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SOME NONPERTURBATIVE TECHNIQUES FOR THE STUDY OF FIELD THEORIES WITH NONABELIAN SYMMETRY

by

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SOME NONPERTURBATIVE TECHNIQUES FOR THE STUDY OF FIELD THEORIES WITH NONABELIAN SYMMETRY

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Abstract

This investigation is concerned with the development of non-perturbative techniques for studying field theories with non-abelian symmetries.

The aim of part A is to formulate nonabelian lattice field theories in terms of rotation or gauge invariant variables. This possibly constitutes a first step towards a useful duality transformation for nonabelian lattice theories. Starting point for our formulation is Rühl's boson representation.

For the case of the O(4) symmetric Heisenberg ferromagnet in 2 dimensions the integration over all rotation variant variables is performed after introducing rotation invariant auxiliary variables. The resulting new Hamiltonian involves a sum over closed loops. It is complex and invariant under U(1) gauge transformations. The model is rewritten as a local gauge theory, the gauge group being a semidirect product of U(1) and the Weyl group of SU(2). A high temperature expansion can be derived without recourse to Clebsch-Gordan series.

The formalism is also applied to SU(2) lattice gauge theories in 3 dimensions. The result is an expansion into closed loops formulated entirely in terms of gauge invariant variables. Each of these loops is confined to a cube of the dual lattice.

In part B, the device of nonperturbative normal ordering is introduced into the analysis of the O(n+1) symmetric nonlinear σ-model in 2 and $2+\varepsilon$ dimensions. In the limit $n \to \infty$, this allows a unified treatment of the symmetric (massive) and the spontaneously broken (massless) phase of the theory.

The problem of cancellation of quadratic divergences is investigated. In the present theory with derivative couplings, not all quadratically divergent graphs appear, for finite n, in the form of tadpole diagrams (which can be treated by normal ordering). All quadratic divergences can be taken into account by a dilute gas approximation which retains only "self-interactions", i.e. interactions at one space-time point. Within this approximation, an equation determining the mass gap of the theory is derived for finite n.

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INTRODUCTION

It is commonly believed that nonabelian gauge theories are the most promising candidate for a correct description of hadronic matter. However, the existence of the essential property of confinement has up to now only been conjectured. The eventual solution to the problem of quark confinement will no doubt involve an interplay of many ideas and techniques. In particular it is of major importance to develop tools that allow the investigation of the phase structure of gauge theories.

Due to the complexity of the problem it often seems advisable to resort in the beginning to simpler theories that share at least some of the vital properties of nonabelian gauge theories. In the present work we will discuss the Heisenberg ferromagnet in 2 dimensions, its field-theoretic counterpart, the nonlinear σ-model in 2 and $2+\varepsilon$ dimensions, and the SU(2) lattice gauge theory in 3 dimensions.

Since confinement is a property inaccessible by perturbation theory the development of nonperturbative methods is of utmost importance.

The nonperturbative technique we aim at in part A of our work is that of duality transformations for nonabelian lattice theories. Up to now, such transformations have only been developed for abelian systems. There it turned out to be important to formulate the theory in terms of gauge or rotation invariant variables. It is the analogue of this step which is carried out here for theories with nonabelian symmetry group, developed first for the O(4) symmetric Heisenberg ferromagnet in 2 dimensions and then applied to the SU(2) lattice gauge theory in 3 dimensions. Our starting point is Rühl's boson representation [1,2]. Our result has the undesirable, but possibly inevitable feature that the new Hamiltonian is complex. Therefore, one is no longer dealing with a system of statistical mechanics.

In part B, we introduce the device of nonperturbative normal ordering into the analysis of the O(n+1) symmetric nonlinear σ-model in 2 or $2+\varepsilon$ dimensions. This provides a key to a unified treatment of the two phases of the theory with and without magnetization in the limit $n \to \infty$.
Our corresponding investigation for finite n is as yet incomplete. In this context, a dilute gas approximation will possibly be of value. The idea of this approximation is to retain only "self-interactions", i.e. interactions at one space-time point. In the present theory with derivative couplings, it is not true that such self-interactions correspond to tadpole diagrams only, but there are contributions from other (quadratically divergent) diagrams as well, except to leading order in $1/n$.

Part A: FORMULATION OF LATTICE FIELD THEORIES IN TERMS OF GAUGE OR ROTATION INVARIANT VARIABLES

I. INTRODUCTION AND SUMMARY OF RESULTS

In various guises, duality transformations have proved to be a useful tool in the investigation of field theoretical models and systems of statistical mechanics.

Duality transformations which were originally discussed in the study of critical phenomena in solid-state physics [3] map a system of statistical mechanics into an equivalent system with inverse temperature. Thus, solving one system implies solving the other. But even apart from the existence of exact solutions, this correspondence between two systems with relative inverse temperature can yield valuable information. In particular, the phase structure of two systems dual to each other has to be the same. Moreover, the relation between the low-temperature region of one system and the high-temperature region of the other can be exploited to much advantage.

In a field theoretical context, high and low temperature regions correspond to strong and weak coupling regions, respectively. A duality transformation offers the possibility to investigate a model in the strong-coupling region by studying the dual model in the weak-coupling region where a perturbative treatment may be legitimate.

The interest in duality transformations for gauge theories is partly motivated by the search for a quark confining mechanism. In fact, in a superconductor of type II, magnetic charges are confined [4]. But since in quantum chromodynamics quarks are minimally coupled to the gauge field they have to be regarded as electric rather than magnetic. It is the idea of Mandelstam-t'Hooft duality [5] to reverse the rôle of electric and magnetic interactions to obtain a magnetic superconductor where electric charges or, more generally, the color charges of a nonabelian gauge theory will be confined.

The magnetically superconducting medium would be realized by the condensation of magnetic objects such as magnetic monopoles or vortices. Since a duality transformation is hoped to interchange electric and magnetic field strengths it may be essential for the implementation of such a picture. It thus seems highly desirable to develop a machinery for duality transformations in nonabelian gauge theories.

To this end, it is advantageous to study Euclidean field theories on a space-time lattice. The lattice spacing provides an UV cutoff, and we are really dealing with systems of statistical mechanics.

In statistical mechanics, duality transformations for systems with abelian symmetry group are quite well understood. (A review of duality transformations for abelian lattice gauge theories and spin systems was given by Savit [6].) Usually, such a transformation is performed according to the following recipe.

A system with abelian symmetry group is described by a partition function $Z = \int e^{\beta S}$, where S is the action of the system, and the integral is over all variables associated to the sites or links of a lattice. The coupling constant β corresponds to the inverse temperature. Since the representations of an abelian group are one-dimensional, the integrand can be expanded into characters of the symmetry group. This implies the introduction of gauge or rotation invariant auxiliary variables which have to fulfill a set of constraints produced by the integration over the original variables of the system. One then tries to solve these constraints by introducing a set of new variables defined in a natural manner on a dual lattice. (For a hypercubic lattice, one arrives at the dual lattice by shifting the original lattice by half a lattice spacing in every direction.) These dual variables then play the rôle of random variables in a new system of statistical mechanics described by a partition function $Z = \int e^{\beta_d S_d}$, where $\beta_d \to 0$ if $\beta \to \infty$, and vice versa.

For the more relevant case of nonabelian gauge theories, however, it is not yet known how to perform a duality transformation. Lacking any better proposal, one is tempted to imitate the procedure outlined above for the abelian case.

A first attempt in this direction has been undertaken by Matsui [7] who considered Heisenberg ferromagnets with O(3) and O(4) symmetry in 2 dimensions. The representations of nonabelian groups not being one-dimensional, considerable technical difficulties are encountered in following the above recipe. In Matsui's ansatz they appear in the form of vector coupling coefficients, and he only arrives at a possibly sensible dual theory after applying some rather involved approximations. The appearance of these cumbersome vector coupling coefficients is caused by the fact that Matsui's method is tied to a special choice of basis for the representation space of the symmetry group so that the structure of the duality transformation remains somewhat blurred. Moreover, we are mainly interested in lattice gauge theories, and it is not at all obvious whether Matsui's approach allows an extension to these theories.

One might try to circumvent the difficulties just mentioned by remembering that a main ingredient of duality transformations for abelian systems is the formulation of the partition function in terms of rotation respectively gauge invariant variables. For instance, the elimination of gauge freedom has proved crucial in the investigation of the 3-dimensional U(1) lattice gauge theory [8] where mass generation turned out to be a perturbative effect in the dually transformed system. It is the aim of this work to answer the question whether nonabelian theories, too, may be formulated entirely in rotation respectively gauge invariant variables and thus to establish a first step towards a useful duality transformation for theories with nonabelian symmetry group.

To avoid the appearance of vector coupling coefficients, use will be made of a formalism recently introduced by Rühl in his investigation of SU(N) lattice gauge theories [1,2]. Its main ingredient is the Bargmann space realization of group representations of SU(N) [9].

In the present investigation, this formalism is employed to study two of the simplest systems with nonabelian symmetry: the O(4) symmetric Heisenberg ferromagnet in 2 dimensions (part II) and SU(2) lattice gauge theory in 3 dimensions (part III).

In chapter 1, the partition function of the O(4) symmetric Heisenberg ferromagnet is reformulated in terms of SU(2) variables and expanded into characters. The integration over the group variables is then carried out using Rühl's formalism. One advantage of this method, as mentioned above, is that no Clebsch-Gordan coefficients arise, and the summation over the irreducible unitary representations of SU(2) may be performed explicitly. This causes the rotation invariant U(1) content of the initial SU(2) variables to reappear, U(1) being the maximal torus of SU(2). Thus, only the rotation variant variables are eliminated. The integration over the group variables implies the introduction of new rotation variant spins. They are $\mathbb{C}^2$ variables with quartic interaction. By introducing auxiliary rotation invariant complex variables this is brought into quadratic form so that a Gaussian integration can be carried out.

The expansion of the resulting determinant leads to a system of closed loops in chapter 2. As intended, the theory is now formulated entirely in rotation invariant variables. The new Hamiltonian is complex and invariant under U(1) gauge transformations.

It is even possible as will be shown in chapter 3 to formulate the theory as a local gauge theory where the gauge group is a semidirect product of U(1) and the Weyl group of SU(2). This gauge invariance reminds one of the equivalence between the O(3) symmetric Heisenberg ferromagnet and the CP^1 lattice model, as the latter exhibits a local U(1) symmetry not visible in the Heisenberg model [1o]. Perhaps this hidden U(1) invariance is a common feature of O(N) symmetric Heisenberg ferromagnets?

In chapter 4, the exact solution of the 1-dimensional model is given.

Chapter 5 shows how to reproduce the standard high temperature expansion without recourse to Clebsch-Gordan series.

In contrast to other methods of introducing rotation invariant variables (see e.g. [11]), our approach is applicable as well to nonabelian lattice gauge theories as part III will show for the SU(2) lattice gauge theory in 3 dimensions. The extension to 4 dimensions is straightforward.

Chapter 6 starts with a character expansion of the partition function. The integration over the group variables is again performed in the Bargmann space formalism, and again the final result is an expansion into closed loops formulated entirely in terms of gauge invariant variables. But in contrast to the Heisenberg ferromagnet, these loops are not allowed to extend over the whole lattice but each loop is confined to a cube of the dual lattice.

In chapter 7, the exact solution of the 2-dimensional SU(2) gauge theory is rederived as an illustration.

Finally, chapter 8 will show that our formalism may be used to reproduce the standard high temperature expansion of the SU(2) lattice gauge theory in 3 dimensions.

II. THE O(4) SYMMETRIC HEISENBERG FERROMAGNET IN 2 DIMENSIONS

1. Formulation in terms of rotation invariant variables

1.1 Definition of the model; character expansion

The Euclidean action $L(\vec{s})$ of the O(4) symmetric Heisenberg ferromagnet in two dimensions is a function of spins $\vec{s}_x \in S^3$ which are attached to the sites of a two-dimensional quadratic (or hexagonal) lattice $\Lambda \subset \mathbb{Z}^2$,

$$L(\vec{s}) = \beta \sum_{b=\langle xy \rangle} \vec{s}_x \cdot \vec{s}_y \qquad (1\text{-}1)$$

where b is a directed link between nearest neighbor vertices $x,y \in \Lambda$. The coupling constant β plays the rôle of inverse temperature.

The partition function of the system is given by

$$Z = \int \prod_{x \in \Lambda} \frac{d^4 s_x}{\pi^2} \, \delta(\vec{s}_x^{\,2} - 1) \, e^{L(\vec{s})} \qquad (1\text{-}2)$$

Periodic boundary conditions are assumed.

Making use of the isomorphism between S^3 and SU(2):

$$\vec{s} \in S^3 \longleftrightarrow u = \begin{pmatrix} s_1 + i s_2 & s_3 + i s_4 \\ -s_3 + i s_4 & s_1 - i s_2 \end{pmatrix} \in SU(2) \qquad (1\text{-}3)$$

the action can be rewritten as

$$L(u) = \sum_b L(u_b) \qquad (1\text{-}4)$$

with

$$L(u_b) = \frac{\beta}{2}\,\mathrm{tr}\,u_b \tag{1-5}$$

$$u_b = u_x u_y^{-1} \quad \text{for} \quad b = \langle x y \rangle \tag{1-6}$$

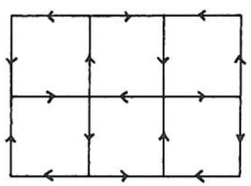
The partition function now reads

$$Z = \int \prod_x du_x \; e^{L(u)} \tag{1-7}$$

where du is the normalized Haar measure on SU(2).

There are several ways to perform the integration over the group variables.

One possible approach starts with transforming the partition function into an integral over link variables $u_b = u_x u_y^{-1}$. To this end it is useful to orient the links of the lattice in the way shown in figure 1.



- Figure 1 -

The orientation of links is always clockwise or counterclockwise around a given plaquette.

The link variables u_b are not independent: For each plaquette p consisting of four links they have to satisfy the constraint

$$\prod_{b\in\partial p} u_b = 1 \tag{1-8}$$

∂p denotes the boundary of the plaquette p.

The partition function is then

$$Z = \int \prod_b du_b \; \prod_b e^{L(u_b)} \; \prod_p \delta\Big(\prod_{b\in\partial p} u_b - 1\Big) \tag{1-9}$$

Since the integrand is a product of class functions we may expand each factor into characters of irreducible unitary representations of SU(2) which are labelled by half integers 0, 1/2, 1 ,...

$$\delta(u-1) = \sum_{\ell\in\frac{1}{2}\mathbb{Z}_+} (2\ell+1)\,\chi_\ell(u) \tag{1-10}$$

$$e^{\frac{\beta}{2}\mathrm{tr}\,u} = \sum_{j\in\frac{1}{2}\mathbb{Z}_+} c_j\,\chi_j(u) \tag{1-11}$$

The expansion coefficients c_j are given in terms of modified Bessel functions $I_{2j+1}(\beta)$.

$$c_j \equiv c_j(\beta) = (2j+1)\frac{2}{\beta}\, I_{2j+1}(\beta) \tag{1-12}$$

Thus, each link carries a half integer j, each plaquette a half integer l.

The partition function now reads

$$Z = \sum_{\{j\}} \sum_{\{\ell\}} \int \prod_b du_b \; \prod_b c_{j_b}\,\chi_{j_b}(u_b) \cdot \prod_p (2\ell_p+1)\,\chi_{\ell_p}\Big(\prod_{b\in\partial p} u_b\Big) \tag{1-13}$$

After decomposing the characters

$$\chi_j(u_b) = \sum_{m=-j}^{j} \mathcal{D}^j_{mm}(u_b) \tag{1-14}$$

$$\chi_\ell\Big(\prod_{b\in\partial p} u_b\Big) = \sum_{\mu_1\ldots\mu_4=-\ell}^{\ell} \mathcal{D}^\ell_{\mu_1\mu_2}(u_1)\,\mathcal{D}^\ell_{\mu_2\mu_3}(u_2)\,\mathcal{D}^\ell_{\mu_3\mu_4}(u_3)\,\mathcal{D}^\ell_{\mu_4\mu_1}(u_4) \tag{1-15}$$

where the D's denote representation matrices of SU(2), the integration over the group variables can be performed. Each integral involves three matrix elements and yields a product of two 3j-symbols [12].

$$\int du_b\, \mathcal{D}^j_{mm}(u_b)\, \mathcal{D}^{\ell_1}_{\mu_1\mu_1'}(u_b)\, \mathcal{D}^{\ell_2}_{\mu_2\mu_2'}(u_b) = \begin{pmatrix} j & \ell_1 & \ell_2 \\ m & \mu_1 & \mu_2 \end{pmatrix}\begin{pmatrix} j & \ell_1 & \ell_2 \\ m & \mu_1' & \mu_2' \end{pmatrix} \tag{1-16}$$

The angular momenta l_1 and l_2 belong to the two plaquettes sharing the link b.

The partition function is now given as a product of 3j-symbols.

$$Z = \sum_{\{j\}}\sum_{\{\ell\}}\sum_{\{m\}}\sum_{\{\mu\}} \prod_p (2\ell_p+1) \cdot \prod_b c_{j_b} \begin{pmatrix} j_b & \ell_1 & \ell_2 \\ m & \mu_1 & \mu_2 \end{pmatrix}\begin{pmatrix} j_b & \ell_1 & \ell_2 \\ m & \mu_1' & \mu_2' \end{pmatrix} \tag{1-17}$$

With a more cumbersome method, the same expression has been derived by Matsui [7] . After summing over the j and m variables in eq. (1-17) he derives a partition function which is written in terms of dual variables defined on a dual lattice. In order to obtain a system of statistical mechanics, he applies various rather involved approximations valid for low temperatures, e.g. the classical vector model.

A main disadvantage of Matsui's method is that it depends strongly on the choice of a special basis for the representation space of SU(2) so that the structure of the duality transformation remains rather blurred. For instance, it is far from obvious whether this approach can be extended to lattice gauge theories.

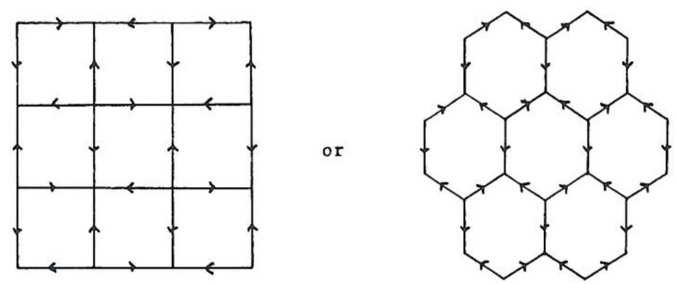
In this respect, Rühl's proposal [1,2] to use the Bargmann space formalism for the representation of SU(2) appears to be much more promising since it leads to expressions not dependent on a choice of basis. In the following, we will therefore adopt Rühl's approach.

We begin with a character expansion of the partition function (1-7) using eqs. (1-11) and (1-12).

$$\begin{aligned} Z &= \int \prod_x du_x \prod_b \sum_{j_b} c_{j_b}\, \chi_{j_b}(u_b) \\ &= \int \prod_x du_x \prod_{b\in\langle xy\rangle} \sum_{j_b} c_{j_b} \sum_{m_b, m_b'=-j_b}^{j_b} \mathcal{D}^{j_b}_{m_b m_b'}(u_x)\, \mathcal{D}^{j_b}_{m_b' m_b}(u_y^{-1}) \end{aligned} \tag{1-18}$$

As mentioned above, the integration over the group variables will be done in the Bargmann space formalism [9] which will be described in the next section.

It will prove convenient to orient the links in alternating order, as shown in figure 2.



- Figure 2 -

Consequently, the sites of the lattice fall into a set Λ_i of starting points of links and a set Λ_f of end points.
Then, all links are labelled by $b = \langle xy \rangle$ with $x \in \Lambda_i$, $y \in \Lambda_f$.

In order to be consistent with the periodic boundary conditions, it is necessary to assume that the lattice contains an even number of sites in every direction.

1.2 The Bargmann space formalism

In the Bargmann space formalism [9], a Hilbert space of entire analytic functions over $\mathbb{C}^2$ is introduced with Gaussian measure

$$d\mu(z) = \frac{1}{\pi^2} \prod_{i=1,2} dx_i\, dy_i\; e^{-x_i^2 - y_i^2}$$

$$\equiv dz\, dz^+\; e^{-z^+ z} \tag{1-19}$$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2 \;; \qquad z_i = x_i + i y_i$$

This space carries the unitary representation of SU(2)

$$T_u f(z) = f(u^T z), \quad u \in SU(2) \tag{1-2o}$$

which contains every unitary irreducible representation of SU(2) once. The analytic homogeneous polynomials of degree 2j form a subspace carrying the irreducible representation D^j of spin $j \in 1/2\, \mathbb{Z}_+$. A basis of this subspace is given by the homogeneous polynomials

$$v_m^j(z) = \frac{z_1^{j+m}\, z_2^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}}, \quad -j \le m \le j \tag{1-21}$$

The representation matrices D^j take the form

$$D^j_{mm'}(u) = (v^j_m, T_u v^j_{m'})$$

$$= \int d\mu(z)\, d\mu(z')\, \bar{v}^j_m(z)\, K(u; z, z')\, v^j_{m'}(z') \tag{1-22}$$

where the kernel

$$K(u;z,z') \equiv e^{z'^{+}\cdot u^T z} \tag{1-23}$$

satisfies the equation [1,9]

$$\begin{aligned}&\int d\mu(z')\,K(1;z',z)\,f(z')\\ &= \int d\mu(z')\,e^{z^{+}\cdot z'} f(z') = f(z)\end{aligned} \tag{1-24}$$

If we denote the $\mathbb{C}^2$ variables associated with m by ζ, the others by z, the partition function becomes

$$\begin{aligned}Z = \int \prod_x du_x \prod_{b=\langle xy\rangle} \Big\{ &\int d\mu(\zeta_{b,x})\,d\mu(z_{b,x})\,d\mu(\zeta_{b,y})\,d\mu(z_{b,y})\\ &\cdot \sum_{j_b} c_{j_b} \sum_{m_b,m_b'} \bar{v}^{j_b}_{m_b}(\zeta_{b,x})\,K(u_x;\zeta_{b,x},z_{b,x})\,v^{j_b}_{m_b'}(z_{b,x})\\ &\cdot \bar{v}^{j_b}_{m_b'}(z_{b,y})\,K(u_y^{-1};z_{b,y},\zeta_{b,y})\,v^{j_b}_{m_b}(\zeta_{b,y})\Big\}\end{aligned} \tag{1-25}$$

By means of the formula [2]

$$\sum_{m=-j}^{j} \bar{v}^j_m(z)\,v^j_m(z') = \frac{(z^+z')^{2j}}{(2j)!} \equiv Q^j(z,z') \tag{1-26}$$

the summations over m, m' can be performed,

$$\begin{aligned}Z = \int \prod_x du_x\, \mathcal{D}\mu(\zeta,z) \prod_{b=\langle xy\rangle} \Big\{ &\sum_{j_b} c_{j_b}\, Q^{j_b}(\zeta_{b,x},\zeta_{b,y})\\ &\cdot Q^{j_b}(z_{b,y},z_{b,x})\,K(u_x;\zeta_{b,x},z_{b,x})\,K(u_y^{-1};z_{b,y},\zeta_{b,y})\end{aligned} \tag{1-27}$$

with the abbreviation

$$\mathcal{D}\mu(\zeta,z) = \prod_b d\mu(\zeta_{b,x})\,d\mu(z_{b,x})\,d\mu(\zeta_{b,y})\,d\mu(z_{b,y}) \tag{1-28}$$

At this stage, the partition function (1-27) shows no longer any dependence on a choice of basis for the representation space.

The integrals over the SU(2) elements are evaluated with the help of the formula

$$\begin{aligned}&\int du\, \exp \sum_i z_i^{+}\cdot u^T z_i'\\ &= \frac{i}{2\pi}\oint dv\, e^{-\frac{1}{v}} \exp v \sum_{(ij)} (z_i'\,\varepsilon\, z_j')(z_i^{+}\,\varepsilon^{-1} z_j^{+})\end{aligned} \tag{1-29}$$

which is proved in appendix A. Summation is over unordered pairs (ij), i.e. (12) and (21) are not counted separately.

ε is the totally antisymmetric tensor in 2 dimensions, with normalization $\varepsilon_{12} = 1$.

The result of the integration is

$$\begin{aligned}Z = \int \prod_x \Big[\frac{i}{2\pi} dv_x\, e^{-\frac{1}{v_x}}\Big]\, &\mathcal{D}\mu(\zeta,z)\\ &\cdot \prod_b \sum_{j_b} c_{j_b}\, Q^{j_b}(\zeta_{b,x},\zeta_{b,y})\,Q^{j_b}(z_{b,y},z_{b,x})\\ &\cdot \prod_{x\in\Lambda_i} \exp\Big[v_x \sum_{(b,b')\wedge x} (\zeta_{b,x}\,\varepsilon\,\zeta_{b',x})(z^{+}_{b,x}\,\varepsilon^{-1} z^{+}_{b',x})\Big]\\ &\cdot \prod_{y\in\Lambda_f} \exp\Big[v_y \sum_{(b,b')\wedge y} (z_{b,y}\,\varepsilon\, z_{b',y})(\zeta^{+}_{b,y}\,\varepsilon^{-1} \zeta^{+}_{b',y})\Big]\end{aligned} \tag{1-3o}$$

$(b,b')\wedge x$ denotes an unordered pair of links touching at the site x.

1.3 Integration over all rotation variant variables

By introducing rotation invariant auxiliary variables η, $\bar{\eta}$ the quartic terms in the exponent of eq. (1-3o) can be brought into quadratic form.

$$\exp \upsilon (\varrho_i \varepsilon \varrho_j)(z_i^+ \varepsilon^{-1} z_j^+) = \frac{1}{\pi} \int_{\mathbb{C}} d\eta \, d\bar{\eta} \, e^{-\eta\bar{\eta}} \exp\{\varrho_i \varepsilon \varrho_j \eta + \upsilon z_i^+ \varepsilon^{-1} z_j^+ \bar{\eta}\} \tag{1-31}$$

We associate a complex variable

$$\eta_{bb'} = -\eta_{b'b} \tag{1-32}$$

with each pair of distinct links (b,b') that share a site.

The partition function is now

$$\begin{aligned} Z = & \int D\upsilon \, D\mu(\eta) D\mu(\varrho, z) \\ & \cdot \prod_b \sum_{j_b} c_{j_b} Q^{j_b}(\varrho_{b,x}, \varrho_{b,y}) Q^{j_b}(z_{b,y}, z_{b,x}) \\ & \cdot \prod_{x\in\Lambda_i} \exp\{\tfrac{1}{2} \sum_{(b,b')\wedge x} (\varrho_{b,x} \varepsilon \varrho_{b',x} \eta_{bb'} + \upsilon_x z_{b,x}^+ \varepsilon^{-1} z_{b',x}^+ \bar{\eta}_{bb'})\} \\ & \cdot \prod_{y\in\Lambda_f} \exp\{\tfrac{1}{2} \sum_{(b,b')\wedge y} (\varrho_{b,y}^+ \varepsilon^{-1} \varrho_{b',y}^+ \bar{\eta}_{bb'} + \upsilon_y z_{b,y} \varepsilon z_{b',y} \eta_{bb'})\} \end{aligned} \tag{1-33}$$

with the abbreviations

$$D\upsilon \equiv \prod_x \frac{i}{2\pi} d\upsilon_x \, e^{-\frac{1}{\upsilon_x}} \tag{1-34}$$

and

$$D\mu(\eta) \equiv \prod_{(b,b')} \frac{1}{\pi} d\eta_{bb'} d\bar{\eta}_{bb'} \, e^{-\eta_{bb'}\bar{\eta}_{bb'}} \tag{1-35}$$

The product $\prod_{(b,b')}$ runs over unordered pairs, whereas the sum $\sum_{(b,b')}$ in the exponent is now over ordered pairs of links. This is compensated by the factor 1/2.

The projector Q^j can be represented by a complex contour integral [2]

$$Q^j(z,z') = \frac{1}{2\pi i} \oint \frac{d\tau}{\tau^{2j+1}} \, e^{\tau z^+ z'} \tag{1-36}$$

Thus

$$Q^j(z,z') Q^j(\varrho,\varrho') = \frac{1}{(2\pi i)^2} \oint d\tau \, d\tau' \, e^{\tau z^+ z' + \tau' \varrho^+ \varrho'} \cdot \left[(\tau\tau')^{-2j-1} - (\tau\tau')^{2j+1}\right] \tag{1-37}$$

The term proportional to $(\tau\tau')^{2j+1}$ does not contribute to the integral but is inserted to make the summation over j feasible. The sum to evaluate is

$$B(\tau) \equiv \sum_j c_j (\tau^{-2j-1} - \tau^{2j+1}) \tag{1-38}$$

If $|\tau|$ is chosen equal to unity, $\tau = e^{i\varphi/2}$, then

$$\begin{aligned} B(\tau) &= -2i \sum_j c_j \sin(2j+1)\varphi/2 \\ &= -2i \sin\varphi/2 \sum_j c_j \chi_j(u) \\ &= -2i \sin\varphi/2 \, e^{L(u)} \end{aligned} \tag{1-39}$$

with u being a rotation by the angle of φ.

$$u = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \tag{1-4o}$$

Analytic continuation leads to

$$B(\tau) = \left(\frac{1}{\tau} - \tau\right) e^{\frac{\beta}{2}\left(\tau + \frac{1}{\tau}\right)} \qquad (1\text{-}41)$$

Essentially, the link variable $\tilde{\tau}_b \equiv \tau_b \tau'_b$ corresponds to the invariant rotation angle of the SU(2) element u_b. That means: Some U(1) variables, U(1) being the maximal torus of SU(2), survive, whereas the rotation variant parts of the field variables are integrated out.

The partition function now reads

$$Z = \int \mathcal{D}\upsilon \, \mathcal{D}\mu(\eta) \, \mathcal{D}\mu(\varrho, z)$$
$$\cdot \prod_b \oint \frac{d\tau_b \, d\tau'_b}{(2\pi i)^2} B(\tau_b \tau'_b) \, e^{\tau_b z^+_{b,y} z_{b,x} + \tau'_b \varrho^+_{b,x} \varrho_{b,y}} \qquad (1\text{-}42)$$
$$\cdot \prod_{x \in \Lambda_i} \exp\left\{\frac{1}{2} \sum_{(b,b')\wedge x} \left(\varrho_{b,x} \varepsilon \varrho_{b',x} \eta_{bb'} + \upsilon_x z^+_{b,x} \varepsilon^{-1} z^+_{b',x} \bar{\eta}_{bb'}\right)\right\}$$
$$\cdot \prod_{y \in \Lambda_f} \exp\left\{\frac{1}{2} \sum_{(b,b')\wedge y} \left(\varrho^+_{b,y} \varepsilon^{-1} \varrho^+_{b',y} \bar{\eta}_{bb'} + \upsilon_y z_{b,y} \varepsilon z_{b',y} \eta_{bb'}\right)\right\}$$

The formula (1-24) of the reproducing kernel for the Bargmann space

$$\int d\mu(z') \, e^{z^+ \cdot z'} f(z') = f(z) \qquad (1\text{-}43)$$

allows to perform the integrations over $z_{b,x}$ and $\zeta_{b,y}$ with $x \in \Lambda_i$, $y \in \Lambda_f$. This amounts to the substitution

$$z_{b,x} \rightarrow \bar{\tau}_b z_{b,y} \; ; \; \varrho_{b,y} \rightarrow \bar{\tau}'_b \varrho_{b,x} \qquad (1\text{-}44)$$

Afterwards, each link carries just one z and one ζ.

With the notation

$$\mathcal{D}'\mu(\varrho, z) \equiv \prod_b d\mu(\varrho_b) \, d\mu(z_b)$$
$$z_b \equiv z_{b,y} \; ; \; \varrho_b \equiv \varrho_{b,x} \qquad (1\text{-}45)$$

the result of the integration is

$$Z = \int \mathcal{D}\upsilon \, \mathcal{D}\mu(\eta) \, \mathcal{D}'\mu(\varrho, z) \oint \prod_b \frac{d\tau_b \, d\tau'_b}{(2\pi i)^2} B(\tau_b \tau'_b) \qquad (1\text{-}46)$$
$$\cdot \prod_{x \in \Lambda_i} \exp\left\{\frac{1}{2} \sum_{(b,b')\wedge x} \left(\varrho_b \varepsilon \varrho_{b'} \eta_{bb'} + \upsilon_x \tau_b \tau_{b'} z^+_b \varepsilon^{-1} z^+_{b'} \bar{\eta}_{bb'}\right)\right\}$$
$$\cdot \prod_{y \in \Lambda_f} \exp\left\{\frac{1}{2} \sum_{(b,b')\wedge y} \left(\tau'_b \tau'_{b'} \varrho^+_b \varepsilon^{-1} \varrho^+_b \bar{\eta}_{bb'} + \upsilon_y z_b \varepsilon z_{b'} \eta_{bb'}\right)\right\}$$

We define antisymmetric matrices λ, λ' and κ, κ', the elements of which are labelled by the links of the lattice.

$$\lambda_{bb'} = \begin{cases} \eta_{bb'} & \text{if } (b,b')\wedge x \text{ for any } x \in \Lambda_i \\ 0 & \text{otherwise} \end{cases}$$

$$\kappa_{bb'} = \begin{cases} \tau'_b \tau'_{b'} \bar{\eta}_{bb'} & \text{if } (b,b')\wedge y \text{ for any } y \in \Lambda_f \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda'_{bb'} = \begin{cases} \upsilon_y \eta_{bb'} & \text{if } (b,b')\wedge y \text{ for any } y \in \Lambda_f \\ 0 & \text{otherwise} \end{cases} \qquad (1\text{-}47)$$

$$\kappa'_{bb'} = \begin{cases} \upsilon_x \tau_b \tau_{b'} \bar{\eta}_{bb'} & \text{if } (b,b')\wedge x \text{ for any } x \in \Lambda_i \\ 0 & \text{otherwise} \end{cases}$$

In appendix B, we prove the formula [13]

$$\int \prod_k d\mu(z_k) \exp \frac{1}{2} \sum_{ij} \left(z_i \varepsilon z_j \lambda_{ij} + z_i^+ \varepsilon^{-1} z_j^+ \kappa_{ij} \right) = \det (1 - \lambda\kappa)^{-1} \tag{1-48}$$

valid for antisymmetric matrices λ and κ. Application of this formula leads to the partition function

$$Z = \int \mathcal{D}v \, \mathcal{D}\mu(\eta) \oint \prod_b \frac{d\tau_b \, d\tau_b'}{(2\pi i)^2} \, B(\tau_b \tau_b') \cdot \det (1 - \lambda\kappa)^{-1} \det (1 - \lambda'\kappa')^{-1} \tag{1-49}$$

which involves only rotation invariant variables.

2. Loop expansion

The formula

$$\det (1 - \lambda\kappa)^{-1} = e^{-\mathrm{tr} \ln (1 - \lambda\kappa)} \tag{2-1}$$

can be used to derive a loop expansion. The circular path of the line integrals over τ variables should be chosen such that no cut of the logarithm is crossed. Since κ, $\kappa' \to 0$ when τ, $\tau' \to 0$ this is possible, but it requires that the integrations over the η's are performed after the integrations over the τ's only. Keeping this restriction in mind, we may expand

$$\det (1 - \lambda\kappa)^{-1} = \exp \sum_{n \geq 1} \frac{1}{n} \mathrm{tr} (\lambda\kappa)^n \tag{2-2}$$

The contributions to $(\lambda\kappa)^n$ can be represented by graphs composed of 2n double links which belong alternatingly to points of Λ_i and Λ_f. Therefore, the graph of lowest, i.e. second, order is a plaquette.

Consider an oriented closed path C of order n_C consisting of double links (b_1,b_2), (b_2,b_3) ... (b_{2n},b_1). The algebraic expression corresponding to C involves the product $\prod_{b \in C} \tau_b'$ and

$$\eta(C) \equiv \eta_{b_1 b_2}(x_1) \, \bar{\eta}_{b_2 b_3}(y_2) \ldots \bar{\eta}_{b_{2n} b_1}(y_{2n}) \tag{2-3}$$

The same path C appears in $(\lambda'\kappa')^n$, but τ_b' is replaced by τ_b, $\eta_{bb'}(x)$ by $\bar{\eta}_{bb'}(x)$, and each site carries a factor v_x.

The determinants are thus replaced by the loop expansion

$$\det (1 - \lambda\kappa)^{-1} \det (1 - \lambda'\kappa')^{-1} = \exp \sum_C \frac{1}{n_C} \left\{ \prod_{b \in C} \tau_b' \eta(C) + \prod_{b \in C} \tau_b \prod_{x \in C} v_x \, \bar{\eta}(C) \right\} \tag{2-4}$$

where $\bar{\eta}(C)$ is the complex conjugate of $\eta(C)$.

The sum extends over all oriented closed paths C that visit sites of Λ_i and Λ_f alternatingly, i.e. no spikes like $\perp$ are possible. Paths which contain the same double links but start at different sites or have opposite orientation are not identified.

Setting $|\tau_b v_x/\tau_b'| = 1$, we perform the variable transformations

$$\eta_{bb'}(x) \longrightarrow \begin{cases} \left(\dfrac{\tau_b v_x}{\tau_b'}\right)^{1/2} \eta_{bb'}(x) & x \in \Lambda_i \\ \left(\dfrac{\tau_b v_x}{\tau_b'}\right)^{-1/2} \eta_{bb'}(x) & x \in \Lambda_f \end{cases} \tag{2-5}$$

and

$$\tau_b \longrightarrow \tau_b \tau_b'^{-1} \tag{2-6}$$

Having done the now trivial integration over τ', we finally arrive at

$$Z = \int \mathcal{D}v\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau \cdot \exp \sum_C \frac{1}{n_C} \tau(C) v(C) \{\eta(C) + \bar{\eta}(C)\} \tag{2-7}$$

where

$$\tau(C) \equiv \prod_{b \in C} \tau_b^{1/2} \tag{2-8}$$

$$v(C) \equiv \prod_{x \in C} v_x^{1/2} \tag{2-9}$$

$$\mathcal{D}\tau \equiv \prod_b \frac{d\tau_b}{2\pi i} B(\tau_b) \tag{2-10}$$

We are thus led to a system of closed non-backtracking loops and a Hamiltonian which is complex and, surprisingly, invariant under U(1) gauge transformations

$$\eta_{bb'}(x) \longrightarrow \eta_{bb'}(x)\, g_b(x)\, g_{b'}(x) \tag{2-11}$$

If we define two closed paths to be equivalent if they differ only in initial point or direction we may replace the sum over paths C in the partition function (2-7) by a sum over equivalence classes of paths.

Every path C can be represented uniquely in the form

$$C = C_0 \circ C_0 \circ \ldots \circ C_0 \equiv C_0^{p_C} \tag{2-12}$$

where C_0 is a simple closed path, i.e. cannot be expressed as a power of a path of lower order. This defines an integer p_C for every closed path C. The corresponding equivalence class $\tilde{C}$ contains $2n_C/p_C$ paths; the factor 2 results from identifying paths with opposite direction. For functions f(C) which depend on C only through its equivalence class $\tilde{C}$ we may thus write

$$\sum_C \frac{1}{n_C} f(C) = \sum_{\tilde{C}} \frac{2}{p_C} f(C) \tag{2-13}$$

We use eq. (2-13) to rewrite the partition function (2-7) in the form

$$Z = \int \mathcal{D}v\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau \cdot \exp \sum_{\tilde{C}} \frac{2}{p_C} \tau(C) v(C) \{\eta(C) + \bar{\eta}(C)\} \tag{2-14}$$

Finally, we may replace the sum over all equivalence classes $\tilde{C}$ by a sum over equivalence classes $\tilde{C}_o$ of simple paths only.

$$\sum_{\tilde{C}} \frac{2}{p_c} \tau(C)\sigma(C)\{\eta(C)+\bar{\eta}(C)\}$$

$$= 2\sum_{\tilde{C}_o}\sum_{p=1}^{\infty}\frac{1}{p}[\tau(C_o)\sigma(C_o)]^p\{\eta(C_o)^p+\bar{\eta}(C_o)^p\} \qquad (2\text{-}15)$$

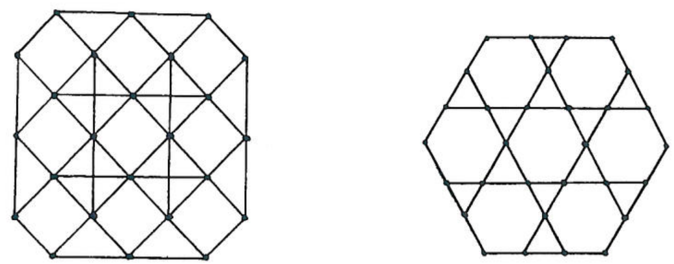
$$= -2\sum_{\tilde{C}_o} \ln\{1-\tau(C_o)\sigma(C_o)\eta(C_o)\}\{1-\tau(C_o)\sigma(C_o)\bar{\eta}(C_o)\}$$

Thus we obtain the partition function

$$Z = \int \mathcal{D}\sigma\,\mathcal{D}\mu(\eta)\,\mathcal{D}\tau \prod_{\tilde{C}_o}\{1-\tau(C_o)\sigma(C_o)\eta(C_o)\}^{-2} \cdot \{1-\tau(C_o)\sigma(C_o)\bar{\eta}(C_o)\}^{-2} \qquad (2\text{-}16)$$

3. Formulation of the Heisenberg ferromagnet as a local gauge theory

One might be tempted to think of the phase of $\eta_{bb'}$ as a parallel transporter of a lattice gauge theory on a lattice whose sites are our links (see figure 3). However, this is not possible because the η satisfy the antisymmetry condition (1-32) instead of $\eta_{bb'}(x) = \bar{\eta}_{b'b}(x)$. Therefore, the parallel transporter would not go over into its inverse under reversal of the direction of the link.



- Figure 3 -

However, it is possible to transform our system into a local gauge theory with a nonabelian but solvable gauge group $Z_2 \otimes U(1) \equiv T^*$.

Separating the η phase

$$\eta \equiv r\vartheta, \quad \vartheta \equiv e^{i\varphi}$$
$$\mathcal{D}\mu(\eta) = \mathcal{D}\mu(r)\,\mathcal{D}\vartheta \qquad (3\text{-}1)$$

with

$$\mathcal{D}\mu(r) \equiv \prod_{(bb')} \frac{1}{2} r_{bb'}\, dr_{bb'}\, e^{-r_{bb'}^2} \qquad (3\text{-}2)$$

$$\mathcal{D}\vartheta \equiv \prod_{(bb')} \frac{1}{2\pi}\, d\varphi_{bb'} \qquad (3\text{-}3)$$

the partition function (2-7) reads

$$Z = \int \mathcal{D}v\, \mathcal{D}\mu(r)\, \mathcal{D}\vartheta\, \mathcal{D}\tau \cdot \exp \sum_c \frac{1}{n_c} \tau(C)\, v(C)\, r(C) \left\{ \vartheta(C) + c.c. \right\} \qquad (3\text{-}4)$$

where

$$r(C) = |\eta(C)| \qquad (3\text{-}5)$$

$$\vartheta(C) = \vartheta_{b_1 b_2}(x_1)\, \bar{\vartheta}_{b_2 b_3}(y_2) \dots \bar{\vartheta}_{b_{2n} b_1}(y_{2n}) \qquad (3\text{-}6)$$

If we define 2x2 matrices

$$t_{bb'} = \begin{pmatrix} \vartheta_{bb'} & 0 \\ 0 & \bar{\vartheta}_{bb'} \end{pmatrix} \qquad (3\text{-}7)$$

which are elements of the maximal torus T = U(1) of SU(2) we may write

$$\vartheta(C) + c.c. = \mathrm{tr}\, t(C) \qquad (3\text{-}8)$$

$$t(C) \equiv t_{b_1 b_2}(x_1)\, t^*_{b_2 b_3}(y_2) \dots t^*_{b_{2n} b_1}(y_{2n}) \qquad (3\text{-}9)$$

The Weyl group W = T^*/T of SU(2), where T^* is the normalizer of T, consists of two elements

$$\omega = \{ 1, \hat{\tau} \} \qquad (3\text{-}1o)$$

where

$$1 = T\,, \quad \hat{\tau} = T\tau\,, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (3\text{-}11)$$

τ corresponds to complex conjugation.

$$\tau\, t\, \tau^{-1} = t^*\,, \qquad \tau^{-1} = -\tau \qquad (3\text{-}12)$$

We may thus write

$$t(C) = (-1)^{n_c}\, t_{b_1 b_2}(x_1)\, \tau\, t_{b_2 b_3}(y_2)\, \tau \dots \tau\, t_{b_{2n} b_1}(y_{2n})\, \tau \qquad (3\text{-}13)$$

We regard the variables $t\tau \in T^*$ as the new variables of our system. They show the desired behavior under U(1) gauge transformations.

$$(t\tau)_{bb'} \longrightarrow g_b\, t_{bb'}\, g_{b'}\, \tau = g_b\, (t\tau)_{bb'}\, g_{b'}^{-1}\,, \quad g \in T \qquad (3\text{-}14)$$

and

$$(t\tau)_{bb'} = (t\tau)^{-1}_{b'b} \qquad (3\text{-}15)$$

We introduce T^* variables

$$u_{bb'} \equiv t_{bb'}\, \sigma_{bb'}\,, \quad t \in T\,, \quad \sigma \in \{1, \tau\} \qquad (3\text{-}16)$$

In contrast to $\eta_{bb'}$, they satisfy

$$u_{bb'} = u^{-1}_{b'b} \qquad (3\text{-}17)$$

The partition function in terms of these new variables is

$$Z = \int \mathcal{D}v \, \mathcal{D}\mu(r) \, \mathcal{D}'u \, \mathcal{D}\tau \cdot \exp \sum_c \frac{(-1)^{n_c}}{n_c} \tau(c) v(c) r(c) \, \mathrm{tr} \, u(c) \tag{3-18}$$

where

$$u(c) \equiv \prod_{(bb')\in c} u_{bb'} \tag{3-19}$$

and

$$\mathcal{D}'u \equiv \prod_{(bb')} du_{bb'} \, \delta_\tau(u_{bb'}) \tag{3-20}$$

du is a Haar measure on T , viz.

$$\int du \, f(u) = \int dt \sum_{\sigma=1,\tau} f(t\sigma) \tag{3-21}$$

and

$$\delta_\sigma(u) = \begin{cases} 1 & \text{if } \sigma' = \sigma, \\ 0 & \text{if } \sigma' \neq \sigma, \end{cases} \qquad u = t\sigma' \tag{3-22}$$

The exponent of eq. (3-18) is invariant under T^* transformations so that each configuration which is obtained from the original one by a W transformation

$$u_{bb'} \to \sigma_b u_{bb'} \sigma_{b'}^{-1} \quad , \quad \sigma \in \{1, \tau\} \tag{3-23}$$

gives the same contribution.

If we started on a quadratic lattice the allowed configurations are determined by the gauge invariant constraints

$$\begin{aligned} &\delta_\tau(u_{b_1b_2} u_{b_2b_3} u_{b_3b_1}) = 1 && \text{for each triangle} \\ &\delta_1(u_{b_1b_2} u_{b_2b_3} u_{b_3b_4} u_{b_4b_1}) = 1 && \text{for each quadrangle} \end{aligned} \tag{3-24}$$

Thus we arrive at the partition function of a lattice gauge theory with local T^* invariance

$$Z = \int \mathcal{D}v \, \mathcal{D}\mu(r) \, \mathcal{D}u \, \mathcal{D}\tau \cdot \exp \sum_c \frac{(-1)^{n_c}}{n_c} \tau(c) v(c) r(c) \, \mathrm{tr} \, u(c) \tag{3-25}$$

with

$$\mathcal{D}u \equiv 2^{-2N} \prod_{(bb')} du_{bb'} \prod_{\Delta} \delta_\tau(uuu) \prod_{\diamond} \delta_1(uuuu) \tag{3-26}$$

where N is the number of sites of the original lattice.

4. Exact solution of the 1-dimensional model

On a 1-dimensional lattice of N sites, N even, with periodic boundary conditions, a simple loop has to contain every link of the lattice once. Thus, only one equivalence class C of simple loops contributes to the partition function (2-16).

$$Z = \int \mathcal{D}\sigma\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau \left[1 - \tau(C)\sigma(C)\eta(C)\right]^{-2} \cdot \left[1 - \tau(C)\sigma(C)\bar{\eta}(C)\right]^{-2} \tag{4-1}$$

We expand Z into a power series.

$$Z = \int \mathcal{D}\sigma\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} (k+1)(k'+1) \cdot \left[\tau(C)\sigma(C)\right]^{k+k'} \eta^{k}(C)\, \bar{\eta}^{k'}(C) \tag{4-2}$$

The integration over the η phase yields the constraint k = k'.

$$Z = \sum_{k=0}^{\infty} (k+1)^2 \int \mathcal{D}\sigma\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau \left[\tau(C)\sigma(C)|\eta(C)|\right]^{2k} \tag{4-3}$$

Reinserting eq. (1-38) into eq. (4-3) we are able to perform the integrations over the τ variables.

$$\int \mathcal{D}\tau \left[\tau(C)\right]^{2k} = \prod_{b\in\Lambda} \oint \frac{d\tau_b}{2\pi i} B(\tau_b)\, \tau_b^{k} = \prod_{b} \left\{ \sum_{j\in 1/2\,\mathbb{Z}_+} c_j \oint \frac{d\tau}{2\pi i} \tau^{-2j-1} \tau^{k} \right\} = c_{k/2}^{N} \tag{4-4}$$

Since

$$\int \mathcal{D}\sigma\, \sigma(C)^{2k} = \prod_{x\in\Lambda} \left\{ \frac{i}{2\pi} \oint d\sigma\, e^{-\frac{1}{\sigma}} \sigma^{k} \right\} = \left[(k+1)!\right]^{-N} \tag{4-5}$$

and

$$\int \mathcal{D}\mu(\eta)\, |\eta(C)|^{2k} = \prod_{(b,b')\in\Lambda} \left\{ \int_0^\infty \tfrac{1}{2} r\, dr\, r^{2k} e^{-r^2} \right\} = (k!)^{N} \tag{4-6}$$

we get

$$Z = \sum_{k=0}^{\infty} c_{k/2}^{N} (k+1)^{2-N} \tag{4-7}$$

Inserting the definition (1-12) we obtain the exact solution of the 1-dimensional Heisenberg ferromagnet

$$Z = \left(\frac{2}{\beta}\right)^{N} \sum_{k=1}^{\infty} k^2\, I_k^{N}(\beta) \tag{4-8}$$

5. High temperature expansion

The partition function (2-14) may serve as starting point for a high temperature expansion. In the standard derivation of this expansion [14] one uses the character expansion (1-18) to perform the integration over the group variables. One then has to deal with Clebsch-Gordan series, and Clebsch-Gordan coefficients are involved in the computation of complicated graphs. They will not appear in our formalism.

If we sum over equivalence classes C of paths which differ only in their initial points but not in direction the partition function (2-14) takes the form

$$Z = \int \mathcal{D}\sigma\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau \cdot \exp \sum_C \frac{1}{p_C}\, \tau(C)\, \sigma(C) \left\{ \eta(C) + \bar{\eta}(C) \right\} \tag{5-1}$$

We expand the exponential into a product of power series.

$$Z = \int \mathcal{D}\sigma\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau \sum_{\{k_C, k'_{C'}\}} \cdot \prod_C \frac{1}{k_C!} \left\{ \frac{1}{p_C}\, \tau(C)\, \sigma(C)\, \eta(C) \right\}^{k_C} \cdot \prod_{C'} \frac{1}{k'_{C'}!} \left\{ \frac{1}{p_{C'}}\, \tau(C')\, \sigma(C')\, \bar{\eta}(C') \right\}^{k'_{C'}} \tag{5-2}$$

We rearrange the sums over k, k' by introducing sets A, A' of directed double links. Then

$$\sum_{\{k_C, k'_{C'}\}} \longrightarrow \sum_{A, A'} \sum_{\substack{\{k_C, k'_{C'}\} \\ \Sigma C^{k_C} = A \\ \Sigma C'^{k'_{C'}} = A'}} \tag{5-3}$$

In an obvious way, A and A' specify sets of sites and of links. We denote by $n_x(A)$ the number of times the site x occurs in A, by $n_b(A)$ resp. $\bar{n}_b(A)$ the number of times the undirected resp. directed link b occurs in A, and by $n_{bb'}(A)$ resp. $\bar{n}_{bb'}(A)$ the number of times the undirected resp. directed double link (bb') occurs in A.

Since the integration over the phase of the η variables annihilates all terms with $A \neq A'$, the partition function may be written as

$$Z = \sum_A \kappa_A^2\, Z_A \tag{5-4}$$

with

$$\begin{aligned} Z_A &= \int \mathcal{D}\sigma\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau \prod_{x \in A} \sigma_x^{n_x(A)} \prod_{b \in A} \tau_b^{n_b(A)} \cdot \prod_{(bb') \in A} |\eta_{bb'}|^{2 n_{bb'}(A)} \\ &= \prod_{b \in A} c_{\frac{1}{2} n_b(A)} \prod_{(bb') \in A} n_{bb'}(A)! \Big/ \prod_{x \in A} (n_x(A)+1)! \end{aligned} \tag{5-5}$$

and the combinatorial factor

$$\kappa_A \equiv \sum_{\substack{\{k_C\} \\ \Sigma C^{k_C} = A}} \prod_C \frac{1}{k_C!}\, p_C^{-k_C} \tag{5-6}$$

If A allows no decomposition of the form $A = \Sigma\, C^{k_C}$, k_A is set equal to zero.

We define a combinatorial quantity

$$k_A' \equiv \frac{\prod_b \bar{n}_b(A)}{\prod_{(bb')} \bar{n}_{bb'}(A)} \tag{5-7}$$

The computation would simplify if we could assume the validity of the equality

$$k_A = k_A' \tag{5-8}$$

which we were not able to prove in general. As a conjecture, however, (5-8) is supported by several examples some of which are given in appendix C.

Using eqs. (5-5) and (5-6) it is now straightforward to perform a high temperature expansion of the partition function. In fact, the coefficients $c_j = c_j(\beta)$ which appear in eq. (5-5) are given in terms of modified Bessel functions (see eq. (1-12)) and thus allow a power series expansion for small β. The leading term is

$$c_j(\beta) = \frac{1}{(2j)!}\left(\frac{\beta}{2}\right)^{2j} \qquad \text{for} \quad \beta \to 0 \tag{5-9}$$

Thus, to leading order in β, we find the behavior

$$z_A \sim \beta^{\sum_{b\in A} n_b(A)} \qquad \text{for} \quad \beta \to 0 \tag{5-10}$$

Proceeding along these lines, one can reproduce the results of the standard high temperature expansion.

III. SU(2) LATTICE GAUGE THEORY IN 3 DIMENSIONS

6. Formulation in terms of gauge invariant variables

6.1 Definition of the model

We consider Wilson's action for an SU(2) gauge theory on a 3-dimensional cubic lattice $\Lambda \subset \mathbb{Z}^3$,

$$L(u) = \frac{\beta}{2} \sum_p \operatorname{tr} u_p \tag{6-1}$$

where u_p is the product of four SU(2) variables u_b attached to the links of the plaquette p.

$$u_p = \prod_{b\in\partial p} u_b^{(-1)} \tag{6-2}$$

The partition function of the system is

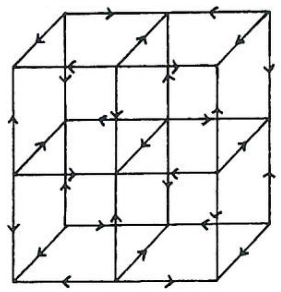
$$Z = \int \prod_{b\in\Lambda} du_b \, e^{L(u)} \tag{6-3}$$

du_b is the normalized Haar measure on SU(2).

We assume periodic boundary conditions. If we further assume that the lattice contains an even number of sites in every direction it is possible to orient the links in alternating order (see figure 4) so that the lattice consists of a set Λ_i of starting points of links and a set Λ_f of end points. Then each link is denoted by $b = \langle xy \rangle$, $x \in \Lambda_i$, $y \in \Lambda_f$.

On this lattice, each plaquette variable u_p takes the form

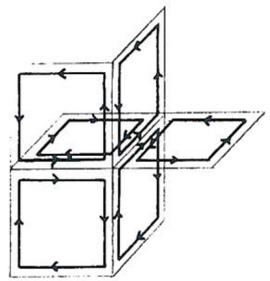
$$u_p = u_{b_1} u_{b_2}^{-1} u_{b_3} u_{b_4}^{-1} \tag{6-4}$$



- Figure 4 -

Since $\mathrm{tr}\, u_p = \mathrm{tr}\, u_p^{-1}$, we have an additional freedom of choice for the orientation of each plaquette. We will use the notation $b \in \partial p$ resp. $b^{-1} \in \partial p$ if a given link b belonging to the boundary of the plaquette p has an orientation parallel resp. antiparallel to the orientation of the plaquette p. If $b^{-1} \in \partial p$ then u_p will contain the inverse u_b^{-1} rather than u_b.

In the present case, it is convenient to choose alternating orientation for the plaquettes, as shown in figure 5.



- Figure 5 -

6.2 Character expansion, integration over the group variables

The partition function (6-3) is expanded into characters of irreducible unitary representations of SU(2).

$$Z = \int \prod_b du_b \prod_p \sum_{j_b} c_{j_p} \chi_{j_p}(u_p) \tag{6-5}$$

The expansion coefficients c_j are given in terms of modified Bessel functions $I_{2j+1}(\beta)$.

$$c_j = \frac{2}{\beta}(2j+1)\, I_{2j+1}(\beta) \tag{6-6}$$

Each plaquette carries a representation j = 0, 1/2, 1, ... of SU(2).

The characters are now decomposed into

$$\chi_j(u_p) = \sum_{\substack{m_1 m_1' \\ m_2 m_2'}} \mathcal{D}^j_{m_1 m_1'}(u_{b_1})\, \mathcal{D}^j_{m_1' m_2}(u_{b_2}^{-1}) \cdot \mathcal{D}^j_{m_2 m_2'}(u_{b_3})\, \mathcal{D}^j_{m_2' m_1}(u_{b_4}^{-1}) \tag{6-7}$$

In complete analogy to our treatment of the Heisenberg ferromagnet in chapter 1, the integration over the field variables is performed in the Bargmann space formalism. (See section 1.2 for further details.) The representation matrices are now given by

$$\mathcal{D}^j_{mm'}(u) = \int d\mu(\zeta)\, d\mu(z')\, \bar{\omega}^j_m(\zeta) \cdot K(u;\zeta,z')\, \omega^j_{m'}(z') \tag{6-8}$$

Using

$$\mathcal{D}^{j}_{m'm}(u^{-1}) = (-1)^{2j-m-m'}\,\mathcal{D}^{j}_{-m,-m'}(u) \qquad (6\text{-}9)$$

we get

$$\mathcal{D}^{j}_{m'm}(u^{-1}) = (-1)^{2j-m-m'} \int d\mu(\zeta')\, d\mu(z) \cdot \bar{\sigma}^{j}_{-m'}(z)\, K(u;\zeta'^{+}, z^{+})\, \sigma^{j}_{-m}(\zeta') \qquad (6\text{-}10)$$

For each link b = ‹xy› of each plaquette p we have introduced two $\mathbb{C}^2$ vectors $\zeta^{(')}_{p,x}$ and $z^{(')}_{p,y}$, where the ζ resp. z variables are associated with the m resp. m' variables and therefore with Λ_i resp. Λ_f.

The partition function now reads

$$Z = \int \prod_b du_b \prod_p \Big\{ \sum_{j_p} c_{j_p} \prod_{\substack{x\in\partial p\\ x\in\Lambda_i}} \sum_{m_{p,x}} (-1)^{j_p - m_{p,x}} \cdot \prod_{\substack{y\in\partial p\\ y\in\Lambda_i}} \sum_{m_{p,y}} (-1)^{j_p - m_{p,y}} \prod_{\substack{b\in\partial p\\ b=\langle xy\rangle}} \Big[\int d\mu(\zeta_{p,x})\, d\mu(z'_{p,y}) \cdot \bar{\sigma}^{j_p}_{m_{p,x}}(\zeta_{p,x})\, K(u_b;\zeta_{p,x}, z'_{p,y})\, \sigma^{j_p}_{m'_{p,y}}(z'_{p,y}) \Big] \cdot \prod_{\substack{b^{-1}\in\partial p\\ b=\langle xy\rangle}} \Big[\int d\mu(\zeta'_{p,x})\, d\mu(z_{p,y})\, \bar{\sigma}^{j_p}_{-m'_{p,y}}(z_{p,y}) \cdot K(u_b;\zeta'^{+}_{p,x}, z^{+}_{p,y})\, \sigma^{j_p}_{-m_{p,x}}(\zeta'_{p,x}) \Big] \Big\} \qquad (6\text{-}11)$$

Doing the summation over m

$$\sum_{m=-j}^{j} (-1)^{j-m}\, \bar{\sigma}^{j}_{m}(\zeta)\, \sigma^{j}_{-m}(\zeta') = \frac{(\zeta^{+}\varepsilon\,\zeta')^{2j}}{(2j)!} \qquad (6\text{-}12)$$

and over m'

$$\sum_{m'=-j}^{j} (-1)^{j-m'}\, \sigma^{j}_{m'}(z')\, \bar{\sigma}^{j}_{-m'}(z) = \frac{(z'\varepsilon\, z^{+})^{2j}}{(2j)!} \qquad (6\text{-}13)$$

we get

$$Z = \int \prod_b du_b\, \mathcal{D}\mu(\zeta,z)\, \mathcal{D}\mu(\zeta',z') \prod_p \Big\{ \sum_{j_p} c_{j_p} \cdot \prod_{\substack{x\in\partial p\\ x\in\Lambda_i}} \frac{(\zeta'_{p,x}\,\varepsilon\,\zeta^{+}_{p,x})^{2j_p}}{(2j_p)!} \prod_{\substack{y\in\partial p\\ y\in\Lambda_f}} \frac{(z'_{p,y}\,\varepsilon\, z^{+}_{p,y})^{2j_p}}{(2j_p)!} \cdot \prod_{b\in\partial p} K(u_b;\zeta_{p,x}, z'_{p,y}) \prod_{b^{-1}\in\partial p} K(u_b;\zeta'^{+}_{p,x}, z^{+}_{p,y}) \Big\} \qquad (6\text{-}14)$$

where

$$\mathcal{D}\mu(\zeta,z) \equiv \prod_p \prod_{\substack{x\in\partial p\\ x\in\Lambda_i}} d\mu(\zeta_{p,x}) \prod_{\substack{y\in\partial p\\ y\in\Lambda_f}} d\mu(z_{p,y}) \qquad (6\text{-}15)$$

Setting

$$\gamma_{p,x}(b) = \begin{cases} \zeta_{p,x} & \text{if } b\in\partial p \\ \zeta'^{+}_{p,x} & \text{if } b^{-1}\in\partial p \end{cases} \qquad (6\text{-}16)$$

$$\alpha_{p,y}(b) = \begin{cases} z'_{p,y} & \text{if } b\in\partial p \\ z^{+}_{p,y} & \text{if } b^{-1}\in\partial p \end{cases} \qquad (6\text{-}17)$$

the integrals over the field variables u_b can be done by means of the formula (cp. eq. (1-29))

$$\int du \, \exp \sum_i \alpha_i^+ \cdot u^T \gamma_i = \frac{i}{2\pi} \oint dv \, e^{-\frac{1}{v}} \exp v \sum_{(ij)} (\gamma_i \varepsilon \gamma_j)(\alpha_i^+ \varepsilon^{-1} \alpha_j^+) \tag{6-18}$$

The summation is over unordered pairs (ij).

The partition function is now

$$\begin{aligned} Z = \int \prod_b \left(\frac{i}{2\pi} dv_b \, e^{-\frac{1}{v_b}} \right) D\mu(\rho, z) D\mu(\rho', z') \\ \cdot \prod_p \left\{ \sum_{j_p} c_{j_p} \prod_{\substack{x \in \partial p \\ x \in \Lambda_i}} \frac{(\rho'_{p,x} \varepsilon \rho^+_{p,x})^{2j_p}}{(2j_p)!} \right. \\ \left. \cdot \prod_{\substack{y \in \partial p \\ y \in \Lambda_f}} \frac{(z'_{p,y} \varepsilon z^+_{p,y})^{2j_p}}{(2j_p)!} \right\} \\ \cdot \prod_{b=\langle xy \rangle} \exp v_b \sum_{(p,p') \wedge b} (\gamma_{p,x} \varepsilon \gamma_{p',x})(\alpha^+_{p,y} \varepsilon^{-1} \alpha^+_{p',y}) \end{aligned} \tag{6-19}$$

$(p,p') \wedge b$ denotes an unordered pair of plaquettes with common link b.

The ε products in the exponential relate variables belonging to the same site but to different plaquettes.

With each corner of each plaquette we associate a complex variable $\tau_{p,x}$ by means of the complex contour integral

$$\frac{(z' \varepsilon z^+)^{2j}}{(2j)!} = \frac{1}{2\pi i} \oint \frac{d\tau}{\tau^{2j+1}} \, e^{\tau z' \varepsilon z^+} \tag{6-2o}$$

The sum over j yields a factor

$$B(\tau_p) = (\tau_p^{-1} - \tau_p) \, e^{\frac{\beta}{2}(\tau_p + \tau_p^{-1})} \tag{6-21}$$

$$\tau_p \equiv \prod_{z \in \partial p} \tau_{p,z} \tag{6-22}$$

for each plaquette (see section 1.3).

Thus we arrive at the partition function

$$\begin{aligned} Z = \int Dv \, D\mu(\rho, z) \, D\mu(\rho', z') \\ \cdot \prod_p \left\{ \prod_{\substack{x \in \partial p \\ x \in \Lambda_i}} \oint \frac{d\tau_{p,x}}{2\pi i} \, e^{\tau_{p,x} \rho'_{p,x} \varepsilon \rho^+_{p,x}} \right. \\ \left. \cdot \prod_{\substack{y \in \partial p \\ y \in \Lambda_f}} \oint \frac{d\tau_{p,y}}{2\pi i} \, e^{\tau_{p,y} z'_{p,y} \varepsilon z^+_{p,y}} \right\} \cdot B(\tau_p) \\ \cdot \prod_{b=\langle xy \rangle} \exp v_b \sum_{(p,p') \wedge b} (\gamma_{p,x} \varepsilon \gamma_{p',x})(\alpha^+_{p,y} \varepsilon^{-1} \alpha^+_{p',y}) \end{aligned} \tag{6-23}$$

with the abbreviation

$$Dv \equiv \prod_b \frac{i}{2\pi} dv_b \, e^{-\frac{1}{v_b}} \tag{6-24}$$

6.3 Elimination of all gauge variant variables - loop expansion

The quartic terms in the exponential of eq. (6-23) are transformed into quadratic ones by introducing one gauge invariant complex variable

$$\eta_{pp'} = -\eta_{p'p} \qquad \text{(6-25)}$$

for each pair of plaquettes (p,p') with a common link, and using the identity

$$\begin{aligned} &\exp \sigma(\gamma\varepsilon\gamma')(\alpha^+\varepsilon^{-1}\alpha'^+) \\ &= \frac{1}{\pi}\int_{\mathbb{C}} d\eta\, d\bar\eta\, e^{-\eta\bar\eta} \exp \sigma^{1/2}(\gamma\varepsilon\gamma'\eta + \alpha^+\varepsilon^{-1}\alpha'^+\bar\eta) \end{aligned} \qquad \text{(6-26)}$$

The partition function then becomes

$$\begin{aligned} Z = \int \mathcal{D}\sigma\, \mathcal{D}\mu(\eta)\, \mathcal{D}\mu(\xi, z)\, \mathcal{D}\mu(\xi', z') \\ \cdot \prod_p \Bigg\{ \prod_{\substack{x\in\partial p \\ x\in\Lambda_i}} \oint \frac{d\tau_{p,x}}{2\pi i}\, e^{\tau_{p,x}\xi'_{p,x}\varepsilon\xi^+_{p,x}} \\ \cdot \prod_{\substack{y\in\partial p \\ y\in\Lambda_f}} \oint \frac{d\tau_{p,y}}{2\pi i}\, e^{\tau_{p,y} z'_{p,y}\varepsilon z^+_{p,y}} \Bigg\} \cdot B(\tau_p) \\ \cdot \prod_{b=\langle xy\rangle} \exp \sigma_b^{1/2} \sum_{(p,p')\wedge b} (\gamma_{p,x}\varepsilon\gamma_{p',x}\eta_{pp'} + \alpha^+_{p,y}\varepsilon^{-1}\alpha^+_{p',y}\bar\eta_{pp'}) \end{aligned} \qquad \text{(6-27)}$$

with

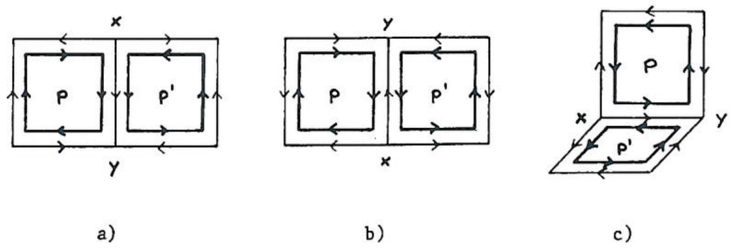
$$\mathcal{D}\mu(\eta) \equiv \prod_{(p,p')} \frac{1}{\pi}\, d\eta_{pp'}\, d\bar\eta_{pp'}\, e^{-\eta_{pp'}\bar\eta_{pp'}} \qquad \text{(6-28)}$$

The integration over ξ', z' amounts to the substitution (remember eq. (1-24))

$$\begin{aligned} \bar z'^{\,i}_{p,y} &\longrightarrow \varepsilon^{ij}\,\tau_{p,y}\,\bar z^{j}_{p,y} \\ \bar\xi'^{\,i}_{p,x} &\longrightarrow \varepsilon^{ij}\,\tau_{p,x}\,\bar\xi^{j}_{p,x} \end{aligned} \qquad \text{(6-29)}$$

within the exponent of the last line of eq. (6-27).

Due to our choices for the orientation of links and plaquettes (see section 6.1) we have to distinguish three cases which are illustrated in figure 6.



- Figure 6 -

The substitution (6-29) then yields

$$\gamma_{p,x}\,\varepsilon\,\gamma_{p',x} \longrightarrow \begin{cases} \xi_{p,x}\,\varepsilon\,\xi_{p',x} & \text{if } b\in\partial p, \partial p' \quad \text{(6-30a)} \\ \tau_{p,x}\tau_{p',x}\xi^+_{p,x}\,\varepsilon\,\xi^+_{p',x} & \text{if } b^{-1}\in\partial p, \partial p' \quad \text{(6-30b)} \\ \tau_{p',x}\,\xi_{p,x}\cdot\xi^+_{p',x} & \text{if } b\in\partial p, b^{-1}\in\partial p' \quad \text{(6-30c)} \end{cases}$$

$$\alpha^{+}_{\rho,y}\,\varepsilon^{-1}\alpha^{+}_{\rho',y} \rightarrow \begin{cases} \tau_{\rho,y}\,\tau_{\rho',y}\,z^{+}_{\rho,y}\,\varepsilon^{-1} z^{+}_{\rho',y} & \text{if } b\in\partial\rho,\partial\rho' \quad (6\text{-}31a) \\ z_{\rho,y}\,\varepsilon^{-1} z_{\rho',y} & \text{if } b^{-1}\in\partial\rho,\partial\rho' \quad (6\text{-}31b) \\ \tau_{\rho,y}\,z^{+}_{\rho,y}\cdot z_{\rho',y} & \text{if } b\in\partial\rho, b^{-1}\in\partial\rho' \quad (6\text{-}31c) \end{cases}$$

Evidently, only variables belonging to the same site interact. Consequently, the ζ and z integrals factorize.

For each site $x \in \Lambda_i$, we introduce two antisymmetric matrices λ and κ.

$$\lambda_{\rho\rho'}(x) = \begin{cases} v_b^{1/2}\,\eta_{\rho\rho'} & \text{if } x\in\partial\rho,\partial\rho' \text{ and } b\in\partial\rho,\partial\rho' \text{ for any b} \\ 0 & \text{otherwise} \end{cases} \tag{6-32}$$

$$\kappa_{\rho\rho'}(x) = \begin{cases} v_b^{1/2}\,\tau_{\rho,x}\,\tau_{\rho',x}\,\eta_{\rho\rho'} & \text{if } x\in\partial\rho,\partial\rho' \text{ and } b^{-1}\in\partial\rho,\partial\rho' \text{ for any b} \\ 0 & \text{otherwise} \end{cases} \tag{6-33}$$

and a matrix ρ

$$\varrho_{\rho\rho'}(x) = \begin{cases} -v_b^{1/2}\,\tau_{\rho',x}\,\eta_{\rho\rho'} & \text{if } x\in\partial\rho,\partial\rho' \text{ and } b\in\partial\rho,\ b^{-1}\in\partial\rho' \text{ for any b} \\ 0 & \text{otherwise} \end{cases} \tag{6-34}$$

Analogously for each $y \in \Lambda_f$:

$$\lambda'_{\rho\rho'}(y) = \begin{cases} v_b^{1/2}\,\bar\eta_{\rho\rho'} & \text{if } y\in\partial\rho,\partial\rho' \text{ and } b^{-1}\in\partial\rho,\partial\rho' \text{ for any b} \\ 0 & \text{otherwise} \end{cases} \tag{6-35}$$

$$\kappa'_{\rho\rho'}(y) = \begin{cases} v_b^{1/2}\,\tau_{\rho,y}\,\tau_{\rho',y}\,\bar\eta_{\rho\rho'} & \text{if } y\in\partial\rho,\partial\rho' \text{ and } b\in\partial\rho,\partial\rho' \text{ for any b} \\ 0 & \text{otherwise} \end{cases} \tag{6-36}$$

$$\varrho'_{\rho\rho'}(y) = \begin{cases} v_b^{1/2}\,\tau_{\rho',y}\,\bar\eta_{\rho\rho'} & \text{if } y\in\partial\rho,\partial\rho' \text{ and } b^{-1}\in\partial\rho, b\in\partial\rho' \text{ for any b} \\ 0 & \text{otherwise} \end{cases} \tag{6-37}$$

Then the partition function takes the form

$$\begin{aligned} Z = \int \mathcal{D}v\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau\, \mathcal{D}\mu(\zeta, z) \\ \cdot \prod_{x\in\Lambda_i} \exp \sum_{\rho\rho'} \Big(\tfrac{1}{2}\lambda_{\rho\rho'}\zeta_\rho\,\varepsilon\,\zeta_{\rho'} \\ + \tfrac{1}{2}\kappa_{\rho\rho'}\zeta^{+}_\rho\,\varepsilon^{-1}\zeta^{+}_{\rho'} + \varrho_{\rho\rho'}\zeta_\rho\cdot\zeta^{+}_{\rho'} \Big) \\ \cdot \prod_{y\in\Lambda_f} \exp \sum_{\rho\rho'} \Big(\tfrac{1}{2}\lambda'_{\rho\rho'} z_\rho\,\varepsilon\, z_{\rho'} \\ + \tfrac{1}{2}\kappa'_{\rho\rho'} z^{+}_\rho\,\varepsilon^{-1} z^{+}_{\rho'} + \varrho'_{\rho\rho'} z_\rho\cdot z^{+}_{\rho'} \Big) \end{aligned} \tag{6-38}$$

Note that we have made a transformation $\eta_{pp'} \rightarrow -\eta_{pp'}$ in the terms involving κ and λ'.

Furthermore, we have introduced the abbreviation

$$\mathcal{D}\tau \equiv \prod_{\rho} \Big\{ \prod_{z\in\partial\rho} \frac{d\tau_{\rho,z}}{2\pi i} \Big\}\, B(\tau_\rho) \tag{6-39}$$

The gaussian integrals over z and ζ are evaluated in appendix D. The result is the loop expansion

$$Z = \int \mathcal{D}\upsilon \, \mathcal{D}\mu(\eta) \, \mathcal{D}\tau$$

$$\cdot \prod_{x \in \Lambda_i} \exp \sum_{C} \frac{(-1)^{s_C}}{n_C} \, \upsilon(C) \, \tau(C) \, \eta(C) \tag{6-40}$$

$$\cdot \prod_{y \in \Lambda_f} \exp \sum_{C} \frac{(-1)^{s_C}}{n_C} \, \upsilon(C) \, \tau(C) \, \bar{\eta}(C)$$

where

$$\upsilon(C) \equiv \prod_{\substack{b \wedge (p,p') \\ (p,p') \in C}} \upsilon_b^{1/2} \tag{6-41}$$

$$\tau(C) \equiv \prod_{p \in C} \tau_{p,z} \tag{6-42}$$

$$\eta(C) \equiv \prod_{(p,p') \in C} \eta_{pp'} \tag{6-43}$$

s_C is the number of ordered pairs $(p,p') \in C$ with $b \in \partial p$, $b^{-1} \in \partial p'$, and n_C is the length of the path C in units of the lattice constant. For a given site $z \in \Lambda$, the sum in the exponent extends over all closed paths C consisting of ordered pairs (p_1,p_2), $(p_2,p_3) \ldots (p_n,p_1)$ with $(p_k,p_{k+1}) \wedge b$ for any $b \ni z$, and obeying the restriction that p_k, p_{k+1}, and p_{k+2} do not share a common link. Paths which are cyclic permutations of one another are not identified.

Again we have to keep in mind that we are allowed to integrate over the η variables only after integration over the τ variables (see chapter 2).

In a manner analogous to that of chapter 2, the sums over paths in eq. (6-40) can be replaced by sums over equivalence classes $\tilde{C}$ of paths. Again, we call two closed paths equivalent if they differ only in initial point or direction. Also, every path C can again be represented uniquely in the form $C = C_0^{p_C}$, where C_0 is a simple closed path. Then on account of eq. (2-13), the partition function can be rewritten as

$$Z = \int \mathcal{D}\upsilon \, \mathcal{D}\mu(\eta) \, \mathcal{D}\tau$$

$$\cdot \prod_{x \in \Lambda_i} \exp 2 \sum_{\tilde{C}} \frac{(-1)^{s_C}}{p_C} \, \upsilon(C) \, \tau(C) \, \eta(C) \tag{6-44}$$

$$\cdot \prod_{y \in \Lambda_f} \exp 2 \sum_{\tilde{C}} \frac{(-1)^{s_C}}{p_C} \, \upsilon(C) \, \tau(C) \, \bar{\eta}(C)$$

Using a relation similar to eq. (2-15), the partition function eventually becomes

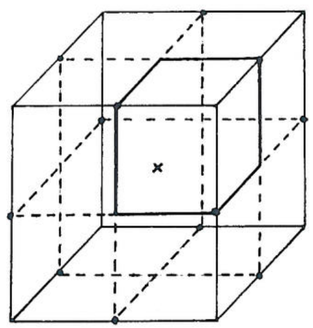
$$Z = \int \mathcal{D}\upsilon \, \mathcal{D}\mu(\eta) \, \mathcal{D}\tau$$

$$\cdot \prod_{x \in \Lambda_i} \left\{ \prod_{\tilde{C}_0} \left[1 - (-1)^{s_C} \tau(C_0) \, \upsilon(C_0) \, \eta(C_0) \right]^{-2} \right\} \tag{6-45}$$

$$\cdot \prod_{y \in \Lambda_f} \left\{ \prod_{\tilde{C}_0} \left[1 - (-1)^{s_C} \tau(C_0) \, \upsilon(C_0) \, \bar{\eta}(C_0) \right]^{-2} \right\}$$

In contrast to the O(4) symmetric Heisenberg model, the paths are localized, i.e. they cannot extend over the whole lattice. In fact, the allowed paths may be visualized on a "dual" lattice whose sites lie in the centers of the old plaquettes. If we draw a cube around each original site the new sites lie in the middle of the edges as shown in figure 7. An allowed path is confined to the surface of a single such "dual" cube. It connects "dual" sites in such a way that three subsequent sites never lie in a plane. If the paths are drawn along the dashed lines in figure 7, we have the equivalent condition that no backtracking paths are allowed.

The path displayed by a solid line in figure 7 is of lowest possible, i.e. third, order.



x original site

• "dual" sites

- Figure 7 -

7. Exact solution of the 2-dimensional model

On a 2-dimensional lattice Λ, there is just one equivalence class $\tilde{C}_z$ of simple loops for every site $z \in \Lambda$. Thus, the partition function (6-45) takes the form

$$Z = \int \mathcal{D}\sigma\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau \cdot \prod_{x \in \Lambda_i} \left[1 - \tau(C_x)\,\sigma(C_x)\,\eta(C_x)\right]^{-2} \cdot \prod_{y \in \Lambda_f} \left[1 - \tau(C_y)\,\sigma(C_y)\,\bar{\eta}(C_y)\right]^{-2} \tag{7-1}$$

We expand each factor of Z into a power series.

$$Z = \int \mathcal{D}\sigma\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau \cdot \prod_{x \in \Lambda_i} \sum_{k_x=0}^{\infty} (k_x+1)\left[\tau(C_x)\,\sigma(C_x)\,\eta(C_x)\right]^{k_x} \cdot \prod_{y \in \Lambda_f} \sum_{k_y=0}^{\infty} (k_y+1)\left[\tau(C_y)\,\sigma(C_y)\,\bar{\eta}(C_y)\right]^{k_y} \tag{7-2}$$

The integration over the η phase gives the constraint that all k_z, $z \in \Lambda$, have to be equal. Thus

$$Z = \sum_{k=0}^{\infty} (k+1)^N \int \mathcal{D}\sigma\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau \cdot \prod_b \sigma_b^k \prod_p \tau_p^k \prod_{(p,p')} |\eta_{pp'}|^{2k} \tag{7-3}$$

where N is the number of sites of the lattice.

Using eqs. (1-38) and (6-39), the integrations over the τ variables can be done.

$$\int \mathcal{D}\tau \prod_p \tau_p^k = \prod_p \oint \frac{d\tau_p}{2\pi i} B(\tau_p)\, \tau_p^k$$

$$= \prod_p \sum_{j\in \frac{1}{2}\mathbb{Z}_+} c_{j_p} \oint \frac{d\tau_p}{2\pi i} \tau_p^{-2j-1} \tau_p^k \tag{7-4}$$

$$= c_{k/2}^{\mathcal{N}}$$

The integral over v yields

$$\int \mathcal{D}v \prod_b v_b^k = \prod_b \frac{i}{2\pi} \oint dv_b\, e^{-\frac{1}{v_b}} v_b^k \tag{7-5}$$

$$= [(k+1)!]^{-2\mathcal{N}}$$

Finally, we perform the integrals over the η variables.

$$\int \mathcal{D}\mu(\eta) \prod_{(p,p')} |\eta_{pp'}|^{2k} = \prod_{(p,p')} \int_0^\infty dr\, r^k e^{-r} \tag{7-6}$$

$$= [k!]^{2\mathcal{N}}$$

Inserting eqs. (7-4), (7-5), and (7-6) into the partition function (7-3), we obtain

$$Z = \sum_{k=0}^\infty c_{k/2}^{\mathcal{N}} (k+1)^{-\mathcal{N}} \tag{7-7}$$

Using eq. (6-6), we get the exact solution of the 2-dimensional SU(2) lattice gauge theory.

$$Z = \left(\frac{2}{\beta}\right)^{\mathcal{N}} \sum_{k=1}^\infty I_k^{\mathcal{N}}(\beta) \tag{7-8}$$

8. <u>High temperature expansion</u>

In analogy to the O(4) Heisenberg ferromagnet (cp. chapter 5), one may derive a high temperature expansion for the SU(2) lattice gauge theory starting from the partition function (6-45).

$$Z = \int \mathcal{D}v\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau$$

$$\cdot \prod_{x\in\Lambda_i} \left\{ \prod_{\tilde{C}_0} \left[1 - (-1)^{s_c} \tau(C_0) v(C_0) \eta(C_0)\right]^{-2} \right\} \tag{8-1}$$

$$\cdot \prod_{y\in\Lambda_f} \left\{ \prod_{\tilde{C}_0} \left[1 - (-1)^{s_c} \tau(C_0) v(C_0) \bar{\eta}(C_0)\right]^{-2} \right\}$$

For each equivalence class $\tilde{C}_0$, we perform a power series expansion to obtain

$$Z = \int \mathcal{D}v\, \mathcal{D}\mu(\eta)\, \mathcal{D}\tau \tag{8-2}$$

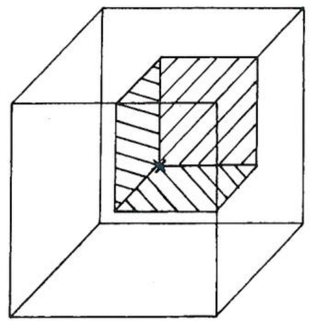
$$\cdot \prod_{x\in\Lambda_i} \left\{ \prod_{\tilde{C}_0} \sum_{k_{C_0}=0}^\infty (k_{C_0}+1) \left[(-1)^{s_c} \tau(C_0) v(C_0) \eta(C_0)\right]^{k_{C_0}} \right\}$$

$$\cdot \prod_{y\in\Lambda_f} \left\{ \prod_{\tilde{C}_0} \sum_{k_{C_0}=0}^\infty (k_{C_0}+1) \left[(-1)^{s_c} \tau(C_0) v(C_0) \bar{\eta}(C_0)\right]^{k_{C_0}} \right\}$$

We want to derive a diagrammatic expansion of the partition function.

The integration over the phase of the η variables annihilates all terms which contain any $\eta_{pp'}$ that is not compensated by its complex conjugate $\bar{\eta}_{pp'}$. Since $\eta_{pp'}$ and $\bar{\eta}_{pp'}$ are associated with paths belonging to two adjacent "dual" cubes the resulting diagrams are no longer local but may extend over the whole lattice.

These diagrams are easier to visualize if the loops C_o are depicted as surfaces. Each loop C_o may be regarded as the contour of a surface containing the center of the "dual" cube, i.e. a site of the original lattice. As an example, fig. 8 shows the surface belonging to the loop of fig. 7.



- Figure 8 -

Then, every contribution to the partition function that does not vanish after the integration over the η phase can be represented by a closed surface. The smallest closed surface is made up of eight loops of the form shown in fig. 8; this is just a cube of the original lattice.

Evidently, each closed surface is composed of plaquettes of the original lattice. Therefore, the variables $\tau_{p,z}$ belonging to the corners of one given plaquette appear an equal number of times so that the algebraic expression corresponding to a given diagram will contain only the plaquette variable $\tau_p = \prod_{\ell \in \partial p} \tau_{p,z}$.

Let G be a set of closed surfaces. Obviously, G specifies sets of links, of plaquettes and of pairs of plaquettes. We denote by $n_b(G)$ the number of times the link b appears in G, by $n_p(G)$ the number of times the plaquette p appears in G, and by $n_{pp'}(G)$ the number of times the pair of plaquettes (p,p') appears in G.

The partition function (8-2) can now be rewritten as a sum over sets G of closed surfaces,

$$Z = \sum_G k_G Z_G \tag{8-3}$$

with

$$Z_G \equiv \int \mathcal{D}\sigma \, \mathcal{D}\mu(\eta) \, \mathcal{D}\tau \prod_b \sigma_b^{n_b(G)} \prod_p \tau_p^{n_p(G)} \cdot \prod_{(pp')} |\eta_{pp'}|^{2n_{pp'}(G)} \tag{8-4}$$

$$= \prod_p c_{\frac{1}{2}n_p(G)} \prod_{(pp')} n_{pp'}(G)! \Big/ \prod_b [n_b(G)+1]!$$

The combinatorial factor k_G is

$$k_G = \sum_{\substack{\{k_{C_o}\} \\ \sum C_o^{k_{C_o}} = G}} (k_{C_o} + 1) \tag{8-5}$$

As a check of our result, we compute the first three terms of the partition function.

$$Z = c_0^{3N}\left\{1 + N\,2^{-4}\frac{c_{1/2}^{6}}{c_0^{6}} + 3N\,2^{-8}\frac{c_{1/2}^{10}}{c_0^{10}} + \dots\right\} \tag{8-6}$$

$$= \left[\frac{2}{\beta} I_1(\beta)\right]^{3N}\left\{1 + 4N\frac{I_2^{6}(\beta)}{I_1^{6}(\beta)} + 12N\frac{I_2^{10}(\beta)}{I_1^{10}(\beta)} + \dots\right\}$$

for $\beta \to 0$

N is the number of sites of the original lattice.

Continuing along these lines, one reproduces the results of the standard high temperature expansion [15].

APPENDIX A :

Proof of equation (1-29)

If we parametrize the elements of SU(2) by

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1 \tag{A-1}$$

the integral we are interested in takes the form

$$I \equiv \int du \, \exp \sum_i z_i^{+} \cdot u^T z_i'$$

$$= \frac{1}{\pi^2}\int d\alpha\, d\beta\, \delta(|\alpha|^2 + |\beta|^2 - 1)\, e^{\alpha U + \beta X + \bar{\alpha} V + \bar{\beta} Y} \tag{A-2}$$

with

$$U = \sum_i \bar{z}_i^{1} z_i'^{1}, \quad V = \sum_i \bar{z}_i^{2} z_i'^{2}$$

$$X = \sum_i \bar{z}_i^{2} z_i'^{1}, \quad Y = -\sum_i \bar{z}_i^{1} z_i'^{2} \tag{A-3}$$

We expand the exponential and write the integral in polar coordinates $\alpha = r\, e^{i\varphi}$, $\beta = s\, e^{i\delta}$.

$$I = \frac{1}{\pi^2}\sum_{k_1=0}^{\infty}\dots\sum_{k_4=0}^{\infty}(k_1!\,k_2!\,k_3!\,k_4!)^{-1}\int_0^{2\pi} d\varphi\, d\vartheta$$

$$\cdot \int_0^1 r\,dr\, s\,ds\, \delta(r^2+s^2-1)\,(r e^{i\varphi} U)^{k_1} \tag{A-4}$$

$$(s e^{i\vartheta} X)^{k_2}\,(r e^{-i\varphi} V)^{k_3}\,(s e^{-i\vartheta} Y)^{k_4}$$

We perform the integrations to obtain

$$I = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} (k_1!\, k_2!)^{-2} \int_0^1 dr^2\, r^{2k_1} (1-r^2)^{k_2} \cdot (uv)^{k_1} (XY)^{k_2}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(uv)^{n-k} (XY)^k}{k!\,(n-k)!} A(n,k) \qquad \text{(A-5)}$$

with

$$A(n,k) \equiv \frac{1}{k!} \frac{1}{(n-k)!} \int_0^1 d\alpha\, \alpha^{n-k} (1-\alpha)^k$$

$$= \frac{1}{(n-k)!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(k-m)!\,(n-k+m+1)} \qquad \text{(A-6)}$$

Since

$$A(n,0) = \frac{1}{(n+1)!}$$

and

$$A(n,k) = A(n,k+1) \quad \text{for} \quad k < n$$

we have

$$A(n,k) = \frac{1}{(n+1)!} \quad \text{for} \quad 0 \leq k \leq n \qquad \text{(A-7)}$$

so that in agreement with [2]

$$I = \sum_{n=0}^{\infty} \frac{(uv + XY)^n}{n!\,(n+1)!} \qquad \text{(A-8)}$$

By introducing a complex variable v

$$\frac{1}{(n+1)!} = \frac{i}{2\pi} \oint dv\, e^{-\frac{1}{v}} (-v)^n \qquad \text{(A-9)}$$

we get

$$I = \frac{i}{2\pi} \oint dv\, e^{-\frac{1}{v} - v(uv + XY)} \qquad \text{(A-10)}$$

Since

$$uv + XY = \sum_{(ij)} (z_i' \varepsilon z_j')(z_i^+ \varepsilon z_j^+) \qquad \text{(A-11)}$$

where (ij) denotes an unordered pair of links, we obtain the final expression (1-29).

$$\int du\, \exp \sum_i z_i^+ \cdot u^T z_i'$$

$$= \frac{i}{2\pi} \oint dv\, e^{-\frac{1}{v}} \exp v \sum_{(ij)} (z_i' \varepsilon z_j')(z_i^+ \varepsilon^{-1} z_j^+) \qquad \text{(A-12)}$$

APPENDIX B :

Proof of equation (1-48)

We want to compute the integral

$$I \equiv \int \prod_k d\mu(z_k) \exp \frac{1}{2} \sum_{ij} \left(z_i \varepsilon z_j \lambda_{ij} + z_i^+ \varepsilon^{-1} z_j^+ \kappa_{ij} \right) \qquad \text{(B-1)}$$

for antisymmetric matrices λ and κ.
If the complex variables z_k are replaced by real variables $r_k \in R^4$ via

$$z = \begin{pmatrix} r_1 + i r_2 \\ r_3 + i r_4 \end{pmatrix} \qquad \text{(B-2)}$$

the integral may be rewritten as

$$I = \int \prod_k d^4 r_k \, e^{-\sum_e (r_e, r_e) + \sum_{ee'} (r_e, C_{ee'} r_{e'})} \qquad \text{(B-3)}$$

with $(r_l, r_k) = \sum_{i=1}^{4} r_l^i r_k^i$. C denotes an antisymmetric block matrix consisting of antisymmetric 4x4-matrices of the form

$$C_{ee'} = \begin{pmatrix} 0 & B_{ee'} \\ -B_{ee'} & 0 \end{pmatrix} = -C_{e'e} \qquad \text{(B-4)}$$

with

$$B_{ee'} = \frac{1}{2} \lambda_{ee'} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} + \frac{1}{2} \kappa_{ee'} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \qquad \text{(B-5)}$$

Performing the gaussian integral (B-3) we obtain

$$I = \det (1 - C)^{-1/2} = \exp \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr} C^n \qquad \text{(B-6)}$$

Since the trace of all odd powers of C vanishes and

$$\operatorname{tr} C^{2n} = 2(-1)^n \operatorname{tr} B^{2n} \qquad \text{(B-7)}$$

we get

$$I = \exp \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \operatorname{tr} B^{2n} \qquad \text{(B-8)}$$

Making use of

$$\operatorname{tr} B^{2n} = 2 \operatorname{tr} (\lambda \kappa)^n \qquad \text{(B-9)}$$

we arrive at the final result (1-48)

$$\int \prod_k d\mu(z_k) \exp \frac{1}{2} \sum_{ij} \left(z_i \varepsilon z_j \lambda_{ij} + z_i^+ \varepsilon^{-1} z_j^+ \kappa_{ij} \right) = \det (1 + \lambda \kappa)^{-1} \qquad \text{(B-1o)}$$

APPENDIX C :

Examples supporting conjecture (5-8)

The simplest set A_1 which contributes to the partition function (5-4) consists of four directed double links forming a plaquette. We depict A_1 symbolically by

$A_1 =$  (C-1)

To A_1 there belongs only one equivalence class $C(A_1)$ of paths, graphically represented by

$C(A_1) =$  $, \quad k_C = \rho_C = 1$ (C-2)

Obviously,

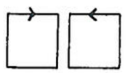
$$k_{A_1} = k'_{A_1} = 1 \tag{C-3}$$

Evidently, all sets A which do not contain any directed link more than once give the same result.

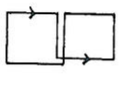
Now consider

$A_2 =$  $\quad k'_{A_2} = 2$ (C-4)

Out of A_2, we may compose two different sets of paths, either

$A_2 \mathrel{\hat{=}} C_1 + C_2 =$  $, \quad k_{C_1} = k_{C_2} = \rho_{C_1} = \rho_{C_2} = 1$ (C-5)

or

$A_2 \mathrel{\hat{=}} C =$  $\quad k_C = \rho_C = 1$ (C-6)

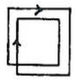
so that

$$k_{A_2} = 2 = k'_{A_2} \tag{C-7}$$

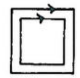
For

$A_3 =$  $, \quad k'_{A_3} = \frac{(2!)^4}{(2!)^4} = 1$ (C-8)

we have again two possibilities:

$A_3 \mathrel{\hat{=}} C' =$  $, \quad k_{C'} = 1, \quad \rho_{C'} = 2$ (C-9)

and

$A_3 \mathrel{\hat{=}} C^2 =$  $, \quad k_C = 2, \quad \rho_C = 1$ (C-1o)

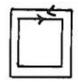
so that

$$k_{A_3} = \frac{1}{2!} + \frac{1}{2} = 1 = k'_{A_3} \tag{C-11}$$

In contrast, the set

$A_4 =$  $, \quad k'_{A_4} = 1$ (C-12)

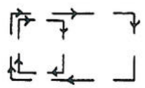
offers only the possibility

$A_4 \mathrel{\hat{=}} C_1 + C_2 =$  $, \quad k_{C_i} = \rho_{C_i} = 1$ (C-13)

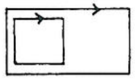
Thus

$$k_{A_4} = 1 = k'_{A_4} \tag{C-14}$$

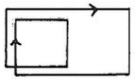
A further example is

$A_5 =$  $, \quad k'_{A_5} = \frac{(2!)^3}{(2!)^2} = 2$ (C-15)

We find

$$A_S \hat{=} C_1 + C_2 =$$  $$, \; k_{C_i} = \rho_{C_i} = 1 \qquad \text{(C-16)}$$

and

$$A_S \hat{=} C =$$  $$, \; k_C = \rho_C = 1 \qquad \text{(C-17)}$$

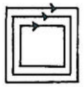
Consequently,

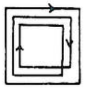
$$k_{A_S} = 2 = k'_{A_S} \qquad \text{(C-18)}$$

As a final example, consider

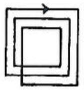
$$A_6 =$$  $$, \; k'_{A_6} = 1 \qquad \text{(C-19)}$$

We get three possible decompositions

$$A_6 \hat{=} C^3 =$$  $$, \; k_C = 3, \; \rho_C = 1 \qquad \text{(C-2o)}$$

$$A_6 \hat{=} C_1 + C_2 =$$  $$, \; k_{C_i} = 1, \; \rho_{C_1} = 1, \; \rho_{C_2} = 2 \qquad \text{(C-21)}$$

and

$$A_6 = C' =$$  $$, \; k_{C'} = 1, \; \rho_{C'} = 3 \qquad \text{(C-22)}$$

Thus

$$k_{A_6} = \frac{1}{3!} + \frac{1}{2} + \frac{1}{3} = 1 = k'_{A_6} \qquad \text{(C-23)}$$

APPENDIX D :

Proof of the loop expansion (6-4o)

Consider one gaussian integral of the partition function (6-38) belonging to a given site $x \in \Lambda_i$.

$$I \equiv \int \prod_{\substack{p \\ x \in \partial p}} d\mu(\varsigma_p) \exp \sum_{pp'} \Big(\frac{1}{2} \lambda_{pp'} \varsigma_p \, \varepsilon \, \varsigma_{p'} + \frac{1}{2} \kappa_{pp'} \varsigma_p^+ \varepsilon^{-1} \varsigma_{p'}^+ + \varsigma_{pp'} \varsigma_p \cdot \varsigma_{p'}^+ \Big) \qquad \text{(D-1)}$$

It is transformed into an integral over real variables by introducing vectors $r_p \in R^4$ via

$$\varsigma_p = \begin{pmatrix} r_1 + i r_2 \\ r_3 + i r_4 \end{pmatrix} \qquad \text{(D-2)}$$

Furthermore, we introduce an antisymmetric block matrix C consisting of 4x4 blocks

$$C_{pp'} \equiv \lambda_{pp'} \begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix} + \kappa_{pp'} \begin{pmatrix} 0 & -F^* \\ F^* & 0 \end{pmatrix} = -C_{p'p} \qquad \text{(D-3)}$$

with

$$F \equiv \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \qquad \text{(D-4)}$$

and a block matrix D consisting of antisymmetric 4x4 blocks

$$\mathcal{D}_{pp'} \equiv \rho_{pp'} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} + \rho_{p'p} \begin{pmatrix} E^* & 0 \\ 0 & E^* \end{pmatrix} \qquad \text{(D-5)}$$

with

$$E \equiv \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \qquad \text{(D-6)}$$

The result of the gaussian integration is

$$\begin{aligned} I &= \int \prod_p d^4 r_p \exp\Big\{-\sum_p (r_p, r_p) + \sum_{pp'} (r_p, [C+\mathcal{D}]_{pp'} r_{p'})\Big\} \\ &= \det(1 - C - \mathcal{D})^{-1/2} \qquad \text{(D-7)} \\ &= \exp \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr} (C+\mathcal{D})^n \end{aligned}$$

with $(r_p, r_q) = \sum_{i=1}^{4} r_p^i r_q^i$.

The powers of D are given by

$$(\mathcal{D}^n)_{pp'} = (\rho^n)_{pp'} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} + (\rho^n)_{p'p} \begin{pmatrix} E^* & 0 \\ 0 & E^* \end{pmatrix} \qquad \text{(D-8)}$$

We have to distinguish even and odd powers of C:

$$(C^{2n})_{pp'} = (\lambda\kappa)^n_{pp'} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} + (\kappa\lambda)^n_{pp'} \begin{pmatrix} E^* & 0 \\ 0 & E^* \end{pmatrix} \qquad \text{(D-9)}$$

$$\begin{aligned} (C^{2n+1})_{pp'} &= [(\lambda\kappa)^n \lambda]_{pp'} \begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix} \\ &\quad + [(\kappa\lambda)^n \kappa]_{pp'} \begin{pmatrix} 0 & -F^* \\ F^* & 0 \end{pmatrix} \end{aligned} \qquad \text{(D-1o)}$$

For the trace in (D-7) not to vanish C has to appear an even number of times.

From the relations (D-9), (D-1o), and

$$\begin{aligned} [C\mathcal{D}^n C]_{pp'} &= (\lambda \rho^{Tn} \kappa)_{pp'} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \\ &\quad + (\kappa \rho^n \lambda) \begin{pmatrix} E^* & 0 \\ 0 & E^* \end{pmatrix} \end{aligned} \qquad \text{(D-11)}$$

we see that λ and κ have to appear alternatingly where λ may be followed by powers of ρ^T and κ by powers of ρ. Thus, the nonvanishing terms in the exponential of (D-7) take the form

$$\operatorname{tr} \ldots \kappa \rho^{n_1} \lambda (\rho^T)^{n_2} \kappa \rho^{n_3} \lambda \ldots \qquad \text{(D-12)}$$

These contributions may be represented by closed paths connecting the centers of neighboring plaquettes which have the point x in common. With each oriented closed path C consisting of n_c ordered pairs of plaquettes (p_1, p_2), (p_2, p_3) ... (p_{n_c}, p_1) we associate the algebraic expressions

$$\sigma(C) \equiv \prod_{\substack{b \wedge (p,p') \\ (p,p') \in C}} \sigma_b^{1/2} \qquad \text{(D-13)}$$

$$\tau(C) \equiv \prod_{p \in C} \tau_{p,x} \qquad \text{(D-14)}$$

and

$$\eta(C) \equiv \prod_{(p,p') \in C} \eta_{pp'} \tag{D-15}$$

Moreover, we define an integer s_c which is the number of pairs $(p,p') \in C$ with $b \in \partial p$, $b^{-1} \in \partial p'$.

Then, (D-7) can be rewritten as

$$I = \exp \sum_C \frac{(-1)^{s_c}}{n_c} \, \upsilon(C)\, \tau(C)\, \eta(C) \tag{D-16}$$

where the sum is over all closed paths C

$$\{C\} = \{ (p_1,p_2) \dots (p_n,p_1) \mid \text{for all } k,\ (p_k,p_{k+1}) \wedge b,$$
$$b = \langle xy \rangle \text{ with } x \in \Lambda_i \text{ fixed, and } p_k, p_{k+1}, p_{k+2} \text{ do not share a common link} \}$$

The integrals belonging to sites $y \in \Lambda_f$ are being treated in a similar way.

Part B: MASS GENERATION BY SUITABLE NORMAL ORDERING IN THE NONLINEAR σ-MODEL

1. Introduction and summary of results

In a general discussion of the confinement problem, nonlinear σ-models in 2 and 2+ε dimensions have become a focus of interest because in some essential features they resemble nonabelian gauge theories in 4 and 4+ε dimensions, respectively. Among these features are, in particular, asymptotic freedom, a nonlinear realization of the underlying symmetry, and the existence of phase transitions. Furthermore, the O(n+1) symmetric σ-models allow a 1/n expansion which is simpler than in the SU(n) gauge theories. It is therefore hoped that even though 2-dimensional field theoretical models are in their own right of academic interest only they still provide a laboratory for the development of techniques to be ultimately of use in the more relevant 4-dimensional gauge theories.

The existence of a phase transition of the nonlinear σ-model in 2+ε dimensions has been discussed by Brézin and Zinn-Justin [16]. Conventionally, the quantum properties of this model in its two phases are treated by two different methods: perturbation theory for the spontaneously broken phase, and 1/n expansion for the symmetric phase. However, is has been found by Lowe [17] that by summing a suitable set of Feynman diagrams the result of the 1/n expansion can be reproduced. This observation gives rise to the expectation that a unified treatment of both phases might be possible. In the present investigation, some steps in this direction will be taken.

We have found that a useful parametrization of the model for our purposes is in terms of stereographic coordinates as introduced by Schwinger [18] and used also by Bardeen et al. [19]. We mention in passing that the introduction of these coordinates complicates the discussion of the renormalization properties of the model considerably [2o] as can be seen from the rather involved and non-explicit form of the Ward identities given by Bardeen et al. [19]. We have not made an effort to exploit these identities although in principle one could try

The main technique newly introduced here into the discussion is the device of nonperturbative normal ordering which enables us to write arbitrary functions in normal ordered form without recourse to expansions in powers of fields or coupling constants. Nonperturbative normal ordering is defined with respect to the covariance of a Gaussian measure which is determined by the free part of the action. This allows for the introduction of two arbitrary parameters, a coupling constant f and a mass parameter m.

We then rewrite the potential which is nonpolynomial in normal ordered form. The parameters m and f are determined as functions of the initial coupling constant f_B by requiring that no terms quadratic in the fields or their derivatives appear in the potential.

In the large n limit, the equations determining f and m become tractable. In $d > 2$ dimensions we find for small values of f_B a massless phase with two solutions for the coupling parameter f corresponding to two values of the spontaneous magnetization and for large f_B a massive phase with a unique solution for f and vanishing spontaneous magnetization. We have therefore achieved in this limit the desired objective of a unified treatment for both phases.

For finite n, however, our method runs into difficulties. They manifest themselves for instance through the appearance of quadratically divergent terms in the equation determining the mass parameter m. Nevertheless, for $d > 2$ a weak-coupling limit ($f_B \to 0$) is studied for finite n assuming that quadratic divergences cancel in the full theory. For $n > 4$, we again find that the spontaneously broken phase is characterized by a vanishing mass and two solutions for the coupling constant f.

We then turn to a closer inspection of the problem of quadratic divergences. Assuming the validity of certain conjectures we can show the cancellation of quadratic divergences for the full theory.

We conclude with a consideration of the dilute gas approximation which unlike our approximation based on normal ordering takes into account all quadratic divergences for finite n. Based on a selection of graphs which is motivated by our investigation of quadratic divergences we derive within the dilute gas approximation a modified equation for the mass parameter m. In the large n limit this equation coincides with the one derived by nonperturbative normal ordering in the $1/n$ expansion. We also find that this approximation effectively reduces the theory to a linear σ-model.

2. The nonlinear σ-model in stereographic coordinates

2.1 Continuum formulation

The variables of the compact nonlinear σ-model are (n+1)-dimensional spins $\vec{\sigma}(x)$ obeying the constraint

$$\sum_{\alpha=0}^{n} \sigma_\alpha(x)\,\sigma_\alpha(x) = 1 \tag{2-1}$$

The action

$$L(\sigma) = -\frac{1}{2}\frac{n}{f_0}\int d^d x\ \nabla_\mu \sigma_\alpha(x)\,\nabla_\mu \sigma_\alpha(x) \tag{2-2}$$

is invariant under O(n+1) rotations.

We will study this model in d = 2 and d = 2+ε dimensions.

Note that our coupling parameter differs from usual conventions by a factor n/(n+1).

The partition function of the model is given by

$$Z = \int \prod_x d\sigma(x)\ e^{L(\sigma)} \tag{2-3}$$

with the invariant measure

$$d\sigma \equiv \frac{d\sigma_1 \dots d\sigma_n}{\sigma_0}\ \theta(\sigma_0^2) \tag{2-4}$$

$$\sigma_0 = \sqrt{1 - \sum_{i=1}^{n} \sigma_i^2}$$

To obtain a formulation in terms of unconstrained variables ranging from $-\infty$ to $+\infty$ we introduce stereographic coordinates [18,19].

$$\sigma_i = \frac{2\xi_i}{1+\vec{\xi}^2}\ , \quad i = 1 \dots n \tag{2-5}$$

so that

$$\sigma_0 = \frac{1-\vec{\xi}^2}{1+\vec{\xi}^2} \tag{2-6}$$

The partition function (2-3) is rewritten as

$$Z = \int \prod_x d\xi(x)\ e^{L(\xi)} \tag{2-7}$$

with the action

$$L(\xi) = -\frac{2n}{f_B}\int d^d x\ \frac{\nabla_\mu \xi_i\,\nabla_\mu \xi_i}{(1+\vec{\xi}^2)^2} \tag{2-8}$$

In terms of these coordinates, the constraint $\sum_{i=1}^{n} \sigma_i^2 \leq 1$ is automatically fulfilled, and the measure takes the form

$$d\xi(x) = \frac{2^n d^n \xi(x)}{(1+\vec{\xi}^2(x))^n} \tag{2-9}$$

2.2 Lattice formulation

The spins $\vec{\sigma}(x)$ are attached to the sites of a d-dimensional lattice with lattice spacing a.

The differential operator ∇_μ is defined as

$$\nabla_\mu \sigma_\alpha(x) = \frac{1}{a}\left\{ \sigma_\alpha(x+a\hat{\mu}) - \sigma_\alpha(x)\right\} \qquad (2\text{-}10)$$

where $\hat{\mu}$ is a unit vector in the direction μ.

Inserting this definition into eq. (2-2), we obtain the action of an O(n+1) symmetric Heisenberg ferromagnet.

$$L(\sigma) = -\frac{n}{2f_B} a^d \sum_x \nabla_\mu \sigma_\alpha(x) \nabla_\mu \sigma_\alpha(x)$$

$$= \frac{n}{f_B} a^{d-2} \sum_{x,\mu} \vec{\sigma}(x)\cdot\vec{\sigma}(x+a\hat{\mu}) + \mathrm{const} \qquad (2\text{-}11)$$

The partition function in terms of stereographic coordinates can again be written in the form

$$Z = \int \prod_x d\xi(x)\, e^{L(\xi)} \qquad (2\text{-}12)$$

with the measure (2-9) and the action

$$L(\xi) = -\frac{2n}{f_B} a^d \sum_{x,\mu} \frac{\nabla_\mu \xi_i(x)\, \nabla_\mu \xi_i(x)}{[1+\vec{\xi}^2(x+a\hat{\mu})][1+\vec{\xi}^2(x)]} \qquad (2\text{-}13)$$

3. Mass generation

We want to rewrite the partition function (2-7) in a form where the action is the sum of the action of a free field with mass m and an interaction which becomes weak in suitably chosen limits.

One notices that the model already contains a mass term generated by the measure (2-9). In fact, we can replace eq. (2-7) by

$$Z = \int \prod_x d^n\xi(x)\, e^{L_{tot}(\xi)} \qquad (3\text{-}1)$$

where the total action $L_{tot}(\xi)$ is given as

$$L_{tot}(\xi) = -\frac{2n}{f_B}\int d^dx\, \frac{\nabla_\mu \xi_i(x)\,\nabla_\mu \xi_i(x)}{(1+\vec{\xi}^2(x))^2}$$

$$- n\,\delta^d(0) \int d^dx\, \ln(1+\vec{\xi}^2(x)) \qquad (3\text{-}2)$$

in the continuum case and

$$L_{tot}(\xi) = -\frac{2n}{f_B} a^d \sum_{x,\mu} \frac{\nabla_\mu \xi_i(x)\,\nabla_\mu \xi_i(x)}{[1+\vec{\xi}^2(x+a\hat{\mu})][1+\vec{\xi}^2(x)]}$$

$$- n \sum_x \ln(1+\vec{\xi}^2(x)) \qquad (3\text{-}3)$$

in the lattice version. The quadratic term in the power series expansion of the logarithm gives a mass term which diverges (quadratically) in the continuum limit. In perturbation theory, cancellations will occur between divergent contributions from the logarithmic term and from $L(\xi)$, respectively.

It would now be highly desirable to obtain a device that actually isolates the physical mass of the theory.

Let us split the total action into a free part and an interaction part,

$$L_{tot}(\xi) = L_0(\xi) + L_I(\xi) \tag{3-4}$$

where we define the free action as

$$L_0(\xi) = -\frac{2n}{f}\int d^d x \left\{ \nabla_\mu \xi_i(x) \nabla_\mu \xi_i(x) + m^2 \vec{\xi}^2(x) \right\} \tag{3-5}$$

The constants m and f are arbitrary to begin with but will eventually be determined as functions of f_B by imposing certain conditions on the form of the interaction part $L_I(\xi)$. For instance, in the following chapters we will investigate the possibility of writing the interaction term in normal ordered form and fixing m and f by requiring that no terms quadratic in ξ or $\nabla_\mu \xi$ appear in the normal ordered potential.
In particular, we will show that to leading order in 1/n the parameter m determined in this way can indeed be identified with the physical mass.

4. Nonperturbative normal ordering

Consider a field $\varphi_i(x)$, i = 1 ... n, of mass m.
The free action

$$\begin{aligned} L_0(\varphi) &= -\frac{\beta}{2}\int d^d x \left\{ \nabla_\mu \varphi_i(x) \nabla_\mu \varphi_i(x) + m^2 \varphi^2(x) \right\} \\ &= -\frac{1}{2}(\varphi, v^{-1}\varphi) \end{aligned} \tag{4-1}$$

defines a functional Gaussian measure

$$d\mu_v(\varphi) = \frac{1}{Z_0} \prod_x d^n \varphi(x)\, e^{L_0(\varphi)} \tag{4-2}$$

The normalization constant Z_0 is chosen such that

$$\int d\mu_v(\varphi) = 1 \tag{4-3}$$

The propagator

$$v = \frac{1}{\beta}(-\Delta + m^2)^{-1} \tag{4-4}$$

is also called the covariance of this Gaussian measure.
Normal ordering with respect to a Gaussian measure with covariance v is defined by [21]

$$\begin{aligned} :e^{i\vec{k}\vec{\varphi}(x)}: &= e^{i\vec{k}\vec{\varphi}(x)} \Big/ \int d\mu_v(\varphi)\, e^{i\vec{k}\vec{\varphi}(x)} \\ &= e^{i\vec{k}\vec{\varphi}(x)}\, e^{-\frac{1}{2}k^2 v(0)} \end{aligned} \tag{4-5}$$

v(0) is the propagator at vanishing distance, $v(0) = \frac{1}{\beta}(-\Delta+m^2)^{-1}(0)$.

To write any tempered distribution $f(\vec{\varphi}(x))$ in normal ordered form,

$$f(\vec{\varphi}(x)) = \,:F(\vec{\varphi}(x)):\tag{4-6}$$

we expand f into a Fourier integral and obtain by means of the definition (4-5)

$$F(\vec{\varphi}) = \left[2\pi v(0)\right]^{-n/2} \int d^n\eta\, f(\vec{\eta})\, e^{-\frac{1}{2v(0)}(\vec{\varphi}-\vec{\eta})^2}\tag{4-7}$$

F is an entire analytic function of φ and can thus be expanded into a convergent power series for every value of the field, even though f does not necessarily have this property.

With the above definition of normal ordering, Wick's theorem holds. For instance, for the example of $f(\vec{\varphi}) = \vec{\varphi}^2$ eq. (4-7) yields

$$\vec{\varphi}^2(x) = \,:\vec{\varphi}^2(x): + n\,v(0)\tag{4-8}$$

Note that this technique of normal ordering does not make use of expansions in powers of fields or coupling constants.

With the help of eq. (4-7) we will eventually be able to write $L_I(\xi)$ in normal ordered form,

$$L_I(\xi) = \,:V(\xi, \nabla\xi):\tag{4-9}$$

where normal ordering is with respect to the free action (3-5). Then the covariance of the Gaussian measure is

$$v = \frac{f}{4n}\, v_m\tag{4-10}$$

with

$$v_m \equiv (-\Delta + m^2)^{-1}\tag{4-11}$$

Since using the lattice action (2-13) would imply normal ordering of functions which depend on the field at several points we restrict ourselves to the continuum case for the time being.

In the following, we will need the normal ordered form of the function

$$f_\alpha(\vec{\xi}) \equiv (1 + \vec{\xi}^2)^{-\alpha}\tag{4-12}$$

For brevity, we will use the notation $\vec{\xi} \equiv \vec{\xi}(x)$ and $v \equiv v(0)$.

Inserting the integral representation

$$f_\alpha(\vec{\xi}) = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt\, t^{\alpha-1}\, e^{-t(1+\vec{\xi}^2)}\tag{4-13}$$

into eq. (4-7) we obtain

$$f_\alpha(\vec{\xi}) = \,:F_\alpha(n, \xi^2):\tag{4-14}$$

with

$$F_\alpha(n, \xi^2) = \frac{1}{\Gamma(\alpha)} \int_0^\infty dt\, t^{\alpha-1} (1 + 2vt)^{-n/2} \cdot e^{-t\left(1 + \frac{\xi^2}{1+2vt}\right)}\tag{4-15}$$

The logarithmic term in eq. (3-2) may be treated as a limiting case of eq. (4-12),

$$\ln(1 + \vec{\xi}^2) = -\frac{\partial}{\partial\alpha} f_\alpha(\vec{\xi})\Big|_{\alpha=0}\tag{4-16}$$

so that

$$\ln(1 + \vec{\xi}^2) = -\,:\frac{\partial}{\partial\alpha} F_\alpha(n, \xi^2)\Big|_{\alpha=0}:\tag{4-17}$$

The action (2-8) may now be written as

$$L(\xi) = -\frac{2n}{f_B}\int d^d x \; \nabla_\mu \xi_i \nabla_\mu \xi_i : F_2(n, \xi^2) : \tag{4-18}$$

It can be expressed in completely normal ordered form by applying Wick's theorem.

In contrast to the lattice formulation, contractions of the form

$$\begin{aligned} \underbracket{\nabla_\mu \xi_i(x) \cdot \xi_j}(x) &= \nabla_\mu v(0)\, \delta_{ij} \\ &= \frac{f}{4n} \nabla_\mu v_m(0)\, \delta_{ij} \end{aligned} \tag{4-19}$$

will vanish in the continuum case if we use a suitable UV-cutoff as provided by dimensional or Pauli-Villars regularization.

Thus, we get

$$\begin{aligned} \frac{\nabla_\mu \xi_i \nabla_\mu \xi_i}{(1+\xi^2)^2} &= : \nabla_\mu \xi_i \nabla_\mu \xi_i \, F_2(n, \xi^2) : \\ &\quad + \underbracket{\nabla_\mu \xi_i \nabla_\mu \xi_i} : F_2(n, \xi^2) : \end{aligned} \tag{4-20}$$

We have to compute only the contraction

$$\begin{aligned} \underbracket{\nabla_\mu \xi_i(x) \nabla_\mu \xi_i}(x) &= -n \Delta v(0) \\ &= -\frac{f}{4} \Delta v_m(0) \end{aligned} \tag{4-21}$$

Since

$$(-\Delta + m^2)\, v_m(x-y) = \delta^d(x-y) \tag{4-22}$$

we obtain a quadratically divergent expression.

$$\underbracket{\nabla_\mu \xi_i(x) \nabla_\mu \xi_i}(x) = \frac{f}{4}\left[\delta^d(0) - m^2 v_m(0)\right] \tag{4-23}$$

Dropping irrelevant additive constants, we finally arrive at the normal ordered potential

$$\begin{aligned} : V(\xi, \nabla\xi) : = \int d^d x \Big\{ &-\frac{2n}{f_B} : \nabla_\mu \xi_i \nabla_\mu \xi_i \left[F_2(n,\xi^2) - \frac{f_B}{f}\right] : \\ &- \frac{n}{2}\frac{f}{f_B}\left[\delta^d(0) - m^2 v_m(0)\right] : F_2(n, \xi^2) : \\ &+ n\, \delta^d(0) : \frac{\partial}{\partial\alpha} F_\alpha(n, \xi^2)\Big|_{\alpha=0} : \\ &+ \frac{2n}{f} m^2 : \xi^2 : \Big\} \end{aligned} \tag{4-24}$$

This expression simplifies considerably in the large n approximation which will be studied in the next chapter.

5. Large n approximation

Consider the integral representation (4-15) of the function $F_\alpha(n,\xi^2)$. For large n, only small values of 2vt contribute. Neglecting the term 2vt in the exponential and making the approximation

$$(1+2vt)^{-n/2} = e^{-\frac{n}{2}\ln(1+2vt)} \approx e^{-nvt} \tag{5-1}$$

we find

$$F_\alpha(n,\xi^2) = (1+nv+\xi^2)^{-\alpha} \qquad \text{for} \quad n \to \infty \tag{5-2}$$

In this limit, the potential (4-24) takes the form

$$\begin{aligned} :V(\xi,\nabla\xi): = \int d^dx \Big\{ &-\frac{2n}{f_B} :\nabla_\mu\xi_i\nabla_\mu\xi_i\Big[(1+nv+\xi^2)^{-2}-\frac{f_B}{f}\Big]: \\ &-\frac{n}{2}\frac{f}{f_B}\left[\delta^d(0)-m^2 v_m(0)\right] :(1+nv+\xi^2)^{-2}: \\ &-n\delta^d(0) :\ln(1+nv+\xi^2): \\ &+\frac{2n}{f} m^2 :\xi^2: \Big\} \end{aligned} \tag{5-3}$$

Now we expand the potential into a power series in ξ^2 and require the quadratic terms to vanish. The condition

$$\left.\frac{\partial^2 V}{\partial(\nabla\xi)^2}\right|_{\xi=0} \stackrel{!}{=} 0 \tag{5-4}$$

yields the coupling equation

$$\left[1+\frac{f}{4}v_m(0)\right]^{-2} = \frac{f_B}{f} \tag{5-5}$$

The second condition

$$\left.\frac{\partial^2 V}{\partial\xi^2}\right|_{\xi=0} \stackrel{!}{=} 0 \tag{5-6}$$

gives

$$\begin{aligned} &n\delta^d(0)\left[\frac{f}{f_B}(1+nv)^{-3}-(1+nv)^{-1}\right] \\ &= nm^2\left[\frac{f}{f_B}(1+nv)^{-3}v_m(0)-\frac{2}{f}\right] \end{aligned} \tag{5-7}$$

Inserting the coupling equation (5-5) into eq. (5-7) we notice that the quadratically divergent terms cancel. We are left with the gap equation

$$\frac{2}{f}m^2 = m^2\,\frac{v_m(0)}{1+\frac{f}{4}v_m(0)} \tag{5-8}$$

We are now looking for real solutions f and m.

The gap equation (5-8) has two possible solutions:

$$m = 0 \tag{5-9}$$

or

$$v_m(0) = \frac{4}{f} \qquad \text{for} \quad m \neq 0 \tag{5-10}$$

We have to check whether these solutions are compatible with the coupling equation (5-5).

Let us start with the massive case. We insert eq. (5-10) into (5-5) to get the relation

$$\frac{f}{4} = f_B = v_m(0)^{-1} \qquad \text{for} \quad m \neq 0 \tag{5-11}$$

Since v_m increases with decreasing m a massive solution is only possible if

$$f_B > v_0(0)^{-1} \tag{5-12}$$

In 2 dimensions, a solution $m \neq o$ exists for all values of f_B since the massless propagator given by the Coulomb potential is infrared divergent. Introducing a large momentum cutoff Λ we get by means of eq. (5-11)

$$\begin{aligned} v_m(0) &= \int_0^\Lambda \frac{d^2p}{(2\pi)^2} \frac{1}{p^2+m^2} \approx \frac{1}{4\pi} \ln \frac{\Lambda^2}{m^2} \\ &= \frac{1}{f_B} \end{aligned} \tag{5-13}$$

so that we obtain the mass gap

$$m = \Lambda e^{-\frac{2\pi}{f_B}} \qquad \text{in 2 dimensions} \tag{5-14}$$

In $2+\varepsilon$ dimensions, we have $v_0(0) < \infty$ with a suitable UV-cutoff. Therefore, we will possibly find a massless solution.

For m = 0, the coupling equation (5-5) has two solutions

$$f = \frac{8}{f_B v_0^2(0)} \left(1 - \frac{1}{2} f_B v_0(0) \pm \sqrt{1 - f_B v_0(0)}\right) \tag{5-15}$$

In order to get a real solution we have to require

$$f_B \leq v_0(0)^{-1} \qquad \text{for} \qquad m = 0 \tag{5-16}$$

For $f_B = v_0(0)^{-1}$ the two solutions coincide.

For small coupling constant f_B, we may expand the square root in eq. (5-15) keeping only the first nonvanishing term. Then the two solutions show the reciprocal behavior

$$f = \begin{cases} \dfrac{16}{f_B v_0^2(0)} & \to \infty \\ f_B & \to 0 \end{cases} \qquad \text{as} \qquad f_B \to 0 \tag{5-17}$$

As we will show below the two solutions (5-15) for the coupling parameter f correspond to two values of the spontaneous magnetization.

But before being able to do so, we have to draw some further conclusions from the representation of the potential in the form (5-3). In fact, normal ordering of the potential in such a way that the quadratic terms vanish implies that, to leading order in 1/n, no corrections to the free n-point functions will appear. [17,19]

In particular, the potential yields no contributions to the propagator so that in this approximation our mass parameter m is equal to the physical mass of the theory.

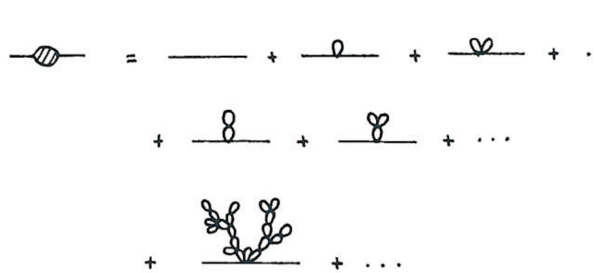
To see this, let us look at the action (3-2) without normal ordering. In conventional perturbation theory, the nonpolynomial interaction is expanded into a power series

$$\begin{aligned} L_{tot}(\xi) = &- \frac{2n}{f_B} \int d^dx \; \nabla_\mu \xi_i \nabla_\mu \xi_i \sum_{k=0}^{\infty} (k+1)(-\xi^2)^k \\ &+ n\,\delta^d(0) \int d^dx \sum_{k=1}^{\infty} \frac{1}{k} (-\xi^2)^k \end{aligned} \tag{5-18}$$

which, however, is not convergent for $\xi^2 \geq 1$.

We note in passing that it is one advantage of our device of nonperturbative normal ordering that it cures this disease of nonconvergence.

The dominant 1particle-irreducible graphs contributing to the propagator are of the form



(5-19)

It follows from eq. (5-18) that each vertex carries a factor n. Since $v(x-y) = f/4n\ v_m(x-y)$ we get a factor 1/n for each line. We finally obtain a factor $\sum_i \delta_{ii} = n$ for each closed loop. According to these rules, all the above cactus graphs are of order 1/n.

Let us now turn to the normal ordered potential (5-3). Since normal ordering eliminates all tadpole graphs none of the above diagrams will appear. To leading order in 1/n, we are left with the free propagator. Thus, normal ordering has the nonperturbative effect of summing infinite sets of diagrams.

In next to leading order, an infinite number of graphs will contribute to the propagator. They are of the form

Consequently, the physical mass of the theory is given by

$$m_{phys} = m + O\left(\frac{1}{n}\right) \qquad (5\text{-}20')$$

Note that the above discussion only makes sense under the assumption that all quadratically divergent graphs cancel. The discussion of this particular problem is deferred to chapter 7. Remember, however, that we have already seen above that to leading order in 1/n the quadratic divergences actually do cancel.

We are now in position to compute the spontaneous magnetization.

Since the Lagrangian in stereographic coordinates is manifestly O(n) invariant but not the spin components $\sigma_i = 2\xi_i/(1+\xi^2)$ a spontaneous magnetization can only point in ±0-direction, i.e. it is given by the expectation value of σ^0. One might be surprised at the existence of a preferred direction but this is merely a consequence of our requirement that the free action be O(n) symmetric.

Because of translation invariance it is sufficient to compute $\langle\sigma^0(x)\rangle$ at the point x = 0.

$$\begin{aligned}\langle\sigma^0(0)\rangle &= \left\langle \frac{1-\xi^2(0)}{1+\xi^2(0)} \right\rangle \\ &= -1 + 2\langle : F_1(n, \xi^2(0)) : \rangle\end{aligned} \qquad (5\text{-}21)$$

In the large n limit, this is

$$\langle\sigma^0(0)\rangle = -1 + 2\left\langle : \frac{1}{1+n v + \xi^2(0)} : \right\rangle \quad \text{for} \quad n \to \infty \qquad (5\text{-}22)$$

As we have seen the corrections to the expectation value from the potential can be neglected to leading order in 1/n. Thus, if we expand $\sigma^0(0)$ into a power series in $\xi^2(0)$ only the constant term will survive in this approximation.

$$\langle \sigma^0(0) \rangle = -1 + \frac{2}{1 + \frac{f}{4} v_m(0)} + O\left(\frac{1}{n}\right) \qquad (5\text{-}23)$$

Because of eq. (5-11) the spontaneous magnetization vanishes in the massive case.

$$\langle \sigma^0(0) \rangle = 0 \quad \text{for} \quad m \neq 0 \qquad (5\text{-}24)$$

In the massless phase, we get by means of eq. (5-15)

$$\langle \sigma^0(0) \rangle = \frac{1 - \frac{f}{4} v_0(0)}{1 + \frac{f}{4} v_0(0)} = \frac{-1 + f_B v_0(0) \mp \sqrt{1 - f_B v_0(0)}}{1 \pm \sqrt{1 - f_B v_0(0)}} \qquad (5\text{-}25)$$

so that

$$\langle \sigma^0(0) \rangle = \mp \sqrt{1 - f_B v_0(0)} \quad \text{for} \quad m = 0 \qquad (5\text{-}26)$$

In conclusion, the following picture emerges:

If $f_B > v_0(0)^{-1}$ we are in a symmetric phase with vanishing spontaneous magnetization and nonvanishing mass $m > 0$. The coupling constant f is uniquely determined as $f = 4f_B$.

If $f_B < v_0(0)^{-1}$ we are in a massless phase with spontaneously broken symmetry. Two solutions for the coupling constant f correspond to two possible values of the spontaneous magnetization.

At least in the limit $n \to \infty$, we have thus succeeded in deriving a unified description of the nonlinear σ-model where the mass m serves as an order parameter to distinguish between the symmetric and the spontaneously broken phase.

In the next chapter, we will consider whether this formulation based on normal ordering of the potential can be of any use for finite n.

6. Gap and coupling equation for finite n

We return to the normal ordered potential (4-24)

$$\begin{aligned} :V(\xi, \nabla\xi): = \int d^dx \Big\{ &-\frac{2n}{f_B} :\nabla_\mu \xi_i \nabla_\mu \xi_i \left[F_2(n,\xi^2) - \frac{f_B}{f}\right]: \\ &- \frac{n}{2}\frac{f}{f_B}\left[\delta^d(0) - m^2 v_m(0)\right] : F_2(n,\xi^2): \\ &+ n\,\delta^d(0) : \frac{\partial}{\partial\alpha} F_\alpha(n,\xi^2)\Big|_{\alpha=0} : \\ &+ \frac{2n}{f} m^2 :\xi^2: \Big\} \end{aligned} \qquad (6\text{-}1)$$

and try, in analogy with the developments of chapter 5, to derive gap and coupling equations for finite n by requiring the quadratic terms in the potential to vanish.

To this end, we will make use of the power series expansion of the functions $F_\alpha(n,\xi^2)$.

$$F_\alpha(n,\xi^2) = \sum_{k=0}^{\infty} (-\xi^2)^k \binom{\alpha+k-1}{k} F_{\alpha+k}(n+2k, 0) \qquad (6\text{-}2)$$

Then, the matching condition

$$\frac{\partial^2 V}{\partial(\nabla\xi)^2}\Big|_{\xi=0} \overset{!}{=} 0 \qquad (6\text{-}3)$$

yields the coupling equation

$$F_2(n,0) = \frac{f_B}{f} . \qquad (6\text{-}4)$$

From the second condition

$$\frac{\partial^2 V}{\partial \xi^2}\Big|_{\xi=0} \overset{!}{=} 0 \qquad (6\text{-}5)$$

we obtain

$$n \frac{f}{f_B} \left[\delta^d(0) - m^2 v_m(0) \right] F_3(n+2,0)$$
$$= n \delta^d(0) F_1(n,0) - \frac{2n}{f} m^2 \tag{6-6}$$

In contrast to the large n approximation, the quadratically divergent terms do not cancel after insertion of the coupling equation. Since the derivative term of the potential (6-1) produces quadratically divergent graphs we may still hope that all quadratic divergences cancel in each order of perturbation theory. This is in the spirit of the dimensional regularization scheme [22] where $\delta^d(o)$ is formally set equal to zero. Postponing a discussion of this problem to chapter 7, we will for the time being follow this proposal and proceed by dropping all terms proportional to $\delta^d(o)$.

Thus, instead of eq. (6-1), we consider the potential

$$:V(\xi,\nabla\xi): = \int d^dx \left\{ -\frac{2n}{f_B} : \nabla_\mu \xi : \nabla_\mu \xi : \left[F_2(n,\xi^2) - \frac{f_B}{f} \right] : \right.$$
$$+ \frac{n}{2} \frac{f}{f_B} m^2 v_m(0) : F_2(n,\xi^2) :$$
$$\left. + \frac{2n}{f} m^2 : \xi^2 : \right\}$$
$$= \int d^dx \left\{ -\frac{2n}{f_B} : \nabla_\mu \xi : \nabla_\mu \xi : \right. \tag{6-7}$$
$$\cdot \left[\sum_{k=0}^{\infty} (k+1)(-\xi^2)^k F_{2+k}(n+2k,0) - \frac{f_B}{f} \right] :$$
$$+ \frac{n}{2} \frac{f}{f_B} m^2 v_m(0) \sum_{k=1}^{\infty} (k+1) F_{2+k}(n+2k,0) : (-\xi^2)^k :$$
$$\left. + \frac{2n}{f} m^2 : \xi^2 : \right\}$$

Consequently, eq. (6-6) is replaced by the gap equation

$$m^2 \frac{f}{f_B} v_m(0) F_3(n+2,0) = \frac{2}{f} m^2 \tag{6-8}$$

Inserting the coupling equation (6-4) into eq. (6-8) we derive

$$m^2 v_m(0) F_3(n+2,0) = \frac{2}{f} m^2 F_2(n,0) \tag{6-9}$$

After application of the recursion formula

$$2v\alpha F_{\alpha+1}(n,\xi^2) = F_\alpha(n-2,\xi^2) - F_\alpha(n,\xi^2) \tag{6-10}$$

the gap equation assumes the form

$$m^2 F_2(n+2,0) = \frac{n-2}{n} m^2 F_2(n,0) \tag{6-11}$$

Again, there are possibly two solutions,

$$m = 0 \tag{6-12}$$

or

$$F_2(n+2,0) = \frac{n-2}{n} F_2(n,0) \quad \text{for} \quad m \neq 0 \tag{6-13}$$

Recall that F_α is a function of v(o). Due to the infrared divergence of the massless propagator and $F_2(n,0) \to 0$ for $v \to \infty$, there is no massless solution to the coupling equation (6-4) in 2 dimensions.

On the other hand, since the functions F_α take on only positive values eq. (6-13) shows that there will be no massive solution for any value of the coupling constant f_B in the cases n = 1 and n = 2 in any dimension $d \geq 2$.

Evidently, in 2 dimensions we find no solution at all for $n \leq 2$.

In the case n = 1, the symmetry of the system becomes abelian. By choosing a suitable parametrization, namely

$$\sigma = \cos \varphi \tag{6-14}$$

the model is transformed into a massless free field theory,

$$\begin{aligned} L &= -\frac{1}{2f_B} \int d^d x \left\{ \nabla_\mu \sigma \nabla_\mu \sigma + \nabla_\mu \sqrt{1-\sigma^2}\, \nabla_\mu \sqrt{1-\sigma^2} \right\} \\ &= -\frac{1}{2f_B} \int d^d x \; \nabla_\mu \varphi \nabla_\mu \varphi \end{aligned} \tag{6-15}$$

the measure being simply $\Pi d\varphi(x)$, with $-\pi \leq \varphi \leq \pi$.

Since in 2 dimensions a massless free field theory presumably does not exist [23] we are not surprised at the previous observation that our method yields no solution for this case.

Note that using a lattice regularization the case n = 1 corresponds to the xy-model which is known [24] to undergo a phase transition in 2 dimensions and to possess a nonvanishing mass gap for strong coupling. The deviation from the properties of the continuum theory may technically have its origin in the fact that the relation $\nabla_\mu v(o) = 0$ and Leibniz' rule do not hold on a lattice.

In contrast to the situation for n = 1, no obvious explanation can be given for the failure of our method in the case n = 2. In fact, this model corresponds to the O(3) symmetric Heisenberg model which is known to possess a symmetric and a spontaneously broken phase for d > 2 [16,25]. We would therefore have expected a massless solution for small f_B and a massive solution for large f_B.

Let us next study the weak-coupling region for n not too small. The large n approximation suggests that in d > 2 dimensions we will find a massless phase with broken symmetry. We further expect two solutions for the coupling parameter f one of which will go to zero, the other to infinity, as $f_B \to 0$. We now want to check whether this picture is consistent with the coupling equation (6-4) and the gap equation (6-11).

For small f and finite propagator, we compute $F_2(n,o)$ by means of the approximation

$$(1 + 2vt)^{-n/2} \approx 1 - nvt \qquad v \to 0 \tag{6-16}$$

since large values of t are suppressed by the exponential in eq. (4-15). The result is

$$F_\alpha(n,0) \approx 1 - \alpha n v \qquad v \to 0 \tag{6-17}$$

The coupling equation (6-4) hence yields for zero mass

$$1 - 2nv = 1 - \frac{f}{2} v_0(0) = \frac{f_B}{f} \tag{6-18}$$

so that

$$f = f_B + O(f_B^2) \tag{6-19}$$

Then, all corrections to the free n-point functions are suppressed by powers of f_B. In the calculation of the spontaneous magnetization up to order f_B contributions from the potential can therefore be neglected, and we obtain

$$\begin{aligned} \langle \sigma^0 \rangle &= -1 + 2F_1(n,0) \\ &= -1 + 2\left[1 - \frac{f}{4} v_0(0)\right] \end{aligned} \tag{6-20}$$

Because of eq. (6-18), the spontaneous magnetization is

$$\langle \sigma^0 \rangle = 1 - \frac{f_B}{2} v_0(0) + O(f_B^2) \qquad (6\text{-}21)$$

Let us now look for a solution $f \to \infty$ if $f_B \to 0$.

For large values of v, only small values of t contribute to the integral (4-15). Approximating the exponential by 1 we obtain for $n > 2\alpha$

$$F_\alpha(n,0) = \frac{(2v)^{-\alpha}}{(\frac{n}{2}-1)(\frac{n}{2}-2)\cdots(\frac{n}{2}-\alpha)} + O(v^{-\alpha-1/2}) \qquad (6\text{-}22)$$

In particular

$$F_2(n,0) = \frac{1}{v^2(n-2)(n-4)} + O(v^{-5/2}) \quad \text{for } n>4 \qquad (6\text{-}23)$$

This leads to the coupling equation

$$\frac{16 n^2}{f^2 v_0^2(0)(n-2)(n-4)} = \frac{f_B}{f} \qquad (6\text{-}24)$$

which has the solution

$$f = \frac{16 n^2}{f_B v_0^2(0)(n-2)(n-4)} \quad \text{for } n>4,\ f_B \to 0 \qquad (6\text{-}25)$$

Note that for $n \to \infty$ the two solutions (6-19) and (6-25) coincide with the result (5-17) of the 1/n expansion.

Since, in the given approximation of small f_B and large f, the functions F_α are of order

$$F_\alpha(n,0) \sim f^{-\alpha} n^0 \sim f_B^{\alpha} n^0 \qquad (6\text{-}26)$$

the coefficient of $:\xi^{2k}:$ in the power series expansion (6-7) of the potential is proportional to $f_B^k n^0$.

Since the propagator v(x-y) contains a factor f_B^{-1} all corrections to the n-point functions are of the same order in f_B. Consequently, perturbation theory in f_B is not possible for the large-f solution. But the effect of normal ordering is again (see chapter 5) that corrections from the potential are suppressed by powers of 1/n.

Thus, if we compute the spontaneous magnetization neglecting the potential and using eqs. (6-22) and (6-24) we get

$$\begin{aligned} \langle \sigma^0 \rangle &= -1 + 2F_1(n,0) \\ &= -1 + \frac{1}{2} f_B v_0(0) \left(1 - \frac{4}{n}\right) \end{aligned} \qquad (6\text{-}27)$$

This gives the expected result, namely the negative of the right-hand side of eq. (6-21), only to leading order in 1/n.

In conclusion, in the weak-coupling region for $n > 4$, we reproduce the picture of the large n approximation.
If $d > 2$ and f_B sufficiently small the symmetry is spontaneously broken. Setting the mass m equal to zero we find two solutions for the coupling parameter f corresponding to the two possible values of the spontaneous magnetization

$$\langle \sigma^0 \rangle = \pm \left[1 - \frac{1}{2} f_B v_0(0)\right] + O(f_B, \tfrac{1}{n}) \qquad (6\text{-}28)$$

Hence for $n > 4$, normal ordering of the potential with respect to arbitrary parameters f and m and requiring the quadratic terms to vanish seems to be a useful approach provided the problem of quadratic divergences can be treated satisfactorily.

For small n, $n \le 4$, however, this formulation seems to be inadequate. A possible modification worth studying might be to replace the condition $\partial^2 V/\partial\xi^2|_{\xi=0} = 0$ by the requirement that, in the spirit of Goldstone's theorem, either the mass or the spontaneous magnetization has to vanish.

7. On the cancellation of quadratic divergences

In the existing literature on the renormalization properties of the nonlinear σ-model, e.g. [16] , the discussion of quadratic divergences seems fairly incomplete. In fact, the mutual cancellation of quadratic divergences arising from different parts of the total action is usually assumed rather than being proven.

For instance, Bardeen et al. [19] discard all quadratic divergences by relying on the dimensional regularization scheme where $\delta^d(0)$ is formally set equal to zero. Apart from the fact that the cancellation of all quadratic divergences is not actually proven dimensional regularization appears also to be rather unsatisfactory because it masks the presence of an explicit mass term in the action which originates in the measure.

As far as we know, only Gerstein et al. [26] have attempted an actual proof for the mutual cancellation of quadratic divergences. They find that the logarithmic term coming from the measure can be dropped if the Feynman rules are modified in a noncovariant way. However, there will still appear quadratically divergent graphs even though the action does no longer contain any divergent piece. Gerstein et al. do not actually show that these graphs cancel.

Here we address ourselves to the problem of quadratic divergences in a different way. Whereas Gerstein et al. use perturbation theory based on canonical quantization our formulation is in terms of functional integrals.

Consider the action of a real field φ

$$L_{tot}(\varphi) = L_0(\varphi) + V(\varphi) \tag{7-1}$$

with derivative couplings

$$V(\varphi) = \int_x \sigma[\varphi(x), \nabla\varphi(x)] \tag{7-2}$$

where $\int_x$ means $\int d^d x$.

In the functional integral formalism, the perturbation series is obtained from the partition function Z(J) in the presence of a source term,

$$\begin{aligned} Z(J) &= \int \mathcal{D}\varphi \, e^{L_{tot}(\varphi) + \int_x \varphi(x) J(x)} \\ &= \int \mathcal{D}\varphi \, e^{V\left(\frac{\delta}{\delta J}\right)} e^{L_0(\varphi) + (J,\varphi)} \\ &= Z_0 \, e^{V\left(\frac{\delta}{\delta J}\right)} e^{\frac{1}{2}(J, v J)} \end{aligned} \tag{7-3}$$

with

$$Z_0 = \int \mathcal{D}\varphi \, e^{L_0(\varphi)} \tag{7-4}$$

$\mathcal{D}\varphi$ means functional integration with respect to the field φ. The propagator v(x-y) is determined by the free action L_o.

The disconnected Greens function $\langle P(\varphi)\rangle$ of a polynomial P involving the field at different space-time points but containing no derivatives is given by

$$Z(0) \langle P(\varphi)\rangle = Z_0 \, P\left(\frac{\delta}{\delta J}\right) e^{V\left(\frac{\delta}{\delta J}\right)} e^{\frac{1}{2}(J, vJ)} \Big|_{J=0} \tag{7-5}$$

with Z(0) = Z(J=0).

Since

$$\frac{\delta}{\delta J(x)} f(J) = \left[\frac{\delta}{\delta J(x)}, f(J)\right]_- \tag{7-6}$$

for any functional f it is sufficient to know the commutator

$$\left[\frac{\delta}{\delta J(x)}, J(y)\right]_- = \delta(x-y) \tag{7-7}$$

in order to compute the right-hand side of eq. (7-5).

Because of this we are allowed to replace J, $\delta/\delta J$ by the operators

$$J \mathrel{\hat{=}} \frac{\overleftarrow{\delta}}{\delta\varphi} \;, \quad \frac{\delta}{\delta J} \mathrel{\hat{=}} \varphi \tag{7-8}$$

which satisfy the same commutation relation. The derivative $\overleftarrow{\delta}/\delta\varphi$ is meant to act to the left.

Interchanging the factors in eq. (7-5) we get

$$Z(0) \langle P(\varphi) \rangle = Z_0 \, e^{\frac{1}{2}\left(\frac{\delta}{\delta\varphi},\, v \frac{\delta}{\delta\varphi}\right)} e^{V(\varphi)} P(\varphi) \Big|_{\varphi=0} \tag{7-9}$$

Application of the chain rule

$$\frac{\delta}{\delta\varphi(x)} F(\varphi(z), \nabla\varphi(z)) = \left[\delta(z-x)\frac{\delta}{\delta\varphi(z)} + \nabla_\mu^z \delta(z-x) \frac{\delta}{\delta\eta_\mu(z)}\right] \cdot F(\varphi(z), \eta_\mu(z)) \Big|_{\eta_\mu = \nabla_\mu \varphi} \tag{7-10}$$

yields

$$Z(0) \langle P(\varphi) \rangle = Z_0 \exp \int_x \int_y \left\{ -\frac{1}{2} \frac{\delta}{\delta\eta_\mu(x)} \nabla_\mu \nabla_\nu v(x-y) \frac{\delta}{\delta\eta_\nu(y)} + \frac{\delta}{\delta\eta_\mu(x)} \nabla_\mu v(x-y) \frac{\delta}{\delta\varphi(y)} + \frac{1}{2}\frac{\delta}{\delta\varphi(x)} v(x-y) \frac{\delta}{\delta\varphi(y)} \right\} \cdot e^{V(\varphi,\, \eta)} P(\varphi) \Big|_{\varphi=\eta=0} \tag{7-11}$$

We are interested in the special case where the potential is quadratic in the derivatives of the field.

$$V(\varphi, \nabla\varphi) = -\frac{1}{2} \nabla_\mu\varphi \nabla_\mu \varphi \, G(\varphi) + V'(\varphi) \tag{7-12}$$

We will perform some of the differentiations in eq. (7-11), i.e. evaluate the expression

$$I \equiv \exp\left\{ -\frac{1}{2} \int_x \int_y \frac{\delta}{\delta\eta_\mu(x)} \nabla_\mu \nabla_\nu v(x-y) \frac{\delta}{\delta\eta_\nu(y)} \right\} \exp\left\{ -\frac{1}{2} \int_z \eta_\mu(z)\eta_\mu(z)\, G(\varphi(z)) \right\} \tag{7-13}$$

Introducing a vector

$$\eta \equiv (\eta_\mu(x)) \tag{7-14}$$

and matrices

$$\tilde{G} \equiv (-\nabla_\mu \nabla_\nu v(x-y)) \tag{7-15}$$

$$G \equiv (G(\varphi(x))\, \delta_{\mu\nu}\, \delta(x-y)) \tag{7-16}$$

we rewrite eq. (7-13) in a self-explanatory compact notation as

$$I = e^{\frac{1}{2}\left\langle \frac{\delta}{\delta\eta},\, \tilde{G} \frac{\delta}{\delta\eta} \right\rangle} e^{-\frac{1}{2}\langle \eta, G\eta \rangle} \tag{7-17}$$

The first factor in eq. (7-17) is replaced by a functional Gaussian integral,

$$I = c \int \mathcal{D}q \; e^{\left\langle q, \frac{\delta}{\delta\eta} \right\rangle} e^{-\frac{1}{2}\langle \eta, G\eta\rangle} \cdot e^{-\frac{1}{2}\langle q, \tilde{G}^{-1} q \rangle} \tag{7-18}$$

where the constant c is given by

$$c^{-1} = \int \mathcal{D}q \; e^{-\frac{1}{2}\langle q, \tilde{G}^{-1} q\rangle} \tag{7-19}$$

Since the differentiation with respect to η is now equivalent to a translation by q we get

$$I = c \int \mathcal{D}q \; e^{-\frac{1}{2}\langle \eta+q, G(\eta+q)\rangle} e^{-\frac{1}{2}\langle q, \tilde{G}^{-1} q\rangle} \tag{7-20}$$

As long as G is small enough we can integrate over q to obtain

$$\begin{aligned} I &= \det(1+\tilde{G}G)^{-1/2} \, e^{\frac{1}{2}\langle G\eta, (G+\tilde{G}^{-1})G\eta\rangle} \\ &\quad \cdot e^{-\frac{1}{2}\langle \eta, G\eta\rangle} \\ &= \det(1+\tilde{G}G)^{-1/2} \, e^{-\frac{1}{2}\langle \eta, (G^{-1}+\tilde{G})^{-1}\eta\rangle} \end{aligned} \tag{7-21}$$

In the case of n field components φ_i, G and $\tilde{v}$ should be read as $G \otimes 1$ and $\tilde{v} \otimes 1$ where 1 is the nxn-unit matrix so that

$$I = \det(1+\tilde{G}G)^{-n/2} \, e^{-\frac{1}{2}\langle \eta, (G^{-1}+\tilde{G})^{-1}\eta\rangle} \tag{7-22}$$

For the nonlinear σ-model in stereographic coordinates we have

$$L_0(\xi) = -\frac{2n}{f} \int_x \left\{ \nabla_\mu \xi_i(x) \nabla_\mu \xi_i(x) + m^2 \xi^2(x) \right\} \tag{7-23}$$

and

$$\begin{aligned} V(\xi, \nabla\xi) &= -\frac{1}{2} \int_x \nabla_\mu \xi_i(x) \nabla_\mu \xi_i(x) \, G(\xi(x)) \\ &\quad + \frac{2n}{f} \int_x m^2 \xi^2(x) - n\delta^d(0) \int_x \ln\left[1+\xi^2(x)\right] \end{aligned} \tag{7-24}$$

with

$$G(\xi) = \frac{4n}{f_B}(1+\xi^2)^{-2} - \frac{4n}{f} \tag{7-25}$$

We rewrite the logarithmic term in the form

$$-n \ln(1+\xi^2) = \frac{n}{2} \ln\left[1+\frac{f}{4n} G(\xi)\right] + \text{const} \tag{7-26}$$

Eq. (7-11) thus assumes the form

$$\begin{aligned} Z(0)\langle P(\xi)\rangle &= Z_0 \exp \int_x \int_y \left\{ \frac{\delta}{\delta \eta_\mu^i(x)} \nabla_\mu v(x-y) \frac{\delta}{\delta \xi^i(y)} \right. \\ &\quad \left. + \frac{1}{2} \frac{\delta}{\delta \xi^i(x)} v(x-y) \frac{\delta}{\delta \xi^i(y)} \right\} \\ &\quad \cdot \exp\left\{ -\frac{1}{2}\langle \eta, (G^{-1}+\tilde{G})^{-1}\eta\rangle + \frac{2n}{f}\int_x m^2 \xi^2(x) \right. \\ &\quad \left. + \frac{n}{2}\delta^d(0) \int_x \ln\left[1+\frac{f}{4n}G(\xi)\right] - \frac{n}{2} \operatorname{tr} \ln(1+\tilde{G}G) \right\} \\ &\quad \cdot P(\xi) \Big|_{\xi=\eta=0} \end{aligned} \tag{7-27}$$

in agreement with the corresponding result of Gerstein et al. [26] .

After performing the differentiations with respect to η_μ we finally arrive at

$$\begin{aligned} Z(0)\langle P(\xi)\rangle &= Z_0 \exp \int_x \int_y \frac{1}{2} \frac{\delta}{\delta \xi^i(x)} v(x-y) \frac{\delta}{\delta \xi^i(y)} \\ &\quad \cdot \exp\left\{ -\frac{1}{2} \int_x\int_y\int_z\int_{z'} \nabla_\mu v(x-z) \nabla_\nu v(y-z') \frac{\delta}{\delta \xi^i(z)} \frac{\delta}{\delta \xi^i(z')} (G^{-1}+\tilde{G})^{-1}_{\mu\nu}(x-y) \right. \\ &\quad + \frac{2n}{f}\int_x m^2 \xi^2(x) + \frac{n}{2}\delta^d(0)\int_x \ln\left[1+\frac{f}{4n}G(\xi)\right] \\ &\quad \left. - \frac{n}{2} \operatorname{tr} \ln(1+\tilde{G}G) \right\} \cdot P(\xi) \Big|_{\xi=0} \end{aligned} \tag{7-28}$$

All the quadratic divergences of the theory are now contained in the two logarithmic terms, but it is not at all obvious whether the divergent parts of these two terms will cancel each other, and as yet we have not been able to find a complete proof for this.

In order to develop an intuition for the cancellation of quadratic divergences we will first study perturbation theory up to second order in the coupling constant.

Our perturbation expansion will be based on the action

$$L_{tot}(\xi) = -\frac{2n}{f_B}\int_x \{\nabla_\mu \xi : \nabla_\mu \xi : + m^2 \xi^2\} + V(\xi, \nabla\xi) \qquad (7\text{-}29)$$

with the potential (cp. eqs. (3-2) and (5-18))

$$\begin{aligned} V(\xi,\nabla\xi) &= -\frac{2n}{f_B}\int_x \nabla_\mu \xi : \nabla_\mu \xi : \left[(1+\xi^2)^{-2} - 1\right] \\ &\quad - n\,\delta^d(0)\int_x \ln(1+\xi^2) \qquad (7\text{-}3o) \\ &= -\frac{2n}{f_B}\int_x \nabla_\mu \xi : \nabla_\mu \xi : \sum_{k=1}^{\infty}(k+1)(-\xi^2)^k \\ &\quad + n\,\delta^d(0)\int_x \sum_{k=1}^{\infty}\frac{1}{k}(-\xi^2)^k \end{aligned}$$

In order to avoid infrared divergences we have introduced an arbitrary mass m.

Let us now consider 1particle-irreducible graphs contributing to the amputated 2-point function $\Gamma^{(2)}(p)$ in momentum space at zero external momentum, p = 0. We discard overall factors in a way that corresponds to a normalization of the free amputated propagator by $\Gamma_o^{(2)}(p=0) = m^2$.

To first order in f_B, the only nonvanishing contributions at zero external momentum are given by a mass insertion diagram  from the logarithmic term and a tadpole diagram  where the dots on the closed loop denote derivatives.

We thus have the following graphical representation

$\Gamma^{(2)}(p=0)$ =  (7-31)

= $- 2n$  $- \frac{f_B}{2}\,\delta^d(0)$ 

+ terms of higher order

where stands for the free amputated propagator $\Gamma_o^{(2)}(0) = m^2$, and

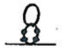
 $= 1$ (7-32a)

 $= \nabla_\mu \nabla_\mu v(0) = \frac{f_B}{4n}\left[-\delta^d(0) + m^2 \sigma_m(0)\right]$ (7-32b)

Evidently, the quadratically divergent parts cancel so that

$$\Gamma_1^{(2)}(0) = m^2\left[1 - \tfrac{1}{2} f_B \sigma_m(0)\right] \qquad (7\text{-}33)$$

To second order in f_B, the following 1particle-irreducible graphs contribute.

$$\Gamma_2^{(2)}(0) = \Gamma_1^{(2)}(0) + \frac{1}{4} f_B^2$$  $$- \frac{3}{8}\left(1+\frac{2}{n}\right) f_B^2$$ 

$$+ \frac{1}{4} f_B^2$$  $$+ \frac{1}{2n} f_B^2$$  (7-34)

$$+ \frac{1}{n^2} f_B^2$$  $$+ \frac{1}{8} f_B^2 \delta^d(0)\left(1+\frac{2}{n}\right)$$ 

$$- \frac{1}{4} f_B^2 \delta^d(0)$$ 

The diagrams in (7-34) represent the contributions

 $$= v_m(0) \int dp \frac{p^4}{(p^2+m^2)^2} \qquad (7\text{-}35a)$$

$$= \delta^d(0)\, v_m(0) - 2m^2 v_m^2(0) + m^4 v_m(0)\, I_2(m)$$

 $$= v_m(0) \int dp \frac{p^2}{p^2+m^2}$$

$$= \delta^d(0)\, v_m(0) - m^2 v_m^2(0) \qquad (7\text{-}35b)$$

 $$= \int dp \frac{p^2}{(p^2+m^2)^2} \int dq \frac{q^2}{q^2+m^2}$$

$$= \delta^d(0)\left[v_m(0) - m^2 I_2(m)\right] - m^2 v_m^2(0) \qquad (7\text{-}35c)$$

$$+ m^4 v_m(0)\, I_2(m)$$

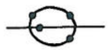
 $$= \int dp\, dq \frac{(p\cdot q)^2}{(p^2+m^2)(q^2+m^2)[(p+q)^2+m^2]}$$

$$= \int dp\, dq \frac{\frac{1}{2} p\cdot q\, [(p+q)^2 - p^2 - q^2]}{(p^2+m^2)(q^2+m^2)[(p+q)^2+m^2]}$$

$$= - \int dp\, dq \frac{p\cdot q}{(p^2+m^2)[(p+q)^2+m^2]} \qquad (7\text{-}35d)$$

$$+ \frac{m^2}{4} \int dp\, dq \frac{(p+q)^2 - p^2 - q^2}{(p^2+m^2)(q^2+m^2)[(p+q)^2+m^2]}$$

$$= \delta^d(0)\, v_m(0) - \frac{5}{4} m^2 v_m^2(0) + \frac{m^4}{4} I_3(m)$$

 $$= - \int dp\, dq \frac{q\cdot p\;\; q\cdot(p+q)}{(p^2+m^2)(q^2+m^2)[(p+q)^2+m^2]}$$

$$= - \int dp\, dq \frac{\frac{1}{2} q\cdot p\, [(p+q)^2 - p^2 + q^2]}{(p^2+m^2)(q^2+m^2)[(p+q)^2+m^2]}$$

$$= \frac{m^2}{4} \int dp\, dq \frac{(p+q)^2 - p^2 - q^2}{(p^2+m^2)(q^2+m^2)[(p+q)^2+m^2]} \qquad (7\text{-}35e)$$

$$= - \frac{m^2}{4} v_m^2(0) + \frac{m^4}{4} I_3(m)$$

 $$= v_m(0) \qquad (7\text{-}35f)$$

 $$= \int dp \frac{p^2}{(p^2+m^2)^2} \qquad (7\text{-}35g)$$

$$= v_m(0) - m^2 I_2(m)$$

We have used the abbreviation

$$dp \equiv \frac{d^d p}{(2\pi)^d} \tag{7-36}$$

The terms $I_2(m)$ and $I_3(m)$ are finite.

$$I_2(m) = \int dp\, (p^2+m^2)^{-2} \tag{7-37}$$

$$I_3(m) = \int dp\, dq \frac{1}{(p^2+m^2)(q^2+m^2)[(p+q)^2+m^2]} \tag{7-38}$$

As expected the quadratically divergent contributions from the graphs (7-35c) and (7-35g) cancel each other since these diagrams contain quadratically divergent subgraphs which are just those of eq.(7-32). We are left with four primitive quadratically divergent graphs (7-35a), (7-35b), (7-35d), and (7-35f) whose quadratic divergences indeed cancel. The final result contains only logarithmic divergences.

$$\Gamma_2^{(2)}(0) = m^2 \Big\{ 1 - \frac{f_B}{2} \sigma_m(0) - \frac{f_B^2}{8n^2} (3n^2+n+2)\sigma_m^2(0) + \frac{f_B^2}{2} m^2 \sigma_m(0) I_2(m) + \frac{f_B^2}{8n^2} (n+2) m^2 I_3(m) \Big\} \tag{7-39}$$

The only other quadratically divergent 1particle-irreducible diagrams of the theory up to second order in the coupling constant are given by corrections to the 4-point function. They are of the form

$4n\, \delta^d(0)$  $= 4n\, \delta^d(0)$ (7-4oa)

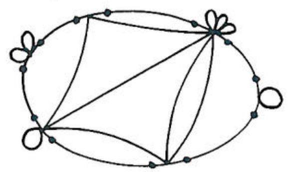
$8n$  $= 8n \{ \delta^d(0) - 2m^2 \sigma_m(0) \}$ + finite (7-4ob)

$-12n$  $= -12n \{ \delta^d(0) - m^2 \sigma_m(0) \}$ (7-4oc)

where we have put the sum of the initial momenta equal to zero, and again their quadratically divergent parts are seen to cancel.

To summarize our results so far, we have seen that all quadratic divergences up to second order in the coupling constant cancel. This observation strengthens our hope that the quadratic divergences will actually cancel in the full theory.

Further, our analysis of the second-order contributions to the 2- and 4-point functions seems to indicate that the primitive quadratically divergent diagrams involving derivative couplings display a particular topology. In fact, we conjecture that, omitting external legs, all these graphs exhibit a structure similar to figure 1.



- Figure 1 -

The essential feature of such a graph is that it contains just one "derivative loop", i.e. a closed loop made up of lines each carrying two derivatives (graphically displayed by two dots). Any two points of this loop may be connected by lines without derivatives. Moreover, an arbitrary number of tadpoles may be attached to each vertex. (They can be treated by normal ordering.) Note that no vertices appear inside the loop.

One sees from the previous discussion that the quadratically divergent primitive graphs of second order in the coupling constant comply with this conjecture.

Within the generic diagram of figure 1, a line with two dots represents a doubly differentiated propagator, $\nabla_\mu \nabla_\nu v(x-y)$. Consider one vertex of figure 1. Partial integration transfers a given derivative attached to this vertex successively to all other lines ending in this vertex which are not tadpole lines. In this way we obtain possibly a number of diagrams containing two lines which carry just one derivative. For these diagrams we conjecture that they will not be quadratically divergent, and we will neglect them. The dilute gas approximation to be discussed in the next chapter will possibly be of use in proving this conjecture which is supported by the second-order diagrams (7-35).
Thus the only diagram we keep after partial integration is the one that features one line carrying three derivatives. To this diagram we again apply partial integration and again retain only one diagram according to our assumption.
Repeating this procedure for each vertex of the derivative loop we eventually obtain a single diagram that looks just like the original one but the doubly dotted lines now represent

$$-\Delta v(x-y) = \frac{f}{4\pi}\left[\delta^d(x-y) - m^2 v(x-y)\right] \tag{7-41}$$

Assuming the validity of this construction we will now demonstrate the cancellation of quadratic divergences in eq. (7-28).

We begin by expanding the logarithmic term generating the derivative loops into a power series.

$$\begin{aligned} I &\equiv -\frac{n}{2}\,\mathrm{tr}\,\ln(1+\tilde{G}G) \\ &= \frac{n}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\,\mathrm{tr}\,(\tilde{G}G)^k \\ &= \frac{n}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_{x_1}\dots\int_{x_k}\nabla_{\mu_1}\nabla_{\mu_2}v(x_1-x_2)\dots \\ &\qquad \dots\nabla_{\mu_k}\nabla_{\mu_1}v(x_k-x_1)\prod_{i=1}^{k}G(x_i) \end{aligned} \tag{7-42}$$

Each term of this sum represents one derivative loop with k vertices.

According to our assumption above this expansion can be re-written as

$$\begin{aligned} I = \frac{n}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_{x_1}\dots\int_{x_k}\Delta v(x_1-x_2)\dots\Delta v(x_k-x_1) \\ \cdot\prod_{i=1}^{k}G(x_i) \end{aligned} \tag{7-43}$$

where only terms which are less than quadratically divergent have been neglected.

Inserting eq. (7-41) and keeping only quadratically divergent terms we obtain from eq. (7-43)

$$I = \frac{n}{2}\,\delta^d(0)\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int d^dx\left[\frac{f}{4n}\,G(x)\right]^k$$
$$= -\frac{n}{2}\,\delta^d(0)\int d^dx\;\ln\left[1+\frac{f}{4n}\,G(x)\right] \qquad (7\text{-}44)$$

Cancellation of this term against the divergent term coming from the measure in eq. (7-28) is now obvious.

8. Dilute gas approximation

In chapter 5, we have derived an equation for the mass gap of the nonlinear σ-model in the limit $n \to \infty$. To leading order in $1/n$, the relevant graphs are found to be tadpole diagrams. Technically, our essential tool was to rewrite the potential in normal ordered form which in effect amounts to a summation of all the tadpole diagrams just mentioned.
In chapter 6, we attempted to apply this method to the case of finite n. This led to difficulties, however, which manifested themselves in the noncancellation of quadratic divergences in the gap equation (6-6). This seems to suggest that normal ordering does not take into account all relevant diagrams.
In fact, we have seen in the last chapter that not all quadratically divergent graphs appear in the form of tadpole diagrams. Consequently, these graphs are not summed by normal ordering. Thus, for finite n, we would like to find a different approximation scheme that includes all quadratically divergent contributions.

A device of this form is the dilute gas approximation. Its essential ingredient is that it takes into account only interactions at the same space-time point, just like the normal-ordering approximation. Formally, this is achieved by approximating the propagator by

$$\upsilon(x-y) = \upsilon(0)\,\delta^d(0)^{-1}\,\delta^d(x-y) \qquad (8\text{-}1)$$

The point is that in the present theory with derivative couplings the quadratically divergent diagrams other than tadpole graphs survive when one retains only interactions at the same point.

In the previous chapter we argued that to show the cancellation of quadratic divergences it is sufficient to consider only diagrams with one derivative loop of the form of figure 1. It is tempting to ask whether a calculation based on these graphs only can lead to sensible results, say, for the mass gap of the theory.

It is therefore these derivative loop graphs which we will now study in the dilute gas approximation. This means in particular that we neglect all diagrams containing lines with only one derivative attached to it. To phrase this in a more formal way, we discard all terms involving $\eta_\mu \equiv \nabla_\mu \xi$ in eq. (7-27). That is, we consider

$$Z(0) \langle P(\xi) \rangle = Z_0 \exp \int_x \int_y \frac{1}{2} \frac{\delta}{\delta \xi^i(x)} v(x-y) \frac{\delta}{\delta \xi^i(y)}$$

$$\cdot \exp \left\{ \frac{n}{2} \delta^d(0) \int_x \ln \left[1 + \frac{f}{4n} G(\xi) \right] \right. \qquad (8\text{-}2)$$

$$\left. - \frac{n}{2} \operatorname{tr} \ln (1 + \check{v} G) + \frac{2n}{f} \int_x m^2 \xi^2 \right\} \cdot P(\xi) \Big|_{\xi = 0}$$

We now use the expansion (7-42) for the term $-n/2 \operatorname{tr} \ln (1+\check{v}G)$. Again we apply partial integration and assume that in the dilute gas approximation the only surviving diagrams are those where the derivatives remain on the original derivative loop which reproduces eq. (7-43).

$$I \equiv - \frac{n}{2} \operatorname{tr} \ln (1 + \check{v} G)$$
$$(8\text{-}3)$$
$$= \frac{n}{2} \sum_{k=1}^{\infty} \frac{1}{k} \int_{x_1} \cdots \int_{x_k} \Delta v(x_1 - x_2) \dots \Delta v(x_k - x_1) \prod_{i=1}^{k} G(x_i)$$

A proof of the above assumption would probably make use of Leibniz' rule and the relation $\nabla_\mu v(0) = 0$. Since both do not hold true on a lattice a proof there would likely be more difficult.

In the dilute gas approximation, we replace Δv by

$$- \Delta v(x-y) = \frac{f}{4n} \delta^d(x-y) \left\{ 1 - m^2 v_m(0) \delta^d(0)^{-1} \right\} \qquad (8\text{-}4)$$

whence eq. (8-3) becomes

$$I = \frac{n}{2} \delta^d(0) \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left\{ 1 - m^2 v_m(0) \delta^d(0)^{-1} \right\}^k$$
$$\cdot \int_x \left[\frac{f}{4n} G(x) \right]^k \qquad (8\text{-}5)$$

We expand the curly brackets keeping only the first two terms since the remainder is suppressed by negative powers of $\delta^d(0)$. Thus

$$I = \frac{n}{2} \sum_{k=1}^{\infty} (-1)^k \left\{ \frac{1}{k} \delta^d(0) - m^2 v_m(0) \right\} \int_x \left[\frac{f}{4n} G(x) \right]^k \qquad (8\text{-}6)$$

We now perform the summation over k to get

$$I = - \frac{n}{2} \delta^d(0) \int_x \ln \left[1 + \frac{f}{4n} G(x) \right]$$
$$(8\text{-}7)$$
$$- \frac{n}{2} m^2 v_m(0) \int_x \left[\frac{1}{1 + \frac{f}{4n} G(x)} - 1 \right]$$

As before, the first term cancels against the divergent contribution from the measure. Since (cp. eq. (7-25))

$$G(x) = \frac{4n}{f_B} \left[1 + \xi^2(x) \right]^{-2} - \frac{4n}{f} \qquad (8\text{-}8)$$

the second term in eq. (8-7) takes the form

$$- \frac{n}{2} m^2 v_m(0) \int_x \left[\frac{f_B}{f} (1 + \xi^2)^2 - 1 \right] \qquad (8\text{-}9)$$

Consequently, eq. (8-2) is replaced by

$$Z(0) < P(\xi) > = Z_0 \exp \frac{1}{2} \int_x \int_y \frac{\delta}{\delta \xi^i(x)} \sigma(x-y) \frac{\delta}{\delta \xi^i(y)}$$

$$\cdot \exp \left\{ -\frac{n}{2} m^2 \sigma_m(0) \int_x \left[\frac{f_B}{f} (1+\xi^2)^2 - 1 \right] \right.$$

$$\left. + \frac{2n}{f} \int_x m^2 \xi^2 \right\} \cdot P(\xi) \Big|_{\xi=0} \qquad (8\text{-}10)$$

Surprisingly, the interaction has become polynomial in this approximation.

By Wick's theorem, eq. (8-1o) reads in normal ordered form

$$Z(0) < P(\xi) > = Z_0 \exp \frac{1}{2} \int_x \int_y \frac{\delta}{\delta \xi^i(x)} \sigma(x-y) \frac{\delta}{\delta \xi^i(y)}$$

$$\cdot \exp \left\{ -n m^2 \sigma_m(0) \frac{f_B}{f} \int_x \left[(1 + (n+2)\sigma) : \xi^2 : \right. \right.$$

$$\left. \left. + \frac{1}{2} : (\xi^2)^2 : \right] + \frac{2n}{f} \int_x m^2 : \xi^2 : \right\}$$

$$\cdot P(\xi) \Big|_{\xi=0} \qquad (8\text{-}11)$$

By requiring that the quadratic terms have to disappear we readily obtain the gap equation

$$\frac{2}{f} m^2 = \frac{f_B}{f} m^2 \sigma_m(0) \left[1 + (n+2)\sigma \right] \qquad (8\text{-}12)$$

Our theory is now reduced to a linear σ-model.

$$Z(0) < P(\xi) > = Z_0 \exp \frac{1}{2} \int_x \int_y \frac{\delta}{\delta \xi^i(x)} \sigma(x-y) \frac{\delta}{\delta \xi^i(y)}$$

$$\cdot \exp \left[-\frac{1}{2} n m^2 \sigma_m(0) \frac{f_B}{f} \int_x : \xi^4 : \right]$$

$$\cdot P(\xi) \Big|_{\xi=0} \qquad (8\text{-}13)$$

This is not too surprising as the equivalence of linear and nonlinear σ-models in some respects (for instance long-distance behavior, scaling properties) has been shown before. [16]

The coupling equation (6-4) derived for finite n using normal ordering does not involve quadratic divergences and may therefore be taken over in unaltered form.

$$\frac{f_B}{f} = F_2(n,0) \qquad (8\text{-}14)$$

We are motivated to do so by the presumption that in the dilute gas approximation corrections to the inverse propagator that are proportional to p^2 are given entirely by tadpole diagrams.

We conclude with the observation that for $n \to \infty$ the results of the 1/n expansion (eqs. (5-5) and (5-8)) are reproduced.

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