^*_y(s_o)\,e^{-i\delta}\} \qquad (4.53a)$$

or if we put

$$\vec{w}_y = \vec{c}_1^{(y)} - i\,\vec{c}_2^{(y)}$$

(4.53) is given by

$$\begin{pmatrix} y(s_o,\,\delta) \\ p_y(s_o,\,\delta) \end{pmatrix} = \sqrt{2\,\varepsilon_y}\;\{\,\vec{c}_1^{(y)}(s_o)\cdot\cos\delta + \vec{c}_2^{(y)}(s_o)\cdot\sin\delta\,\} \qquad (4.53b)$$

$$(o \leq \delta < 2\pi).$$

This ellipse in the phase-plane which is assumed to describe an ensemble of particles is generated by the two linearly independent vectors $\vec{c}_1(s_o)$ and $\vec{c}_2(s_o)$.

During the motion of the particles the ellipse (4.53) is transformed into another phase-plane ellipse of the form

$$\begin{pmatrix} y(s,\,\delta) \\ p_y(s,\,\delta) \end{pmatrix} = \sqrt{\frac{\varepsilon_y}{2}}\cdot\{\,\vec{w}_y(s)\,e^{i\delta} + \vec{w}^*_y(s)\,e^{-i\delta}\,\} \qquad (4.54a)$$

or

$$\begin{pmatrix} y(s,\,\delta) \\ p_y(s,\,\delta) \end{pmatrix} = \sqrt{2\varepsilon_y}\,\{\vec{c}_1^{(y)}(s)\cdot\cos\delta + \vec{c}_2^{(y)}(s)\cdot\sin\delta\,\} \qquad (4.54b)$$

where the vectors $\vec{w}_y(s)$, $\vec{w}^*_y(s)$ and $\vec{c}_1^{(y)}(s)$, $\vec{c}_2^{(y)}(s)$ are defined by

$$\vec{w}_y(s) = \underline{m}_y(s,\,s_o)\;\vec{w}_y(s_o)$$
$$\vec{w}^*_y(s) = \underline{m}_y(s,\,s_o)\;\vec{w}^*_y(s_o) \qquad (4.55a)$$

and

$$\vec{c}_1^{(y)}(s) = \underline{m}_y(s,\,s_o)\;\vec{c}_1^{(y)}(s_o)$$
$$\vec{c}_2^{(y)}(s) = \underline{m}_y(s,\,s_o)\;\vec{c}_2^{(y)}(s_o)\,. \qquad (4.55b)$$

Using the relationship

$$\underline{m}_y(s_o + L,\,s_o)\;\vec{w}_y(s_o) = e^{-i\,2\pi\,Q_y}\cdot\vec{w}_y(s_o) \qquad (4.56)$$

one finds that (4.53) transforms into itself after one revolution.

The "generating vectors" $\vec{c}_1(s)$ and $\vec{c}_2(s)$ represent two special particle trajectories (comp. with eq. (4.55b)). In this way these two trajectories determine the complete motion of the particles defined by the phase-plane ellipse (4.53).

Because of (4.50) and (4.55b) we have

$$\vec{c}_1(s) = \begin{pmatrix} \sqrt{\frac{\beta(s)}{2}} \cos\psi(s) \\ -\frac{\alpha(s)}{\sqrt{2\beta(s)}} \cos\psi(s) - \frac{1}{\sqrt{2\beta(s)}} \sin\psi(s) \end{pmatrix} ; \tag{4.57a}$$

$$\vec{c}_2(s) = \begin{pmatrix} \sqrt{\frac{\beta(s)}{2}} \sin\psi(s) \\ -\frac{\alpha(s)}{\sqrt{2\beta(s)}} \sin\psi(s) + \frac{1}{\sqrt{2\beta(s)}} \cos\psi(s) \end{pmatrix} \tag{4.57b}$$

which implies that

$$\beta(s) = 2\,[c_{11}^2(s) + c_{21}^2(s)] \tag{4.58a}$$

$$\gamma(s) = 2\,[c_{12}^2(s) + c_{22}^2(s)] \tag{4.58b}$$

$$\alpha(s) = -2\,[c_{11}(s)\cdot c_{12}(s) + c_{21}(s)\cdot c_{22}(s)] \tag{4.58c}$$

with

$$(c_{11},\, c_{12}) = \vec{c}_1^{\,T}$$

$$(c_{21},\, c_{22}) = \vec{c}_2^{\,T}$$

so that the determination of the particle trajectories $\vec{c}_1(s)$ and $\vec{c}_2(s)$ allows to calculate the functions $\alpha(s)$, $\beta(s)$ and $\gamma(s)$.
Knowing $\beta(s)$ we can determine the remaining function $\psi(s)$ with the help of

$$\psi(s) = \psi(s_o) + \int_{s_o}^{s} \frac{F(s')}{\beta(s')}\, ds' \tag{4.59}$$

which follows from (4.52a).
Eliminating the parameter δ in (4.54b) and taking into account (4.57) we can also find the following expression for the phase-plane ellipse (4.54)

$$\gamma_y \cdot y^2 + \beta_y \cdot p_y^2 + 2\alpha_y \cdot y p_y = \varepsilon_y . \tag{4.60}$$

Fig. 1 shows a plot of this ellipse and it illustrates the significance of the quantities α, β and γ.

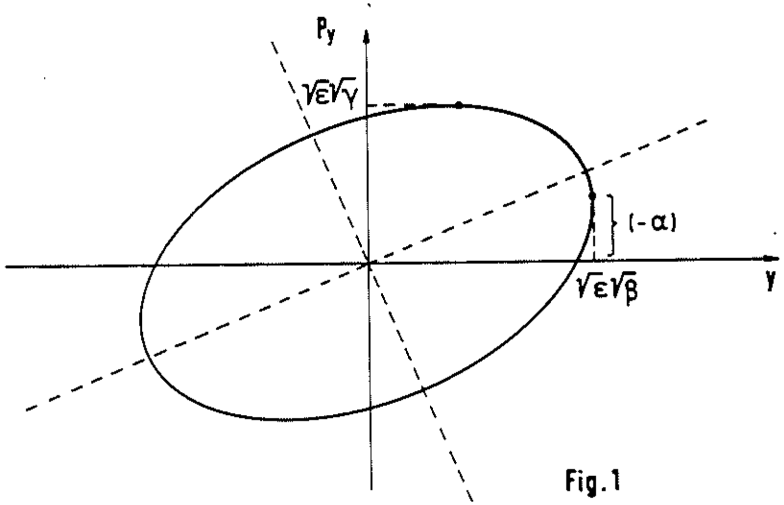

Fig. 1

According to fig. 1 the maximum extension of the ensemble of particles described by the phase-plane ellipse (4.54) or (4.60) at position s is given by

$$E_y(s) = \sqrt{\varepsilon_y}\;\sqrt{\beta_y(s)} . \tag{4.61}$$

E is called beam envelope.
The area of the ellipse (4.60) is given by

$$J = \pi\varepsilon . \tag{4.62}$$

Since ε is independent of s, J is a constant of the motion. Eq. (4.60) and (4.61) now imply that once the area of the phase-plane ellipse is given the maximum amplitude of the oscillation of the particle $E_y(s)$ becomes the smaller, the smaller $\beta_y(s)$ is. The maximum of the function $\beta_y(s)$ around the ring

$$\varkappa = \max_s \{\beta_y(s)\} \tag{4.63}$$

can therefore be considered as a measure for the focussing strength of the lens system.
Since the phase-plane ellipse of fig. 1 transforms into itself after one revolution the functions α, β and γ must be periodic:

$$\alpha(s_o + L) = \alpha(s_o)\,;$$
$$\beta(s_o + L) = \beta(s_o)\,; \qquad (4.64)$$
$$\gamma(s_o + L) = \gamma(s_o)\,.$$

Using (4.50) and (4.64) it follows from (4.56)

$$\psi(s_o + L) - \psi(s_o) = 2\pi\, Q\,. \qquad (4.65)$$

Finally we want to mention that because of (4.50) and (4.54a) the general solution of the equation of motion (4.42) can also be written in the form

$$y(s) = \sqrt{\varepsilon_y}\,\sqrt{\beta_y(s)}\,\cos\,[\psi_y(s) - \delta] \qquad (4.66)$$

where ε_y and δ are constants of integration and where $\beta_y(s)$ and $\psi_y(s)$ are determined by (4.58a) and (4.59). β is called amplitude function and ψ is called phase function. This equation implies that once the "emittance" ε_y is given the maximum amplitude of the oscillation y(s) is given by the beam envelope $E_y(s)$ which has been defined in eq. (4.61).
A similar expression with another, generally different constant $\tilde{\delta}$ is obtained for the momentum p_y

$$p_y(s) = \sqrt{\varepsilon_y}\,\sqrt{\gamma_y(s)}\cdot\cos(\psi_y(s) - \tilde{\delta}) \qquad (4.67)$$

from which one gets

$$|p_y| \le \sqrt{\varepsilon_y}\,\sqrt{\gamma_y(s)}$$

so that the function

$$A_y(s) = \sqrt{\varepsilon_y}\,\sqrt{\gamma_y(s)}$$

could be called "momentum envelope" (see fig.1).

Appendix I

<u>A necessary and sufficient condition for the canonical structure of the equations of motion (symplecticity of the Jacobian matrix).</u>

Consider a mechanical system with n degrees of freedom described by the 2 n variables

$$(q_1, p_1, q_2, p_2 \ldots\ldots q_n, p_n) \equiv (x_1, x_2 \ldots\ldots x_{2n-1}, x_{2n}) \qquad (I.1)$$

$$x_i = x_i(s) \qquad (I.2)$$

(s = motion variable).

Using the Jacobian matrix

$$\underline{J}\,(s+\ell, s) = ((J_{ik}\,(s+\ell, s))),$$
$$J_{ik}\,(s+\ell, s) := \frac{\partial x_i\,(s+\ell)}{\partial x_k\,(s)} \qquad (I.3)$$

we want to show that one can find a necessary and sufficient condition for the canonical structure of the corresponding equations of motion.

It follows from the chain rule

$$\frac{\partial x_i\,(s'')}{\partial x_k\,(s)} = \frac{\partial x_i\,(s'')}{\partial x_m(s')}\cdot\frac{\partial x_m\,(s')}{\partial x_k\,(s)}$$

or

$$J_{ik}(s'', s) = J_{im}(s'', s')\cdot J_{mk}(s', s)$$

that the Jacobian matrix obeys the same decomposition law as the transfer matrix

$$\underline{J}\,(s'', s) = \underline{J}\,(s'', s')\cdot\underline{J}\,(s', s). \qquad (I.4)$$

For linear equations of motion the Jacobian matrices are equivalent to the transfer matrices :

$$x_i\,(s+\ell) = M_{ik}\cdot x_k\,(s)\,; \qquad \text{(I.5a)}$$

$$J_{ik}\,(s+\ell, s) = M_{ik}\,. \qquad \text{(I.5b)}$$

We first prove

Theorem 1: The canonical structure of the equations of motion implies the symplecticity of the Jacobian matrices.

Supposition: There is a Hamiltonian

$$\mathcal{H} = \mathcal{H}(q_i, p_i, s)$$

so that the equations of motion can be written in the form

$$\frac{d}{ds}\, q_i\,(s) = \frac{\partial \mathcal{H}}{\partial p_i}\,; \qquad \text{(I.6a)}$$

$$\frac{d}{ds}\, p_i\,(s) = -\frac{\partial \mathcal{H}}{\partial q_i}\,; \qquad \text{(I.6b)}$$

$(i = 1, 2, \ldots n)$.

Proposition: The Jacobian matrices describing the motion of the mechanical system are symplectic, which means (see eq. (4.2))

$$\underline{J}^T(s+\ell, s)\cdot \underline{S}\cdot\underline{J}\,(s+\ell, s) = \underline{S}\,. \qquad \text{(I.7)}$$

Proof: It is sufficient to prove the symplecticity for infinitesimal matrices $\underline{J}\,(s+\Delta s, s)$

because a finite matrix can always be represented by a product of infinitesimal matrices and because a product of symplectic matrices is also symplectic. Therefore we first show that the following Poisson-bracket relationships

$$(q_i, q_k) = (p_i, p_k) = 0\,;$$
$$(p_i, q_k) = \delta_{ik} \qquad \text{(I.8)}$$

with

$$(\xi_\alpha, \xi_\beta) = \sum_{k=1}^{n} \left(\frac{\partial \xi_\alpha}{\partial p_k}\frac{\partial \xi_\beta}{\partial q_k} - \frac{\partial \xi_\alpha}{\partial q_k}\frac{\partial \xi_\beta}{\partial p_k}\right)$$

$$= \sum_{\ell=1}^{n} \left(\frac{\partial \xi_\alpha}{\partial x_{2\ell}}\,\frac{\partial \xi_\beta}{\partial x_{(2\ell-1)}} - \frac{\partial \xi_\alpha}{\partial x_{(2\ell-1)}}\,\frac{\partial \xi_\beta}{\partial x_{2\ell}}\right)$$

$$= \sum_{i,k=1}^{n} S_{ik}\,\frac{\partial \xi_\alpha}{\partial x_i}\,\frac{\partial \xi_\beta}{\partial x_k} \qquad \text{(I.9)}$$

are invariant under the infinitesimal transformation

$$q_k\,(s) \longrightarrow q_k\,(s+\Delta s) = q_k\,(s) + \Delta s\cdot\frac{dq_k}{ds}$$
$$= q_k\,(s) + \Delta s\cdot\frac{\partial \mathcal{H}}{\partial p_k} \quad \text{(eq.I.6a)}$$
$$p_k\,(s) \longrightarrow p_k\,(s+\Delta s) = p_k\,(s) + \Delta s\cdot\frac{dp_k}{ds} \qquad \text{(I.10)}$$
$$= p_k\,(s) - \Delta s\,\frac{\partial \mathcal{H}}{\partial q_k} \quad \text{(eq.I.6b)}$$

which means (21)

$$(q_i(s+\Delta s), q_k\,(s+\Delta s)) = 0\,;$$
$$(p_i(s+\Delta s), p_k\,(s+\Delta s)) = 0\,; \qquad \text{(I.11)}$$
$$(p_i\,(s+\Delta s), q_k\,(s+\Delta s)) = \delta_{ik}\,.$$

In fact, substituting (I.10) into the left hand side of eq. (I.11) and taking into account (I.8) we get

$$(q_i\,(s+\Delta s), q_k\,(s+\Delta s)) = \left(q_i\,(s) + \Delta s\cdot\frac{\partial \mathcal{H}}{\partial p_i}\,,\; q_k\,(s) + \Delta s\cdot\frac{\partial \mathcal{H}}{\partial p_k}\right)$$

$$= (q_i, q_k) + \Delta s\cdot\left\{\left(\frac{\partial \mathcal{H}}{\partial p_i}\,, q_k\right) + \left(q_i\,, \frac{\partial \mathcal{H}}{\partial p_k}\right)\right\}$$

$$= (q_i, q_k) + \Delta s\cdot\left\{\frac{\partial}{\partial p_k}\,\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial}{\partial p_i}\,\frac{\partial \mathcal{H}}{\partial p_k}\right\}$$

$$= (q_i, q_k) \equiv 0\,;$$

$$(p_i(s+\Delta s),\ p_k(s+\Delta s)) = (p_i(s) - \Delta s\cdot\frac{\partial\chi}{\partial q_i},\ p_k(s) - \Delta s\cdot\frac{\partial\chi}{\partial q_k})$$

$$= (p_i,\ p_k) - \Delta s\cdot\left\{(\frac{\partial\chi}{\partial q_i},\ p_k) + (p_i,\ \frac{\partial\chi}{\partial q_k})\right\}$$

$$= (p_i,\ p_k) - \Delta s\cdot\left\{-\frac{\partial}{\partial q_k}\frac{\partial\chi}{\partial q_i} + \frac{\partial}{\partial q_i}\frac{\partial\chi}{\partial q_k}\right\}$$

$$= (p_i,\ p_k) \equiv 0\ ;$$

$$(p_i(s+\Delta s),\ q_k(s+\Delta s)) = (p_i(s) - \Delta s\cdot\frac{\partial\chi}{\partial q_i},\ q_k(s) + \Delta s\cdot\frac{\partial\chi}{\partial p_k})$$

$$= (p_i,\ q_k) - \Delta s\cdot\left\{(\frac{\partial\chi}{\partial q_i},\ q_k) - (p_i,\ \frac{\partial\chi}{\partial p_k})\right\}$$

$$= (p_i,\ q_k) - \Delta s\cdot\left\{\frac{\partial}{\partial p_k}\frac{\partial\chi}{\partial q_i} - \frac{\partial}{\partial q_i}\frac{\partial\chi}{\partial p_k}\right\}$$

$$= (p_i,\ q_k) \equiv \delta_{ik}\ .$$

We can write the relationship (I.11) in the form

$$(x_\alpha\ (s+\Delta s),\ \ x_\beta(s+\Delta s)) = S_{\alpha\beta}\ . \qquad \text{(I.12)}$$

For the left side of (I.12) we also obtain

$$(x_\alpha\ (s+\Delta s),\ x_\beta(s+\Delta s)) = \sum_{i,k=1}^{2n} S_{ik}\frac{\partial x_\alpha(s+\Delta s)}{\partial x_i(s)}\cdot\frac{\partial x_\beta(s+\Delta s)}{\partial x_k(s)}$$

$$= \sum_{i,k=1}^{2n} S_{ik}\cdot J_{\alpha i}\ (s+\Delta s,s)\cdot J_{\beta k}(s+\Delta s,s)$$

$$= \sum_{i,k=1}^{2n} J_{\alpha i}(s+\Delta s,s)\cdot S_{ik}\cdot J^T_{k\beta}\ (s+\Delta s,s)$$

so that we get from (I.12)

$$\underline{J}\ (s+\Delta s,s)\cdot\underline{S}\ \cdot\underline{J}^T(s+\Delta s,s) = \underline{S}\ . \qquad \text{(I.13)}$$

From this equation we finally obtain the condition of symplecticity (I.7) we wanted to prove, if we take into account

$$(\det\ (\underline{J}\ (s+\Delta s,\ s)))^2 = 1 \neq 0 \qquad \text{(see eq. (I.13))}$$

(existence of the inverse matrix $\underline{J}^{-1}\ (s+\Delta s,s)$).

We only have to multiply eq. (I.13) from the left with $\underline{S}\ \underline{J}^{-1}$ and from the right with $\underline{S}^T\ \underline{J}$ so that we obtain

$$\underline{S}\ \underline{J}^{-1}\cdot\ \underline{J}\ \underline{S}\ \underline{J}^T\cdot\ \underline{S}^T\ \underline{J} = \underline{S}\ \underline{J}^{-1}\cdot\underline{S}\cdot\underline{S}^T\ \underline{J}$$

or (with $\underline{S}^2 = -\underline{1}$, $\underline{S}^T = -\underline{S}$)

$$\underline{J}^T\ (s+\Delta s,\ s)\ \underline{S}\ \underline{J}\ (s+\Delta s,s) = \underline{S} \qquad \text{(I.14)}$$

q.e.d.

The converse of theorem 1 is also true.

<u>Theorem 2:</u> The symplecticity of the Jacobian matrix implies that the equations of motion can be written in canonical form.
<u>Supposition:</u> The Jacobian matrices

$$\underline{J}\ (s+\ell,s)\quad\text{with}\quad J_{ik}\ (s+\ell,s) = \frac{\partial x_i\ (s+\ell,s)}{\partial x_k(s)}$$

satisfy the symplectic condition

$$\underline{J}^T\ (s+\ell,s)\ \cdot\underline{S}\cdot\underline{J}\ (s+\ell,s) = \underline{S} \qquad \text{(I.15)}$$

<u>Proposition:</u> There is a function

$$\chi = \chi(q_i,\ p_i,\ s)$$

so that the equations of motion can be written in the canonical form

$$\frac{d}{ds}\,q_k(s) = \frac{\partial\chi}{\partial p_k}\ ;$$

$$\frac{d}{ds}\,p_k(s) = -\frac{\partial\chi}{\partial q_k}\ . \qquad \text{(I.16)}$$

Proof: Because of $(\det \underline{J}(s+\ell,s))^2 = 1 \neq 0$ (existence of the inverse $\underline{J}^{-1}(s+\ell,s)$) eq. (I.15) implies that

$$\underline{J}\,\underline{S}\cdot\underline{J}^T\,\underline{S}\,\underline{J}\cdot\underline{J}^{-1}\,\underline{S}^T = \underline{J}\,\underline{S}\cdot\underline{S}\cdot\underline{J}^{-1}\,\underline{S}^T$$

or

$$\underline{J}\,\underline{S}\,\underline{J}^T = \underline{S}\,. \qquad (I.17)$$

Taking into account this relationship we obtain for the Poisson-brackets defined in eq. (I.9)

$$(x_\alpha(s+\Delta s),\, x_\beta(s+\Delta s)) = \sum_{i,k=1}^{2n} S_{ik}\cdot\frac{\partial x_\alpha(s+\Delta s)}{\partial x_i(s)}\cdot\frac{\partial x_\beta(s+\Delta s)}{\partial x_k(s)}$$

$$= \sum_{i,k=1}^{2n} S_{ik}\cdot J_{\alpha i}(s+\Delta s,s)\cdot J_{\beta k}(s+\Delta s,s)$$

$$= \sum_{i,k=1}^{2n} J_{\alpha i}(s+\Delta s,s)\cdot S_{ik}\cdot J^T_{k\beta}(s+\Delta s,s)$$

$$= S_{\alpha\beta}$$

or

$$\left\{\begin{aligned} (p_i(s+\Delta s),\, p_k(s+\Delta s)) &= 0\;;\\ (q_i(s+\Delta s),\, q_k(s+\Delta s)) &= 0\;;\\ (p_i(s+\Delta s),\, q_k(s+\Delta s)) &= \delta_{ik}\,. \end{aligned}\right. \qquad (I.18)$$

Putting

$$p_i(s+\Delta s) = p_i(s) + \Delta s\cdot p_i'(s)\;;$$

$$q_i(s+\Delta s) = q_i(s) + \Delta s\cdot q_i'(s)$$

and using (I.9)

$$(p_i(s),\, p_k(s)) = (q_i(s),\, q_k(s)) = 0\;;$$

$$(p_i(s),\, q_k(s)) = \delta_{ik}$$

one gets from (I.18)

$$\begin{aligned} 0 &= (p_i'(s),\, p_k) + (p_i,\, p_k'(s))\\ &= -\frac{\partial}{\partial q_k}p_i'(s) + \frac{\partial}{\partial q_i}p_k'(s)\;; \end{aligned} \qquad (I.19a)$$

$$\begin{aligned} 0 &= (q_i'(s),\, q_k) + (q_i,\, q_k'(s))\\ &= \frac{\partial}{\partial p_k}q_i'(s) - \frac{\partial}{\partial p_i}q_k'(s)\;; \end{aligned} \qquad (I.19b)$$

$$\begin{aligned} 0 &= (p_i'(s),\, q_k) + (p_i,\, q_k'(s))\\ &= \frac{\partial}{\partial p_k}p_i'(s) + \frac{\partial}{\partial q_i}q_k'(s)\,. \end{aligned} \qquad (I.19c)$$

Eq. (I.19a) implies that the n functions $p_i'(s)$ form an irrotational vector field in the space of the q_k and therefore they can be expressed in this space as the gradient of a function $F(q_k, p_k)$ (22):

$$p_k'(s) = \frac{\partial}{\partial q_k}F(q_k, p_k)\,. \qquad (I.20a)$$

Because of eq. (I.19b) a similar expression holds for the n functions $q_i'(s)$ in the space of the p_k

$$q_k'(s) = \frac{\partial}{\partial p_k}G(q_k, p_k). \qquad (I.20b)$$

Substituting (I.20a, b) into the remaining expressions (I.19c) we get:

$$\frac{\partial^2}{\partial p_k \partial q_i}(F + G) = 0$$

which means that (F + G) is a linear function of the variables p_k, q_k:

$$F + G = a + \sum_\mu (b_\mu\cdot q_\mu + c_\mu\cdot p_\mu)\,.$$

In eq. (I.20a) we can express F by G

$$p_k'(s) = b_k - \frac{\partial}{\partial q_k} G .$$

Using the relationships

$$b_k = -\frac{\partial}{\partial q_k}\left(-\sum_\mu b_\mu q_\mu\right) ;$$

$$0 = \frac{\partial}{\partial p_k}\left(-\sum_\mu b_\mu q_\mu\right)$$

we can finally write with a single function

$$\mathcal{H} = G - \sum_\mu b_\mu q_\mu ;$$

$$q_k' = \frac{\partial \mathcal{H}}{\partial p_k} ;$$

$$p_k' = -\frac{\partial \mathcal{H}}{\partial q_k}$$

whence we have proven the canonical structure of the equations of motion.
q.e.d.

Theorems 1 and 2 show that the symplecticity of the Jacobian matrix (I.3) is a necessary and sufficient condition for the canonical structure of the equations of motion. In the special case of linear equations of motion this implies theorems 1 and 2 of chapter 4.2.

Appendix II

Calculation of the eigenvalues and eigenvectors of the transfer matrix

1. Eigenvalues

For the calculation of the eigenvalues of the transfer matrix

$$\underline{M} \equiv \underline{M}(s_o + L, s_o)$$

we use the fact, that besides the eigenvalue equation

$$\underline{M}\,\vec{v} = \lambda\cdot\vec{v} \qquad \text{(II.1)}$$

also the "inverse" equation

$$\underline{M}^{-1}\,\vec{v} = \lambda^{-1}\cdot\vec{v} \qquad \text{(II.2)}$$

is valid ($\lambda \neq 0$).

Because of the symplectic condition

$$\underline{M}^T\,\underline{S}\,\underline{M} = \underline{S}$$

the inverse matrix $\underline{M}^{-1}$ can be expressed in the form

$$\underline{M}^{-1} = -\underline{S}\cdot\underline{M}^T\cdot\underline{S} \qquad \text{(II.3)}$$

where we have used

$$\underline{S}^2 = -\underline{1} .$$

Writing

$$\underline{M} = \begin{pmatrix} \underline{A}_{11} & \underline{A}_{12} & \underline{A}_{13} \\ \underline{A}_{21} & \underline{A}_{22} & \underline{A}_{23} \\ \underline{A}_{31} & \underline{A}_{32} & \underline{A}_{33} \end{pmatrix} \qquad \text{(II.4)}$$

($\underline{A}_{\mu\nu}$ 2 x 2 submatrices)

$$\underline{S} = \begin{pmatrix} \underline{I} & \underline{0}_2 & \underline{0}_2 \\ \underline{0}_2 & \underline{I} & \underline{0}_2 \\ \underline{0}_2 & \underline{0}_2 & \underline{I} \end{pmatrix} \qquad \underline{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{II.5}$$

and

$$\overline{\underline{A}}_{\nu\mu} = - \underline{I}\cdot\underline{A}^T_{\nu\mu}\cdot\underline{I} \tag{II.6}$$

we get from (II.3)

$$\underline{M}^{-1} = \begin{pmatrix} \overline{\underline{A}}_{11} & \overline{\underline{A}}_{21} & \overline{\underline{A}}_{31} \\ \overline{\underline{A}}_{12} & \overline{\underline{A}}_{22} & \overline{\underline{A}}_{32} \\ \overline{\underline{A}}_{13} & \overline{\underline{A}}_{23} & \overline{\underline{A}}_{33} \end{pmatrix} . \tag{II.7}$$

The following relationships are valid for an arbitrary 2 x 2 matrix $\underline{K}$:

$$\underline{K} + \overline{\underline{K}} = \underline{1} \cdot \mathrm{Sp}\, \underline{K} \ ; \tag{II.8a}$$

$$\underline{K} \cdot \overline{\underline{K}} = \overline{\underline{K}} \cdot \underline{K} = \underline{1} \cdot \det \underline{K} \ . \tag{II.8b}$$

Putting

$$\vec{v} = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vec{x}_3 \end{pmatrix} \quad ; \qquad \vec{x}_i^T = (x_{i1}, x_{i2}) \tag{II.9}$$

we get from (II.1) and (II.2)

$$(\underline{A}_{11} - \lambda\cdot\underline{1})\vec{x}_1 + \underline{A}_{12}\,\vec{x}_2 + \underline{A}_{13}\,\vec{x}_3 = 0 \tag{II.10a}$$

$$\underline{A}_{21}\,\vec{x}_1 + (\underline{A}_{22} - \lambda\cdot\underline{1})\,\vec{x}_2 + \underline{A}_{23}\,\vec{x}_3 = 0 \tag{II:10b}$$

$$\underline{A}_{31}\,\vec{x}_1 + \underline{A}_{32}\,\vec{x}_2 + (\underline{A}_{33} - \lambda\cdot\underline{1})\,\vec{x}_3 = 0 \tag{II.10c}$$

and

$$(\overline{\underline{A}}_{11} - \tfrac{1}{\lambda}\cdot\underline{1})\,\vec{x}_1 + \overline{\underline{A}}_{21}\,\vec{x}_2 + \overline{\underline{A}}_{31}\,\vec{x}_3 = 0 \tag{II.11a}$$

$$\overline{\underline{A}}_{12}\,\vec{x}_1 + (\overline{\underline{A}}_{22} - \tfrac{1}{\lambda}\cdot\underline{1})\,\vec{x}_2 + \overline{\underline{A}}_{32}\,\vec{x}_3 = 0 \tag{II.11b}$$

$$\overline{\underline{A}}_{13}\,\vec{x}_1 + \overline{\underline{A}}_{23}\,\vec{x}_2 + (\overline{\underline{A}}_{33} - \tfrac{1}{\lambda}\cdot\underline{1})\,\vec{x}_3 = 0 \tag{II.11c}$$

Taking into account (II.8) and adding (II.10) and (II.11) one finds the following equations

$$(\mathrm{Sp}\,\underline{A}_{11} - \Lambda)\,\vec{x}_1 + (\underline{A}_{12} + \overline{\underline{A}}_{21})\,\vec{x}_2 + (\underline{A}_{13} + \overline{\underline{A}}_{31})\,\vec{x}_3 = 0 \tag{II.12a}$$

$$(\underline{A}_{21} + \overline{\underline{A}}_{12})\,\vec{x}_1 + (\mathrm{Sp}\,\underline{A}_{22} - \Lambda)\,\vec{x}_2 + (\underline{A}_{23} + \overline{\underline{A}}_{32})\,\vec{x}_3 = 0 \tag{II.12b}$$

$$(\underline{A}_{31} + \overline{\underline{A}}_{13})\,\vec{x}_1 + (\underline{A}_{32} + \overline{\underline{A}}_{23})\,\vec{x}_2 + (\mathrm{Sp}\,\underline{A}_{33} - \Lambda)\,\vec{x}_3 = 0 \tag{II.12c}$$

where we have set

$$\Lambda = \lambda + \frac{1}{\lambda} \ . \tag{II.13}$$

With the help of eq. (II.12a, b) we can eliminate the vectors $\vec{x}_2$ and $\vec{x}_3$ and we obtain

$$\vec{x}_2 = \underline{F}_{21}\,\vec{x}_1 \tag{II.14a}$$

with

$$\underline{F}_{21} = - \frac{1}{\left[(\mathrm{Sp}\,\underline{A}_{22} - \Lambda)(\mathrm{Sp}\,\underline{A}_{33} - \Lambda) - \det(\underline{A}_{23} + \overline{\underline{A}}_{32})\right]} \times \tag{II.14b}$$

$$\times \left\{(\mathrm{Sp}\,\underline{A}_{33} - \Lambda)\,(\underline{A}_{21} + \overline{\underline{A}}_{12}) - (\underline{A}_{23} + \overline{\underline{A}}_{32})(\underline{A}_{31} + \overline{\underline{A}}_{13})\right\}$$

and

$$\vec{x}_3 = \underline{F}_{31}\,\vec{x}_1 \tag{II.15a}$$

with

$$\underline{F}_{31} = -\frac{1}{(\mathrm{Sp}\,\underline{A}_{33} - \Lambda)} \cdot \left\{ (\underline{A}_{31} + \overline{A}_{13}) + (\underline{A}_{32} + \overline{A}_{23}) \cdot \underline{F}_{21} \right\} . \qquad \text{(II.15b)}$$

Substituting (II.14) and (II.15) into (II.12a) we finally find

$$F(\Lambda) \cdot \vec{x}_I = 0 \qquad \text{(II.16a)}$$

where $F(\Lambda)$ is defined by

$$\begin{aligned} F(\Lambda) = {} & (\mathrm{Sp}\,\underline{A}_{11} - \Lambda)(\mathrm{Sp}\,\underline{A}_{22} - \Lambda)(\mathrm{Sp}\,\underline{A}_{33} - \Lambda) - \\ & - (\mathrm{Sp}\,\underline{A}_{11} - \Lambda) \det (\underline{A}_{23} + \overline{A}_{32}) - \\ & - (\mathrm{Sp}\,\underline{A}_{22} - \Lambda) \det (\underline{A}_{13} + \overline{A}_{31}) - \\ & - (\mathrm{Sp}\,\underline{A}_{33} - \Lambda) \det (\underline{A}_{12} + \overline{A}_{21}) + \\ & + \mathrm{Sp}\left[(\underline{A}_{12} + \overline{A}_{21})(\underline{A}_{23} + \overline{A}_{32})(\underline{A}_{31} + \overline{A}_{13}) \right] . \end{aligned} \qquad \text{(II.16b)}$$

Together with (II.14a) and (II.15a) equation (II.16a) implies that there are only nonvanishing vectors $\vec{v}$ satisfying (II.12) if

$$F(\Lambda) = 0 \qquad \text{(II.17)}$$

where $F(\Lambda)$ is given by (II.16b).

Equation (II.17) defines the eigenvalues Λ of the matrix

$$\underline{N} = \underline{M} + \underline{M}^{-1} .$$

Since (II.17) represents a cubic equation, we get three different eigenvalues

$$\Lambda_I , \; \Lambda_{II} , \; \Lambda_{III} .$$

Knowing Λ_k according to (II.17) we finally find for the eigenvalues $\lambda_{\pm k}$ of the revolution matrix $\underline{M}$

$$\lambda_{\pm k} = \frac{1}{2}\Lambda_k \pm \sqrt{\frac{1}{4}\Lambda_k^2 - 1} \qquad \text{(II.18)}$$

with

$$\lambda_k \cdot \lambda_{-k} = 1 .$$

2. Eigenvectors

We want to describe two methods for calculating the eigenvectors. We assume that the eigenvalues are known and non-degenerate

<u>Method 1:</u>

Still using the symplectic structure of the transfer matrix $\underline{M}$ we get from (II.14a), (II.15a) and (II.10a)

$$\underline{P} \, \vec{x}_1 = 0 \qquad \text{(II.19a)}$$

with

$$\underline{P} = (\underline{A}_{11} - \lambda \cdot \underline{1}) + \underline{A}_{12} \cdot \underline{F}_{21} + \underline{A}_{13} \cdot \underline{F}_{31} \equiv ((P_{ik})) . \qquad \text{(II.19b)}$$

But from (II.19a) it follows that

$$\vec{x}_1 = \begin{pmatrix} P_{12} \\ -P_{11} \end{pmatrix} \qquad \text{(II.20)}$$

which means that we have determined $\vec{x}_1$. Substituting (II.20) into (II.14a) and (II.15a) we can get the other components of the eigenvector $\vec{v}$.

Method 2:

Here we make use of the Cayley-Hamilton theorem which states that an arbitrary nxn matrix $\underline{M}$ obeys its own characteristic equation

$$\underline{G} \equiv (\underline{M} - \lambda_1 \cdot \underline{1})(\underline{M} - \lambda_2 \cdot \underline{1}) \ldots (\underline{M} - \lambda_n \cdot \underline{1}) = 0 . \quad \text{(II.21)}$$

For the special case that the eigenvalue spectrum is non-degenerate (which means that the system of eigenvectors must form a complete set) one can easily prove this theorem by applying $\underline{G}$ to an arbitrary vector $\vec{y}$

$$\vec{y} = \sum_{\nu=1}^{n} c_\nu \vec{v}_\nu$$

expanded in terms of the eigenvectors $\vec{v}_\nu$.

Defining matrices

$$\underline{G}_\nu = (\underline{M} - \lambda_1 \cdot \underline{1}) \ldots (\underline{M} - \lambda_{\nu-1} \cdot \underline{1})(\underline{M} - \lambda_{\nu+1} \cdot \underline{1}) \ldots (\underline{M} - \lambda_n \cdot \underline{1}) \quad \text{(II.22)}$$

$$(\nu = 1, 2 \ldots\ldots n)$$

differing from $\underline{G}$ by the single factor $(\underline{M} - \lambda_\nu \cdot \underline{1})$

$$\underline{G} = \underline{G}_\nu \cdot (\underline{M} - \lambda_\nu \cdot \underline{1}) \quad \text{(II.23)}$$

we get

$$\underline{G}\,\vec{y} = \sum_{\nu=1}^{n} c_\nu \, \underline{G}_\nu (\underline{M} - \lambda_\nu \cdot \underline{1}) \; \vec{v}_\nu$$

$$= \sum_{\nu=1}^{n} c_\nu \, \underline{G}_\nu (\underline{M}\,\vec{v}_\nu - \lambda_\nu \cdot \vec{v}_\nu)$$

$$= 0$$

because $\underline{M}\,\vec{v}_\nu = \lambda_\nu \cdot \vec{v}$.

From

$$\underline{G}\,\vec{y} = 0 \quad \text{for all } \vec{y}$$

we finally find that $\underline{G}$ must vanish identically q.e.d.

In order to construct the eigenvectors we first show that the matrices $\underline{G}_\nu$ $(\nu = 1, \ldots n)$ are always different from zero. This is shown by applying $\underline{G}_\nu$ to an eigenvector $\vec{v}_\nu$ and one obtains

$$\underline{G}_\nu \vec{v}_\nu = (\underline{M} - \lambda_1 \cdot \underline{1}) \ldots (\underline{M} - \lambda_{\nu-1} \cdot \underline{1})(\underline{M} - \lambda_{\nu+1} \cdot \underline{1}) \ldots (\underline{M} - \lambda_n \cdot \underline{1}) \; \vec{v}_\nu$$

$$= (\underline{M} - \lambda_1 \cdot \underline{1}) \ldots (\underline{M} - \lambda_{\nu-1} \cdot \underline{1})(\underline{M} - \lambda_{\nu+1} \cdot \underline{1}) \ldots (\lambda_\nu - \lambda_n) \; \vec{v}_\nu$$

$$= (\lambda_\nu - \lambda_1) \ldots (\lambda_\nu - \lambda_{\nu-1})(\lambda_\nu - \lambda_{\nu+1}) \ldots (\lambda_\nu - \lambda_n) \; \vec{v}_\nu$$

$$\neq 0$$

because

$$(\lambda_\nu - \lambda_\mu) \neq 0 \quad \text{for} \quad \mu = 1, \ldots \nu-1, \nu+1 \ldots n,$$

and hence

$$\underline{G}_\nu \neq 0 . \quad \text{(II.24)}$$

From (II.24) we can conclude that $\underline{G}_\nu$ has at least one nonvanishing column vector $\vec{g}_{i\nu}$:

$$\underline{G}_\nu = (\vec{g}_{1\nu} , \; \vec{g}_{2\nu} \ldots\ldots \vec{g}_{n\nu}) ; \quad \text{(II.25)}$$

$$\vec{g}_{i\nu} \neq 0 .$$

But such a column vector is just an eigenvector of $\underline{M}$ belonging to the eigenvalue λ_ν, because we have

$$0 \equiv \underline{G} = (\underline{M} - \lambda_\nu \cdot \underline{1}) \, \underline{G}_\nu ;$$

$$0 = (\underline{M} - \lambda_\nu \cdot \underline{1})(\vec{g}_{1\nu} , \; \ldots \vec{g}_{i\nu} \ldots \vec{g}_{n\nu}) \quad \text{(II.26)}$$

and especially

$$(\underline{M} - \lambda_\nu \cdot \underline{1}) \quad \vec{g}_{i\nu} = 0 \qquad \text{q.e.d.}$$

In this way we can construct an eigenvector to each eigenvalue λ_ν.

The method described above is valid for all nxn matrices with non-degenerate eigenvalues. It can also be applied to non-symplectic matrices.

Appendix III

Introduction of the dispersion function into the equations of motion. Amplitude and phase function of the decoupled synchrotron oscillation.

Starting from the general equations for the synchro-betatron oscillations (3.33) and making additional assumptions we want to show that it is possible to derive simpler equations, which are usually used in linear accelerator theory.

Therefore we first eliminate $\tilde{p}_x$ and $\tilde{p}_z$ in eq. (3.33) and we get

$$\tilde{x}'' + G_1 \tilde{x} - (N + H') \tilde{z} - 2H\tilde{z}' - K_x \tilde{\eta} = 0 \quad ; \qquad \text{(III.1a)}$$

$$\tilde{z}'' + G_2 \tilde{z} - (N - H') \tilde{x} + 2H\tilde{x}' - K_z \tilde{\eta} = 0 \quad ; \qquad \text{(III.1b)}$$

$$\tilde{\sigma}' + K_x \cdot \tilde{x} + K_z \cdot \tilde{z} = 0 \quad ; \qquad \text{(III.1c)}$$

$$\tilde{\eta}' - \tilde{\sigma}(s) \cdot \frac{e\hat{V}}{E_0} \cdot k \cdot \frac{2\pi}{L} \cos\phi \cdot \sum_\nu \delta(s - s_\nu) = 0 \,. \qquad \text{(III.1d)}$$

Now we introduce the dispersion functions $D_x(s)$ and $D_z(s)$ defined by

$$\left\{\begin{array}{l} D_x'' + G_1 D_x - (N + H')D_z - 2H D_z' - K_x = 0 \,; \\ D_z'' + G_2 D_z - (N - H')D_x + 2H D_x' - K_z = 0 \,; \end{array}\right. \qquad \text{(III.2a)}$$

$$\left\{\begin{array}{l} D_x(s_0+L) = D_x(s_0); \; D_z(s_0+L) = D_z(s_0); \\ D_x'(s_0+L) = D_x'(s_0); \; D_z'(s_0+L) = D_z'(s_0). \end{array}\right. \qquad \text{(III.2b)}$$

These functions describe the closed (periodic) trajectory of a particle with energy deviation

$$\tilde{\eta} = 1 \,.$$

If $\tilde{\eta}$ is a constant in eq. (III.1a) and (III.1b), we can write the following expressions for the amplitudes $\tilde{x}$ and $\tilde{z}$ of the betatron oscillations:

$$\begin{aligned} \tilde{x} &= \tilde{\eta}\cdot D_x + x_\beta \;; \\ \tilde{z} &= \tilde{\eta}\cdot D_z + z_\beta \;, \end{aligned} \qquad \text{(III.3)}$$

where x_β and z_β are solutions of the homogeneous differential equations

$$\begin{aligned} x_\beta'' + G_1\cdot x_\beta - (N + H')\cdot z_\beta - 2H\cdot z_\beta' &= 0 \;; \\ z_\beta'' + G_2\cdot z_\beta - (N - H')\cdot x_\beta + 2H\cdot x_\beta' &= 0 \;. \end{aligned} \qquad \text{(III.4)}$$

This decomposition of the amplitude $\tilde{x}$ into the terms x_β and $\tilde{\eta}\cdot D_x$ and of the $\tilde{z}$ amplitude into z_β and $\tilde{\eta}\cdot D_z$ is also possible if $\tilde{\eta}(s)$ is varying slowly compared with $D_x(s)$ and $D_z(s)$ (which is true in most cases).

The coupled betatron oscillations of eq. (III.2) and (III.4) without the variable $\tilde{\eta}$ have been investigated in an earlier report (7). So we can restrict the following considerations to the synchrotron oscillation alone, described by eq. (III.1c) and (III.1d).

Putting (III.3) into (III.1c) we get

$$\tilde{\varepsilon}' = - K_x\cdot(D_x\tilde{\eta} + x_\beta) - K_z\cdot(D_z\tilde{\eta} + z_\beta).$$

The rapidly oscillating amplitudes x_β and z_β, having positive and negative values, do not cause a systematic change of the phase $\tilde{\varepsilon}$, so that we can approximately write down the equations for the synchrotron oscillations in the form:

$$\frac{d}{ds}\begin{pmatrix}\tilde{\varepsilon}\\ \tilde{\eta}\end{pmatrix} = \begin{pmatrix} 0 & - K_x D_x - K_z D_z \\ \frac{e\hat{V}}{E_0}\cdot k\cdot\frac{2\pi}{L}\cdot\cos\phi\cdot\sum_\nu\delta(s-s_\nu) & 0 \end{pmatrix}\begin{pmatrix}\tilde{\varepsilon}\\ \tilde{\eta}\end{pmatrix} . \qquad \text{(III.5)}$$

In this equation the synchrotron oscillation is completely decoupled from the betatron oscillation and the coefficient matrix has the form

(4.46) with

$$R(s) = 0 \;; \qquad \text{(III.6a)}$$

$$F(s) = - K_x D_x - K_z D_z \;; \qquad \text{(III.6b)}$$

$$G(s) = - \frac{e\hat{V}}{E_0}\cdot k\cdot\frac{2\pi}{L}\cdot\cos\phi\cdot\sum_\nu\delta(s-s_\nu) \qquad \text{(III.6c)}$$

so that the transfer matrix for eq. (III.5) fulfills the condition (4.44a) or (4.45), which means that the transfer matrix is symplectic.

For the infinitesimal transfer matrix $\underline{m}_\sigma(s+\Delta s, s)$ we get with (III.5)

$$\underline{m}_\sigma(s+\Delta s, s) = \begin{pmatrix} 1 & -(K_x D_x + K_z D_z)\cdot\Delta s \\ \frac{e\hat{V}}{E_0}\cdot k\cdot\frac{2\pi}{L}\cdot\cos\phi\cdot\sum_\nu\delta(s-s_\nu)\cdot\Delta s & 1 \end{pmatrix} . \qquad \text{(III.7)}$$

From this we obtain the transfer matrix for a cavity ($s = s_\nu$; $K_x = K_z = 0$)

$$\underline{m}_\sigma(s_\nu+0, s_\nu-0) = \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix} \qquad \text{(III.8a)}$$

where A is defined by

$$A = \frac{e\hat{V}}{E_0}\cdot k\cdot\frac{2\pi}{L}\cdot\cos\phi \;. \qquad \text{(III.8b)}$$

The transfer matrix $\underline{m}_\sigma(s'', s')$ for an interval

$$s' \leqq s \leqq s''$$

without cavities is given by

$$\underline{m}_\sigma(s'', s') = \begin{pmatrix} 1 & B(s'', s') \\ 0 & 1 \end{pmatrix} \qquad \text{(III.9)}$$

with

$$B\,(s'',\,s') \;=\; -\int_{s'}^{s''} ds\cdot\left\{K_x\,(s)\cdot D_x\,(s) + K_z\,(s)\cdot D_z\,(s)\right\}. \qquad \text{(III.9b)}$$

If $K_x = K_z = 0$ for $s' \leqq s \leqq s''$ we have

$$\underline{m}_\sigma\,(s'',s') \;=\; \underline{1}\;.$$

If there is only a single cavity at the position $s = s_0$, we obtain from (III.8) and (III.9) the transfer matrix for a complete revolution (9), (24):

$$\underline{m}_\sigma\,(s_0-0+L,\; s_0-0) = \underline{m}_\sigma(s_0-0+L,\; s_0+0)\cdot\underline{m}_\sigma(s_0+0,\; s_0-0)$$

$$= \begin{pmatrix} 1-A\cdot L\cdot\varkappa & -L\cdot\varkappa \\ A & 1 \end{pmatrix} \qquad \text{(III.10)}$$

where we have set

$$\varkappa \;=\; \frac{1}{L}\int_{s_0}^{s_0+L} ds\cdot\left\{K_x\,(s)\cdot D_x\,(s) + K_z\,(s)\cdot D_z\,(s)\right\} \qquad \text{(III.11)}$$

(momentum compaction factor).

(If there are several periodically distributed cavities, eq. (III.10) is still valid if we replace L by the periodicity length).

Using the fact that $\varkappa$ is generally > 0 and using eq. (4.51) the stability condition for the synchrotron oscillation reads

$$A > 0 \qquad \text{(stability condition)}\,. \qquad \text{(III.12)}$$

Comparing (III.10) with (4.47a) one finds (23):

$$\cos 2\pi Q_\sigma \;=\; 1 - \frac{1}{2}\,A\cdot L\cdot\varkappa\;; \qquad \text{(III.13a)}$$

$$\sin 2\pi Q_\sigma \;=\; -\sqrt{AL\varkappa\cdot(1-\tfrac{1}{4}\,A\,L\varkappa)}\;; \qquad \text{(III.13b)}$$

$$\alpha_\sigma(s_0-0) \;=\; \frac{-A\,L\,\varkappa}{2\,\sin 2\pi Q_\sigma}\;; \qquad \text{(III.13c)}$$

$$\beta_\sigma\,(s_0-0) \;=\; \frac{-L\cdot\varkappa}{\sin 2\pi Q_\sigma}\;; \qquad \text{(III.13d)}$$

$$\gamma_\sigma\,(s_0-0) \;=\; \frac{-A}{\sin 2\pi Q_\sigma}\;. \qquad \text{(III.13e)}$$

Taking into account the relation

$$\vec{w}_\sigma\;(s') \;=\; \underline{m}_\sigma(s',\,s)\cdot\vec{w}_\sigma\,(s)$$

(see eq. (4.33)) and eq's (4.50), (4.52), (4.57) and (III.13c, d, e) we obtain:

$$\alpha_\sigma\,(s_0+0) \;=\; -\,\alpha_\sigma(s_0-0)\;; \qquad \text{(III.14a)}$$

$$\beta_\sigma\,(s_0+0) \;=\; \beta_\sigma\,(s_0-0) \;\equiv:\; \beta_\sigma\,(s_0)\;; \qquad \text{(III.14b)}$$

$$\gamma_\sigma\,(s_0+0) \;=\; \gamma_\sigma\,(s_0-0) \;\equiv:\; \gamma_\sigma\,(s_0)\;; \qquad \text{(III.14c)}$$

$$\psi_\sigma\,(s_0+0) \;=\; \psi_\sigma\,(s_0-0) \;\equiv:\; \psi_\sigma(s_0)\;. \qquad \text{(III.14d)}$$

(Traversing the cavity the function α changes sign while the other quantities β, γ and ψ remain unchanged).

For $s > s_0$ we find

$$\beta_\sigma\,(s) \;=\; \beta_\sigma\,(s_0) + 2\,B\,(s,s_0)\cdot\alpha_\sigma(s_0-0) + B^2(s,s_0)\cdot\gamma_\sigma(s_0)\;; \qquad \text{(III.15a)}$$

$$\gamma_\sigma\,(s) \;=\; \gamma_\sigma\,(s_0)\;; \qquad \text{(III.15b)}$$

$$\alpha_\sigma(s) \;=\; -\,\alpha_\sigma(s_0-0) - B\,(s,\,s_0)\cdot\gamma_\sigma(s_0)\;; \qquad \text{(III.15c)}$$

$$\psi_\sigma(s) \;=\; \psi_\sigma\,(s_0) - \int_{s_0}^{s} ds'\cdot\frac{K_x(s')\cdot D_x(s') + K_z(s')\cdot D_z(s')}{\beta_\sigma\,(s')} \qquad \text{(III.15d)}$$

with $B(s,\,s_0)$ from eq. (III.9b).

The differential equations for α, β and ψ now take the form

$$\Psi_\sigma' = -\frac{K_x \cdot D_x + K_z \cdot D_z}{\beta_\sigma} \quad ; \qquad \text{(III.16a)}$$

$$\beta_\sigma' = 2 \cdot (K_x \cdot D_x + K_z \cdot D_z) \cdot \alpha_\sigma \quad ; \qquad \text{(III.16b)}$$

$$\alpha_\sigma' + (K_x \cdot D_x + K_z \cdot D_z) \cdot \gamma_\sigma = 0 \quad ; \quad (s \neq s_0) \qquad \text{(III.16c)}$$

It is remarkable that the quantity γ_σ is a constant everywhere in the ring.

Finally we want to mention that the functions $\tilde{\sigma}$ and $\tilde{\eta}$ can be written in the following form:

$$\tilde{\sigma}(s) = \sqrt{\epsilon_\sigma} \cdot \sqrt{\beta_\sigma(s)} \cos(\psi_\sigma(s) - \delta) \quad ; \qquad \text{(III.17a)}$$

$$\tilde{\eta}(s) = \sqrt{\epsilon_\sigma} \cdot \sqrt{\gamma_\sigma(s)} \cos(\psi_\sigma(s) - \tilde{\delta}) \quad . \qquad \text{(III.17b)}$$

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