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Note

Renormalization Problem in Nonrenormalizable Massless Φ^4 Theory

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Abstract. Nonrenormalizable massless Φ^4 theory is made finite by regularization via higher derivatives in the kinetic part of the Lagrangean. The theory is shown to remain finite in the infinite cutoff limit if certain integrals over functions of one variable, with computable Taylor expansion at the origin, are finite. The values of these integrals are the only unknowns in the double series in powers of g and $g^{2/\varepsilon}$ obtained for the Green's functions in massless $(\Phi^4)_{4+\varepsilon}$ with generic ε. For $\varepsilon=1$ and $\varepsilon=2$, these series reduce to double series in powers of g and $\ln g$. The problems of extension to $(\Phi^4)_{4+\varepsilon}$ with mass, of causality and unitarity, of the relation to the BPHZ formalism, and of the indeterminacy of the result are discussed.

0. Introduction

Φ^4 theory in more than four space-time dimension is not renormalizable. This means that there is no choice of bare parameters such that all ultraviolet (UV) divergences in the perturbation theoretical construction are cancelled. Equivalently, construction of the perturbation expansions by BPHZ [1] or Epstein-Glaser [2] methods introduces an infinite number of arbitrary constants, and any finite choice of these constants apparently corresponds to actually employing a Lagrangean that is not the Φ^4 one.

Introducing a (sufficiently strong) cutoff Λ, however, removes all these UV divergences identically in all parameters. Our aim is to analyze the mechanism of this removal for $\Lambda<\infty$, to find sufficient conditions for the cancellation involved to persist in the limit $\Lambda\to\infty$, and to obtain and discuss the $\Lambda=\infty$ result if these conditions are satisfied. For a certain class of regularizations, this program can be carried out.

We introduce the cutoff in terms of higher derivatives in the kinetic part of the Lagrangean as proposed by Pais and Uhlenbeck [3], which is here equivalent to the method of Pauli and Villars [4]. Specifically, we set

$$L=-\tfrac{1}{2}\Phi\Box(1+\Lambda^{-2}\Box)\Phi-(1/4!)g_B\Phi^4-\tfrac{1}{2}m_{B0}^2\Phi^2\,, \tag{0.1a}$$

where

$$m_{B0}^2=\Lambda^2\sum_{k=1}^{\infty}a_k(\varepsilon)\,(g_B\Lambda^{\varepsilon})^k \tag{0.1b}$$

is the bare mass squared of the zero-physical-mass theory, to which we restrict ourselves, for simplicity, in this paper. The number of space-time dimensions is $4+\varepsilon$, such that g_B has dimension $-\varepsilon$, and it is advantageous to keep ε generic [5], e.g. also complex, rather than positive integer, since hereby dimensional degeneracies are lifted that otherwise would require the consideration of logarithms, which are less easily handled than powers, in almost all formulae below. The restriction of ε to positive integers is a special case of the one to positive rationals, which will be discussed.

The connected amputated one-particle-irreducible parts of the Green's functions

$$2^{-l}\langle T\Phi(x_1)\ldots\Phi(x_{2n})\Phi^2(y_1)\ldots\Phi^2(y_l)\rangle$$

are called vertex functions, and their Fourier transform, with a factor $(2\pi)^{4+\varepsilon}\delta(\sum p+\sum q)$ taken out, we denote as

$$\Gamma_{AB}(p_1\ldots p_{2n}, q_1\ldots q_l; g_B, \varepsilon)\equiv\Gamma_{AB}((2n),(l); g_B, \varepsilon).$$

As shown in Appendix A, these functions possess the expansion

$$\Gamma_{AB}((2n),(l); g_B, \varepsilon)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Lambda^{-2j+\varepsilon k}f_{jk}((2n),(l); g_B, \varepsilon). \tag{0.2}$$

Here f_{00} is the function obtained from the Lagrangean (0.1a), with no cutoff and m_{B0}^2 set zero, employing the analytic integration method [5] of Wilson, and 't Hooft and Veltman, and is singular at (in sufficiently high order, all) nonnegative rational ε (in the present massless case, also at negative rational ε). The f_{jk} are power series in g_B with similar properties, whereby $\mathscr{L}$-loop diagrams do not contribute to the f_{jk} with $k>\mathscr{L}$. The expansion coefficients of all f_{jk} are meromorphic in ε.

The mechanism of ε-singularity cancellation (for $0\leqq\varepsilon<3$) is transparent in the expansion (0.2): Singularities in f_{00} at $\varepsilon=R$ are cancelled by singularities in the f_{jk} with $2j/k=R$, giving rise to powers of $\ln\Lambda$. Likewise, the singularities in all f_{jk} with equal $-2j+Rk$ cancel.

We show in Appendix B how by a process of renormalization, functions are obtained for which

$$\Gamma_\Lambda((2n),(l); \mu, g, \varepsilon)=h_{00}((2n),(l); \mu, g, \varepsilon)+\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\Lambda^{-2j+\varepsilon k}h_{jk}((2n),(l); \mu, g, \varepsilon) \tag{0.3}$$

holds, where g is dimensionless and μ a unit of mass, both appearing only in the combination $\mu^{-\varepsilon}g$. The Γ_Λ are proven to be singularity free for $0<\varepsilon<4$. h_{00} has ε-singularities again at positive rational ε, and thus the total residuum of the double sum in (0.3) at the singularity is Λ-independent. Our aim is to convert, in the limit $\Lambda\to\infty$, that double sum into a Λ-free expression that has precisely these same singularities, and thus yields, together with h_{00}, an expression for Γ_∞ without ε-singularities, which we define as the renormalized vertex function of the nonrenormalizable theory. That we start hereto from Γ_Λ of (0.3) rather than from Γ_{AB} of (0.2) is suggested by the success [6] of an analogous program for the construction of massless superrenormalizable Φ^4 theory, where the expansion one starts from is analogous to (0.3) rather than to (0.2).

In Section 1 we state the Feynman rules for direct construction of the Γ_Λ. Normal operator products at zero momentum transfer, as needed later, are introduced in Section 2. Differention and reintegration with respect to Λ, which are the basic process in setting up the expansion (0.3), are treated in Section 3. The sufficient conditions for the limit $\Lambda \to \infty$ to be possible are obtained in Section 4. The result, a quasi-perturbation expansion as well as its relation to BPHZ formalism, and also the unitarity problem, are discussed in Section 5. In Appendix A the expansion (0.2), from which (0.3) follows, is derived. In Appendix B we describe an informative method, alternative to that of Section 1, to construct the functions Γ_Λ.

1. Feynman Rules for Regularized Massless $(\Phi^4)_{4+\varepsilon}$ Theory

We choose the zeroth-order propagator $i(p^2+i0)^{-1}[1-\Lambda^{-2}(p^2+i0)]^{-1}$ and vertex $-ig\mu^{-\varepsilon}$ and apply the well-known integration rules [5] for $4+\varepsilon$ dimensions. Subtractions for divergent (sub)graphs are made just as in 4 dimensions (with one exception, see below), which would suffice for $\mathscr{L}$-loop graphs if $0<\varepsilon<2\mathscr{L}^{-1}$ in case $\Lambda=\infty$, but here always suffices for $0\leqq\varepsilon<4$. Exploiting that infrared (IR) divergences that are only logarithmic in 4 dimensions disappear in $4+\varepsilon$ dimensions, we impose the renormalization conditions (which require $0<\varepsilon<4$) for vertex functions

$$\Gamma_\Lambda(00,;\mu,g,\varepsilon)=0\,, \tag{1.1a}$$

$$[\partial/\partial p^2]\Gamma_\Lambda(p(-p),;\mu,g,\varepsilon)|_{p=0}=i\,, \tag{1.1b}$$

$$\Gamma_\Lambda(0000,;\mu,g,\varepsilon)=-i\mu^{-\varepsilon}g\,, \tag{1.1c}$$

$$\Gamma_\Lambda(00,0;\mu,g,\varepsilon)=1\,, \tag{1.1d}$$

$$\Gamma_\Lambda(,00;\mu,g,\varepsilon)=0\,. \tag{1.1e}$$

Hereby, however, we stipulate that when (1.1b) is implemented, not a term const p^2 but const $p^2(1-\Lambda^{-2}p^2)$ is subtracted. This secures that the total kinetic part of the Lagrangean remains of the form $-\frac{1}{2}Z_3\Phi_{\text{ren}}\Box(1+\Lambda^{-2}\Box)\Phi_{\text{ren}}$ such that there is no "renormalization" of Λ relative to Lagrangean (0.1). It is easy to verify that for $\varepsilon<4$ this modification of subtraction rules does not affect the convergence discussion.

The perturbation theoretical construction using these rules (for $0<\varepsilon<4$) yields vertex functions obeying obviously

$$\mathcal{O}\!\!\!/\,\Gamma_\Lambda((2n),(l);\mu,g,\varepsilon)=0 \tag{1.2a}$$

with

$$\mathcal{O}\!\!\!/\equiv\mu[\partial/\partial\mu]+\varepsilon g[\partial/\partial g]\,. \tag{1.2b}$$

The relation of these functions to more conventionally parametrized ones, defined for $0\leqq\varepsilon<4$, is discussed in Appendix B. While (1.2) and all of the following would remain valid for the functions obtained by usual implementation of (1.1b), this suggestive interpretation of the Γ_Λ would then be less direct.

2. Normal Operator Products at Zero Momentum Transfer

In later sections, we will need insertions of the operators $0_i, i=1\dots6$, whose characteristic form is listed here without regard to counter terms and dimensionless overall factors:

$$\begin{array}{lcccccc} i= & 1 & 2 & 3 & 4 & 5 & 6 \\ 0_i= & \Phi^2 & \Phi\square\Phi & \Phi^4 & \Phi\square^2\Phi & \Phi^3\square\Phi & \Phi^6 \\ \dim 0_i= & 2+\varepsilon & 4+\varepsilon & 4+2\varepsilon & 6+\varepsilon & 6+2\varepsilon & 6+3\varepsilon\,. \end{array}$$

The dimension given,

$$\dim 0_i = 2a_i + \varepsilon b_i\,, \qquad 1 \leqq b_i \leqq a_i$$

is the ordinary one, and we have listed only the operators that are, at zero momentum transfer, linearly independent. As counterterms to 0_i only operators 0_j with $a_j \leqq a_i$ are admitted.

To construct the vertex functions with these insertions,

$$\Gamma_{Ai}(p_1 \dots p_{2n}; \mu, g, \varepsilon) \equiv \langle T(N(0_i(0))\Phi(\tilde{p}_1)\dots(\tilde{p}_{2n}))\rangle^{\text{prop}}$$

in the sense of Zimmermann [7], we choose minimal degree and no extra Φ^2 arguments for simplicity. As renormalization conditions we take

$$\Gamma_{Ai}(00; \mu, g, \varepsilon) = \delta_{i1}\,, \qquad \forall i\,, \tag{2.1a}$$

$$[\partial/\partial p^2]\Gamma_{Ai}(p(-p); \mu, g, \varepsilon)|_{p=0} = \delta_{i2}\,, \qquad i>1\,, \tag{2.1b}$$

$$\Gamma_{Ai}(0000; \mu, g, \varepsilon) = \delta_{i3}\,, \qquad i>1\,, \tag{2.1c}$$

$$[\partial/\partial p^2]^2\Gamma_{Ai}(p(-p); \mu, g, \varepsilon)|_{p=0} = 2\delta_{i4}\,, \qquad i>3\,, \tag{2.1d}$$

$$[\partial/\partial p^2]\Gamma_{Ai}(p(-p)00; \mu, g, \varepsilon)|_{p=0} = \delta_{i5}\,, \qquad i>3\,, \tag{2.1e}$$

$$\Gamma_{Ai}(000000; \mu, g, \varepsilon) = \delta_{i6}\,, \qquad i>3\,. \tag{2.1f}$$

Then all Γ_{Ai} are specified hereby (for $0<\varepsilon<3$, to which we restrict ourselves from here on), admit expansions analogous to (0.3), and obey

$$\mathcal{O}_{\!\not\mu}\Gamma_{Ai}((2n); \mu, g, \varepsilon) = 0 \tag{2.2}$$

since in the Feynman rules and all renormalization conditions (1.1), (2.1) μ and g appear only in the combination $\mu^{-\varepsilon}g$.

On the basis of the Lagrangean (1.1), the counting identity [8]

$$n\Gamma_A((2n); \mu, g, \varepsilon) = \textstyle\sum_{i=1}^{6} e_i(\mu, g, \Lambda, \varepsilon)\Gamma_{Ai}((2n); \mu, g, \varepsilon) \tag{2.3}$$

holds. Because of (1.2) and (2.2), the e_i satisfy $\mathcal{O}_{\!\not\mu} e_i(\mu, g, \Lambda, \varepsilon) = 0$, $\forall i$ i.e.

$$e_i(\mu, g, \Lambda, \varepsilon) = \Lambda^{4+\varepsilon-\dim 0_i} e_i(\Lambda^\varepsilon \mu^{-\varepsilon} g, \varepsilon)\,, \tag{2.4}$$

due to linear independence of the Γ_{Ai} which is obvious from (2.1). (1.1), (2.1), and (2.3) give

$$e_1 = 0,\ e_2(z, \varepsilon) = i,\ e_3(z, \varepsilon) = -2iz \tag{2.5}$$

and, using these, also expressions for the e_4, e_5, e_6 in terms of the $\Gamma_\Lambda, \Gamma_{\Lambda i}$ at zero momenta and derivatives there.

Also

$$[\partial/\partial g]\Gamma_\Lambda((2n);\mu,g,\varepsilon)=\mu^{-\varepsilon}\sum_{i=1}^{6} d_i(\mu,g,\Lambda,\varepsilon)\Gamma_{\Lambda i}((2n);\mu,g,\varepsilon) \tag{2.6}$$

holds, with

$$d_i(\mu,g,\Lambda,\varepsilon)=\Lambda^{4+2\varepsilon-\dim O_i}d_i(\Lambda^\varepsilon\mu^{-\varepsilon}g,\varepsilon)\,, \tag{2.7}$$

$$d_1=d_2=0,\ d_3(z,\varepsilon)=-i\,. \tag{2.8}$$

Using (0.1), formulae of Appendix B give

$$d_i(z,\varepsilon)=-2\varepsilon^{-1}z^{-1}\gamma(\bar g(z,\varepsilon),\varepsilon)e_i(z,\varepsilon),,\qquad i=4,5,6\,. \tag{2.9}$$

3. Differentiation and Reintegration with Respect to the Cutoff

The Schwinger action principle together with the counting identity (2.3) leads, as for Lowenstein's differential vertex operation [8], for the Lagrangean (1.1) to

$$\Lambda[\partial/\partial\Lambda]\Gamma_\Lambda((2n);\mu,g,\varepsilon)=\sum_{i=1}^{6} c_i(\mu,g,\Lambda,\varepsilon)\Gamma_{\Lambda i}((2n);\mu,g,\varepsilon)\,. \tag{3.1}$$

(1.2) and (2.2) give, again due to linear independence, $\mathcal{O}\not{p}c_i(\mu,g,\Lambda,\varepsilon)=0$, $\forall i$ i.e.

$$c_i(\mu,g,\Lambda,\varepsilon)=\Lambda^{4+\varepsilon-\dim O_i}c_i(\Lambda^\varepsilon\mu^{-\varepsilon}g,\varepsilon)\,. \tag{3.2}$$

Furthermore, from (1.1) and (2.1) follows $c_1=c_2=c_3=0$, while the explicit form (0.1) yields for $i=4,5,6$

$$c_i(z,\varepsilon)=-2[1+\gamma(\bar g(z,\varepsilon),\varepsilon)]e_i(z,\varepsilon) \tag{3.3}$$

in terms of functions introduced in Appendix B. Alternatively, from (3.1), (1.1), and (2.1)

$$c_4(\Lambda^\varepsilon\mu^{-\varepsilon}g,\varepsilon)=\tfrac{1}{2}\Lambda^3[\partial/\partial\Lambda][\partial/\partial p^2]^2\Gamma_\Lambda(p(-p);\mu,g,\varepsilon)|_{p=0}\,, \tag{3.4a}$$

$$c_5(\Lambda^\varepsilon\mu^{-\varepsilon}g,\varepsilon)=\Lambda^{3+\varepsilon}[\partial/\partial\Lambda][\partial/\partial p^2]\Gamma_\Lambda(p(-p)00;\mu,g,\varepsilon)|_{p=0}\,, \tag{3.4b}$$

$$c_6(\Lambda^\varepsilon\mu^{-\varepsilon}g,\varepsilon)=\Lambda^{3+2\varepsilon}[\partial/\partial\Lambda]\Gamma_\Lambda(000000;\mu,g,\varepsilon)\,. \tag{3.4c}$$

As a consequence of the validity of (3.1), the perturbation expansion coefficients $c_{ik}(\varepsilon)$ in

$$c_i(z,\varepsilon)=2i\delta_{i4}+\sum_{k_{\min i}}^{\infty} c_{ik}(\varepsilon)z^k\qquad i=4,5,6 \tag{3.5a}$$

are holomorphic outside $(-\infty,0]\cup[3,\infty)$, and this property we ascribe also to the functions $c_i(z,\varepsilon)$. The Feynman rules of Section 1 give

$$k_{\min i}=2,2,3\quad\text{for}\quad i=4,5,6\,. \tag{3.5b}$$

It is worth noting that the vertex functions (or p^2-derivatives of vertex functions) in (3.4) are IR finite only for $\varepsilon>2$. However, the Λ-differentiation removes the IR singularities in $0<\varepsilon\leq 2$, in conformity with the mentioned analytic properties of the functions c_i.

The integral of (3.1), taking (3.2) and (3.5a) into account, is

$$\Gamma_\Lambda((2n); \mu, g, \varepsilon) = h_{00}((2n); \mu, g, \varepsilon) + \sum_{i=4}^{6} \text{ (anal. cont. from } \varepsilon \text{ suffic. small)}$$
$$\cdot \int_\infty^\Lambda dt\, t^{-3-(b_i-1)\varepsilon} c_i(t^\varepsilon \mu^{-\varepsilon} g, \varepsilon) \Gamma_{ti}((2n); \mu, g, \varepsilon). \quad (3.6)$$

Here "sufficiently small" is only defined in perturbation theory: For $\mathscr{L}$-loop diagrams (where $\mathscr{L} = N - n + 1$ at order g^N), $0 < \varepsilon < 2\mathscr{L}^{-1}$. The Λ-free term h_{00} is then the one of (0.3), as follows from the discussion at the beginning of Section 1. In (3.6), partial integration gives

$$\int_\infty^\Lambda dt \ldots = \Gamma_{\infty i}((2n); \mu, g, \varepsilon) \int_\infty^\Lambda dt\, t^{-3-(b_i-1)\varepsilon} c_i(t^\varepsilon \mu^{-\varepsilon} g, \varepsilon)$$
$$- \int_\infty^\Lambda dt \{[\partial/\partial t] \Gamma_{ti}((2n); \mu, g, \varepsilon)\} \int_\Lambda^t ds\, s^{-3-(b_i-1)\varepsilon} c_i(s^\varepsilon \mu^{-\varepsilon} g, \varepsilon). \quad (3.7)$$

The analytic continuation, as indicated before, of the first part is

$$-i\Lambda^{-2} h_{004}((2n); \mu, g, \varepsilon) \delta_{i4}$$
$$+ h_{00i}((2n); \mu, g, \varepsilon) \cdot \sum_{k_{\min i}}^{\infty} [-2 + (k+1-b_i)\varepsilon]^{-1} c_{ik}(\varepsilon) \Lambda^{-2+(k+1-b_i)\varepsilon} \mu^{-\varepsilon k} g^k, \quad (3.8)$$

where the h_{00i} are constructed without cutoff as h_{00} is and have singularities at ε (zero and) positive-rational. The evaluation and analytic continuation of the last term in (3.7) requires to analyze the functions $\Lambda[\partial/\partial\Lambda]\Gamma_{\Lambda i}$, which we defer to a later publication. It is easy to see, however, that the $j = 1$ terms in (0.3) are precisely those obtained by inserting (3.8) into the sum in (3.6), and continuation of the procedure would yield the terms in (0.3) with $j = 2, 3$, etc. Since the $c_{ik}(\varepsilon)$ are holomorphic as described before, the ε singularities of the h_{1k} in (0.3) are, besides those of the functions h_{00i}, the ones explicit in (3.8) stemming from the integration.

4. Removal of the Cutoff

For $\Lambda \to \infty$, the last term in (3.6) does not vanish since the analytic continuation destroys the form of the integral for any $\varepsilon > 0$ if one considers perturbation theoretical orders $N > n - 1 + 2\varepsilon^{-1}$. It is possible, however, to replace that term by an expression whose evaluation prescription does not depend on the order considered.

We write instead of (3.7)

$$\int_\infty^\Lambda dt \ldots = \Gamma_{\Lambda i}((2n); \mu, g, \varepsilon) \int_\infty^\Lambda dt\, t^{-3-(b_i-1)\varepsilon} c_i(t^\varepsilon \mu^{-3} g, \varepsilon)$$
$$- \int_\infty^\Lambda dt \{[\partial/\partial t] \Gamma_{ti}((2n); \mu, g, \varepsilon)\} \int_\infty^t ds\, s^{-3-(b_i-1)\varepsilon} c_i(s^\varepsilon \mu^{-\varepsilon} g, \varepsilon). \quad (4.1)$$

The analytic continuation in ε of $\Gamma_{\Lambda i}$ is trivial since it is singularity free. For the corresponding continuation of the first integral we write, instead of (3.8), in view of (3.5),

$$-i\Lambda^{-2}\delta_{i4} + \text{(anal. cont. from } \varepsilon > 2) \int_0^\Lambda dt\, t^{-3-(b_i-1)\varepsilon} [c_i(t^\varepsilon \mu^{-\varepsilon} g, \varepsilon) - 2i\delta_{i4}]$$
$$= -i\Lambda^{-2}\delta_{i4}$$
$$+ \mu^{-2-(b_i-1)\varepsilon} g^{(2/\varepsilon)+b_i-1} \{ \sum_{k_{\min i}}^{\infty} [-2 + (k+1-b_i)\varepsilon]^{-1} c_{ik}(\varepsilon)$$
$$+ \text{(anal. cont. from } \varepsilon > 2)\, \varepsilon^{-1} \int_1^{\Lambda^\varepsilon \mu^{-\varepsilon} g} dz\, z^{-(2/\varepsilon)-b_i} [c_i(z, \varepsilon) - 2i\delta_{i4}] \}. \quad (4.2)$$

Here the $\Lambda \to \infty$ limit is trivial provided the integral converges at ∞ for $2<\varepsilon<3$ (or ε in some subinterval), and the analytic continuation is then formally given by the integral from 1 to ∞ also for $0<\varepsilon<2$ since the $c_i(z,\varepsilon)$ were argued before to be holomorphic in an environment of the $0<\varepsilon<3$ interval. In any case, at this stage we have

$$\begin{aligned}&\Gamma_\infty((2n);\mu,g,\varepsilon)=h_{00}((2n);\mu,g,\varepsilon)\\&+\textstyle\sum_{i=4}^{6}\mu^{-2-(b_i-1)\varepsilon}g^{(2/\varepsilon)+b_i-1}\{\sum_{k_{\min i}}^{\infty}[-2+(k+1-b_i)\varepsilon]^{-1}c_{ik}(\varepsilon)\\&+c_i(\varepsilon)\}\Gamma_{\infty i}((2n);\mu,g,\varepsilon)+\text{higher terms},\end{aligned} \tag{4.3}$$

where

$$c_i(\varepsilon)=(\text{anal. cont. from } 2<\varepsilon<3)\varepsilon^{-1}\textstyle\int_1^\infty dz\, z^{-(2/\varepsilon)-b_i}[c_i(z,\varepsilon)-2i\delta_{i4}]\,, \tag{4.4}$$

and where the "higher terms" are to be obtained from the analysis of the last term in (4.1) along lines analogous to the ones followed here, and are characterized by having factors $g^{4/\varepsilon}, g^{6/\varepsilon}$ etc. Also for the $\Gamma_{\infty i}$ in (4.3) an expansion analogous to (4.3) should be inserted, and so forth.

Thus, our result can be written

$$\Gamma_\infty((2n);\mu,g,\varepsilon)=\textstyle\sum_{j=0}^{\infty}\mu^{-2j}g^{2j/\varepsilon}h_j((2n);\mu,g,\varepsilon)\,, \tag{4.5}$$

where the h_j are (integer-) power series in $\mu^{-\varepsilon}g$, with h_0 the h_{00} of before. These series, however, except h_0 contain constants, like the $c_i(\varepsilon)$ of (4.4), that are not computable by finite-order perturbation theory, but are not expected to be singular in $0<\varepsilon<3$. There are ε-singularities, however, in the expansions of the functions h_j, computable apart from the constants $c_i(\varepsilon)$ and, possibly, similar ones, at ε positive rational; however, at such ε, all singularities to any finite order of perturbation theory will cancel, leading to logarithms of g. Namely, any surviving singularity would be in contradiction to the regularity in ε of Γ_Λ, as the comparison of (4.2) with (3.8) and extension to "higher terms" shows, the regularity of the $c_i(\varepsilon)$ being supposed. In particular, at $\varepsilon=1$ and $\varepsilon=2$, (4.5) reduces to a double power series in g and $\ln g$,

$$\Gamma_\infty((2n);\mu,g,\varepsilon=1\text{ or }2)=\textstyle\sum_{j=0}^{\infty}(\ln g)^j\quad H_j((2n);\mu,g,\varepsilon=1\text{ or }2) \tag{4.6}$$

with functions H_j that are obtained linearly from the complete set of functions h_j of (4.5).

Insertion of (3.4a)–(3.4c) into (4.4) yields

$$\begin{aligned}&c_4(\varepsilon)=\tfrac{1}{2}\mu^2 g^{-2/\varepsilon}\\&\quad\cdot(\text{anal. cont. from } 2<\varepsilon<3)\lim_{\Lambda\to\infty}\{[\partial/\partial p^2]^2\Gamma_{\mu g^{-1/\varepsilon}}(p(-p);\mu,g,\varepsilon)|_{p=0}\\&\quad-[\partial/\partial p^2]^2\Gamma_\Lambda(p(-p);\mu,g,\varepsilon)|_{p=0}\}\end{aligned} \tag{4.7a}$$

$$\begin{aligned}&c_5(\varepsilon)=\mu^{2+\varepsilon}g^{-2/\varepsilon)-1}\\&\quad\cdot(\text{anal. cont. from } 2<\varepsilon<3)\lim_{\Lambda\to\infty}\{[\partial/\partial p^2]\Gamma_{\mu g^{-1/\varepsilon}}(p(-p)00;\mu,g,\varepsilon)|_{p=0}\\&\quad-[\partial/\partial p^2]\Gamma_\Lambda(p(-p)00;\mu,g,\varepsilon)|_{p=0}\}\,,\end{aligned} \tag{4.7b}$$

$$\begin{aligned}&c_6(\varepsilon)=\mu^{2+2\varepsilon}g^{-(2/\varepsilon)-2}\\&\cdot(\text{anal. cont. from } 2<\varepsilon<3)\lim_{\Lambda\to\infty}\{\Gamma_{\mu g^{-1/\varepsilon}}(000000;\mu,g,\varepsilon)-\Gamma_\Lambda(000000;\mu,g,\varepsilon)\}\end{aligned} \tag{4.7c}$$

such that existence of these constants is equivalent to existence of the limites herein. The separate terms in the curly brackets are (in perturbation theory) infrared divergent unless $\varepsilon > 2$; these divergences however, cancel in the difference as they did in (3.4), Thus, a limit zero for the Γ_Λ in (4.7) might upset the cancellation of ε-singularities in (4.5) discussed before.

5. Discussion and Outlook

In (4.5) and, in particular, (4.6) we have obtained a quasi-perturbation expansion for Γ_∞. In (4.5) all functions h_j satisfy $\mathcal{O}_{\not{p}} h_j = 0$; in (4.6) the PDE $\mathcal{O}_{\not{p}} \Gamma_\infty = 0$ reduces to a coupled system in an obvious way. These PDEs take a less trivial form if the parametrization is changed to a Gell-Mann-Low one [9], i.e. one normalizes, rather than by (1.1), away from the origin in momentum space. Then there will arise for generic ε broken powers, or for $\varepsilon = 1$ and $\varepsilon = 2$ logarithms, also in the familiar parametric functions [9, 10], and the result appears then to be related to the one of Blokhintsev, Efremov, and Shirkov [11]. These authors found broken power, or for integer ε logarithmic, terms to be necessary in the parametric functions of nonrenormalizable Φ^4 theory for consistency with the renormalization group. We have shown the origin of such terms, and that there remains no arbitrariness, in principle, given the manner of regularization.

If one pursues the present approach, the following problems arise:

1) Definition and Λ-differential relations for vertex functions with insertions so far as not given here, in particular, for $\Gamma_{\Lambda ij}$, $\Gamma_{\Lambda ijk}$, etc. This is relatively straightforward, given Zimmermann's analysis [7].

2) The number of constants, uncomputable in perturbation theory like the $c_i(\varepsilon)$ of (4.4), if one carries the analysis to higher j in (0.3). It is easy to see that the $c_i(\varepsilon)$ will then reappear in higher powers, but the occurrence of further constants is to be expected, as comparison of our approach with the BPHZ one, given later, will show.

3) Extension to massive Φ^4 theory. The most suggestive way hereto appears to be the addition of a mass to the Lagrangean (1.1), in the manner of t' Hooft and Weinberg [12]. The nonregularity in the mass [13], for integer dimension, is avoided by the use of $4+\varepsilon$ dimensions with ε generic. It would be necessary to carry out point 1) with arbitrarily many Φ^2-insertions at zero momentum. Inspite of potential IR difficulties, we do not see an obstacle of principle hereby. – Alternatively, one may carry out the analysis directly for the massive theory. Because of the inhomogeneous PDEs then [if one does not want to have extra mass ratios appear in constants such as the $c_i(\varepsilon)$], the problem of insertions of several operators, see 1) above, arises already with the $j=1$ terms of (0.3).

4) The problematics of considering $\Lambda \to \infty$ for seperate j in (0.3). For generic ε, the j^{th} line obtains an overall factor $g^{2j/\varepsilon}$ and thus is functionally different for different j. Thus, if the limit $\Lambda \to \infty$ should exist for generic ε and a range of values of g, it appears mandatory that each line possess a $\Lambda \to \infty$ limit separately, and, as we displayed for $j=1$, the different momenta dependence of the h_{00i}, $i=4, 5, 6$, then requires also the $c_i(\varepsilon)$ to be finite separately. This argument is supported by the formulae (4.7): if the limit $\Lambda \to \infty$ of the Γ_Λ exists, the $c_i(\varepsilon)$, $i=4, 5, 6$ should

exist at least for $2<\varepsilon<3$. For integer (or positive rational) ε, on the other hand, the requirement of separate $\Lambda\to\infty$ existence for the j-lines appears possibly too strong. This could be discussed and, probably, excluded if the momenta dependence in the different lines, which factorizes in each line similarly as for $j=1$, were analyzed after cancelling the Λ-independent singularities for integer ε.

5) The dependence of the renormalized theory (4.6) on the manner of regularization. We have in mind here replacing $(1+\Lambda^{-2}\Box)$ in (1.1) by $\prod_{r=1}^{R}(1+a_r^2\Lambda^{-2}\Box)$. In that case, the sum in (3.1) will have to go over more terms, and the detailed form of (4.5) and perhaps also (4.6) might change. In particular, for fixed R, a dependence on the ratios of the a_r might appear in the "noncomputable" constants. While a direct treatment of this question appears difficult, the discussion of the relation to the BPHZ approach, given below, will be revealing. – Note that in (3.5a), (4.2), (4.3) the $c_{ik}(\varepsilon)$ are also regularization dependent, only their values at $\varepsilon=2/(k+1-b_i)$ are not, in order that the cancellation of ε-singularities in the regularization independent function $h_{00}((2n);\mu,g,\varepsilon)\equiv f_{00}((2n);\mu^{-\varepsilon}g,\varepsilon)$ take place. Clearly, the replacement of the $c_{ik}(\varepsilon)$ by these regularization independent values is compensated by a redefinition of the $c_i(\varepsilon)$, which in the following we consider to have been done. – It must also be remarked that in the transition from (3.8) or (4.1) to (4.2) or (4.3), possible parts in the functions $c_i(t^\varepsilon\mu^{-\varepsilon}g,\varepsilon)$ with vanishing asymptotic expansion in g are in general not treated correctly. E.g.,

$$\int_\infty^\Lambda dt\,t^{-3}\exp(-t^{-\varepsilon}\mu^\varepsilon g^{-1})\neq\int_0^\Lambda dt\,t^{-3}\exp(-t^{-\varepsilon}\mu^\varepsilon g^{-1})$$

and a fictitious (nonsingular) contribution to $c_4(\varepsilon)$ of (4.4) would arise from such a term. We are obviously unable to control such terms by perturbation theoretical methods and shall not discuss them here.

6) Finiteness, and computation, of the $c_i(\varepsilon)$ and similar constants. This also requires non-perturbation theoretical methods, possibly to be applied to the formulae (4.7). It is presumably only in special approximations (comparable to e.g. the vertex approximation of Arbuzov and Filippov [14]) that the $\Lambda\to\infty$ limit can be controlled directly. There are, of course, brute-force summation procedures, such as the Padé one, that could produce from a finite number of terms in (3.5) functions integrable in the sense required in (4.4). – A different approach to these problems is the Parisi one [15] mentioned later.

7) Causality and unitarity, i.e. validity of a nonlinear system of integral equations for the Γ_∞ such as the one of Glaser, Lehmann, and Zimmermann [16] (GLZ) (modified, if convenient, along the lines described in Refs. [17] and adapted to time-ordered functions [2]). This will require to perform the $\Lambda\to\infty$ limit carefully on the GLZ system satisfied by the Γ_Λ by virtue of their construction in terms of the Feynman rules of Section 1. Only if in the phase space integrations merely the contributions from $\theta(p_0)\delta(p^2)$ survive will indefinite metric have been avoided and a viable theory been obtained. It is to be expected that such check of causality and unitarity will also give insight into the role of the constants like $c_i(\varepsilon)$ though that role might be exhausted by the formulae (4.7). In principle, as is well known (see, e.g., Ref. [18]), any solution of the GLZ system represents a candidate for a "physical" theory. – Closely related to the check of the GLZ

equations is the problem of the precise relation to the BPHZ construction, handled in the way of Ref. [11] where broken powers, or logarithms, of the renormalized coupling constant are introduced via BPHZ subtraction coefficients. It is known [2, 16] that any formal BPHZ-type series yields a formal solution of the GLZ system or equivalent ones. In fact, without need of carrying out the details of our construction for $j>1$, the result is directly obtained by performing the limit $\Lambda\to\infty$ in Lagrangean (A.8) as described there, and is as follows:

For generic ε, the theory yielding the expansions (4.5) is described by the Lagrangean

$$L_\infty=\tfrac{1}{2}\,\partial_\mu\Phi\partial^\mu\Phi-(1/4!)\mu^{-\varepsilon}g\Phi^4$$
$$+\sum_{\substack{r=0\\ r+s\geq 3}}^{\infty}\sum_{s=1}^{\infty}\sum_{v=1}^{n_{rs}} c_{rsv}(\varepsilon)(\text{``}D^{2r}\Phi^{2s}\text{''})_v\cdot g^{s-1+(r+s-2)(2/\varepsilon)}\mu^{-\varepsilon(s-1)-2(r+s-2)}\,. \quad (5.1)$$

Here the v-sum goes over the n_{rs} scalar monomials, involving $2r$ derivatives and $2s$ factors Φ, that are linearly independent at zero momentum transfer ($n_{rs}=1$ for $r+s=3$). The operator products are ordinary ones, and the integrations with the obvious Feynman rules, all terms in the triple sum being treated as insertions, are the usual analytic ones [5]. Then (5.1) yields vertex functions Γ_∞ satisfying $\mathcal{O}\!\!\!/\,\Gamma_\infty=0$ and also the conditions (1.1a)–(1.1c). The $c_{rsv}(\varepsilon)$ are meromorphic in ε, with poles at (zero and) positive rational ε such that at $\varepsilon\to R$ positive rational, in the Γ_∞ all ε-singularities cancel:

Consider the $2s$-point vertex function. At order N, its superficial divergence will be $D_{Ns}(\varepsilon)=2[\tfrac{1}{2}\varepsilon(N-s+1)]+4-2s$. At that ε at which this divergence first appears, and thus the square bracket can be omitted, $D_{Ns}(\varepsilon)$ takes the value $2r$ which allows it to be cancelled by a singular term of order N in the triple sum in (5.1). The subdivergences being cancelled by lower-order terms in (5.1), finiteness of the Γ_∞ at $\varepsilon\to R$ can thus be brought about. The validity of $\mathcal{O}\!\!\!/\,\Gamma_\infty=0$ hereby has the consequence that each operator in the triple sum in (5.1) has as coefficient a definite power of g rather than a polynomial or power series, which means that the nonleading final divergences do not actually require subtractions.

For ε close to an integer, $\varepsilon=2$ say, one may write in (5.1)

$$g^{s-1+(r+s-2)(2/\varepsilon)}=g^{r+2s-3}+g^{r+2s-3}(g^{(r+s-2)[(2/\varepsilon)-1]}-1)$$

such that for a first-order pole in $c_{rsv}(\varepsilon)$, the first term on the r.h.s. would still give the required singularity cancellation at $\varepsilon=2$, while the second term then leads to a factor $\ln g$, as appears in (4.6), already on the Lagrangean level. (Presumably, higher-order poles also occur.) Thus, the necessity of $\ln g$ terms for consistency with the renormalization group, as observed in Ref. [11], is a direct consequence of (5.1). – The theory should admit also for integer ε a Lagrangean L_∞ to be written down, involving, however, normal products similar to Zimmermann's [7] explicitly, and leading directly to the expansions (4.6).

Not embodied in (5.1), however, is the particular choice of finite parts of the coefficients $c_{rsv}(\varepsilon)$ in (5.1) arrived at by starting from the Lagrangean (0.1) in combination with the renormalization described in Appendix B, rather than from a general nonrenormalizable Lagrangean subjected merely to renormalization group validity. Namely, the procedure of Sections 1–4 leads not only to a construction of the correct singular and, apart from induced effects, regularization independent parts of the $c_{rsv}(\varepsilon)$ but also to definite finite parts, if one assumes that the limites discussed in Section 4 and Point 6) above do exist. We shall try to discuss what is involved here and forward a conjecture.

Consider the GLZ system as described under Point 7) above, and modified by using full propagators as described in Ref. [17]. The $\Lambda \to \infty$ limit hereon will lead to a GLZ system for the Γ_∞ functions (4.5), constructed e.g. from Lagrangean (5.1), in which as propagators to be cut the functions

$$-\Gamma_\infty(p(-p); \mu, g, \varepsilon)^{-1} \equiv i(p^2 + i\varepsilon)^{-1} f(p^2 \mu^{-2} g^{2/\varepsilon}, \varepsilon)^{-1}$$

appear, the only scale for these functions, as for all Γ_∞ functions, being the mass $\mu g^{-1/\varepsilon}$. If $f(z, \varepsilon)$, which satisfies $f(0, \varepsilon) = 1$, has a first zero at $z = z_0 > 0$, then

$$-\Gamma_\infty(p(-p); \mu, g, \varepsilon)^{-1} = i(p^2 + i\varepsilon)^{-1} + i(p^2 - z_0 \mu^2 g^{-2/\varepsilon})^{-1} [z_0 f'(z_0)]^{-1} + \dots$$

i.e. there will be, due to $f'(z_0) < 0$, a negative-metric particle [or, if $f'(z_0) = 0$, a dipole ghost], and unitarity will be violated. Whether or not such a zero exists cannot be decided knowing only the expansion (4.5), since at such zero all terms in $\Gamma_\infty(p(-p); \mu, g, \varepsilon)$ will have the same order of magnitude (namely, unity). Generally, for momenta much smaller than $\mu g^{-1/\varepsilon}$, the quasi-perturbation expansion (4.5) will be usable, but not for momenta $\gtrsim \mu g^{-1/\varepsilon}$, and the specific physical content of the theory can only be seen at large momenta.

Nevertheless, there are some indications that the theory derived from (0.1) might make physical sense. What is its large-momenta limit, under conventional assumptions? In the Zinn-Justin parametrization, $\bar{\Gamma}_\infty$, which is related to Γ_∞ in the Λ-independent manner described in (B.13), (B.4) holds and suggests that if $\beta(\bar{g}, \varepsilon)$ has a first zero $\bar{g}_\infty(\varepsilon)$ with negative slope, then the theory becomes at large (nonexceptional) momenta a scale invariant one with dimension $1 + (\varepsilon/2) + \gamma(\bar{g}_\infty(\varepsilon), \varepsilon)$ for Φ and $2 + \varepsilon + \eta(\bar{g}_\infty(\varepsilon), \varepsilon)$ for Φ^2. That these dimensions are not suggested by the equivalent PDE (1.2) satisfied by the Γ_∞ is due to a pathology in the reparametrization (B.13): For the scale-invariant limit theory itself, the parametrization by (1.1) is unsuitable due to infrared divergences. [A finite $g(\bar{g}_\infty(\varepsilon), \varepsilon)$, as would be caused by a zero of $\beta(\bar{g}, \varepsilon)$ of less than first order at $\bar{g}_\infty(\varepsilon)$, would lead to a paradox related to the one Landau [19] arrived at, on the basis of effectively the assumption that the function $\beta(g)$ of a renormalizable theory is well represented by its second order term alone, for all g.]

The scale (and conformal [20]) invariant theory for $\bar{g} = \bar{g}_\infty(\varepsilon)$ is expected to be the one that one aims to construct more directly by exploiting conformal invariance ([15, 21], and references given there) for integer and, by suitable analytic interpolation, also generic ε. The $g < \infty$ or $\bar{g} < \bar{g}_\infty(\varepsilon)$ theory would be the corresponding praeasymptotic [22] one.

If $\gamma(\bar{g}(g, \varepsilon), \varepsilon)$ for $g \to \infty$ is bounded away from minus one (it should, for real ε, have a nonnegative limit value if the discussion before applies), then, because of $c_i(\mu, g, \infty, \varepsilon) = 0$ in (3.1) due to the assumed existence of the Γ_∞ and $\Gamma_{\infty i}$, (3.3) yields $e_i(\mu, g, \infty, \varepsilon) = 0$ for $i = 4, 5, 6$, and then (2.9) yields $d_i(\mu, g, \infty, \varepsilon) = 0$ for $i = 4, 5, 6$. This means that the counting identity (2.3) and the DVO [8] identity (2.6) take the form they would in regularization – free Φ^4 theory. We take this as a support of the conjecture that the "noncomputable" constants arrived at from (0.1) lead to a more physical theory than one with arbitrary, albeit renormalization-group-conform [11], "noncomputable" constants in (5.1).

If one would have started with a (nonlocal) Lagrangean with a space-momentum cutoff, or with a Euclidean "Lagrangean" (exponent in a function-space integral) with absolute-momentum cutoff [23], the method of this paper becomes unavailable since for the derivatives with respect to the cutoff no simple formulae like (3.1) exist, and thus an expansion like (0.2) is not valid. Still, the renormalization procedure discussed in Appendix B can be carried out (at least in the covariant case), since the functions $g(\bar{g}, \varepsilon)$, $\gamma(\bar{g}, \varepsilon)$, $\eta(\bar{g}, \varepsilon)$, and $\kappa(\bar{g}, \varepsilon)$ remain computable from formulae (B.11a)–(B.11d), as discussed there, without reference to the precise manner of the cutoff. The question of the $\Lambda \rightarrow \infty$ limit can be posed, and since (at least in the noncovariant case) no indefinite metric is involved, one would not expect indefinite metric to appear if that limit does exist. This argument merely is to make plausible the possibility of a choice of constants in (5.1) (or, at least, in its integer-ε form) such that indefinite metric is avoided.

Finally, assuming that in particular the unitarity conjecture holds, the question must be answered as to what the theory might be useful for. It appears certain that the large-momenta behaviour, also on the mass shell, of the terms in the double series expansion obtained from (4.6) violates bounds such as Froissart's [24] the more strongly the higher the order. A meaningful theory would require a summation to be performed, by which e.g. acceptable high-energy behaviour should be brought about. A nonrenormalizable theory with the multitude of arbitrary parameters as e.g. the BPHZ method introduces, as explained above in detail, appears to have little chance to yield a meaningful result with any summation procedure (unless it is the summation procedure itself that enforces physically acceptable behaviour, in which case that procedure would outmode Lagrangean quantum field theory). The more the arbitrary constants are constrained, the better are the chances for imaginable summation procedures to be successful. The $\Lambda \rightarrow \infty$ procedure discussed here appears to be the most efficient systematic way to obtain meaningful constraints and, as we discussed, to lead to a unitary theory.

Mentioning summation procedures, Parisi's proposals [15] for constructing nonrenormalizable theories come to mind. These proposals are predominantly made for theories that are "asymptotically free" in some space-time dimension, such that they can be expected to have an UV fixed point close to the origin if the number of dimensions is increased somewhat. (In the case relevant here, $(\Phi^4)_4$ theory, $g<0$ would be required [25].) While Parisi's methods, which rely on anomalous dimensions, might lead more easily to physically meaningful approximations, the method appears restricted in applicability, and its principles (cp. also Ref. [26]) have not yet been described in detail. However, Parisi's idea [15] that the problems of massless superrenormalizable theories and of nonrenormalizable theories are technically related is the backbone also of our approach.

Lee's work [27] is, like ours, based on the principle of using a cutoff and obtaining nonregular behaviour in the coupling constant in the process of cutoff removal. We are able, in the framework of our approach, to confirm arguments advanced by Lee in connection with the ξ-limiting procedure, whereby ξ is analogous to Λ^{-2} in our case. (See the end of Appendix A).

The method of the present paper applies to all theories that are strictly renormalizable in some space-time dimension, provided the continous increase of dimension does not meet (e.g. the well-known γ_5) difficulties. In particular, $((\bar{\psi}\psi)^2)_{2+2}$ theories would be of greater physical interest than $(\Phi^4)_{4+1}$ and $(\Phi^4)_{4+2}$. The method might apply also to nonpolynomial chiral invariant theories [28] in 2+2 dimensions. It would also be instructive to compare $(\Phi^6)_{3+1}$ theory (e.g., with respect to large-momenta behaviour) with $(\Phi^4)_4$ theory.

Acknowledgment. That the method the author used to analyze the $m\to 0$ limit of $(\Phi^4)_{4-\varepsilon}$ theory [6] might be applicable to nonrenormalizable theory was suggested to him by G. Parisi some time ago. A discussion with J. Zinn-Justin, in which he explained to the author his results on the lattice-cutoff dependence of Green's function in statistical mechanics proved to be most helpful and stimulating. The author is greatly indebted to G. Parisi and J. Zinn-Justin for these discussions. He also thanks H. Trute for discussions.

Appendix A

Vertex Functions Expansions for Large Cutoff

In this appendix we derive the expansions (0.2) for $\Gamma_{AB}((2n),(l);g_B,\varepsilon)$. For brevity, we shall only consider $l=0$ and shall furthermore start from two properties of Γ_{AB}, which will be proven elsewhere:

1) For large Λ, graphs contributing to Γ_{AB} possess an asymptotic expansion with terms proportional $\Lambda^{-2j+\varepsilon k}(\ln\Lambda)^l$, with j,k,l integers, $j\geqq 0, 0\leqq k\leqq \mathscr{L}$(the number of loops), and $l\geqq 0$ bounded, and coefficients meromorphic in ε.

2) The $k\doteq 0$ terms in this expansion have $l=0$ only, and are obtained by treating the term $-\frac{1}{2}\Lambda^{-2}\phi\square^2\phi$ in (0.1a) as a perturbation, i.e. inserting it j times into the graphs defined by Feynman rules without regularization, and integrating analytically. Both these properties will be used only very weakly, i.e. they will appear self-evident at the end of this appendix.

Concerning 2), we add that vertex functions with insertions as described obey

$$\Gamma(00;g_B,\varepsilon)=0\,, \tag{A.1a}$$

$$[\partial/\partial p^2]\Gamma(p(-p);g_B,\varepsilon)|_{p=0}=i\,, \tag{A.1b}$$

$$\Gamma(0000;g_B,\varepsilon)=-ig_B\,, \tag{A.1c}$$

since for nonrational $\varepsilon>0$ all corrections to the lowest-order terms on the r.h.s. vanish for dimensional reasons.

Starting from the Lagrangean (0.1a) with the self mass term omitted, we construct renormalized vertex functions as follows: We determine the superficial divergence degree ϑ of any vertex function graph as if it were to be computed in $4+E$ dimensions, E real positive nonrational, and there were no regularisation. This yields $\vartheta=4-2n+\mathscr{L}E$.We now insert into the Lagrangian the counter terms for corresponding minimal subtraction at the origin of momentum space. If $\mathrm{Re}\,\varepsilon$ is sufficiently close to E, these subtractions yield vertex functions that admit $\Lambda\to\infty$. (We use here an extension of Zimmermann's convergence proof [1] to complex dimension, which has been discussed by several authors, e.g., [29], and to massless propagators, see [30].) These subtractions also leave the vertex functions IR convergent: the counter terms are, due to minimality, UV divergent for $\Lambda\to\infty$ and thus IR convergent for dimensional reasons.

The Lagrangean yielding these vertex functions can be written

$$\begin{aligned} L_{\Lambda,E} = &-\tfrac{1}{2}\phi\Box(1+\Lambda^{-2}\Box)\phi-(1/4!)g_B\phi^4 \\ &-\textstyle\sum_{s=1}^{\infty}\sum_{\mathscr{L}=1}^{\infty}\sum_{r=0}^{2-s+[\frac{1}{2}E\mathscr{L}]}\sum_{v=1}^{n_{rs}} f_{rs\mathscr{L}v}(\varepsilon,E) \\ &\cdot(\text{"}D^{2r}\phi^{2s}\text{"})_v g_B^{s-1+\mathscr{L}}\Lambda^{4-2r-2s+\varepsilon\mathscr{L}}\,. \end{aligned} \tag{A.2}$$

The v-sum goes over the n_{rs} scalar monomials, involving $2r$ derivatives and $2s$ factors ϕ, that are linearly independent at zero momentum transfer. Each $f_{rs\mathscr{L}v}(\varepsilon, E)$ is piecewise constant in E in intervals of length $2\mathscr{L}^{-1}$, and is holomorphic in ε in the strip

$$2\mathscr{L}^{-1}[\tfrac{1}{2}E\mathscr{L}]<\mathrm{Re}\,\varepsilon<2\mathscr{L}^{-1}([\tfrac{1}{2}E\mathscr{L}]+1)\,.$$

For ε to the left of this strip, $f_{rs\mathscr{L}v}(\varepsilon, E)$ is meromorphic with (IR) singularities at $\varepsilon-2\mathscr{L}^{-1}[\frac{1}{2}E\mathscr{L}]=0,\ -2\mathscr{L}^{-1},\ -4\mathscr{L}^{-1}\dots$.

Moreover, for $E>2\mathscr{L}^{-1}[\frac{1}{2}\mathscr{L}\,\mathrm{Re}\,\varepsilon]$, $f_{rs\mathscr{L}v}(\varepsilon, E)$ is independent of E. This is due to the fact that oversubtraction on subgraphs, the origin of E-dependence, does not affect the counter terms in the final, analytically [5] computed, integration, again for dimensional reasons.

The vertex functions computed with Lagrangean (A.2) we denote as $\Gamma_{\Lambda,E}$. [The functions $\Gamma_{\Lambda,0}((2n);g\mu^{-\varepsilon},\varepsilon)$ will be denoted $\Gamma_\Lambda((2n),;\mu,g,\varepsilon)$ in Section 1.] They are meromorphic in ε.

The properties 1) and 2) described at the beginning imply that for $E>\mathrm{Re}\,\varepsilon$, due to the existence of the $\Lambda\to\infty$ limit,

$$\Gamma_{\Lambda,E}((2n);g_B,\varepsilon)=\sum_{j=0}^{\infty}\Lambda^{-2j}f_{j0}((2n);g_B,\varepsilon)+\sum_{\substack{j,k\\ -2j+Ek<0\\ k\geq 1}}0(\Lambda^{-2j+\varepsilon k})\,. \tag{A.3}$$

Here the $0(\Lambda^{-2j+\varepsilon k})$ are a sum of terms having factors $\Lambda^{-2j+\varepsilon k}(\ln\Lambda)^l$, with coefficients meromorphic in ε. The f_{j0} terms in (A.3) are the ones in (0.2) since the quadruple sum in (A.2) involves only terms with factors $\Lambda^{-2j+\varepsilon k}$, $k\geq 1$, and graphs with ("$D^{2r}\phi^{2s}$") insertions have properties analogous to 1), 2) before.

We now can write for the Lagrangean (0.1)

$$\begin{aligned} L=&L_{\Lambda,E}-\tfrac{1}{2}m_{B0}^2\phi^2 \\ &+\textstyle\sum_{r=0}^{\infty}\sum_{s=1}^{\infty}\sum_{v=1}^{n_{rs}} f_{rsv}(g_B\Lambda^{\varepsilon},\varepsilon,E)(\text{"}D^{2r}\phi^{2s}\text{"})_v\, g_B^{s-1}\Lambda^{4-2r-2s}\,, \end{aligned} \tag{A.4a}$$

where

$$f_{rsv}(g_B\Lambda^{\varepsilon},\varepsilon,E)\equiv\textstyle\sum_{\mathscr{L}=\mathrm{Max}(1,-[2E^{-1}(2-r-s)])}^{\infty} f_{rs\mathscr{L}v}(\varepsilon,E)(g_B\Lambda^{\varepsilon})^{\mathscr{L}}\,. \tag{A.4b}$$

For E sufficiently large, in the meaning explained later, (A.3) becomes

$$L_{\Lambda,E}\xrightarrow[E\to\infty]{}L_{\Lambda,\infty}=-\tfrac{1}{2}\phi\Box\phi-(1/4!)g_B\phi^4-\tfrac{1}{2}\Lambda^{-2}\phi\Box^2\phi\,. \tag{A.5}$$

The difference between $L_{\Lambda,\infty}$ and L of (0.1a), disregarding the mass term, is that in (A.5), the last term is meant as insertion into the Lagrange function, to be treated as a perturbation i.e. as repeated insertion into vertex functions, while in (0.1a) that term modifies the bare propagator. Thus, $L_{\Lambda,\infty}$ requires no self mass counter terms, and indeed its vertex functions, computed analytically [5], satisfy (A.1a)–(A.1c), while for the Γ_{AB} the quite different relations (B.11a) and (B.11b) hold, and $\Gamma_{AB}(00,;g_B,\varepsilon)=0$ requires the self mass term (0.1b).

Using (A.5) in (A.4a), and in view of the $E-$ independence of the $f_{rs\mathscr{L}v}(\varepsilon,E)$ for E large, we can write

$$L\cong L_\Lambda=-\tfrac{1}{2}\phi\Box\phi-(1/4!)g_B\phi^4-\tfrac{1}{2}m_{B0}^2\phi^2-\tfrac{1}{2}\Lambda^{-2}\phi\Box^2\phi$$
$$+\sum_{r=0}^{\infty}\sum_{s=1}^{\infty}\sum_{v=1}^{n_{rs}}f_{rsv}(g_B\Lambda^\varepsilon,\varepsilon,\infty)(\text{“}D^{2r}\phi^{2s}\text{”})_v g_B^{s-1}\Lambda^{4-2r-2s} \tag{A.6}$$

whereby the first two terms on the r.h.s. define the Feynman rules and all other terms are to be treated as insertions. It is this proviso which demasks L_Λ as an effective Lagrangean, equivalent to the genuine Lagrangean L only in the limited sense of yielding the asymptotic expansions (0.2) directly. Hereby we must have

$$m_{B0}^2=2\Lambda^2 f_{011}(g_B\Lambda^\varepsilon,\varepsilon,\infty) \tag{A.7}$$

since all other insertions, upon analytic computation, leave self energy parts vanishing at zero momentum, for dimensional reasons.

We now explain what was meant by “E sufficiently large”. We want to compute the expansion (0.2) from Lagrangean (A.6) to a prescribed accuracy. We should more correctly start from (A.4a); hereby insertions are to be made in the vertex functions (A.3). Since the counter terms give rise to at most a factor $\Lambda^{(\mathscr{L}-1)\varepsilon}$, in view of (A.7), using them as insertions in “correction graphs” yields at most a factor $\Lambda^{(\mathscr{L}-1)\varepsilon-2j+\varepsilon k}$ with $-2j+Ek<0, k\geqq 1$. Thus the factor is smaller than $\Lambda^{(\mathscr{L}-1)\varepsilon-(E-\varepsilon)k}$, which is arbitrarily small for large E due to $k\geqq 1$. This means that (A.6) is adequate to generate the whole expansion (0.2) correctly.

We will now use results of Appendix B to rewrite (A.6). The passage from Γ_{AB} via (B.1) to $\bar{\Gamma}_\Lambda$ and from there via (B.13) to Γ_Λ amounts to introducing renormalized operators $\phi_{\text{ren}}=Z_3^{-1/2}\phi$ with [using (B.9b)]

$$Z_3\equiv\exp\left[-2\int_0^{g_B\Lambda^\varepsilon}d\bar{g}\beta(\bar{g},\varepsilon)^{-1}\gamma(\bar{g},\varepsilon)\right]\equiv Z_3(g_B\Lambda^\varepsilon,\varepsilon)$$

and to write (B.9a) $\quad g_B=\Lambda^{-\varepsilon}\bar{g}(g\Lambda^\varepsilon\mu^{-\varepsilon},\varepsilon)$.
Then we obtain

$$L_\Lambda=-\tfrac{1}{2}\phi_{\text{ren}}\Box\phi_{\text{ren}}-(1/4!)g\mu^{-\varepsilon}\phi_{\text{ren}}^4$$
$$+\sum_{\substack{r=0\\r+s\geqq 3}}^{\infty}\sum_{s=1}^{\infty}\sum_{v=1}^{n_{rs}}c_{rsv}(g\Lambda^\varepsilon\mu^{-\varepsilon},\varepsilon)(\text{“}D^{2r}\phi_{\text{ren}}^{2s}\text{”})_v\cdot g^{s-1+(r+s-2)(2/\varepsilon)}\mu^{-\varepsilon(s-1)-2(r+s-2)} \tag{A.8}$$

$$Z_3(t,\varepsilon)[1-2f_{111}(t,\varepsilon,\infty)]=1\,, \tag{A.9a}$$

$$tZ_3(t,\varepsilon)^2[1-4!f_{021}(t,\varepsilon,\infty)]=g(t,\varepsilon)\,, \tag{A.9b}$$

and set

$$f_{rsv}(t,\varepsilon,\infty)Z_3(t,\varepsilon)^s t^{s-1}g(t,\varepsilon)^{1-s+(2-r-s)(2/\varepsilon)}$$
$$-\tfrac{1}{2}\delta_{r2}\delta_{s1}Z_3(t,\varepsilon)g(t,\varepsilon)^{-2/\varepsilon}\equiv c_{rsv}(g(t,\varepsilon),\varepsilon)\,,\qquad r+s\geqq 3\,. \tag{A.10}$$

The identifications (A.9a) and (A.9b) are necessary for the conditions (1.1b) and (1.1c) of Section 1 to be satisfied, as (A.7) was necessary for (1.1a) to hold. [In writing (A.9a) we have chosen "$D^2\phi^2$" $=\phi\Box\phi$.] The coefficients c_{211}, c_{121}, and c_{031} in (A.8) are closely related to c_4, c_5, and c_6, respectively, of Eq. (3.2).

It is now manifest that the limit L_∞ of (A.8) yields (5.1) [ϕ_{ren} in (A.8) is the ϕ of Sections 1 through 5] if $\lim_{t\to\infty} c_{rsv}(t,\varepsilon)=c_{rsv}(\varepsilon)$. The condition for existence of this limit can be written in a similar way as (4.4), by differentiating with respect to t, reintegrating to infinity and splitting the integration region into two parts.

Formula (A.6) allows to read off the composition of the contributions to $\Gamma_{AB}((2n),; g_B, \varepsilon)$ in (0.2) that are "leading", "next-to-leading" etc., i.e. that have in a given order of g_B the highest, next-to-highest ect. power of Λ. For ε such that degeneracies occur, there will in addition result $\ln\Lambda$, $(\ln\Lambda)^2$ etc. factors as discussed in the Introduction.

In order to obtain (0.3) from (A.8), one writes

$$c_{rsv}(g\Lambda^\varepsilon\mu^{-\varepsilon},\varepsilon)=c'_{rsv}(g\Lambda^\varepsilon\mu^{-\varepsilon},\varepsilon)g^{-(r+s-2)(2/\varepsilon)}\mu^{2(r+s-2)}\Lambda^{4-2r-2s},$$

where c'_{rsv} is an (integer) power series in its first argument. The leading, next-to-leading etc. terms can then be identified as for (A.6). One observes that for $\Lambda\to\infty$ all terms, including highly nonleading ones, survive to yield (5.1). However, to generate the expansion (4.5) up to a given power of g, only the corresponding "not too nonleading" terms of the rewritten (A.8) need be kept. A similar remark applies for $\varepsilon=1$ or 2 to (4.6). This confirms arguments advanced by Lee [27] in connection with the ξ-limiting procedure.

Appendix B
Zinn-Justin Parametrization

To the vertex function (0.2) calculated directly from Lagrangean (0.1) we associate a renormalized vertex-function $\bar\Gamma_\Lambda$ by

$$\begin{aligned}\bar\Gamma_\Lambda((2n),(l);\mu,\bar g,\varepsilon)&\\ &=Z_3(\bar g,\Lambda/\mu,\varepsilon)^n Z_2(\bar g,\Lambda/\mu,\varepsilon)^l\Gamma_{\Lambda B}((2n),(l);g_B(\bar g,\mu,\Lambda,\varepsilon),\varepsilon)\\ &\quad-i\delta_{n0}\delta_{l2}\Lambda^\varepsilon Z_2(\bar g,\Lambda/\mu,\varepsilon)^2K(\bar g,\Lambda/\mu,\varepsilon)\,.\end{aligned}\tag{B.1}$$

Herein we set

$$g_B(\bar g,\mu,\Lambda,\varepsilon)=\mu^{-\varepsilon}\{\bar g+\textstyle\sum_{k=1}^\infty[(\Lambda/\mu)^{\varepsilon k}-1]a_k(\bar g,\varepsilon)\}\,,\tag{B.2a}$$

$$Z_3(\bar g,\Lambda/\mu,\varepsilon)=1+\textstyle\sum_{k=1}^\infty[(\Lambda/\mu)^{\varepsilon k}-1]z_{3k}(\bar g,\varepsilon)\,,\tag{B.2b}$$

$$Z_2(\bar g,\Lambda/\mu,\varepsilon)=1+\textstyle\sum_{k=1}^\infty[(\Lambda/\mu)^{\varepsilon k}-1]z_{2k}(\bar g,\varepsilon)\,,\tag{B.2c}$$

$$K(\bar g,\Lambda/\mu,\varepsilon)=\mu^\varepsilon\Lambda^{-\varepsilon}\textstyle\sum_{k=1}^\infty[(\Lambda/\mu)^{\varepsilon k}-1]b_k(\bar g,\varepsilon)\,,\tag{B.2d}$$

with functions a_k, z_{3k}, z_{2k}, b_k chosen uniquely (as power series in $\bar g$) such that (except for $n=0, l=1$) in the Λ-expansion analogous to (0.3) no $\Lambda^{\varepsilon k}, k>0$, terms arise:

$$\bar\Gamma_\Lambda((2n),(l);\mu,\bar g,\varepsilon)=\bar h_{00}((2n),(l);\mu,\bar g,\varepsilon)+\textstyle\sum_{j=1}^\infty\sum_{k=0}^\infty\Lambda^{-2j+\varepsilon k}\bar h_{jk}((2n),(l);\mu,\bar g,\varepsilon)\,.\tag{B.3}$$

That this is possible is implied in the demonstration in Appendix A.

Applying $\mu[\partial/\partial\mu]|_{g_B,\Lambda}$ to (B.1) yields

$$\mathcal{O}\!\!\!/\,_{2n,l}\bar{\Gamma}_\Lambda((2n),(l);\mu,\bar{g},\varepsilon)=-i\delta_{n0}\delta_{l2}\mu^\varepsilon\kappa(\bar{g},\varepsilon) \tag{B.4a}$$

with

$$\mathcal{O}\!\!\!/\,_{2n,l}\equiv\mu[\partial/\partial\mu]+\beta(\bar{g},\varepsilon)[\partial/\partial\bar{g}]-2n\gamma(\bar{g},\varepsilon)+l\eta(\bar{g},\varepsilon)\,, \tag{B.4b}$$

where

$$\begin{aligned}\beta(\bar{g},\varepsilon)&=\mu[\partial/\partial\mu]\bar{g}|_{g_B,\Lambda}\\ &=-\{[\partial/\partial\bar{g}]g_B|_{\mu,\Lambda}\}^{-1}\mu[\partial/\partial\mu]g_B|_{\bar{g},\Lambda}=\varepsilon\bar{g}+b_0(\varepsilon)\bar{g}^2+\ \dots\,,\end{aligned} \tag{B.5a}$$

$$\gamma(\bar{g},\varepsilon)=\tfrac{1}{2}\,\mu[\partial/\partial\mu]\ln Z_3|_{g_B,\Lambda}=\tfrac{1}{2}\,\mathcal{O}\!\!\!/\,_{00}\ln Z_3=c_0(\varepsilon)\bar{g}^2+\dots\,, \tag{B.5b}$$

$$\eta(\bar{g},\varepsilon)=-\,\mu[\partial/\partial\mu]\ln Z_2|_{g_B,\Lambda}=-\,\mathcal{O}\!\!\!/\,_{00}\ln Z_2=h_0(\varepsilon)\bar{g}+\dots\,, \tag{B.5c}$$

and

$$\kappa(\bar{g},\varepsilon)=\mu^{-\varepsilon}\Lambda^\varepsilon Z_2^2\mu[\partial/\partial\mu]K|_{g_B,\Lambda}=\mu^{-\varepsilon}\Lambda^\varepsilon Z_2^2\mathcal{O}\!\!\!/\,_{00}K=k_0(\varepsilon)+k_1(\varepsilon)\bar{g}+\dots\,. \tag{B.5d}$$

The Λ-independence of the functions β,γ,η, and κ follows thereby from (B.2) and (B.4): the first implies that no factors $\Lambda^{-2j+\varepsilon k}$, $j>0$, may occur, and the second, with (B.3), excludes the occurrence of $\Lambda^{\varepsilon k}$, $k>0$. – This argument was first used by Zinn-Justin [31] in the case $\varepsilon=0$, where $\ln\Lambda$ replaces Λ^ε, and later extended [32] to $\varepsilon<0$.

We now introduce the function

$$g(\bar{g},\varepsilon)\equiv\bar{g}\exp\{\varepsilon\textstyle\int_0^{\bar{g}}d\bar{g}'[\beta(\bar{g}',\varepsilon)^{-1}-(\varepsilon\bar{g}')^{-1}]\} \tag{B.6}$$

which obeys

$$[\partial/\partial\bar{g}]g(\bar{g},\varepsilon)=\varepsilon g(\bar{g},\varepsilon)\beta(\bar{g},\varepsilon)^{-1}\,, \tag{B.7a}$$

and

$$g(\bar{g},\varepsilon)=\bar{g}-\varepsilon^{-1}b_0(\varepsilon)\bar{g}^2+\dots\,. \tag{B.7b}$$

It is a monotonically increasing function of $\bar{g}$ in the interval $0<\bar{g}<\bar{g}_\infty(\varepsilon)$ where $\bar{g}_\infty(\varepsilon)$ is the first positive zero of $\beta(\bar{g},\varepsilon)$, if any, and we will only consider $\bar{g}$ in this interval. The inverse function we denote by $\bar{g}(g,\varepsilon)$:

$$\bar{g}(g,\varepsilon)=g+\varepsilon^{-1}b_0(\varepsilon)g^2+\dots\,. \tag{B.8}$$

Then from (B.5), together with the boundary conditions incorporated in the forms (B.2), follows

$$g_B=\Lambda^{-\varepsilon}\bar{g}(g(\bar{g},\varepsilon)\Lambda^\varepsilon\mu^{-\varepsilon},\varepsilon)\,, \tag{B.9a}$$

$$Z_3=\exp[2\textstyle\int_{g_B\Lambda^\varepsilon}^{\bar{g}}d\bar{g}'\beta(\bar{g}',\varepsilon)^{-1}\gamma(\bar{g}',\varepsilon)]=\exp[2\varepsilon^{-1}\int_{g(\bar{g},\varepsilon)\Lambda^\varepsilon\mu^{-\varepsilon}}^{g(\bar{g},\varepsilon)}dg\,g^{-1}\eta(\bar{g}(g,\varepsilon),\varepsilon)]\,, \tag{B.9b}$$

$$Z_2=\exp[-\textstyle\int_{g_B\Lambda^\varepsilon}^{\bar{g}}d\bar{g}'\beta(\bar{g}',\varepsilon)^{-1}\eta(\bar{g}',\varepsilon)]=\exp[-\varepsilon^{-1}\int_{g(\bar{g},\varepsilon)\Lambda^\varepsilon\mu^{-\varepsilon}}^{g(\bar{g},\varepsilon)}dg\,g^{-1}\eta(\bar{g}(g,\varepsilon),\varepsilon)]\,, \tag{B.9c}$$

and

$$K=g(g_B\Lambda^\varepsilon,\varepsilon)^{-1}\int_{g_B\Lambda^\varepsilon}^{\bar g} d\bar g'\beta(\bar g',\varepsilon)^{-1}g(\bar g',\varepsilon)\kappa(\bar g',\varepsilon)\exp\left[2\int_{g_B\Lambda^\varepsilon}^{\bar g'} d\bar g''\beta(\bar g'',\varepsilon)\eta(\bar g'',\varepsilon)\right]$$
$$=\varepsilon^{-1}\Lambda^{-\varepsilon}\mu^\varepsilon g(\bar g,\varepsilon)^{-1}$$
$$\cdot\int_{g(\bar g,\varepsilon)\Lambda^\varepsilon\mu^{-\varepsilon}}^{g(\bar g,\varepsilon)} dg\kappa(\bar g(g,\varepsilon),\varepsilon)\exp\left[2\varepsilon^{-1}\int_{g(\bar g,\varepsilon)\Lambda^\varepsilon\mu^{-\varepsilon}}^{g} dg'(g')^{-1}\eta(\bar g(g',\varepsilon),\varepsilon)\right]. \quad \text{(B.9d)}$$

Applying $\Lambda[\partial/\partial\Lambda]|_{\mu,\bar g}$ to (B.1) and using (B.3) and (B.9) yields

$$\{\Lambda[\partial/\partial\Lambda]|_{g_B\Lambda^\varepsilon}+\beta(g_B\Lambda^\varepsilon,\varepsilon)[\partial/\partial(g_B\Lambda^\varepsilon)]|_\Lambda$$
$$-2n\gamma(g_B\Lambda^\varepsilon,\varepsilon)+l\eta(g_B\Lambda^\varepsilon,\varepsilon)\}\Gamma_{\Lambda B}((2n),(l);g_B,\varepsilon)+i\delta_{n0}\delta_{l2}\Lambda^\varepsilon\kappa(g_B\Lambda^\varepsilon,\varepsilon)$$
$$=Z_3^{-n}Z_2^{-l}\Lambda[\partial/\partial\Lambda]|_{\mu,\bar g}\bar\Gamma_\Lambda((2n),(l);\mu,\bar g,\varepsilon)=0(\Lambda^{-2+\varepsilon k}), \quad \text{(B.10)}$$

the PDE for unrenormalized vertex functions due to Zinn-Justin [31, 32]. For $\varepsilon\leqq 0$, the r.h.s. in (B.10) vanishes for $\Lambda\to\infty$ (we do not discuss the IR problem [6] here); for $\varepsilon>0$ it vanishes in this limit if the assumption of Section 5 applies.

It follows easily from (B.10) and regularity in $g_B\Lambda^\varepsilon$ that

$$\Gamma_{\Lambda B}(0000,;g_B,\varepsilon)=-i\Lambda^{-\varepsilon}g(g_B\Lambda^\varepsilon,\varepsilon)\exp\left[4\int_0^{g_B\Lambda^\varepsilon} d\bar g\beta(\bar g,\varepsilon)^{-1}\gamma(\bar g,\varepsilon)\right], \quad \text{(B.11a)}$$

$$[\partial/\partial p^2]\Gamma_{\Lambda B}(p(-p),;g_B,\varepsilon)|_{p=0}=i\exp\left[2\int_0^{g_B\Lambda^\varepsilon} d\bar g\beta(\bar g,\varepsilon)^{-1}\gamma(\bar g,\varepsilon)\right], \quad \text{(B.11b)}$$

$$\Gamma_{\Lambda B}(00,0;g_B,\varepsilon)=\exp\left\{\int_0^{g_B\Lambda^\varepsilon} d\bar g\beta(\bar g,\varepsilon)^{-1}[2\gamma(\bar g,\varepsilon)-\eta(\bar g,\varepsilon)]\right\}, \quad \text{(B.11c)}$$

and

$$\Gamma_{\Lambda B}(,00;g_B,\varepsilon)=-i\Lambda^\varepsilon[g(g_B\Lambda^\varepsilon,\varepsilon)]^{-1}$$
$$\cdot\int_0^{g_B\Lambda^\varepsilon} d\bar g\beta(\bar g,\varepsilon)^{-1}g(\bar g,\varepsilon)\kappa(\bar g,\varepsilon)\exp\left[2\int_{g_B\Lambda^\varepsilon}^{\bar g} d\bar g'\beta(\bar g',\varepsilon)^{-1}\eta(\bar g',\varepsilon)\right], \quad \text{(B.11d)}$$

where $g(\cdot,\varepsilon)$ is the function (B.6), and we have used that (B.3) also holds for the momenta sets appearing in (B.11) on the l.h. sides since the, for $\varepsilon=0$ only logarithmic, IR divergences disappear in $4+\varepsilon$ dimensions. These l.h. sides are computable in a relatively straightforward way for $0<\varepsilon<3$, and thus also the functions $\beta(\cdot,\varepsilon)$, $\gamma(\cdot,\varepsilon)$, $\eta(\cdot,\varepsilon)$, and $\kappa(\cdot,\varepsilon)$ are; moreover, these are manifestly regular in ε (at least in their power series expansion in $\bar g$) for $0<\varepsilon<3$, and actually exist also for $\varepsilon=0$.

It now follows from (B.1) that also the functions $\bar\Gamma_\Lambda$ themselves are regular in $0<\varepsilon<3$, since the functions $\Gamma_{\Lambda B}$ are and the substitution for g_B and multiplication by factors Z_3 and Z_2 does not introduce singularities there as follows from their forms (B.9a)–(B.9c) and the just proved regularity of the parametric functions therein. This also holds in the case $n=0$, $l=2$. The functions $\bar\Gamma_\Lambda$ exist, as do the parametric functions β,γ,η, and κ, also for $\varepsilon=0$; in this case (B.9a) must be replaced by its $\varepsilon\to 0$ limit

$$g_B=\varrho^{-1}[\ln(\Lambda/\mu)+\varrho(\bar g)]=\exp\{\ln(\Lambda/\mu)\beta(\bar g,0)[\partial/\partial\bar g]\}\bar g, \quad \text{(B.12a)}$$

where $\varrho(\bar g)$ is the monotonically increasing function

$$\varrho(\bar g)\equiv\int^{\bar g} d\bar g'\beta(\bar g',0)^{-1}. \quad \text{(B.12b)}$$

Finally, we perform the finite (i.e., Λ-independent) renormalization

$$\Gamma_\Lambda((2n),(l);\mu,g,\varepsilon)$$
$$=\exp\{\int_0^{\bar g(g,\varepsilon)} d\bar g'\beta(\bar g',\varepsilon)^{-1}[-2n\gamma(\bar g',\varepsilon)+l\eta(\bar g',\varepsilon)]\}\,\bar\Gamma_\Lambda((2n),(l);\mu,\bar g(g,\varepsilon),\varepsilon)$$
$$+i\delta_{n0}\delta_{l2}\mu^\varepsilon g^{-1}\varepsilon^{-1}\int_0^g dg'\kappa(\bar g(g',\varepsilon),\varepsilon)\exp[2\varepsilon^{-1}\int_0^{g'} dg''(g'')^{-1}\eta(\bar g(g'',\varepsilon),\varepsilon)]\,. \tag{B.13}$$

One easily verifies that these functions obey the conditions (1.1) and the PDE (1.2) of Section 1. Thus, the Γ_Λ are identical with the functions defined there by Feynman rules, since also the bare propagators that introduce Λ, by virtue of our convention concerning the implementation of (1.1b), are the same. The regularity of the Γ_Λ for $0<\varepsilon<3$, which actually extends till $\varepsilon<4$, proven in this appendix confirms the deduction of this property from the Feynman rules in Section 1.

We add two remarks:

1) The $\varepsilon\to+0$ limit in (B.11a)–(B.11d) is subtle: the l.h. sides do not have limites termwise in the power series expansion in $g_B\Lambda^\varepsilon$ since the coefficients become infrared divergent (like $\varepsilon^{-1}, \varepsilon^{-2}$ etc.). This can be seen by writing out the r.h. sides, whereby the $\varepsilon\bar g$ term in (B.5a) is crucial. The same term, however, allows a resummation on the r.h. sides of (B.11a)–(B.11d) such that for $\varepsilon=+$ r.h. sides regular in g_B are obtained, without Λ dependence for dimensional reasons. [Only the manner of cutoff remains reflected in the higher terms of the $\beta, \gamma, \eta, \kappa$ functions. It is presupposed that $g_B\Lambda^\varepsilon<\bar g_\infty(\varepsilon)$, and $g_B<\bar g_\infty(0)$ for $\varepsilon=0$.] A representative example of such resummation is the one of the function $g(\bar g,\varepsilon)$ in (B.6) and (B.7b). Let (B.5a) be

$$\beta(\bar g,\varepsilon)=\varepsilon\bar g+b_0\bar g^2+\varepsilon b_{01}\bar g^2+b_1\bar g^3+0(\varepsilon^2\bar g^2,\varepsilon\bar g^3,\bar g^4)\,, \tag{B.14}$$

where b_0 and b_1 are the usual universal ones but b_{01} is convention dependent and here depends on the particular way of regularisation. Then

$$g(\bar g,\varepsilon)=b_0^{-1}\varepsilon+b_0^{-3}b_1\varepsilon^2\ln(\varepsilon/b_0)+\varepsilon^2(b_0^{-3}b_1-b_0^{-2}b_{01})$$
$$+b_0^{-1}\varepsilon^2(2\pi i)^{-1}\int_{\bar g+i0}^{\bar g-i0} dz\beta(z,0)^{-1}\ln(-z)+O(\varepsilon^3\ln\varepsilon,\varepsilon^3)\,, \tag{B.15}$$

where the integration path encircles the segment $[0,\bar g]$in the z-plane counter-clockwise but no poles of $\beta(z,0)^{-1}$ other than the double one at the origin. It is the ε^2-terms in (B.15) that lead from (B.9a) for $\varepsilon=+0$ to (B.12). – Other apparent paradoxes than the one in (B.11) for $\varepsilon=+0$ arise upon interchange of the limits $\Lambda\to\infty$ and $\varepsilon\to+0$.

2) If one uses a regularisation more general than the one in (0.1), e.g. by absolute-momentum cutoff [23], then the double sum in (B.3) must be replaced by $O(\Lambda^{-2+\varepsilon k})$, and terms with non-power Λ-dependence, e.g. involving logarithms of Λ or oscillatory factors, can be expected in the asymptotic expansions. (B.10) remains valid as it stands, however [31, 32]. For the functions (B.11a)–(B.11d) the given formulae remain valid since the r.h.s. in (B.10) vanishes identically in these cases for reasons of regularity in $g_B\Lambda^\varepsilon$.

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