Large momentum behavior of the Feynman amplitudes in the ϕ_4^4-theory

K. Pohlmeyer
Fakultät für Physik der Albert-Ludwigs-Universität, Hermann-Herder-Str. 3, D-7800 Freiburg i.Br., West Germany

(Received 29 October 1981; accepted for publication 19 March 1982)

The complete asymptotic expansion of the Feynman amplitudes for large values of the scale parameter is derived in the ϕ_4^4-theory for Euclidean and Minkowski metrics.

PACS numbers: 11.20.Dj, 11.10.Jj, 11.10.Gh

I. INTRODUCTION

The large-momentum behavior of Feynman amplitudes has attracted attention since the early days of renormalization theory.[1] Weinberg's power counting theorem,[2] proved for convergent graphs and Euclidean metrics, found innumerable applications in Lagrangian field theory. In 1968, Fink[3] obtained more detailed information concerning the logarithms which accompany the leading power of the scale parameter Λ. In 1973, Slavnov[4] showed that every (inverse) power of the overall scale parameter Λ in the asymptotic expansion of the Feynman amplitude is accompanied by a polynomial in ln Λ—and nothing else. In 1974, Bergère and Lam[5] determined all the coefficients going along with the leading power of Λ.

In the context of local gauge theories and chiral dynamics the transition from massive to massless propagators is of interest. On the other hand, the behavior of a Feynman amplitude when all masses in the propagators shrink to zero, the mass ratios (and the external momenta) being kept fixed, is intimately related to that of the same Feynman amplitude when all (external) momenta are rescaled by a common factor Λ tending to infinity, the masses in the propagators being kept fixed. Hence, there is additional motivation for the analysis of the large-momentum behavior of individual Feynman amplitudes. In particular, our analysis includes Feynman amplitudes of massless theories.

The present work gives the complete large-momentum (and small mass) behavior of Feynman amplitudes for individual vertex graphs and thereby for arbitrary individual graphs in the ϕ_4^4-theory to which we restrict ourselves for the sake of transparency.

It should be pointed out here that no restrictions are imposed on the momenta carried by the external lines of the graph in question, either linear ones (of the kind of vanishing partial sums of the momenta) or quadratic ones (mass shell conditions). Restrictions of this or a similar kind require special consideration because we view the asymptotic expansion in the context of distributions and not for every configuration of the external momenta separately. This distribution-theoretical formulation of the problem turns out to be both adequate and helpful for Minkowski metrics.

The appropriate frame for the derivation of the asymptotic expansion of the Feynman amplitudes appears to be the analytic renormalization scheme[6–8] (and possibly the renormalization formalism which is based on dimensional regularization[9]). In this scheme integrations over contours in the complex plane achieving analytic continuation take the place of the cumbersome Taylor operator in the Bogoliubov–Parasiuk–Hepp–Zimmermann renormalization formalism (cf., e.g., Ref. 5). Also, the concept of labeled (singularity-) s-families $(\mathbb{E},\sigma)$ in the analytic renormalization scheme[7] corresponding to the resolution of the ultraviolet singularities of the Feynman integrand[10] lends itself in a natural way to a generalization: the concept of labeled s_∞-families $(\mathbb{E}_\infty, \sigma_\infty)$ (explained below in Sec. IV) corresponding to the resolution of the combined ultraviolet and infrared singularities. In order to derive the complete asymptotic expansion for the scale parameter Λ tending to plus infinity, the degeneracies of the quadratic form of the external momenta entering the Feynman integrand need to be extracted completely. This is achieved by diagonalization of the quadratic form (Sec. III), subdivision of the integration domain of the Feynman parameters (Sec. IV), and parametrization of the resulting subdomains (Sec. V). Section VI recalls the analytic renormalization procedure. In Sec. VII the asymptotic expansion is derived and stated in a form which allows one to read off the error committed when it is truncated after a finite number of terms.

Actually, if one is interested only in the first few terms of the asymptotic expansion and the error committed when truncating the asymptotic expansion already after those, a partial diagonalization of the quadratic form and a less detailed subdivision of the integration domain of the Feynman parameters will be sufficient.

For the convenience of the reader some basic notations and definitions of the theory of Feynman graphs—so far as they are relevant in the present context—are collected in Sec. II. The material is taken from Refs. 11 and 12.

The present article reports work done in 1974.[13] This work has not been published up to now with the exception of a short introduction to the problem and an announcement of the results.[14] Nevertheless, meanwhile the basic ideas and methods have been applied in the literature (cf., e.g., Ref. 15). The actual presentation differs from the original one by the incorporation of Trute's graphical rules for the diagonalization of the relevant quadratic form, very appealing and efficient rules, indeed, which replace the author's own less elegant diagonalization techniques.

Since August 1974, several articles[16–19] appeared dealing with the (pointwise) asymptotic behavior of Feynman amplitudes. On the one hand, they carried the analysis of the asymptotic behavior beyond the limitations of the present paper establishing, e.g., the (pointwise) asymptotics when an arbitrary subset of the invariant momenta is scaled[17,18] or the Regge pole behavior of the four body scattering amplitude for the ϕ_4^3-theory.[19] On the other hand, the procedure described there for constructing a required number of terms of the asymptotic expansion and for determining the error committed by truncation is in general much more time consuming than the algorithm presented here, where the coefficients of the asymptotic expansion are expressed in terms of subgraphs and reduced graphs; only the minimal amount of partial ordering of the Feynman parameters is employed and the dimension of the multiple Mellin transformation is independent of the order of perturbation theory. Moreover, the analysis of Refs. 16–19 requires Euclidean metrics. It applies to Minkowski metrics only in special situations.

It is recommended that the reader look up Ref. 14 for a general orientation before going through the following detailed analysis.

II. BASIC GRAPH-THEORETICAL NOTATIONS AND DEFINITIONS

A *graph* G is a triplet consisting of a (finite) collection $\mathscr{V}(G)$ of $|\mathscr{V}(G)|$ vertices v, a (finite) collection $\mathscr{L}(G)$ of $|\mathscr{L}(G)|$ (internal) lines l and a mapping $\phi_G: \mathscr{L}(G) \to \mathscr{V}(G) \times \mathscr{V}(G)$, $l \overset{\phi_G}{\to} (\phi_i(l), \phi_f(l))$, where $\phi_i(l)$, $\phi_f(l)$ are the initial and final vertices of the line l.

$$G = (\mathscr{V}(G), \mathscr{L}(G), \phi_G).$$

The *union* G of two graphs G_1 and G_2: $G = G_1 \cup G_2$ defined by $\mathscr{V}(G) = \mathscr{V}(G_1) \cup \mathscr{V}(G_2)$, $\mathscr{L}(G) = \mathscr{L}(G_1) \cup \mathscr{L}(G_2)$,

$$\phi_G(l) = \begin{cases} \phi_{G_1}(l) & \text{if} \quad l \in \mathscr{L}(G_1), \\ \phi_{G_2}(l) & \text{if} \quad l \in \mathscr{L}(G_2), \end{cases}$$

provided $\phi_{G_1}(l) = \phi_{G_2}(l)$ for every $l \in \mathscr{L}(G_1) \cap \mathscr{L}(G_2)$ is again a graph.

A *subgraph* H of a graph G is a graph $(\mathscr{V}(H), \mathscr{L}(H), \phi_H)$ such that $\mathscr{V}(H) \subset \mathscr{V}(G)$, $\mathscr{L}(H) \subset \mathscr{L}(G)$, $\phi_H = \phi_G | \mathscr{L}(H)$ [in words, ϕ_G restricted to $\mathscr{L}(H)$].

A subgraph F of a subgraph H of a graph G is a subgraph of G.

If H is a subgraph of G we shall write $H \subset G$.

Let v be an element of $\mathscr{V}(G)$. We define sets of lines $\mathscr{S}(v)$ and $\mathscr{L}(v)$ according to

$$\mathscr{S}(v) = \{ l \in \mathscr{L}(G) / \phi_i(l) = v \quad \text{or} \quad \phi_f(l) = v \},$$

$$\mathscr{L}(v) = \{ l \in \mathscr{L}(G) / \phi_i(l) = v = \phi_f(l) \}.$$

Two distinct vertices v and v' are called *adjacent* if $\mathscr{S}(v) \cap \mathscr{S}(v') \neq \varnothing$. Two vertices v and v' are *connected* if there is a sequence of vertices $v = v_0, v_1, \ldots, v_k = v'$ such that v_j and v_{j+1} $(j = 0, 1, \ldots, k-1)$ are adjacent.

A graph G is said to be *connected* if any two of its vertices are connected. Otherwise, G is said to be *disconnected*.

A graph G can uniquely be decomposed into a union of connected subgraphs H_i of G $i = 1, 2, \ldots, p(G)$:

$$G = \bigcup_{i=1}^{p(G)} H_i.$$

The H_i's $i = 1, 2, \ldots, p(G)$ are called the *connected components* of G.

For any subset $\mathscr{I}$ of $\mathscr{L}(G)$ we define the subgraph $G - \mathscr{I}$ of the graph G according to

$$G - \mathscr{I} = (\mathscr{V}(G), \quad \mathscr{L}(G) \setminus \mathscr{I}, \phi_G | \mathscr{L}(G) \setminus \mathscr{I}).$$

For an arbitrary line $l_0 \in \mathscr{L}(G)$ consider the subgraph $G - \{l_0\}$ obtained from G by deleting the line l_0. The line l_0 is said to be a *cut-line* if $p(G - \{l_0\}) > p(G)$.

A graph G is called *one-line-irreducible* (or strongly connected) if it is connected and if it does not contain a cut-line. Otherwise, G is called *one-line-reducible*.

Any connected component H_i of a graph G can be decomposed uniquely into $c(H_i)$ one-line-irreducible components joined by $(c(H_i) - 1)$ cut-lines.

A vertex $v \in \mathscr{V}(G)$ is said to be a *cut-vertex* of the graph G if a connected component H of G has two subgraphs F_1 and F_2 such that $\mathscr{L}(F_1) \neq \varnothing$, $\mathscr{L}(F_2) \neq \varnothing$, $\mathscr{L}(F_1) \cap \mathscr{L}(F_2) = \varnothing$, $\mathscr{L}(F_1) \cup \mathscr{L}(F_2) = \mathscr{L}(H)$, and $\mathscr{V}(F_1) \cap \mathscr{V}(F_2) = \{v\}$.

A graph G is said to be *one-vertex-irreducible* if $\mathscr{V}(G)$ contains no cut-vertex. Otherwise, G is said to be *one-vertex-reducible*.

Any connected component H_i of G can be decomposed uniquely into $k(H_i)$ one-vertex-irreducible components joined by $(k(H_i) - 1)$ or fewer cut-vertices.

A graph G is said to be *irreducible* if it is both one-line-irreducible and one-vertex-irreducible. Otherwise, G is said to be *reducible*.

Let $\mathscr{k}$ be a subset of $\mathscr{V}(G)$ containing more than one element. We define $G(\mathscr{k})$ to be the graph $((\mathscr{V}(G) \setminus \mathscr{k}) \cup \{[\mathscr{k}]\}$, $\mathscr{L}(G)$, $\phi_{G(\mathscr{k})})$ obtained by identifying the vertices in $\mathscr{k}$

$$\phi_{G(\mathscr{k})}(l) = \begin{cases} (\phi_i(l), \phi_f(l)) & \text{if} \quad \phi_i(l), \phi_f(l) \in \mathscr{V}(G) \setminus \mathscr{k}, \\ (\phi_i(l), [\mathscr{k}]) & \text{if} \quad \phi_i(l) \in \mathscr{V}(G) \setminus \mathscr{k}, \ \phi_f(l) \in \mathscr{k}, \\ ([\mathscr{k}], \phi_f(l)) & \text{if} \quad \phi_i(l) \in \mathscr{k}, \phi_f(l) \in \mathscr{V}(G) \setminus \mathscr{k}, \\ ([\mathscr{k}], [\mathscr{k}]) & \text{if} \quad \phi_i(l), \phi_f(l) \in \mathscr{k}. \end{cases}$$

If $v \in \mathscr{V}(G)$, then $v^{\mathscr{k}}$ will denote the corresponding vertex in $G(\mathscr{k})$.

Let $\mathscr{k}' = \{v_1, \ldots, v_{|\mathscr{k}'|}\}$ be a subset of $\mathscr{V}(G)$ such that $|\{v_1^{\mathscr{k}}, \ldots, v_{|\mathscr{k}'|}^{\mathscr{k}}\}| \geqslant 2$. Then the symbol $G(\mathscr{k}|\mathscr{k}')$ denotes the graph

$$(G(\mathscr{k}))(\{v_1^{\mathscr{k}}, \ldots, v_{|\mathscr{k}'|}^{\mathscr{k}}\}),$$

obtained from the graph $G(\mathscr{k})$ by identifying the vertices $v_1^{\mathscr{k}}, \ldots, v_{|\mathscr{k}'|}^{\mathscr{k}}$. We define similarly $G(\mathscr{k}|\mathscr{k}'|\mathscr{k}'')$, and so on.

Let $H \subset G$ be a subgraph of G. For any $v \in \mathscr{V}(G)$ define a number

$$D(v, H) = |\mathscr{S}(v) \cap \mathscr{L}(H)| + |\mathscr{L}(v) \cap \mathscr{L}(H)|.$$

A *path* P between two distinct vertices v_1 and v_2 is a minimal connected subgraph of G such that for any $v \in \mathscr{V}(G)$

$$D(v, P) = \begin{cases} 0 \quad \text{or} \quad 2 & \text{if} \quad v \neq v_1, v_2, \\ 1 & \text{if} \quad v = v_1 \ \text{or} \ v = v_2. \end{cases}$$

A *loop* C is a minimal connected nonempty subgraph of G such that for any $v \in \mathscr{V}(G)$

$$D(v, C) = 0 \quad \text{or} \quad 2.$$

Let G be a graph. The *number of independent loops* of G will be denoted by $N(G)$. The numbers $|\mathscr{V}(G)|$, $|\mathscr{L}(G)|$, $p(G)$, and $N(G)$ are related by the equation

$$N(G) = |\mathscr{L}(G)| + p(G) - |\mathscr{V}(G)|.$$

From now on, the graph G will always denote a connected graph.

A subgraph $T_1 = T$ of G is called a 1-tree or a *tree* of G if $\mathcal{V}(T) = \mathcal{V}(G)$, $|\mathcal{L}(T)| = |\mathcal{V}(G)| - 1$, $p(T) = 1$; i.e., if T connects all vertices of G to each other, and if $\mathcal{L}(T)$ does not form loops $(N(T) = 0)$.

The set of all trees of G will be denoted by $\mathbb{T}_G$.

A subgraph T_r of G is called an *r-tree* of G if $\mathcal{V}(T_r) = \mathcal{V}(G)$, $|\mathcal{L}(T_r)| = |\mathcal{V}(T_r)| - r$, $p(T_r) = r$, i.e., if T_r effects a partition of the vertices of G into r mutually disjoint sets any two vertices of the same set being connected in T_r, and if $\mathcal{L}(T_r)$ does not form loops.

The set of all 2-trees of G effecting a partition of the disjoint subsets h_1, h_2 of $\mathcal{V}(G)$ from each other will be denoted by $\mathbb{T}_G(h_1|h_2)$.

A subgraph T'_r of G, related to some r-tree T_r of G as follows:

$$\mathcal{V}(T'_r) = \{v/v = \phi_i(l) \text{ or } v = \phi_f(l) \text{ for some } l \in \mathcal{L}(G) \setminus \mathcal{L}(T_r)\},$$
$$\mathcal{L}(T'_r) = \mathcal{L}(G) \setminus \mathcal{L}(T_r),$$
$$\phi_{T'_r} = \phi_{G/\mathcal{L}(T'_r)},$$

is called a *co-r-tree* of G. Among the various subsets of $\mathcal{V}(G)$ we distinguish *the set g_G of all $|g_G|$ external vertices*. If h_g is a subset of g_G, h'_g will denote the complementary subset $g_G \setminus h_g$ of g_G.

Next, following Trute,[12] we introduce the concept of an *m-family* of subsets of g_G (the latter m stands for "momentum"):

A collection $\mathbb{H}_G$ of subsets h of g_G is called an m-family if it meets the following requirements:

(α) If $h_1, h_2 \in \mathbb{H}_G$, then either $h_1 \subset h_2$ or $h_2 \subset h_1$ or $h_1 \cap h_2 = \varnothing$;

(β) $\varnothing \notin \mathbb{H}_G$. If $\mathbb{F} \subset \mathbb{H}_G$, then $g_G \setminus \bigcup_{h \in \mathbb{F}} h \notin \mathbb{H}_G$;

(γ) If $h \in \mathbb{H}_G$, then $|h| = |\bigcup_{h_1 \in \mathbb{H}_G,\, h_1 \subsetneq h} h_1| + 1$;

(δ) $\mathbb{H}_G$ is maximal.

Given an m-family $\mathbb{H}_G$, then the following statements are true:

(i) $|\mathbb{H}_G| = |g_G|$;

(ii) $g_G \in \mathbb{H}_G$;

(iii) Each $h \in \mathbb{H}_G$ can be labeled by the unique vertex $v_h \in g_G$, which is contained in h and is not contained in any of its proper subsets from $\mathbb{H}_G$.

(iv) Each $v = v_h$, $h \in \mathbb{H}_G$, $h \neq g_G$ possesses a unique predecessor $v_{\hat{h}}$ such that

$$h \subsetneq \hat{h} \in \mathbb{H}_G, \ \hat{h} \text{ is minimal.}$$

III. DIAGONALIZATION OF THE RELEVANT QUADRATIC FORMS

The integrand of the Feynman-parametric integral corresponding to the (connected) Feynman graph G involves the external momenta by means of the quadratic form

$$V_G = U_G^{-1} \times \sum_{\{h, h'\} = g_G} W_G^{(h|h')} \times \Big(\sum_{v \in h} p_v\Big)^2, \quad p_v \in \mathbb{R}^4,$$

$$\sum_{v \in g_G} p_v = 0.$$

Here the quantities U_G and $W_G^{(h|h')}$ are defined by

$$U_G = \sum_{T_1 \in \mathbb{T}_G} \prod_{l \in T'_1} \alpha_l, \quad W_G^{(h|h')} = \sum_{T_2 \in \mathbb{T}_G(h|h')} \prod_{l \in T'_2} \alpha_l.$$

The quadratic form V_G can be diagonalized. In this context, Trute[12] derived the following important statement.

Let $\mathbb{H}_G$ be an arbitrary m-family of G. Let the elements of $\mathbb{H}_G$; h_i, $i = 1,\ldots, |g_G|$; $h_{|g_G|} = g_G$ be ordered (in some arbitrary fashion) according to the index i. Define $G(v_{h_0} v_{\hat{h}_0})$ to be the graph G, denote the vertices of $G(v_{h_1} v_{\hat{h}_1} \ldots v_{h_{i-1}} v_{\hat{h}_{i-1}})$ by $v^{h_1 \cdots h_{i-1}}$ and let W_{ij} stand for the following function (in the definition of $W^{(\ldots|\ldots)}_{G(\ldots|\ldots|\ldots)}$ the sum extends over all 2-trees separating the indicated sets of external vertices)

$$W^{(v_{h_i}^{h_1 \cdots h_{i-1}},\, v_{h_j}^{h_1 \cdots h_{i-1}} \,|\, v_{\hat{h}_i}^{h_1 \cdots h_{i-1}},\, v_{\hat{h}_j}^{h_1 \cdots h_{i-1}})}_{G(v_{h_1} v_{\hat{h}_1} | \cdots | v_{h_{i-1}} v_{\hat{h}_{i-1}})} - W^{(v_{h_i}^{h_1 \cdots h_{i-1}},\, v_{\hat{h}_j}^{h_1 \cdots h_{i-1}} \,|\, v_{\hat{h}_i}^{h_1 \cdots h_{i-1}},\, v_{h_j}^{h_1 \cdots h_{i-1}})}_{G(v_{h_1} v_{\hat{h}_1} | \cdots | v_{h_{i-1}} v_{\hat{h}_{i-1}})}.$$

Then the following formula is valid:

$$V_G = \sum_{i=1}^{|g_G| - 1} \frac{U_{G(v_{h_1} v_{\hat{h}_1} | \cdots | v_{h_i} v_{\hat{h}_i})}}{U_{G(v_{h_1} v_{\hat{h}_1} | \cdots | v_{h_{i-1}} v_{\hat{h}_{i-1}})}} q^2_{h_i},$$

with

$$q_{h_i} = \sum_{v \in h_i} p_v + \sum_{j > i}^{|g_G| - 1} \frac{W_{ij}}{U_{G(v_{h_1} v_{\hat{h}_1} | \cdots | v_{h_i} v_{\hat{h}_i})}} \sum_{v \in h_j} p_v.$$

The transformation

$$\{p_1, \ldots p_{|g_G|}\} / \sum_{v \in g_G} p_v = 0 \to \{q_{h_i}\}_{i = 1, \ldots, |g_G| - 1}$$

is a nonsingular linear mapping of $\mathbb{R}^{4|g_G| - 4}$ onto $\mathbb{R}^{4|g_G| - 4}$ depending smoothly (C^∞), on the Feynman parameters α in the range $\alpha_l > 0$ for all $l \in \mathcal{L}(G)$.

In the preceding, we have employed a shorthand notation by writing, for instance, $G(v_{h_1} v_{\hat{h}_1} | \cdots | v_{h_i} v_{\hat{h}_i})$ instead of $G(\{v_{h_1}, v_{\hat{h}_1}\} | \cdots | \{v_{h_i}, v_{\hat{h}_i}\})$. In the sequel we shall continue to use this shorthand notation.

IV. SUBDIVISION OF THE INTEGRATION DOMAIN OF THE FEYNMAN PARAMETERS

Given a (connected) graph of G, $|g_G| \neq 0$. Along with G consider the graph $G(g_G)$

$$G(g_G) = (\mathcal{V}(G) \setminus g_G \cup \{[g_G]\}, \quad \mathcal{L}(G), \ \phi_{G(g_G)}),$$

obtained from G by identifying all external vertices. We shall write

$$v_\infty = [g_G].$$

Consider also the graphs $G(g_G \setminus \{v\})$, $v \in g_G$,

$$G(g_G \setminus \{v\}) = (\mathcal{V}(G) \setminus (g_G \setminus \{v\}) \cup \{[g_G \setminus \{v\}]\}, \ \mathcal{L}(G), \ \phi_{G(g_G \setminus \{v\})}),$$

obtained from G by identifying all external vertices but one: v. We shall write

$$[g_G \setminus \{v\}] = v_\infty(v) \quad \text{if} \quad g_G \setminus \{v\} \neq \varnothing.$$

A subgraph H of G is said to be irreducible "in view of infinity" (I_∞) if the subgraphs $H(\wp_G)$ and $H(\wp_G \setminus \{v\})$ ($v \in \wp_G \cap \mathscr{V}(H)$) of $G(\wp_G)$ and $G(\wp_G \setminus \{v\})$ ($v \in \wp_G \cap \mathscr{V}(H)$), respectively satisfy

$$\mathscr{V}(H(\wp_G)) = \begin{cases} \mathscr{V}(H) \setminus (\wp_G \cap \mathscr{V}(H)) \cap \{v_\infty\} & \text{if } \wp_G \cap \mathscr{V}(H) \neq \varnothing, \\ \mathscr{V}(H) & \text{if } \wp_G \cap \mathscr{V}(H) = \varnothing, \end{cases}$$

and

$$\mathscr{L}(H(\wp_G)) = \mathscr{L}(H), \quad \phi_{H(\wp_G)} = \phi_{G(\wp_G)|\mathscr{L}(H)},$$

and

$$\mathscr{V}(H(\wp_G \setminus \{v\})) = \begin{cases} \mathscr{V}(H) \setminus ((\wp_G \setminus \{v\}) \cap \mathscr{V}(H)) \cup \{v_\infty(v)\}, & \text{if } (\wp_G \setminus \{v\}) \cap \mathscr{V}(H) \neq \varnothing, \\ \mathscr{V}(H), & \text{if } (\wp_G \setminus \{v\}) \cap \mathscr{V}(H) = \varnothing, \end{cases}$$

and

$$\mathscr{L}(H(\wp_G \setminus \{v\})) = \mathscr{L}(H), \phi_{H(\wp_G \setminus \{v\})} = \phi_{G(\wp_G \setminus \{v\})|\mathscr{L}(H)}$$

have the following properties:

(i) $H(\wp_G)$ is one-line-irreducible,

(ii) none of the vertices contained in $\mathscr{V}(H(\wp_G)) \setminus \{v_\infty\}$ a cut-vertex of $H(\wp_G)$,

(iii) $H(\wp_G \setminus \{v\})$, $v \in \wp_G \cap \mathscr{V}(H)$ is connected, and

(iv) no vertex $v \in \wp_G \cap \mathscr{V}(H)$ is a cut-vertex of $H(\wp_G \setminus \{v\})$.

Otherwise, H is called *reducible "in spite of infinity"* (R_∞). Next, we define an s_∞*-family* $\mathbb{E}_\infty$ for G as a maximal collection of I_∞-subgraphs H of G with the following properties (cf. Ref. 13):

$(S_\infty - 0)$ $\quad \mathscr{L}(H) \neq \varnothing$.

$(S_\infty - 1)$ $\quad$ If $H, H' \in \mathbb{E}_\infty$, then either $H \subset H'$, $H' \subset H$, or $\mathscr{L}(H) \cap \mathscr{L}(H') = \varnothing$.

$(S_\infty - 2)$ $\quad$ If $H_1,...,H_r \in \mathbb{E}_\infty$ and $\mathscr{L}(H_i) \cap \mathscr{L}(H_j) = \varnothing$ for any $i \neq j$, then $\cup_{i=1}^r H_i$ is R_∞.

Moreover, we define a labeled s_∞-family for G to be a pair $(\mathbb{E}_\infty, \sigma_\infty)$ where $\mathbb{E}_\infty$ is an s_∞-family for G and σ_∞ a mapping $\sigma_\infty : \mathbb{E}_\infty \to \mathscr{L}(G)$ satisfying

$(S_\infty - 3)$ $\quad \sigma_\infty(H) \in \mathscr{L}(H)$.

$(S_\infty - 4)$ $\quad$ If $H' \in \mathbb{E}_\infty$ is a proper subset of $H \in \mathbb{E}_\infty$, then $\sigma_\infty(H) \notin \mathscr{L}(H')$.

The following statements can be proved along the lines of Ref. 20.

(i) For every $H \in \mathbb{E}_\infty$ there exists a line $l \in \mathscr{L}(H)$ not contained in $\mathscr{L}(H')$ for any $H' \in \mathbb{E}_\infty$, $H' \subsetneq H$.

(ii) $|\{H' \in \mathbb{E}_\infty / H' \subset H\}| = N(H(\wp_G))$.

(iii) Every s_∞-family $\mathbb{E}_\infty$ for a (connected graph G may be labeled, i.e., there exists a mapping $\sigma_\infty : \mathbb{E}_\infty \to \mathscr{L}(G)$ such that $(\mathbb{E}_\infty, \sigma_\infty)$ is a labeled s_∞-family for G.

(iv) If $(\mathbb{E}_\infty, \sigma_\infty)$ is a labeled s_∞-family for G, then $T_{|\wp_G|} = T_{|\wp_G|}(\mathbb{E}_\infty, \sigma_\infty)$; $\{\mathscr{V}(G), \mathscr{L}(G) \setminus \{\sigma_\infty(\mathbb{E}_\infty)\}, \phi_{G/\mathscr{L}(G)\setminus\{\sigma_\infty(\mathbb{E}_\infty)\}}\}$ is a $|\wp_G|$-tree of G, each of the $|\wp_G|$ connected components of $T_{|\wp_G|}$ containing exactly one external vertex. Consequently $T = T(\mathbb{E}_\infty, \sigma_\infty) = \{\mathscr{V}(G) \setminus \wp_G \cup \{v_\infty\}, \mathscr{L}(G) \setminus \{\sigma_\infty(\mathbb{E}_\infty)\}, \phi_{G(\wp_G)/\mathscr{L}(G)\setminus\{\sigma_\infty(\mathbb{E}_\infty)\}}\}$ is a 1-tree of $G(\wp_G)$.

(v) Consider the domain of integration of the Feynman parameters α_l, $l \in \mathscr{L}(G)$: $\{\underline{\alpha} = (\alpha_1,...,\alpha_{|\mathscr{L}(G)|}) / \alpha_l \geqslant 0$ for all $l \in \mathscr{L}(G)\}$.

If $(\mathbb{E}_\infty, \sigma_\infty)$ is a labeled s_∞-family for G we define $\mathscr{D}_\infty = \mathscr{D}_\infty(\mathbb{E}_\infty, \sigma_\infty)$ to be the subset of the above integration domain given by $\mathscr{D}_\infty = \{\underline{\alpha} / \alpha_l \geqslant 0$ for all $l \in \mathscr{L}(G)$, $\alpha_l \leqslant \alpha_{\sigma_\infty}(H)$ for all $l \in \mathscr{L}(H)$, $H \in \mathbb{E}_\infty\}$.

Then it is true that:

(a) $\cup \mathscr{D}_\infty = \cup \mathscr{D}_\infty(\mathbb{E}_\infty, \sigma_\infty) = \{\alpha / \alpha_l \geqslant 0$ for all $l \in \mathscr{L}(G)\}$, where the union extends over all labeled s_∞-families for G.

(b) $\mathscr{D}_\infty(\mathbb{E}_\infty, \sigma_\infty) \cap \mathscr{D}_\infty(\mathbb{E}'_\infty, \sigma'_\infty)$ has Lebesgue measure zero for any two distinct labeled s_∞-families for G: $(\mathbb{E}_\infty, \sigma_\infty)$ and $(\mathbb{E}'_\infty, \sigma'_\infty)$.

(vi) Set $\mathbb{H}_\infty = \{H / H \in \mathbb{E}_\infty, N(H) = N(H - \{\sigma_\infty(H)\})$ and $N(H(\wp_G)) = N(H(\wp_G) - \{\sigma_\infty(H)\}) + 1\}$. There exists a mapping $\tau: H \in \mathbb{H}_\infty \overset{\tau}{\underset{\tau^{-1}}{\leftrightarrow}} \hbar \in \mathbb{H}_G \setminus \{\wp_G\}$ for some m-family $\mathbb{H}_G$ such that the partial sum of external momenta that flow through the line $\sigma_\infty(H)$ in the tree $(\mathscr{V}(G), \mathscr{L}(G) \setminus \{\sigma_\infty(\mathbb{E}_\infty \setminus \mathbb{H}_\infty)\}, \phi_{G/...})$ can be written as $\Sigma_{v \in \hbar} p_v$. The elements of $\mathbb{H}_G$ are totally ordered $\hbar_i$ $i = 1,...,|\wp_G| - 1$, $\hbar_{|\wp_G|} = \wp_G$: $j > i$ if $H_j = \tau^{-1}(\hbar_j) \supseteq H_i = \tau^{-1}(\hbar_i)$.

V. PARAMETRIZATION OF THE SUBDOMAINS OF α-INTEGRATION

For any line $l \in \mathscr{L}(G)$ we define H_l to be the minimal element of $\mathbb{E}_\infty$ containing the line l. With this notation the subset $\mathscr{D}_\infty$ of the domain of α-integration can be parametrized as follows:

$$\alpha_l = \begin{cases} \prod_{H' : \mathbb{E}_\infty \ni H' \supset H_l} t_{H'} & \text{if } l = \sigma_\infty(H) \text{ for some } H \in \mathbb{E}_\infty, \\ \beta_l \prod_{H' : \mathbb{E}_\infty \ni H' \supset H_l} t_{H'} & \text{if } l \neq \sigma_\infty(H) \text{ for any } H \in \mathbb{E}_\infty, \end{cases}$$

where $0 \leqslant t_G < \infty$, $0 \leqslant t_H \leqslant 1$ for any $H \in \mathbb{E}_\infty$, $H \neq G$, $0 \leqslant \beta_l \leqslant 1$, $l \neq \sigma_\infty(H)$ for any $H \in \mathbb{E}_\infty$ or, writing the symbol $\underline{t}$ for $(t_H)_{H \in \mathbb{E}_\infty, H \neq G}$ and the symbol $\underline{\beta}$ for $(\beta_l)_{l \neq \sigma_\infty(H), \forall H \in \mathbb{E}_\infty}$ $0 \leqslant t_G < \infty$, $(\underline{t}, \underline{\beta}) \in I^{|\mathscr{L}(G)| - 1}$ with $I = [0,1]$.

Now, arguing along the lines of Ref. 20, one finds in $\mathscr{D}_\infty$,

(i) $U_G = \prod_{H \in \mathbb{E}_\infty} t^{N(H)} d_{G,0}^{(\mathbb{E}_\infty, \sigma_\infty)}(\underline{t}, \underline{\beta})$,

$$\frac{U_{G(v_{\hbar_i} v_{\hbar_i}^c | \cdots | v_{\hbar_i} v_{\hbar_i}^c)}}{U_{G(v_{\hbar_i} v_{\hbar_i}^c | \cdots | v_{\hbar_{i-1}} v_{\hbar_{i-1}}^c)}} = \frac{U_{G(\{v_{\hbar_1},...,v_{\hbar_{i-1}}\})}}{U_{G(\{v_{\hbar_1},...,v_{\hbar_i}\})}} = \left(\prod_{\substack{H' \in \mathbb{E}_\infty \\ H' \supset H_i}} t_{H'}\right) e_{H_i}(\underline{t}, \underline{\beta}),$$

where

$$e_{H_i}(\underline{t}, \underline{\beta}) = \frac{d_{G,i}^{(\mathbb{E}_\infty, \sigma_\infty)}(\underline{t}, \underline{\beta})}{d_{G,i-1}^{(\mathbb{E}_\infty, \sigma_\infty)}(\underline{t}, \underline{\beta})}$$

and where

$$d_{G,i}^{(\mathbb{E}_\infty,\sigma_\infty)}(\underline{t},\underline{\beta}) = d_{G(\{v_{\mathscr{k}_i},\ldots,v_{\mathscr{k}_i}\})}^{(\mathbb{E}_\infty,\sigma_\infty)}(\underline{t},\underline{\beta}), \quad i = 0,\ldots,|\wp_G| - 1$$

are polynomials in $\underline{t},\underline{\beta}$ larger than or equal to one for $(\underline{t},\underline{\beta}) \in I^{|\mathscr{L}(G)|-1}$;

(ii) the transformation

$$\{p_1,\ldots,p_{|\wp_G|}\}/\sum_{v\in\wp_G} p_v = 0 \rightarrow \{q_H = q_H(\underline{t},\underline{\beta})\}_{H\in\mathbb{H}_\infty} = \underline{q}(\underline{t},\underline{\beta}) = \underline{q},$$

$q_H = q_{\mathscr{k}}$ where $\mathscr{k} = \tau(H)$, $H\in\mathbb{H}_\infty$ is a nonsingular linear mapping of $\mathbb{R}^{4|\wp_G|-4}$ onto $\mathbb{R}^{4|\wp_G|-4}$ depending smoothly (i.e., in an infinite differential manner) on the parameters $\underline{t},\underline{\beta}$ for $(\underline{t},\underline{\beta})$ for $(t,\beta)\in I^{|\mathscr{L}(G)|-1}$.

Thus the quadratic form V_G in $\mathscr{D}_\infty$ is

$$V_G = -t_G E_{\underline{t},\underline{\beta}}(\underline{q}(\underline{t},\underline{\beta}),\underline{q}(\underline{t},\underline{\beta})),$$

$$E_{\underline{t},\underline{\beta}}(\underline{q},\underline{q}) = +\sum_{H\in\mathbb{H}_\infty}\left(\prod_{\substack{H'\in\mathbb{E}_\infty\setminus\{G\}\\ H'\supset H}} t_{H'}\right) e_H(\underline{t},\underline{\beta})(-q_H^2).$$

VI. ANALYTICALLY RENORMALIZED FEYNMAN AMPLITUDES

For the sake of simplicity and definiteness, we shall restrict our discussion to Feynman amplitudes occurring in the perturbation expansion of a $P(\phi)_4$ Lagrangian field theory describing a polynomial self-interaction of one sort of neutral scalar massive (m) particles in one time and three space dimensions. The generalization to theories involving massive particles with spin and derivative coupling in one time and arbitrarily many space dimensions is straightforward (cf., e.g., Ref. 12).

Consider a vertex graph G. Without loss of generality it may be assumed that G is irreducible. Set

$$\mathscr{v}(G) = v, \ \wp_G = \wp, \ \mathscr{L}(G) = \mathscr{L}, \ N(G) = N.$$

With Speer[6] we associate with every line l of the vertex graph G a complex variable λ_l, $\underline{\lambda} = (\lambda_l)_{l\in\mathscr{L}}$, and modify the propagators according to

$$\frac{i}{(2\pi)^2}\frac{1}{k^2 - m^2 + i0} \rightarrow \frac{e^{i\pi\lambda}\Gamma(\lambda)}{i(2\pi)^2}[k^2 - m^2 + i0]^{-\lambda},$$

$$\Delta_F(x_{i(l)} - x_{f(l)};m) \rightarrow \Delta_F^{\lambda_l}(x_{i(l)} - x_{f(l)};m) = \mathscr{F}_k\left\{\frac{e^{i\pi\lambda_l}\Gamma(\lambda_l)}{i(2\pi)^2}[k^2 - m^2 + i0]^{-\lambda_l}\right\} \times (x_{i(l)} - x_{f(l)}).$$

This modification of the propagators results in the replacement of the amplitude

$$i^{|\mathscr{v}|-1}\frac{(4\pi^2)^{|\mathscr{L}|+|\wp|}4^N}{(4\pi^2)^{|\mathscr{v}|+1}}\int\cdots\int\prod_{v\in\mathscr{v}\setminus\wp} d^4x_v \prod_{l\in\mathscr{L}}\Delta_F(x_{i(l)} - x_{f(l)};m),$$

which in general is ill-defined, by the analytically regularized amplitude

$$\mathscr{T}_\lambda((x_v)_{v\in\wp};m) = i^{|\mathscr{v}|-1}\frac{(4\pi^2)^{|\mathscr{L}|+|\wp|}4^N}{(4\pi^2)^{|\mathscr{v}|+1}} \times\int\cdots\int\prod_{v\in\mathscr{v}\setminus\wp} d^4x_v\prod_{l\in\mathscr{L}}\Delta_F^{\lambda_l}(x_{i(l)} - x_{f(l)};m)$$

which is well-defined for $\underline{\lambda}\in\Omega_2 = \{\underline{\lambda}/\mathscr{R}e\lambda_l > 2 \text{ for all } l\in\mathscr{L}\}$.

In Ω_2 the Fourier transform of $\mathscr{T}_{\underline{\lambda}}$ can be expressed with the help of the parameters $\underline{t}$ and $\underline{\beta}$ as follows (cf. Ref. 7)

$$\tilde{\mathscr{T}}_{\underline{\lambda}}(\underline{p};m) = \sum\tilde{\mathscr{T}}_{\underline{\lambda}}^{\mathscr{D}_\infty}(\underline{p};m),$$

where

$$\tilde{\mathscr{T}}_{\underline{\lambda}}^{\mathscr{D}_\infty}(\underline{p};m) = \delta\Big(\sum_{v\in\wp}p_v\Big)\prod_{l\in\mathscr{L}\setminus\{\sigma_\infty(\mathbb{E}_\infty)\}}\left[\int_0^1 d\beta_l\beta_l^{\lambda_l-1}\right] \times\prod_{H\in\mathbb{E}_\infty\setminus\{G\}}\left[\int_0^1 dt_H t_H^{\nu(H)-1}\right] \times\frac{\Gamma(\nu)}{[d(\underline{t},\underline{\beta})]^2}\Big[E_{\underline{t},\underline{\beta}}(\underline{q}(\underline{t},\underline{\beta}),\underline{q}(\underline{t},\underline{\beta})) + m^2\sum_{l\in\mathscr{L}}(\alpha_l/\alpha_{\sigma_\infty(G)}) - i0\Big]^{-\nu},$$

$\nu(H) = \Sigma_{l\in\mathscr{L}(H)}(\lambda_l - 1) + n(H), n(H) = |\mathscr{L}(H)| - 2N(H)$, $\nu = \nu(G)$, $n = n(G)$ and where the sum extends over all labeled s_∞-families $(\mathbb{E}_\infty,\sigma_\infty)$ for G.

Using the fact that $[E_{\underline{t},\underline{\beta}}(\underline{q}(\underline{t},\underline{\beta}),\underline{q}(\underline{t},\underline{\beta})) + m^2\Sigma_{l\in\mathscr{L}}(\alpha_l/\alpha_{\sigma_\infty(G)}) - i0]^{-\nu}$ is an infinitely differentiable distribution-valued function of $\underline{t}$ and $\underline{\beta}$ as long as m is larger than zero, we may convince ourselves that

$$\prod_{H\in\mathbb{E}_\infty}\Gamma(\nu(H))^{-1}\tilde{\mathscr{T}}_{\underline{\lambda}}^{\mathscr{D}_\infty}(\underline{p};m)$$

is an entire distribution-valued function of $\underline{\lambda}$ for every labeled s_∞-family. Hence the distribution-valued function of $\underline{\lambda}$

$$\prod_H\Gamma\Big(\sum_{l\in\mathscr{L}(H)}(\lambda_l - 1) + n(H)\Big)^{-1}\tilde{\mathscr{T}}_{\underline{\lambda}}(\underline{p};m)$$

is an entire distribution-valued function of $\underline{\lambda}$ for every labeled s_∞-family. Hence the distribution-valued function of $\underline{\lambda}$

Speer's generalized evaluator $\mathscr{W} = \{\mathscr{W}_L/L = 1,2,\ldots\}$ is applicable to the amplitudes $\tilde{\mathscr{T}}_{\underline{\lambda}}(\underline{p}:m)$. The result of the application $\mathscr{W}_{|\mathscr{L}|}\tilde{\mathscr{T}}_{\underline{\lambda}}(\underline{p}:m)$ is the analytically renormalized Feynman amplitude of the vertex graph G contributing in $|\mathscr{v}|$th order perturbation theory to the vertex function of the momenta carried by the "external lines" of G.

VII. ASYMPTOTIC EXPANSION OF ANALYTICALLY RENORMALIZED FEYNMAN AMPLITUDES

Now we are in the position to determine the complete asymptotic expansion of the (Λ-) parameter dependent distribution

$$\tilde{\mathscr{Z}}_\Lambda(\underline{p};m) = \mathscr{W}_{|\mathscr{L}|}\tilde{\mathscr{T}}_{\underline{\lambda}}(\Lambda\underline{p};m)$$

for Λ tending to plus infinity. By contrast to other authors having contributed to this subject, we do not discuss the asymptotic behavior in Λ of $\tilde{\mathscr{Z}}_\Lambda(\underline{p}:m)$ pointwise, i.e., for a fixed configuration of the external momenta $(p_v)_{v\in\wp}$. Instead, we establish the asymptotic behavior of the complex-valued function of Λ

$$\langle\tilde{\mathscr{Z}}_\Lambda,\tilde{\varphi}\rangle = \int d^{4|\wp|}p\tilde{\varphi}(\underline{p})\tilde{\mathscr{Z}}_\Lambda(\underline{p};m)$$

for any $\tilde{\varphi}\in\mathscr{S}(\mathbb{R}^{4|\wp|})$. At the first sight, this seems to complicate matters unnecessarily. For Minkowski metrics, however, the latter formulation of the problem turns out to be both adequate and helpful.

In order to establish the asymptotic expansion of the parameter-dependent distribution $\tilde{\mathscr{Z}}_{\Lambda}(\underline{p};m)$ for Λ tending to plus infinity, it suffices to determine the asymptotic behavior of

$$[\Lambda^2]^{n+2}\tilde{\mathscr{Z}}_{\Lambda}^{\mathscr{D}_\infty}(\underline{p};m) = [\Lambda^2]^{n+2}\mathscr{W}_{|\mathscr{L}|}\tilde{\mathscr{T}}_{\underline{\lambda}}^{\mathscr{D}_\infty}(\Lambda\underline{p};m) = \mathscr{W}_{\Lambda,|\mathscr{L}|}\tilde{\mathscr{T}}_{\underline{\lambda}}^{\mathscr{D}_\infty}(\underline{p};m/\Lambda) = \delta\Big(\sum_{v\in g} p_v\Big)\mathscr{W}_{|\mathscr{L}|}$$
$$\times\Big\{[\Lambda^2]^{-\sum_{l\in\mathscr{L}}(\lambda_l-1)}\Gamma(\nu)\prod_{l\in\mathscr{L}\setminus\{\sigma_\infty(\mathbb{E}_\infty)\}}\Big[\int_0^1 d\beta_l\beta_l^{\lambda_l-1}\Big]\prod_{H\in\mathbb{E}_\infty^-(G)}\Big[\int_0^1 dt_H t_H^{\nu(H)-1}\Big]$$
$$\times[E_{\underline{t},\underline{\beta}}(\underline{q}(\underline{t},\underline{\beta}),\underline{q}(\underline{t},\underline{\beta})) + e(\underline{t},\underline{\beta})m^2/\Lambda^2 - i0]^{-\nu}/[d(\underline{t},\underline{\beta})]^2\Big\}$$

for every labeled s_∞-family where we have set

$$e(\underline{t},\underline{\beta}) = e^{(\mathbb{E}_\infty,\sigma_\infty)}(\underline{t},\underline{\beta}) = \sum_{l\in\mathscr{L}}(\alpha_l/\alpha_{\sigma_\infty(G)})\geqslant 1$$

and

$$\mathscr{W}_{\Lambda,|\mathscr{L}|} = \mathscr{W}_{|\mathscr{L}|}[\Lambda^2]^{-\sum_{l\in\mathscr{L}}(\lambda_l-1)}.$$

Without loss of generality we may assume that $|g|$ is larger than or equal to two.

The limit $\Lambda\to+\infty$ corresponds formally to the transition from the Feynman amplitude with massive lines to the Feynman amplitude (for the same vertex graph G) with massless lines. In the zero mass case, however, we are dealing with a complex power of a homogeneous quadratic form: $[E_{\underline{t},\underline{\beta}}(\underline{q}(\underline{t},\underline{\beta}),\underline{q}(\underline{t},\underline{\beta})) - i0]^{-\nu}$ which fails to be an infinitely differentiable function of $\underline{t}$ whenever and wherever the quadratic form $E_{\underline{t},\underline{\beta}}(\underline{q},\underline{q})$ degenerates. It is this lack of infinite differentiability which prevents us from finding the answer to our problem right away, and, moreover, forces us to introduce the subsets of α-space $\mathscr{D}_\infty = \mathscr{D}_\infty(\mathbb{E}_\infty,\sigma_\infty)$ instead of $\mathscr{D}(\mathbb{E},\sigma)$ (cf. Ref. 7).

In order to control the formation of the singularity under consideration, we convert the additive occurrence of $-q_H^2$ and m^2/Λ^2 in

$$\Big[+\sum_{H\in\mathbb{H}_\infty}\Big(\prod_{\substack{H'\in\mathbb{E}_\infty\setminus\{G\}\\ H'\supset H}} t_{H'}\Big)e_H(\underline{t},\underline{\beta})(-q_H^2) + e(\underline{t},\underline{\beta})m^2/\Lambda^2 - i0\Big]^{-\nu}$$

into a multiplicative occurrence with the help of Mellin transforms. The result is

$$\Gamma(\nu)^{-1}\Big[\frac{\Lambda^2}{m^2e(\underline{t},\underline{\beta})}\Big]^{\nu}\prod_{H\in\mathbb{H}_\infty}\Big[\frac{1}{2\pi i}\int_{-\gamma_H-i\infty}^{-\gamma_H+i\infty}ds_H\Gamma(-s_H)\Big(\frac{e_H(\underline{t},\underline{\beta})\Lambda^2}{e(\underline{t},\underline{\beta})m^2}\prod_{H'\in\mathbb{E}_+^\infty(H)\setminus\{G\}}t_{H'}\Big)^{s_H}(-q_H^2-i0)^{s_H}\Big]\Gamma\Big(\nu+\sum_{H\in\mathbb{H}_\infty}s_H\Big),$$

where the γ_H's, $H\in\mathbb{H}_\infty$ are real numbers between zero and two.

Moreover, we have partly employed the following notation:

Let $\mathbb{G}_\infty$ be an arbitrary family of subgraphs of G. Then for a subgraph H of G we define

$$\mathbb{G}_\infty^+(H) = \{F/F\in\mathbb{G}_\infty, F\subset H\},$$
$$\mathbb{G}_\infty^-(H) = \{F/F\in\mathbb{G}_\infty, F\subsetneq H\},$$
$$\mathbb{G}_+^\infty(H) = \{F/F\in\mathbb{G}_\infty, F\supset H\},$$
$$\mathbb{G}_-^\infty(H) = \{F/F\in\mathbb{G}_\infty, F\supsetneq H\}.$$

For $\underline{\lambda}$ contained in a compact subset of $\Omega_{2(|g|-1)} = \{\lambda/\mathscr{R}e\lambda_l > 2(|g|-1)$ for all $l\in\mathscr{L}\}$ and for $(\underline{t},\underline{\beta})$ contained in $I^{|\mathscr{L}|-1}$, the integrations over s_H converge uniformly. In order to prove the uniform convergence, we note the identity

$$(-q^2-i0)^s = \frac{(-q^2-i0)^{s+j}}{[(s+1)\cdots(s+j+1)][(s+2)\cdots(s+j)]}\Big(-\frac{\square_q}{4}\Big)^j$$

for any $j = 0,1,\ldots$. In view of this identity and the above Mellin representation we obtain

$$\Big(\prod_{H\in\mathbb{E}_\infty^-(G)}t_H^{\nu(H)-1}\Big)[E_{\underline{t},\underline{\beta}}(\underline{q},\underline{q}) + e(\underline{t},\underline{\beta})m^2/\Lambda^2 - i0]^{-\nu}$$
$$= \prod_{H\in\mathbb{H}_\infty}\Big[\frac{1}{2\pi i}\int_{-\gamma_H-i\infty}^{-\gamma_H+i\infty}ds_H\frac{\Gamma(-s_H-j_H-1)}{(s_H+2)\cdots(s_H+j_H)}\Big(\frac{\Lambda^2e_H(\underline{t},\underline{\beta})}{m^2e(\underline{t},\underline{\beta})}\prod_{H'\in\mathbb{E}_\infty^-(G)\cap\mathbb{E}_+^\infty(H)}t_{H'}\Big)^{s_H}(-q_H^2-i0)^{s_H+j_H}\Big]$$
$$\times\Gamma\Big(\nu+\sum_{H\in\mathbb{H}_\infty}s_H\Big)\Big(\prod_{F\in\mathbb{E}_\infty^-(G)}t_F^{\nu(F)-1}\Big)\frac{(-1)^{|g|-1}}{\Gamma(\nu)}\Big[\frac{\Lambda^2}{m^2e(\underline{t},\underline{\beta})}\Big]^{\nu}\prod_{H\in\mathbb{H}_\infty}\Big(\frac{\square_{q_H}}{4}\Big)^{j_H},$$

where the following estimate for $\gamma_H\neq-1, j_H\geqslant 2$, $H\in\mathbb{H}_\infty$ can be used

$$\left|\prod_{H\in\mathbb{H}_\infty}\left[\frac{\Gamma(-s_H-j_H-1)}{(s_H+2)\cdots(s_H+j_H)}\left(\frac{\Lambda^2 e_H(\underline{t},\underline{\beta})}{m^2 e(\underline{t},\underline{\beta})}\prod_{H'\in\mathbb{E}_+^\infty(H)\cap\mathbb{E}_\infty^-(G)} t_{H'}\right)^{s_H}(-q_H^2-i0)^{s_H+j_H}\right]\Gamma\left(\nu+\sum_{H\in\mathbb{H}_\infty}s_H\right)\right|\prod_{F\in\mathbb{E}_\infty^-(G)} t_F^{\mathscr{R}e\,\nu(F)-1}$$

$$<\text{const}\times\left[1+\left|\sum_{H\in\mathbb{H}_\infty}\eta_H\right|\right]^{\mathscr{R}e\,\nu-\Sigma\gamma_H-1/2}\prod_{H\in\mathbb{H}_\infty}\frac{[1+\|q_H\|^2]^{j_H}}{[1+|\eta_H|]^{1/2-\gamma_H+2j_H}}$$

for $s_H=-\gamma_H+i\eta_H$.

In the pointwise discussion for Minkowski metrics, on the other hand, even in the case that *all* $q_H^2>0$, $H\in\mathbb{H}_\infty$, the corresponding s-integrations would not converge uniformly in $\underline{\lambda}$, $\underline{t}$, and $\underline{\beta}$ provided that $\mathscr{R}e\,\nu$ is larger than or equal to zero.

For $\underline{\lambda}\in\Omega_{2(|\mathscr{V}|-1)}$ we have shown the following representation to be valid

$$\widetilde{\mathscr{T}}_{\underline{\lambda}}^{\mathscr{G}_\infty}(\Lambda\underline{p};m)=\delta\left(\sum_{v\in\mathscr{V}}p_v\right)\prod_{H\in\mathbb{H}_\infty}\left[\frac{1}{2\pi i}\int_{-\gamma_H-i\infty}^{-\gamma_H+i\infty}ds_H\Gamma(-s_H)\Gamma(s_H+2)\right][\Lambda^2]^{\sum_{H\in\mathbb{E}_\infty}s_H-2}g_{\underline{\lambda};\underline{s}}(\underline{p};\underline{m}),$$

where

$$g_{\underline{\lambda};\underline{s}}(\underline{p};m)=g_{\underline{\lambda};\underline{s}}^{(\mathbb{E}_\infty,\sigma_\infty)}(\underline{p};m)=\prod_{H\in\mathbb{E}_\infty}\left[\Gamma\left(\nu(H)+\sum_{H'\in\mathbb{H}_\infty^-(H)}s_H\right)\right]h_{\underline{\lambda};\underline{s}}(\underline{p};m)$$

with the entire function of $\underline{\lambda}$ and $\underline{s}$

$$h_{\underline{\lambda};\underline{s}}(\underline{p};m)=h_{\underline{\lambda};\underline{s}}^{(\mathbb{E}_\infty,\sigma_\infty)}(\underline{p};m)=\prod_{l\in\mathscr{L}\setminus\{\sigma_\infty(\mathbb{E}_\infty)\}}\left[\int_0^1 d\beta_l\beta_l^{\lambda_l-1}\right]$$
$$\times\prod_{F\in\mathbb{E}_\infty^-(G)}\left[\Gamma\left(\nu(F)+\sum_{F'\in\mathbb{H}_\infty^-(F)}s_{F'}\right)^{-1}\int_0^1 dt_F t_F^{\{\nu(F)+\sum_{F'\in\mathbb{H}_\infty(F)}s_{F'}-1\}}\right][d(\underline{t},\underline{\beta})]^{-2}$$
$$\times[m^2e(\underline{t},\underline{\beta})]^{-\nu}\prod_{H\in\mathbb{H}_\infty}\left[\left(\frac{e_H(\underline{t},\underline{\beta})}{m^2e(\underline{t},\underline{\beta})}\right)^{s_H}\Gamma(2+s_H)^{-1}(-q_H(\underline{t},\underline{\beta})^2-i0)^{s_H}\right]$$

and where the s-integrations converge uniformly for $\underline{\lambda}$ contained in any compact subset of $\Omega_{2(|\mathscr{V}|-1)}$. We define $K=K(\mathbb{E}_\infty,\sigma_\infty)$ to be the minimal element of $\mathbb{H}_\infty$ with the property $n(H)>0$ for every $H\in\mathbb{E}_+^\infty(K)$. Specializing to the quartic self-interaction and to vertex graphs "with more than two external lines" we notice that $n(H)$ is larger than or equal to zero for all $H\in\mathbb{E}_+^\infty(F)$, $F\in\mathbb{H}_\infty$. We shift some of the s-contours to the right and obtain

$$\prod_{H\in\mathbb{E}_\infty}\left[\Gamma\left(\sum_{l\in\mathscr{L}(H)}(\lambda_l-1)+n(H)\right)^{-1}\right]\widetilde{\mathscr{T}}_{\underline{\lambda}}^{\mathscr{G}_\infty}(\Lambda\underline{p};m)$$
$$=\sum_{F\in\mathbb{H}_\infty^+(K)}\delta\left(\sum_{v\in\mathscr{V}}p_v\right)\prod_{H\in\mathbb{H}_\infty^-(F)}\left[\frac{1}{2\pi i}\int_{-\gamma_H-i\infty}^{-\gamma_H+i\infty}ds_H\Gamma(-s_H)\Gamma(2+s_H)\right]\frac{1}{2\pi i}\int_{S_F-i\infty}^{S_F+i\infty}ds_F\Gamma(-s_F)\Gamma(2+s_F)$$
$$\times[\Lambda^2]^{\sum_{H\in\mathbb{H}_+^\infty(F)}s_H-2}\left\{\prod_{H'\in\mathbb{E}_\infty}\left[\Gamma\left(\sum_{l\in\mathscr{L}(H')}(\lambda_l-1)+n(H')\right)^{-1}\right]g_{\underline{\lambda},\underline{s}}(\underline{p};m)\right\}_{/s_H=0,H\in\mathbb{H}_\infty^-(F)},$$

where $0<\gamma_H<1$, $0<\Sigma_{H\in\mathbb{H}^\infty(F)}\gamma_H<S_F<1$ for $F\in\mathbb{H}_\infty^-(K)$ and $0<\Sigma_{H\in\mathbb{H}^\infty(\mathrm{K})}\gamma_H-S_K<-S_K<1$. The s-integrations converge uniformly not only when $\underline{\lambda}$ varies over any compact subset of $\Omega_{2(|\mathscr{V}|-1)}$ but also after analytic continuation of the integrand when $\underline{\lambda}$ varies over any compact subset of $\Omega_{1-\epsilon}=\{\underline{\lambda}/\mathscr{R}e\lambda_l>1-\epsilon$ for all $l\in\mathscr{L}\}$. Thus, for the quartic self-interaction and vertex graphs "with more than two external lines"—we shall restrict the subsequent discussion to this case—we may continue $\widetilde{\mathscr{T}}_{\underline{\lambda}}^{\mathscr{G}_\infty}(\Lambda\underline{p};m)$ analytically from $\Omega_{2(|\mathscr{V}|-1)}$ to a neighborhood of $\lambda_l=1$, $l\in\mathscr{L}$. Again in view of the uniform convergence of the above s-integrations, the generalized evaluator $\mathscr{W}_{|\mathscr{L}|}$ operates directly on the integrand:

$$\mathscr{W}_{|\mathscr{L}|}\widetilde{\mathscr{T}}_{\underline{\lambda}}^{\mathscr{G}_\infty}(\Lambda\underline{p};m)=\sum_{F\in\mathbb{H}_\infty^+(K)}\widetilde{\mathscr{T}}_{F;\mathscr{W}}^{\mathscr{G}_\infty}(\Lambda\underline{p};m).$$

The terms $\widetilde{\mathscr{T}}_{F;\mathscr{W}}^{\mathscr{G}_\infty}(\Lambda\underline{p};m)$ stand for

$$\delta\left(\sum_{v\in\mathscr{V}}p_v\right)\frac{1}{2\pi i}\int_{\rho_{o(F)}-i\infty}^{\rho_{o(F)}+i\infty}dz_{o(F)}\cdots\frac{1}{2\pi i}\int_{\rho_2-i\infty}^{\rho_2+i\infty}dz_2\frac{1}{2\pi i}\int_{\rho_1-i\infty}^{\rho_1+i\infty}dz_1$$
$$\times\prod_{j=1}^{o(F)-1}[\Gamma(z_{j+1}-z_j)$$
$$\times\Gamma(2+z_j-z_{j+1})]\Gamma(-z_{o(F)})\Gamma(2+z_{o(F)})[\Lambda^2]^{z_1-2}$$
$$\times f_{z_1,\ldots,z_{o(F)}}^{(\mathscr{H})}(\underline{p};m)$$

with $o(F)=|\mathbb{H}_+^\infty(F)|$ and, further, with $1>\rho_{o(F)}>\cdots>\rho_2>\rho_1>0$ for $F\in\mathbb{H}_\infty^-(K)$ and $0>\rho_{0(K)}>\cdots>\rho_2>\rho_1>-1$ for $F=K$. We replaced the integration variables s_H and s_F by the new variables z_j, $1\leqslant j\leqslant o(F)$, defined by

$$z_j=\sum_{i=j}^{o(F)}s_{H_i}$$

after having enumerated the elements of $\mathbb{H}_\infty$ according to the inverted previously introduced order, i.e.,

$$H_1 = \text{maximal element of } \mathbb{H}_\infty,$$
$$H_j = \text{maximal element of } \mathbb{H}_\infty^-(H_{j-1}) \quad j=2,\dots o(F).$$

Finally, $f^{(\mathscr{H})}_{z_1,\dots,z_{o(F)}}(\underline{p};m)$ stands for

$$\mathscr{W}_{|\mathscr{S}|}\{g_{\underline{\lambda};s}(\underline{p};m)/_{s_H=0, H\in\mathbb{H}_\infty^-(F)}\},$$

a distribution-valued meromorphic function of z_j, $1\leqslant j\leqslant o(F)$, with poles at $z_j = -n_j, -n_j-1, -n_j-2,\dots$, $n_j = n_j(\mathbb{E}_\infty,\sigma_\infty) = \min\{n(H)/H\in\mathbb{E}_j^\infty\}$ $j=1,2,\dots o(F)$ and

$$\mathbb{E}_j^\infty = \begin{cases} \mathbb{E}_+^\infty(H_j)\cap\mathbb{E}_\infty^-(H_{j-1}) & \text{for } j=2,\dots,o(F),\\ \mathbb{E}_+^\infty(H_1) & \text{for } j=1.\end{cases}$$

The order of the pole at $z_j = -\mu_j$, $\mu_j = n_j, n_j+1, n_j+2,\dots$, for every j separately, is at most equal to

$$|\mathbb{E}^\infty_{j,\mu_j}| + |\{H'/H'\in\mathbb{E}_\infty^-(H)\setminus\mathbb{E}_+^\infty(F)$$

for some

$$H\in\mathbb{E}^\infty_{j,\mu_j}, n(H')\leqslant 0\}|,$$

where

$$\mathbb{E}_{j,\mu_j} = \{H/H\in\mathbb{E}_j^\infty,\ n(H)\leqslant\mu_j\}.$$

We move the z_1-contour to the left and obtain

$$\tilde{\mathscr{T}}^{\mathscr{S}}_{F;\mathscr{Y}}(\underline{\Lambda p};m) = \sum_{\mu=\mu_0(F)}^{M}\frac{1}{2\pi i}\oint_{|z_1+\mu|=\epsilon} dz_1[\Lambda^2]^{z_1-2}\tilde{\mathscr{T}}^{\mathscr{S};z_1}_{F;\mathscr{Y}}(\underline{p};m) + \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} dz_1[\Lambda^2]^{z_1-2}\tilde{\mathscr{T}}^{\mathscr{S};z_1}_{F;\mathscr{Y}}(\underline{p};m)$$

for $F\in\mathbb{H}_\infty^+(K)$, any integer $M\geqslant\mu_0(F)$, $-(M+1)<\rho<-M$, $\epsilon>0$ sufficiently small. Here, for every $F\in\mathbb{H}_\infty^+(K)$, $\tilde{\mathscr{T}}^{\mathscr{S};z_1}_{F;\mathscr{Y}}(\underline{p};m)$ is a distribution-valued meromorphic function of z_1 with poles at $z_1 = -\mu$, $\mu = -\mu_0(F), -\mu_0(F)-1, -\mu_0(F)-2,\dots$.

$$\mu_0(F) = \mu_0(\mathbb{E}_\infty,\sigma_\infty;F) = \min_{1<j\leqslant o(F)}\{n_j + 2(j-1)\}$$

of order $m_\mu(F)$

$$m_\mu(F) = m_\mu(\mathbb{E}_\infty,\sigma_\infty;F) = \left|\bigcup_{j=1}^{o(F)}\mathbb{E}^\infty_{j,\mu_j-2(j-1)}\right|$$
$$+ |\{H'/H'\in\mathbb{E}_\infty^-(H)\setminus\mathbb{E}_+^\infty(F)$$

for some

$$H\in\bigcup_{j=1}^{o(F)}\mathbb{E}^\infty_{j,\mu_j-2(j-1)}, n(H')\leqslant 0\}|$$
$$+\begin{cases}1 & \text{if } \mu\geqslant 2(o(F)-1) \quad \text{for } F\in\mathbb{H}_\infty^-(K),\\ 1 & \text{if } \mu\geqslant 2o(K) \quad\quad\ \text{for } F=K,\\ 0 & \text{otherwise.}\end{cases}$$

For $-\mu > \mathscr{R}e z_1 > -(\mu+1)$, $\tilde{\mathscr{T}}^{\mathscr{S};z_1}_{F;\mathscr{Y}}(\underline{p};m)$ is given by

$$\delta\Big(\sum_{v\in\wp}p_v\Big)\sum_{r=1}^{o(F)}\sum_{\mu_2=2}^{\mu-\theta(2)}(-1)^{\mu_2}(\mu_2-1)\cdots\sum_{\mu_r=2}^{\mu-\theta(r)-[\mu_2+\cdots+\mu_{r-1}]}(-1)^{\mu_r}$$
$$\times(\mu_r-1)\frac{1}{2\pi i}\int_{\rho_{r+1}-i\infty}^{\rho_{r+1}+i\infty}dz_{r+1}\cdots\frac{1}{2\pi i}\int_{\rho_{o(F)}-i\infty}^{\rho_{o(F)}+i\infty}dz_{o(F)}$$
$$\times\Gamma([2+\mu_2+\cdots+\mu_r+z_1]-z_{r+1})$$
$$\times\Gamma(z_{r+1}-[\mu_2+\cdots+\mu_r+z_1])$$
$$\times\prod_{j=r+1}^{o(F)-1}[\Gamma(2+z_j-z_{j+1})\Gamma(z_{j+1}-z_j)]$$
$$\times\Gamma(2+z_{o(F)})\Gamma(-z_{o(F)})f^{(\mathscr{H})}_{z_1,\dots,z_{o(F)}}(\underline{p};m)/_{\substack{z_2=\mu_2+z_1\\ \vdots\\ z_r=\mu_2+\cdots+\mu_r+z_1}}$$

with

$$\theta(j) = \begin{cases}1 & \text{if } j\leqslant o(K)\\ 0 & \text{otherwise}\end{cases}$$

and

$$0<\theta(r+1)+\rho_{r+1}<1+\mu+\mathscr{R}e z_1,$$
$$0<\theta(r+1)+\rho_{r+1}<\cdots<\theta(o(F))+\rho_{o(F)}<+1.$$

This formula may be proved by induction on μ. As to the possible values of $\mu_0(F)$ we note the inequality

$$p(H)\leqslant j \quad \text{for} \quad H\in\mathbb{E}_j^\infty$$

and take into account the following relation valid for the ϕ_4^4-theory

$$n(H) = \frac{1}{2}\sum_{i=1}^{p(H)}\{\#[\text{external lines of } H_i]-4\},$$

where the H_i's denote the connected components of H. From this we infer for the ϕ_4^4-theory

$$n_1 = n,\quad n_j\geqslant n+2-j,\quad j=2,3,\dots,o(F).$$

Actually, these relations are true for all monomial interactions apart from the cubic one. Hence $\mu_0(F) = \mu_0(\mathbb{E}_\infty,\sigma_\infty;F)$ is equal to n for all $F\in\mathbb{H}_\infty^+(K)$ and all labeled s_∞-families $(\mathbb{E}_\infty,\sigma_\infty)$ for G. Moreover, the order of the poles at $z_1 = -\mu = -n, -n-1, -n-2,\dots$ of

(i) $\sum_{F\in\mathbb{H}_\infty^+(K)}\tilde{\mathscr{T}}^{\mathscr{S};z_1}_{F;\mathscr{Y}}(\underline{p};m)$ is equal to $m_\mu(\mathbb{E}_\infty,\sigma_\infty)$

$$= \max_{F\in\mathbb{H}_\infty^+(K)} m_\mu(\mathbb{E}_\infty,\sigma_\infty;F)$$

(ii) $\sum_{(\mathbb{E}_\infty,\sigma_\infty)}\sum_{F\in\mathbb{H}_\infty^+(K)}\tilde{\mathscr{T}}^{\mathscr{S};z_1}_{F;\mathscr{Y}}(\underline{p};m) = \tilde{\mathscr{T}}^{z_1}_{\mathscr{Y}}(\underline{p};m)$ is equal to

$$m_\mu = \max_{(\mathbb{E}_\infty,\sigma_\infty)} m_\mu(\mathbb{E}_\infty,\sigma_\infty).$$

Now, we have all the necessary information at hand to write down the asymptotic expansion for the analytically renormalized Feynman amplitude $\mathscr{W}_{|\mathscr{S}|}\tilde{\mathscr{T}}_{\underline{\lambda}}(\underline{\Lambda p};m)$ of the vertex graph G:

$$\sum_{\mu=n}^{M}\sum_{\kappa=0}^{m_\mu-1}[\Lambda^2]^{-\mu-2}[\ln\Lambda^2]^\kappa\tilde{\mathscr{T}}^{(\mathscr{Y})}_{\mu,\kappa}(\underline{p};m) + \mathscr{R}_m$$

with

$$\tilde{\mathscr{T}}^{(\mathscr{Y})}_{\mu,\kappa}(\underline{p};m) = \frac{1}{\kappa!}\frac{1}{2\pi i}\oint_{|z_1+\mu|=\epsilon}dz_1(z_1+\mu)^\kappa\tilde{\mathscr{T}}^{z_1}_{\mathscr{Y}}(\underline{p};m)$$

and

$$\mathscr{R}_M = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}dz_1[\Lambda^2]^{z_1-2}\tilde{\mathscr{T}}^{z_1}_{\mathscr{Y}}(\underline{p};m),$$

where $-(M+1)<\rho<-M$.

If the number of "external lines" is equal to two (and $|\wp|=2$) we adopt the same procedure as before with the only difference that in the beginning we push the s-contour further to the right. In this way we obtain

$$\mathscr{W}\tilde{\mathscr{T}}_{\underline{\lambda}}(\underline{\Lambda p};m) = [\Lambda^2]^{-1}\delta(p_1+p_2)$$
$$\times\frac{1}{2\pi i}\oint_{|s-1|=\epsilon}ds\frac{\pi(s+1)}{\sin\pi(s-1)}$$
$$\{[\Lambda^2]^{s-1}\mathscr{W}g_{\underline{\lambda};s}(\underline{p};m) - \mathscr{W}g_{\underline{\lambda};1}(\underline{p};m)\} - [\Lambda^2]^{-2}$$
$$\times\delta(p_1+p_2).$$

$$\frac{1}{2\pi i}\oint_{|s|=\epsilon} ds \frac{\pi(s+1)}{\sin \pi s}\{[\Lambda^2]^s \mathscr{W} g_{\underline{\lambda};s}(\underline{p};m) - \mathscr{W} g_{\underline{\lambda};o}(\underline{p};m)\}$$

$$+ \sum_{\mu=1}^{M} \delta(p_1+p_2)\frac{1}{2\pi i}\oint_{|s+\mu|=\epsilon} ds\, \Gamma(2+s)\Gamma(-s)[\Lambda^2]^{s-2}$$

$$\times \mathscr{W} g_{\underline{\lambda};s}(\underline{p};m)$$

$$+ \delta(p_1+p_2)\frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} ds\,\Gamma(2+s)\Gamma(-s)[\Lambda^2]^{s-2}$$

$$\times \mathscr{W} g_{\underline{\lambda};s}(\underline{p};m)$$

for any positive integer M, $-(M+1)<\rho<-M$ and $\epsilon>0$ sufficiently small,

$$\mathscr{W} \equiv \mathscr{W}_{|\mathscr{L}|}.$$

The order of the pole of the integrand at $s=-\mu$, $\mu=-1,0,1,2,\ldots$ is equal to

$$1-\delta_{\mu,1} + |\{H/H\in\mathbb{E}_\infty, \mathscr{G}\subset \mathscr{o}(H),\ n(H)\leqslant\mu\}| + |\{H/H\in\mathbb{E}_\infty, \mathscr{G}\not\subset \mathscr{o}(H),\ n(H)\leqslant 0\}|.$$

Now, the asymptotic expansion in powers of Λ^{-2} and $\ln\Lambda^2$ of the Feynman amplitude corresponding to G which contributes in $|\mathscr{o}|$th order perturbation theory to the two point vertex function can be read off easily.

An arbitrary number j of mass insertions can be incorporated into the above scheme by partitioning j in all possible ways into a sum of $|\mathscr{L}|$ non-negative integers $j_1,\ldots,j_{|\mathscr{L}|}$ replacing the propagator of the line l in the amplitude $\tilde{\mathscr{T}}_{\underline{\lambda}}(\Lambda\underline{p};m)$

$$\frac{e^{i\pi\lambda_l}\Gamma(\lambda_l)}{i(2\pi)^2}[k_l^2-m^2+i0]^{-\lambda_l}$$

by

$$(m^2)^{j_l}\frac{\exp(i\pi\Sigma_{\nu=1}^{j_l+1}\lambda_{l,\nu})\Pi_{\nu=1}^{j_l+1}\Gamma(\lambda_{l,\nu})}{[i(2\pi)^2]^{j_l+1}}$$

$$\times[k_l^2-m^2+i0]^{-\Sigma_{\nu=1}^{j_l+1}\lambda_{l,\nu}}$$

multiplying subsequently by the combinatorial factor $j!/j_1!\cdots j_l!$, summing over all different partitions and applying finally an appropriate generalized evaluator $\mathscr{W}'_{|\mathscr{L}|+j}$.

ACKNOWLEDGMENT

The author profited from numerous discussions with the late Dr. H. Trute.

[1]F. J. Dyson, Phys. Rev. **75**, 1736 (1949); A. Salam, Phys. Rev. **82**, 217 (1951); Phys. Rev. **84**, 426 (1951).
[2]S. Weinberg, Phys. Rev. **118**, 838 (1960).
[3]J. P. Fink, J. Math. Phys. **9**, 1389 (1968).
[4]D. A. Slavnov, Theor. Math. Phys. (USA) **17**, 1055 (1973).
[5]M. C. Bergère and Y-M.P. Lam, Commun. Math. Phys. **39**, 1 (1974/75).
[6]E. R. Speer, J. Math. Phys. **9**, 1404 (1968).
[7]E. R. Speer, *Generalized Feynman Amplitudes* (Princeton U. P., Princeton, NJ, 1969), Chap. III.
[8]V. F. Kovalenko, Teoret. Mat. Fiz. **15**, 340 (1973) (in Russian). In this reference the infrared singularities of the Feynman integrand are totally ignored.
[9]E. R. Speer, J. Math. Phys. **15**, 1 (1974); Commun. Math. Phys. **37**, 83 (1974).
[10]H. Hironaka, Ann. Math. **79**, 109 (1964).
[11]N. Nakanishi, *Graph Theory and Feynman Integrals* (Gordon and Breach, New York, 1970).
[12]H. Trute, Desy Preprint 74/44 (1974).
[13]K. Pohlmeyer, Desy Preprint 74/36 (1974).
[14]K. Pohlmeyer, *Lecture Notes in Physics*, Vol. 39 (Springer, New York, 1975), p. 59.
[15]P. Breitenlohner and D. Maison, Commun. Math. Phys. **52**, 39, 55 (1977).
[16]M. C. Bergère and Y-M.P. Lam, "Asymptotic Expansion of Feynman Amplitudes: Part II - The Divergent Case," Freie Universität Berlin Preprint FUB HEP 74/9.
[17]M. C. Bergère, C. de Calan, and A. P. C. Malbouisson, Commun. Math. Phys. **62**, 137 (1978).
[18]C. de Calan and A. P. C. Malbouisson, Ann. Inst. Henri Poincaré, A **32**, 91 (1980).
[19]M. C. Bergère and C. Gilain, J. Math. Phys. **19**, 1495 (1978).
[20]E. R. Speer and M. J. Westwater, Ann. Inst. H. Poincaré, A **14**, 1 (1971).