Infrared behavior in non-Abelian gauge theories

Richard A. Brandt*
Department of Physics, New York University, New York, New York 10003

Ng Wing-Chiu†‡
Deutsches Elektronen-Synchrotron, 2 Hamburg 52, Notkestieg 1, Germany
and Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794
(Received 10 May 1976)

The knowledge of the renormalization constants in ultraviolet-free non-Abelian gauge theories is exploited in exposing a new symmetry, R: $\vec{A}_\mu(x) \to \vec{A}_\mu(x) + \vec{r}_\mu$, under which the renormalized field equations of such theories are invariant. As a consequence, we derive the low-energy theorem for the renormalized proper vertices $\Gamma_R^{\cdots\mu\cdots}(\ldots,q,\ldots)$ as $0 = \Gamma_R^{\cdots\mu\cdots}(\ldots,q=0,\ldots)$.

I. INTRODUCTION

In contrast to its more prosaic cousin the photon, non-Abelian gauge [Yang-Mills (YM)] particles seem to be afflicted with violent infrared difficulties.[1] There are extra divergences in non-Abelian gauge theories as these massless quanta couple among themselves, yielding more virulent infinities in internal integrations, and the non-Abelian coupling complicates any attempt to order the chaos.

In the language of the renormalization group, the distinction between quantum electrodynamics (QED) and Yang-Mills theory lies in the slope of the Callan-Symanzik function $\beta(g)$ at $g=0$. For QED, $\beta'(0)>0$, and the theory is infrared (IR) free[2] [and ultraviolet (UV) nonfree] for small coupling constant, while for YM theory $\beta'(0)<0$ and the theory is IR nonfree (and UV free). This means that IR behavior in QED is exactly computable in perturbation theory[3] and UV behavior in YM theory is exactly computable in perturbation theory. This circumstance seems to preclude perturbative studies of UV behavior in QED[4] and IR behavior in YM theory.

In this paper we will investigate the IR behavior in YM theories using techniques we have recently developed.[5] We will show that, precisely because of the UV freedom, exact[6] statements can be made about the zero-momentum behavior in many of these theories. The idea is that the UV freedom enables the renormalization constants to be exactly computed.[7] The implied vanishing of relevant ratios of such constants then implies the presence of certain new symmetries in the theory which are not present classically or in finite orders of perturbation theory. These (spontaneously broken perhaps) symmetries then imply exact zero-momentum theorems. The exact behavior implied by these theorems is extremely simple: *The proper YM vertex functions vanish when any boson four-momentum vanishes.*

The relevant symmetry is what we have called R invariance. Under the R transformation, the YM field $\vec{A}_\mu(x)$ transforms as

$$\vec{A}_\mu(x) \to \vec{A}_\mu(x) + \vec{r}_\mu , \qquad (1.1)$$

where the $\vec{r}_\mu$ are constants. Equation (1.1) is a particular case of the Abelian gauge transformation

$$\vec{A}_\mu(x) \to \vec{A}_\mu(x) + \partial_\mu \vec{\Lambda}(x) , \qquad (1.2)$$

for

$$\vec{\Lambda}(x) = \vec{R}(x) \equiv \vec{r}_\mu x^\mu . \qquad (1.3)$$

It has previously been shown[8] that for theories invariant under the scalar version

$$\phi(x) \to \phi(x) + r \qquad (1.4)$$

of (1.1), *if* an S matrix exists, it vanishes at zero four-momentum of any ϕ particle. This result is not useful for our purposes since the existence of the S matrix is the very question we want to investigate. We deduce the consequences of symmetry under (1.1) for the Green's functions and study what these consequences tell us about the existence of an S matrix, among other things.

Our conclusions are that the transverse YM propagator has a singularity at $q^2=0$ and that the proper YM vertices vanish when any four-momentum vanishes. These results suggest there are zero-mass excitations in the mass spectrum of the theory, and that it may be possible to construct an S matrix for this theory along the lines of Zwanziger.[9] We should immediately emphasize that our analysis is not a mathematically rigorous one in that limits are freely interchanged, and functional methods are used cavalierly. Our use of these preocedures is more suspect here than in the usual applications since the effects we study are due to renormalization. Furthermore, we have to assume that any counterterms required by regularization do not destroy the R invariance. Our results should therefore be considered as suggestive but not proved beyond a reasonable doubt.

In Sec. II, we introduce the R transformation and discuss its implications in several field-theoretic models. Section III is devoted to showing that R invariance is present in YM theories as a result of renormalization. In Sec. IV, we deduce the low-energy theorems implied by R invariance in YM theories, and conclude that all proper vertices of YM particles vanish whenever any one of the four-momenta of the external YM particle vanishes. We discuss the relevance of the low-energy theorem to the problem of the existence of an S matrix, confinement, etc., in the final Sec. V.

II. R INVARIANCE

The R transformations have been defined on vector fields in (1.1) and on scalar fields in (1.4). For definiteness, we will illustrate the consequences of R invariance in QED, where the transformations for the Maxwell and Dirac fields are

$$A_\mu(x) \to A_\mu(x) + r_\mu \,, \tag{2.1}$$

$$\psi(x) \to e^{ier\cdot x}\psi(x) \,. \tag{2.2}$$

This is a special case of the general gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \,, \tag{2.3}$$

$$\psi(x) \to \psi(x) e^{ie\Lambda(x)} \,, \tag{2.4}$$

with

$$\Lambda(x) = R(x) \equiv r\cdot x \,. \tag{2.5}$$

As discussed and illustrated in paper I, it is the presence or absence of $R(x)$ in the gauge group $\mathcal{G}$ [$\mathcal{G}$ is the set of $\Lambda(x)$ such that (2.3), (2.4) is a symmetry transformation] that determines whether or not there is a zero-mass excitation in the theory. Namely, if $R \in \mathcal{G}$, there is a singularity at zero mass in the transverse part of the photon propagator.

We have deduced the above and other consequences of R invariance in two ways. The first method uses the transformation property (see also paper I)

$$D_{\mu\nu}(q) \to D'_{\mu\nu}(q) = D_{\mu\nu}(q) + r_\mu r_\nu \delta^4(q) \tag{2.6}$$

of the photon propagator under (2.1) and the invariance of the Green's function

$$\int d^4x\, e^{iq\cdot x}\langle 0|Tj_\mu(x)A_\nu(0)|0\rangle \equiv \Pi_{\mu\kappa}(q)D^\kappa{}_\nu(q) \tag{2.7}$$

under (2.1):

$$\Pi_{\mu\kappa}(q)D^{\kappa\nu}(q) = \Pi_{\mu\kappa}(q)D'^{\kappa\nu}(q) \,. \tag{2.8}$$

Here $j_\mu(x)$ is the electric current and $\Pi_{\mu\kappa}$ is the photon proper self-energy part:

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2) \,. \tag{2.9}$$

Equations (2.6) and (2.8) immediately give

$$\Pi_{\mu\nu}(0) = 0 \,, \tag{2.10}$$

which implies that the transverse part

$$D(q^2) = \frac{1}{q^2 + q^2\Pi(q^2)} \tag{2.11}$$

of $D_{\mu\nu}(q)$ has a singularity at $q^2 = 0$:

$$D(0) = \infty. \tag{2.12}$$

The second method uses formal functional techniques. The invariance of the Lagrangian under (2.1) and (2.2) leads immediately to exact zero-momentum theorems, including (2.10), for all the proper vertices. In particular, the N-photon amplitude vanished whenever any photon four-momentum vanishes.

For conventional four-dimensional QED, one has R invariance in each order of perturbation theory, and so (2.12) obtains and the physical photon is thus interpreted as the Goldstone boson corresponding to the spontaneous breakdown of (2.1) (see paper I). For two-dimensional massless QED (Schwinger model[10]), on the other hand, it is shown in paper I that $R \notin \mathcal{G}$ and that is why the photon can become massive in this model. Other models illustrating our conclusion were also discussed in paper I. For example, in the derivative-coupling model

$$\mathcal{L}_I = g\partial^\mu A B_\mu C \,, \tag{2.13}$$

with A and C scalar and B_μ a vector field, R invariance can be maintained in each order so that A is massless. In gauge-invariant models of the type recently discussed by Cornwall,[11] on the other hand, there occur for example mass terms of the form

$$M^2(A_\mu - \Box^{-1}\partial_\mu \partial\cdot A)^2 \,. \tag{2.14}$$

This expression is invariant under (2.3) for $\Lambda(x)$ such that $\Box\Lambda(x) \neq 0$, but it is not invariant under (2.1). Correspondingly, the vector particle acquires a mass in the model.

In all of the above examples, the R invariance was present classically and order by order in perturbation theory. The R invariance we will study in this paper is not of this type, but is true only in the exact theory. To illustrate the idea, consider a scalar renormalized formal field equation

$$\Box\phi(x) = gZ\phi(x)\phi(x) \,, \tag{2.15}$$

where Z is a renormalization constant and further R-invariant terms may be present. The equation (2.15) is not invariant to (1.4) either classically, where $Z=1$, or in finite orders of perturbation theory, where Z is divergent. In terms of a cutoff K,

$$Z(K)=\sum_{n=0}^{\infty} g^n Z_n(K), \quad Z_n(\infty)=\infty . \tag{2.16}$$

Suppose now that the exact $Z(K)$ satisfies

$$Z=Z(\infty)=0 . \tag{2.17}$$

Then (2.15) becomes R invariant since

$$Z\phi(x)\phi(x) \overset{R}{=} Z\phi(x)\phi(x)+rZ\phi(x)+Zr^2 = Z\phi(x)\phi(x) . \tag{2.18}$$

It is shown in paper II how such arguments can be made more precise when formulated in terms of finite local field equations.[12,13] Ordinary gauge invariance in QED can be mathematically formulated in that way.[12] We assume that similar methods can be used to infer the presence of symmetries even if they depend on a circumstance like (2.17).

We can indicate the reliability of our procedure by illustrating how it works in some diagrammatic-model calculations. We consider a single massless scalar field $\phi(x)$ interacting via $g\phi^4$. Our model is the set of diagrams made of a chain of bubbles, where the single bubble is defined by

$$\Gamma^{(1)}(P=p_1+p_2)=g\int\frac{d^4l}{l^2(P-l)^2} . \tag{2.19}$$

Thus our model (see also Sec. IV) for the unrenormalized four-point function is

$$\Gamma_W(P)=g+g\Gamma^{(1)}(P)+g[\Gamma^{(1)}(P)]^2+\cdots . \tag{2.20}$$

The formula (2.19) for $\Gamma^{(1)}$ is of course UV divergent, and this gives the contribution to the Z_1 renormalization constant

$$Z_1(K)=1+C\ln\frac{K}{\mu}+\left(C\ln\frac{K}{\mu}\right)^2+\cdots , \tag{2.21}$$

where K is the cutoff, and

$$\Gamma^{(1)}(P)\underset{K\to\infty}{\sim} C\ln\frac{K}{(P^2)^{1/2}} , \tag{2.22}$$

with C a constant.

The renormalized $\Gamma_R^{(1)}(P)$ still suffers from IR divergence when $p_1, p_2 \to 0$, so that

$$\Gamma_R^{(1)}(P)\underset{P\sim 0}{\sim} C'\ln\frac{P^2}{\mu^2} , \tag{2.23}$$

where μ is the renormalization point. The IR limit of $\Gamma_W^R(P)$ is then

$$\Gamma_W^R(P)\underset{P\to 0}{\sim} g_R+g_R C'\ln\frac{P^2}{\mu^2}+g_R\left(C'\ln\frac{P^2}{\mu^2}\right)^2+\cdots . \tag{2.24}$$

Thus Z_1 is logarithmically UV divergent in every finite order, and $\Gamma_W^R(P)$ is IR divergent also in finite orders. However, the infinite sum in (2.21) is

$$Z_1(K)=\frac{1}{1-C\ln(K/\mu)} \xrightarrow[K\to\infty]{} 0 . \tag{2.25}$$

At the same time, the infinite sum for $\Gamma_W^R(P)$ also ameliorates the IR divergence:

$$\Gamma_W^R(P)\underset{P\to 0}{\sim}\frac{1}{1-C'\ln(P^2/\mu^2)} \to 0 . \tag{2.26}$$

Thus we have in fact an example of the low-energy theorem here. The field equation is given by

$$\Box\phi_R(x)=\frac{Z_1}{Z_2}g_R\phi_R{}^3(x) . \tag{2.27}$$

(Z_2 is unity here.) Thus precisely as $Z_1=0$, making the equation R invariant, we have the vanishing of the four-point proper vertex as the external momenta tend to zero, as given by (2.26).

We have also previously used[14] R invariance to study the problem of reconciling Bjorken scaling with the singularity structure implied by canonical commutation relations.

To conclude our discussion of nonperturbative R invariance, we note that it might be possible to reverse our procedure; that is, to proceed from an exact zero-momentum behavior (inferred from renormalization-group arguments in IR-free theories[3]) which is consistent with R invariance and to conclude that the exact equation of motion is R invariant. Then the renormalization constants would have to arrange themselves to give R invariance to the equation. One of the ways would presumably be that relevant renormalization constants vanish.

III. *R* INVARIANCE IN NON-ABELIAN GAUGE THEORIES

For definiteness, we consider a theory of non-Abelian gauge fields $A_\mu^a(x)$ interacting with fermions according to the Lagrangian

$$\mathcal{L}(x)=-\tfrac{1}{4}Z_3\left[\partial_\mu A_{\nu R}^a(x)-\partial_\nu A_{\mu R}^a(x)+\frac{Z_1}{Z_3}g_R f^{abc}A_{\mu R}^b(x)A_{\nu R}^c(x)\right]^2 -\frac{1}{2\alpha_R}[\partial\cdot\vec{A}_R(x)]^2+\tilde{Z}_3\partial_\mu c_{1R}^a(x)\left[\delta^{ab}\partial^\mu+\frac{Z_1}{Z_3}g_R f^{acb}A_{cR}^\mu(x)\right]c_{2R}^b(x)+Z_2\bar{\psi}_R(x)\gamma_\mu\left[\partial^\mu-i\frac{Z_1}{Z_3}g_R A_{aR}^\mu(x)T_a\right]\psi_R(x). \tag{3.1}$$

The subscript R denotes that the quantity is renormalized, the Z's are renormalization constants, the f^{abc} are the structure constants of the gauge group, T^a is the fermion representation matrix, c_1^a and c_2^a are ghost fields, and α_R is the renormalized gauge parameter. Variation of the Lagrangian leads to the renormalized field equations

$$0=\left[\delta^{ab}\partial^\nu+\frac{Z_1}{Z_3}g_R f^{acb}A_{cR}^\nu(x)\right]\left[\partial_\mu A_{\nu R}^b(x)-\partial_\nu A_{\mu R}^b(x)+\frac{Z_1}{Z_3}g_R f^{bfg}A_{\mu R}^f(x)A_{\nu R}^g(x)\right]+(Z_3\alpha_R)^{-1}\partial^\mu\partial\cdot A_{aR}(x)$$

$$+\tilde{Z}_1 Z_3^{-1}g_R f^{abc}\partial_\mu c_{1R}^b(x)c_{2R}^c(x)+Z_1Z_2Z_3^{-2}g_R\bar{\psi}_R(x)\gamma_\mu T_a\psi_R(x)\,, \qquad (3.2)$$

$$[\delta^{ab}\partial^\mu+Z_1Z_3^{-1}g_R f^{acb}A_{cR}^\mu(x)]\partial_\mu c_{1R}^b(x)=0\,, \qquad (3.3)$$

$$\partial_\mu[\delta^{ab}\partial^\mu+Z_1Z_3^{-1}g_R f^{acb}A_{cR}^\mu(x)]c_{2R}^b(x)=0\,, \qquad (3.4)$$

$$\gamma_\mu(\partial^\mu-iZ_1Z_3^{-1}g_R A_{aR}^\mu T_a)\psi_R(x)=0\,. \qquad (3.5)$$

Fermion mass terms can be added without changing our conclusions.

The renormalization constants occurring in (3.1)–(3.5) can all be exactly calculated, because of the UV freedom of the theory, from perturbation theory via the renormalization group. If we make the R transformation on the renormalized $\vec{A}^\mu$ field,

$$\vec{A}_{\mu R}(x)\to\vec{A}_{\mu R}(x)+\vec{r}_\mu\,, \qquad (3.6)$$

and leave all other fields unchanged, then in all cases the changes in Eqs. (3.2)–(3.5) are proportional to

$$\Delta_R=\frac{Z_1}{Z_3}\vec{r}^\mu\times\left(\frac{Z_1}{Z_3}AA+\text{finite operators}\right). \qquad (3.7)$$

In all the UV-free theories of the type (3.1), the renormalization-group calculations give

$$\frac{Z_1}{Z_3}=0\,, \qquad (3.8)$$

and the result is gauge independent. Furthermore, for SU(N) with more than N fundamental quark representations,[15]

$$\left(\frac{Z_1}{Z_3}\right)^2 AA\to 0\,, \qquad (3.9)$$

and so

$$\Delta_R=0\,, \qquad (3.10)$$

so that the theory defined by (3.1)–(3.5) is R invariant. We now consider models with sufficiently many quarks so that (3.10) is valid. This corresponds to the most physically interesting case of SU(3) with four or more quark triplets. Extension of our results to other cases will be considered elsewhere.

Similar considerations apply in performing the R transformation (3.6) formally on the Lagrangian (3.1). The rule to be observed here is that any term in the transformed Lagrangian that gives a vanishing contribution to the equation of motion is to be discarded. For example, the ghost term gives

$$\Delta\left\{\frac{Z_1\tilde{Z}_3}{Z_3}g_R[\partial_\mu\vec{c}_{1R}(x)\times\vec{c}_{2R}(x)]\cdot\vec{A}_R^\mu(x)\right\}=\frac{Z_1\tilde{Z}_3}{Z_3}g_R\vec{r}_\mu\cdot[\partial^\mu\vec{c}_{1R}(x)\times\vec{c}_{2R}(x)]\,, \qquad (3.11)$$

and this term is discarded since by $Z_1/Z_3=0$ it does not contribute to the ghost equations of motion upon variation. Proceeding in this way, we see $\mathcal{L}(x)$ is invariant under (3.6).

The R invariance of the renormalized theory can again (see paper II) be directly traced to the renormalization effects on the non-Abelian gauge transformations under which the classical theory is invariant. We write

$$\vec{A}_{\mu R}(x)\to\vec{A}_{\mu R}(x)+2\vec{\omega}_c(x)\times\vec{A}_{\mu R}(x)+\frac{Z_3}{Z_1}\frac{1}{g_R}\partial_\mu\vec{\omega}_c(x)\,, \qquad (3.12)$$

and choose $\vec{\omega}_c(x)$ such that

$$\frac{Z_3}{Z_1}\vec{\omega}_c(x)\equiv\vec{\Lambda}(x)=g_R\vec{r}\cdot x\,, \qquad (3.13)$$

with $\vec{r}$ a finite constant vector. Then making use of $Z_1/Z_3=0$, we see that (3.12) becomes just the R transformation (3.6).

IV. LOW-ENERGY THEOREM FROM R INVARIANCE

From the R invariance discussed in the last section we shall now proceed to unravel consequences for the low-energy behavior of the exact solution of the YM theory. We use three different methods to derive the conclusion; none of them are completely rigorous, but they should suggest the validity of the connection.

The first argument is due to Kramer and Palmer.[8] They consider the S operator (assumed to exist), whose matrix elements between in and out states constitute the S matrix, and expand it in the set of normal products of complete in-fields:

$$S=\sum_{n=0}^{\infty}\int d^4x_1\cdots d^4x_n \sum_{\alpha_1\cdots\alpha_n} S^{(n)}_{\alpha_1\cdots\alpha_n}(x_1,\ldots,x_n):A_{\text{in}}^{\alpha_1}(x_1)\cdots A_{\text{in}}^{\alpha_n}(x_n): , \tag{4.1}$$

where $\{\alpha_i\}$ denotes all attributes of the field. The coefficients $S^{(n)}_{\alpha_1\cdots\alpha_n}$ are then the n-particle S-matrix elements when S is placed between in and out states. The invariance of S under the R transformation of the A_{in} fields,

$$S[A_{\text{in}}]=S[A_{\text{in}}+r] , \tag{4.2}$$

implies, to first order in r,

$$0=\sum_{n=0}^{\infty}\sum_{i=1}^{n}\int d^4x_1\cdots d^4x_i\cdots d^4x_n \sum_{\alpha_1\cdots\alpha_n} S^{(n)}_{\alpha_1\cdots\alpha_n}(x_1,\ldots,x_n):A_{\text{in}}^{\alpha_1}(x_1)\cdots \hat{A}_{\text{in}}^{\alpha_i}(x_i)\cdots A_{\text{in}}^{\alpha_n}(x_n):r^{\alpha_i} , \tag{4.3}$$

where the caret indicates the quantity is omitted. By renaming indices, (4.3) gives

$$0=\sum_{n=0}^{\infty}\sum_{i=0}^{n-1}\int d^4x\, d^4y_1\cdots d^4y_{n-1} \sum_{\alpha,\beta_1,\ldots,\beta_{n-1}} S^{(n)}_{\beta_1\cdots\beta_{i-1}\alpha\beta_{i+1}\cdots\beta_{n-1}}(y_1,\ldots,y_{i-1},x,y_{i+1},\ldots,y_{n-1}) \times :A_{\text{in}}^{\beta_1}(y_1)\cdots A_{\text{in}}^{\beta_{n-1}}(y_{n-1}):r^{\alpha} . \tag{4.4}$$

Completeness of the in-fields then says that all individual coefficients of the expansion must vanish:

$$0=\int d^4x\, S^{(n)}_{\beta_1\cdots\beta_{i-1}\alpha\beta_{i+1}\cdots\beta_{n-1}}(y_1,\ldots,y_{i-1},x,y_{i+1},\ldots,y_{n-1}) \tag{4.5}$$

for all $i=0,\ldots,n-1$. Thus, we have

$$0=\tilde{S}^{(n)}_{\alpha_1\cdots\alpha_i\cdots\alpha_n}(q_1,\ldots,q_i=0,\ldots,q_n) \tag{4.6}$$

for all $i=0,\ldots,n$, where

$$\tilde{S}^{(n)}_{\alpha_1\cdots\alpha_n}(q_1,\ldots,q_n)(2\pi)^4\delta^4(q_1+\cdots+q_n)=\int d^4x_1\cdots d^4x_n\, e^{i(q_1x_1+\cdots+q_nx_n)}\, S^{(n)}_{\alpha_1\cdots\alpha_n}(x_1,\ldots,x_n) . \tag{4.7}$$

The result, Eq. (4.6), thus says that all S-matrix elements must vanish whenever any one of the external momenta vanish.

As we mentioned in the Introduction, this derivation, while quite satisfactory for the usual (IR-free) theories considered in Ref. 8, is, however, inadequate for our purposes. In perturbation theory, the IR divergences for the YM fields are such that it is not certain if the particle states can be defined. In fact that was the rationale behind speculations to confine YM gluons (and quarks) by the IR mechanism. Thus, S-matrix elements need not be well defined, and indeed the S operator need not exist as an expansion in the in-fields.

The second derivation is the one used in paper I for QED, summarized in Sec. II. Equations (2.6)–(2.12) remain valid in the YM theory if color indices are added in the obvious way. Proceeding similarly with the Green's functions $\langle J^a_\mu A^b_\nu A^c_\kappa\rangle$ and $\langle A^a_\mu A^b_\nu A^c_\kappa\rangle$, we further deduce that the proper vertex function $\Gamma^{abc}_{\mu\nu\kappa}(p,q)$ vanishes whenever one of the the boson four-momenta $(p,q,p+q)$ vanishes:

$$\Gamma^{abc}_{\mu\nu\kappa}(0,q)=\cdots=0 . \tag{4.8}$$

Our third derivation uses functional methods. Consider the generating functional for Green's functions

$$W[J^\mu]=e^{Z[J^\mu]} =\int[dA][dc_1][dc_2][d\psi][d\bar{\psi}] \times\exp\left\{i\int d^4x[\mathcal{L}(x)-\vec{J}^R_\mu(x)\cdot\vec{A}^\mu_R(x)]\right\} , \tag{4.9}$$

with $\mathcal{L}(x)$ given by (3.1). The R transformation on $\vec{A}^\mu_R$, as we saw, leaves $\mathcal{L}(x)$ invariant. Thus, under R

$$W[J^\mu]\to\exp\left[-i\int d^4x\vec{J}^R_\mu(x)\cdot\vec{r}^\mu\right]W[J^\mu] . \tag{4.10}$$

But the transformation of the variable $\vec{A}_\mu$ of functional integration does not change the value of the integral, so that we can conclude that

$$\int d^4x\vec{J}^R_\mu(x)W[J_\mu]=0 . \tag{4.11}$$

Notice that a constant $\vec{r}^{\mu}$ necessitates the integrated form in (4.11). Now we make the usual Legendre transformation to the generating functional of proper vertices $\Gamma[\mathcal{A}_\mu]$:

$$\Gamma[\mathcal{A}_\mu]=Z[J_\mu]-\int d^4x \vec{J}^R_\mu(x)\cdot\vec{\mathcal{A}}^\mu_R(x)\,, \tag{4.12}$$

$$\vec{\mathcal{A}}^\mu(x)=\frac{\delta Z[J^\mu]}{\delta \vec{J}_\mu(x)}\,, \tag{4.13}$$

$$-\vec{J}^\mu(x)=\frac{\delta\Gamma[\mathcal{A}^\mu]}{\delta\vec{\mathcal{A}}_\mu(x)}\,. \tag{4.14}$$

Then we obtain from (4.11) effectively

$$\int d^4x\frac{\delta\Gamma[\mathcal{A}_\mu]}{\delta\vec{\mathcal{A}}_\mu(x)}=0\,. \tag{4.15}$$

After taking n functional derivatives of (4.15), we get

$$\int d^4x\,\frac{\delta^{(n+1)}\Gamma[\mathcal{A}_\mu]}{\delta\mathcal{A}_\mu(x)\delta\mathcal{A}_{\alpha_1}(y_1)\cdots\delta\mathcal{A}_{\alpha_n}(y_n)}=0\,. \tag{4.16}$$

We thus obtain

$$\int d^4x\,\tilde{\Gamma}^{\mu\alpha_1\cdots\alpha_n}(x,y_1,\ldots,y_n)=0\,, \tag{4.17}$$

where $\tilde{\Gamma}^{\mu\alpha_1\cdots\alpha_n}$ is the $(n+1)$-point proper vertex for the gauge fields. From (4.17), we can write in momentum space

$$\Gamma^{\alpha_1\cdots\alpha_n}(\ldots,q_i=0,\ldots)=0, \tag{4.18}$$

where

$$\Gamma^{\alpha_1\cdots\alpha_n}(q_1,\ldots,q_n)(2\pi)^4\delta^4(q_1+\cdots+q_n)=\int d^4z_1\cdots d^4z_n\,e^{i(q_1z_1+\cdots+q_nz_n)}\times\tilde{\Gamma}^{\cdots}(z_1,\ldots,z_n)\,. \tag{4.19}$$

Our main result, the low-energy theorem (4.18), states that the proper vertex for the coupling of n gauge fields vanishes whenever any one of the momenta of the particles is set equal to zero. The same low-energy theorem obviously holds in the presence of fermions as external lines.

We shall now show how the bubble-graph model for the scalar theory discussed in Sec. II can be generalized to the YM case at hand. This will provide us with an example in gauge field theory that illustrates the low-energy theorem.

Consider the same set of diagrams as discussed in Sec. II, where now the lines are the YM quanta. We have to be slightly more careful in multiplying the bubble diagrams, thanks to the presence of indices. We shall write the bubble as

$$\Gamma^{(1)bcde}_{\mu\nu\rho\sigma}(P)=\Gamma^{(0)bcb'c'}_{\mu\nu\mu_1\nu_1}\left[\int d^4l\,D^{\mu_1\mu_2}_{(0)}(l)D^{\nu_1\nu_2}_{(0)}(P-l)\right]\times\Gamma^{(0)b'c'de}_{\mu_2\nu_2\rho\sigma}\,, \tag{4.20}$$

where $\Gamma^{(0)}_{\cdots}{}^{\cdots}$ is the momentum-independent bare four-particle vertex (of order g^2), and the free propagator is written as

$$D^{\mu\nu(0)}_{ab}(q)=\delta_{ab}D^{\mu\nu}_{(0)}(q)\,. \tag{4.21}$$

We define the left multiplication, $\otimes$, of the bubble with the bare four-point vertex as

$$\Gamma^{(0)bcb'c'}_{\mu\nu\mu'\nu'}\otimes\Gamma^{(1)b'c'de}_{\mu'\nu'\rho\sigma}(P)\equiv\Gamma^{(1)bcde}_{\mu\nu\rho\sigma}(P)\,, \tag{4.22}$$

and its left multiplication with the bare three-point vertex $\gamma^{(0)abc}_{\lambda\mu\nu}$ (of order g) as

$$\begin{aligned}\gamma^{(0)ab'c'}_{\lambda\mu_1\nu_1}\otimes\Gamma^{(1)b'c'de}_{\mu_1\nu_1\rho\sigma}(P)&\equiv\gamma^{(0)ab'c'}_{\lambda\mu_1\nu_1}\left[\int d^4l\,D^{\mu_1\mu_2}_{(0)}(l)D^{\nu_1\nu_2}_{(0)}(P-l)\right]\Gamma^{(0)b'c'de}_{\mu_2\nu_2\rho\sigma}\\&=\gamma^{(1)ade}_{\lambda\rho\sigma}(P)\,.\end{aligned} \tag{4.23}$$

It is easy to verify that the left multiplication on the elements $\Gamma^{(0)}$ and $\gamma^{(0)}$ defined by (4.22) and (4.23) satisfy the usual axioms of associativity and distributivity with respect to addition for multiplication. Thus if we consider the bubble-graph model for the three-point function

$$\gamma_W(P)=\gamma^{(0)}+\gamma^{(0)}\otimes\Gamma^{(1)}+\gamma^{(0)}\otimes\Gamma^{(1)}\otimes\Gamma^{(1)}+\cdots\,, \tag{4.24}$$

then it sums to

$$\gamma_W(P)=\gamma^{(0)}\otimes(1-\Gamma^{(1)})^{-1} \tag{4.25}$$

or

$$\gamma_W\otimes(1-\Gamma^{(1)})=\gamma^{(0)}\,. \tag{4.26}$$

Thus γ_W would have to vanish if $\Gamma^{(1)}$ diverges. The same considerations as in Sec. II then show that $Z_1=0$, and the low-energy theorem is satisfied.

Such models are of course of dubious merit. Since the sum is not even gauge invariant, it need not have anything to do with the actual behavior of the theory.

V. DISCUSSION

A comparison of our results with that obtained in the usual perturbative approach[16] to IR behavior in QED is in order. The usual approach considers only finite orders in perturbation theory with the usual bare couplings as vertices. By virtue of the relative simplicity of the photon-electron coupling, it is possible to sum (in n) exactly the IR divergences contributed by n real (external legs) and virtual (internal integrations) photons separately. The resulting contributions from the two kinds of IR photons then cancel each other exactly in the observable cross section. Our approach, instead, makes use of the decomposition of a complete Green's function into exact propagators and proper vertices, and exact statements on the vanishing of the proper vertices. Expressed in terms of the full propagators and irreducible vertices, the various Green's functions have a treelike structure. Because of our lack of control over the precise nature of the singularity in the propagators and of the zeros in the proper vertices, it remains to be seen whether the removal of the IR divergence from the S matrix can be accomplished in the usual sense. On the other hand, the Abelian gauge invariance (see paper II) of the theory should perhaps reduce the difficulty in summing treelike structures in YM theory, since much of the difficulty is to be attributed to the noncommutativity of the external line insertions.

In the usual field theories, a low-energy theorem of the type we have derived is sufficient to guarantee the existence of the on-shell S matrix for the zero-mass excitations. In chiral-symmetric field theories,[17] the Adler self-consistency condition decrees the absence of IR divergences for the Goldstone particle. Individual Feynman diagrams show IR divergences, but they cancel in each finite order of perturbation. The situation we encounter is more complex; the amplitudes might vanish as $q_\mu \to 0$ but still diverge as the invariant $q^2 \to 0$. We have not investigated this problem.

Our conclusions are relevant to the question of color[18] confinement in hadron physics.[19] A theory of hadrons based on YM gluons is extremely attractive, largely because of the UV freedom. The catch is that massless YM quanta, or indeed any color nonsinglet states, have never been seen. The by now conventional resolution of breaking the color YM symmetry via the Higgs mechanism does not work here unless the UV freedom is lost,[20] and in any case, the presence of elementary scalar mesons is unattractive. Furthermore, it is desirable to keep the color symmetry exact in order to understand the apparent confinement of quarks in spite of their presumed small effective masses.[21] The hope was that the "violent" IR divergences in the YM theory would dynamically accomplish this color confinement. Our results indicate that the IR behavior is not really so violent, and it therefore is not clear to us that the confinement will occur.

This of course does not mean that YM hadron theory is ruled out. One possibility is to alter the large-distance behavior of the theory, a possibility mentioned in Ref. 18. Another possibility is that nonperturbative solutions of the theory, with the desired properties, exist. Also, it might be possible to obtain a desirable theory by some resummation of the conventional perturbation expansion. As a final possibility, we note that there might be a Hilbert space of bound-state (color-singlet) states which is orthogonal to the color-nonsinglet Hilbert space, and in which there is a unitary S matrix. The trouble with all this is that no one knows how to do calculations in any of these suggested frameworks.

The IR behavior is also relevent to on-shell quantities of theories with mass.[3] To investigate processes like form factors and large-transverse-momentum hadronic collisions,[22] one needs information both at the UV and the IR ends. However, it would be necessary to know precisely how fast the vertices vanish in the IR regime, and that information is lacking. It would be desirable to investigate models which possess both UV freedom and low-energy theorems. It is clear from the derivation that even though the UV freedom gives a logarithmically vanishing value for Z_1/Z_3, this does not mean that the proper vertices vanish at that rate in the IR limit.

Fascinating questions like these must await future efforts for their answer.

ACKNOWLEDGMENT

We acknowledge the hospitality of the Institut für Theoretische Physik der Freien Universität Berlin, and the Deutsches Elektronen-Synchrotron, where this research was completed. We are particularly indebted to Kurt Symanzik for helpful discussions.

---

*Research supported in part by the U. S. National Science Foundation under Grant No. GP-36316X.

†Research supported in Stony Brook by the U. S. National Science Foundation under Grant No. GP-32998X.

‡Present address: Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742.

[1]See the reviews of YM theory: E. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1974); M. Veltman, in *Proceedings of the Sixth International Symposium on Electron and Photon Interactions at High Energy, Bonn, Germany, 1973,* edited by H. Rollnik and W. Pfeil (North-Holland, Amsterdam, 1974); S. Coleman, in *Laws of Hadronic Matter*, proceedings of the 1973 International Summer School "Ettore Majorana," Erice, Italy, edited by A. Zichichi (Academic, New York, 1975).

[2]Actually this is valid only for zero electron mass.

However, see Ref. 3.

[3]K. Symanzik, Commun. Math. Phys. 34, 7 (1973).

[4]An exception is the JBW electrodynamics: K. Johnson and M. Baker, Phys. Rev. D 8, 1110 (1973), and references therein; S. L.Adler, *ibid.* 5, 3021 (1972).

[5]This paper is the third in a series of three investigations on YM theories: R. A. Brandt and W.-C. Ng, Phys. Rev. D 10, 4198 (1974), which we refer to as paper I; and the preceding paper, *ibid.* 15, **2235** (1977), which we refer to as paper II.

[6]Provided that ordinary perturbation series is at least an asymptotic expansion.

[7]W.-C. Ng and K. Young, Phys. Lett. 51B, 291 (1974); T. P. Cheng, W.-C. Ng, and K. Young, Phys. Rev. D 10, 2459 (1974).

[8]K. Nishijima, Nuovo Cimento 11, 698 (1959); G. Kramer and W. F. Palmer, Phys. Rev. 182, 1492 (1969).

[9]D. Zwanziger, Phys. Rev. Lett. 30, 934 (1973), Phys. Rev. D 14, 2570 (1976).

[10]J. Schwinger, Phys. Rev. 128, 2425 (1962); J. Lowenstein and J. A. Swieca, Ann. Phys. (N.Y.) 68, 172 (1971).

[11]J. M. Cornwall, Phys. Rev. D 10, 500 (1974).

[12]R. A. Brandt, Ann. Phys. (N.Y.) 44, 221 (1967); 52, 122 (1969); Fortschr. Phys. 18, 249 (1970).

[13]W. Zimmermann, Commun. Math. Phys. 6, 161 (1967); 10, 325 (1968); in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser *et al.* (MIT, Cambridge, Mass., 1970).

[14]R. A. Brandt and W.-C. Ng, Nuovo Cimento 13A, 1025 (1973); Phys. Rev. D 9, 373 (1974).

[15]R. A. Brandt, W.-C. Ng, and K. Young, Phys. Rev. D (to be published).

[16]S. Weinberg, Phys. Rev. 140, B516 (1965), and references therein.

[17]See, for example, B. W. Lee, *Chiral Dynamics* (Gordon and Breach, New York, 1972).

[18]H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. 47B, 365 (1973).

[19]See also the review of R. A. Brandt, *in High Energy Hadron Interactions,* Proceedings of the IX Balaton Symposium on Particle Physics, edited by I. Montvay *et al.* (Central Research Inst. for Physics, Budapest, 1975), Vol. 1, p. 227.

[20]T. P. Cheng, E. Eichten, and L. F. Li, Phys. Rev. D 9, 2259 (1974).

[21]Related to the presumed small violation of Bjorken scaling.

[22]T. Appelquist and E. Poggio, Phys. Rev. D 10, 3280 (1974); M. Creutz and L.-L. Wang, *ibid.* 10, 3749 (1974).