XII. Classical aspects and fluctuation-behaviour of two dimensional models in statistical mechanics and many body physics

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1. Introduction

As we have witnessed at this conference, there are currently two competing ideas on dynamical problems in high energy physics. On the one hand there is the rapidly developing field of classical solutions of nonlinear equations and their quantization ("soliton"-physics) with many interesting applications and the promise to understand things as "quark entrapment". On the other hand, after the progress at the beginning of the 70ties in critical phenomena, there have been several attempts to use lattice methods for abelian and non-abelian gauge theories with the aim to understand quark confinement via a charge-screening mechanism. In the latter approach quantum fluctuations are right from the beginning very important and one does not envisage classical limits to be helpful.

Some models, for example the two dimensional A^4-theory have been investigated in the quasiclassical domain as well as in the long range fluctuation (= critical) domain (A^4 is believed to have the same critical behaviour as the Lenz–Ising-model).

We will in the following sketch the *quasiclassical aspects* of the $D = 2$ *Lenz–Ising-model* which are similar to those of the A^4-theory. In particular, the conclusion that there are new coherent states [1] which appear to be Majorana fermions are similar to the A^4-theory. It would be tempting to speculate that there is an equivalent description in terms of a Lagrangian which contains a Majorana fermion field in the same way as the sine-Gordon equation and the Thirring field seem to be related. We have not yet been able to establish such an equivalence. The only independent evidence for this speculation comes from the investigation of the *critical region*. In section 3 we give an explicit construction for the scale invariant limit (corresponding to the A^4-theory where the two minima coalesce for the first time) and show that its most simple field theoretical description can be given in terms of a free $D = 2$ Majorana field. All fields, including the basic field variables, can be written as functions of the Majorana field but for some fields (including all odd powers of the basic field) the relation is somewhat nonlocal (involving line-integrals).

We will carry out the same investigation for the Baxter-model, which is a nontrivial generalization of the Lenz–Ising-model and turns out to be described (in the scale invariant region) by a Lagrangian identical to the Lagrangian of the Thirring-model. In section 4 we will briefly discuss the relation of the Thirring-model to the sine-Gordon equation. Even though our conclusions agree with Coleman [2], our method is sufficiently different as to justify our presentation. In the last section we will discuss an interesting problem for a one dimensional electron gas originating from a generalization of the well known Tomonaga–Luttinger-model [3]. The formation of a gap which contains soliton–antisoliton bound states is very similar to the occurrence of this phenomenon in the massive Thirring-model.

2. Stationary, position dependent mean field solutions for the Lenz–Ising-model

The (isotropic) Lenz–Ising-model is defined in terms of the lattice Hamiltonian:

$$\mathcal{H}[\sigma] = -K \sum_{\langle ij \rangle} \sigma_i \sigma_j \,, \tag{2.1}$$

$\langle ij \rangle$ = summation over nearest neighbours

and:

$K = \beta J$,

J = ferromagnetic coupling strength,

$\beta = 1/k_B \cdot T$.

The symbol $[\sigma]$ indicates a configuration, i.e. as assignment of values ± 1 to every lattice spin σ_i.

Correlation functions may be formed with the help of the generating functional:

$$Z[h] = C \sum_{[\sigma]} \exp\left\{ -\mathcal{H}[\sigma] + \sum_i h_i \sigma_i \right\} . \tag{2.2}$$

Here the C is a constant determined in such a way that $Z[0] = 1$. The connected n-point function is obtained in the usual way by taking the nth derivative of $\ln Z$ with respect to the source:

$$\langle \sigma_{i_1} \cdots \sigma_{i_n} \rangle_c = \frac{\partial^n}{\partial h_{i_1} \cdots \partial h_{i_n}} \ln Z \,. \tag{2.3}$$

In order to obtain a formalism which is more closely linked to field theory, it is costumery to trade the discrete spin for a continuous field variable by performing a functional Laplace-transformation: ‡

$$\exp(\tfrac{1}{2}\sigma \boldsymbol{K} \sigma) = \pi^{-N/2} |\boldsymbol{K}|^{-1/2} \int \prod_i \mathrm{d}\Phi_i \exp(-\tfrac{1}{2}\Phi K^{-1} \Phi + \sigma\Phi) \,, \tag{2.4}$$

N = number of lattice spins,

$|K|$ = determinant of $\boldsymbol{K}$,

$$\boldsymbol{K} = (K_{ij}), \qquad K_{ij} = K_{i-j} = \begin{cases} 0, \; |i-j| > 1 \text{ and } i-j=0 \\ K, \; |i-j| = 1 \end{cases} .$$

Here a lattice index i stands for a vector (whose components are integer) which determines a lattice point. The Fourier-transform of K_n:

$$K_n = \frac{1}{(2\pi)^D} \int_{-\pi/a}^{\pi/a} \tilde{K}(k) \exp(\mathrm{i}\boldsymbol{k n}) \, \mathrm{d}^D k \tag{2.5}$$

is the inverse of the "lattice propagator" (a = lattice length):

‡ For the anisotropic coupling we will use K_i, $i = 2, 1$ for the horizontal resp. vertical coupling strength βJ_i in a two-dimensional lattice.

$$\tilde{K}(k) = 2K \sum_{i=1}^{D} \cos ak_i \, . \tag{2.6}$$

Absorbing all k-independent constants into C we have, after doing the sums of $\sigma_i = \pm 1$, the following field theoretical problem in functional integral language:

$$Z[h] = C \int \prod_i \mathrm{d}\Phi_i \exp \left\{ -\tfrac{1}{2} \Phi K^{-1} \Phi + \sum_j \ln \cosh (\Phi_j + h_j) \right\} . \tag{2.7}$$

Passing to the continuous limit:

$$\Phi K^{-1} \Phi \to \frac{1}{2KD} \int \Phi^2 \, \mathrm{d}^D x + \frac{a^2}{2K} \int (\partial_i \Phi)^2 \, \mathrm{d}^D x \ ,$$

we obtain a Euclidean field theory with a non-polynomial interaction:

$$Z[h] = C \int \mathrm{d}[\Phi] \exp(-\mathcal{H}[\sigma]) \ , \tag{2.8}$$

$$\mathcal{H}[\Phi] = \frac{a^2}{2K} \int (\partial_i \Phi)^2 \, \mathrm{d}^D x + \frac{1}{2KD} \int \Phi^2 \, \mathrm{d}^D x - \ln \cosh (\Phi + h) \tag{2.9}$$

which in relativistic language belongs to a relativistic Lagrangian (in zero external field):

$$\mathcal{L} = \tfrac{1}{2} \partial_\mu \Phi \, \partial^\mu \Phi - \frac{1}{2a^2 D} \, \Phi^2 + \frac{K}{a^2} \ln \cosh \Phi + \text{const.} \tag{2.10}$$

Here a positive mass term stabilizes the weakly (asymptotically linear decreasing) falling off interacting part of the effective potential. In using the effective Φ Lagrangian (2.10) we should not forget that the original σ is a point transformation ($\sigma = \tanh \Phi$) of the Φ; this is a direct consequence from (2.3) and (2.8) which lead to:

$$\langle \sigma(x_1) \cdots \sigma(x_n) \rangle_c = \langle (\tanh \Phi(x_1) \cdots \tanh \Phi(x_n) \rangle_c^{\mathcal{L}\Phi} \ . \tag{2.11}$$

For large K (small temperature) we have the well known degenerate "mean field" vacua:

$$\text{Min} \left(\frac{1}{2a^2 D} \, \Phi^2 - \frac{K}{a^2} \log \cosh \Phi \right) :$$

$\Phi = \pm \Phi_0$, solutions of transcendental equation:

$$\Phi_0 = 2KD \tanh \Phi_0 \tag{2.12a}$$

resp.:

$$\tanh^{-1} \sigma_0 = 2KD \, \sigma_0 \ . \tag{2.12b}$$

Just as in the (analytically simpler) case of the A^4 interaction there exists a stable stationary kink-like solution [1] with finite energy relative to the vacuum. The identification of the classical solution as an approximation to the quantum theoretical form factor in a new state would indicate as in the A^4 case the Majorana (selfconjugate) fermion nature of such a state. In contrast to the A^4 case the model has however only one parameter and one can not achieve a weak coupling regime by a suitable choice of parameters. We will not pursue the semiclassical aspects of this model any further but go directly to the investigation of the critical behaviour (i.e. that temperature which corresponds to the confluence of the degenerate mean field minima) and establish rigorously the fundamental role played by (in this case zero-mass) Majorana fermions.

3. Investigation of the critical region, construction of the scale invariant limit

According to a commonly accepted working hypothesis critical phenomena (i.e. second order phase transitions) find their natural explanation in an underlying scale invariant field theory. The validity of this picture has only been rigorously established for those cases for which mean field theory is exact (for example A^4 in $D = 4$ dimensions) and therefore our following construction for the $D = 2$ Lenz–Ising-model is far from being a simple exercise.

It is well known that the computation of the $D = 2$ Lenz–Ising partition function (2.2) for $C = 1$ and $h = 0$ may be reformulated in terms of a transfer matrix (lattice size $N \times M$)

$$Z = \operatorname{tr} V^N , \qquad V = V_2^{1/2} V_1 V_2^{1/2} \tag{3.1}$$

with

$$V_1 = (2 \sinh 2K_1)^{M/2} \exp\left(K_1^* \sum_{m=1}^{M} \sigma_m^z\right), \quad \tanh K_1^* = \exp(-2K_1)$$

$$V_2 = \exp\left(K_2 \sum_{1}^{M} \sigma_m^x \sigma_{m+1}^x\right) .$$

There are several possibilities to simplify this transfer-matrix with the help of "Pauli–Jordan lattice fermions" [5]

$$C_m^+ = \sigma_m^+ \exp\left\{-\mathrm{i}\pi \sum_{m+1}^{M} \sigma_i^- \sigma_i^+\right\}, \qquad C_m^- = (C^+)^+ . \tag{3.2}$$

The relation of these spinors to the spinors used by Kadanoff [6] are:

$$b_{m+} = \tfrac{1}{2} C_{m-1}^x , \qquad b_{m-} = -\tfrac{1}{2} C_m^y \tag{3.3}$$

with

$$C_m^x = C_m^+ + C_m^- , \qquad \mathrm{i} C_m^y = C_m^+ - C_m^- .$$

Further simplification is reached by Fourier decomposition (cyclic boundary conditions for C)

$$C_m^+ = M^{-1/2} \, \mathrm{e}^{-\mathrm{i}\pi/4} \sum_{-\pi<q<\pi} \mathrm{e}^{-\mathrm{i}qm} \eta_q^+ \tag{3.4}$$

and a Bogoliubov–Valatin transformation [5]

$$\xi_q = \cos\Phi_q \eta_q + \sin\Phi_q \eta_{-q}^+ , \qquad \xi_{-q} = \cos\Phi_q \eta_{-q} - \sin\Phi_q \eta_q^+ \tag{3.5}$$

which diagonalizes the transfer matrix:

$$V = (2 \sinh 2K_1)^{M/2} \exp\left\{- \sum_{-\pi<q<\pi} \epsilon_q (\xi_q^+ \xi_q - \tfrac{1}{2})\right\} , \tag{3.6}$$

$$\tan\Phi_q = \frac{2C_q}{B_q - A_q}$$

$$A_q = \exp(2K_1^*)\,(\cosh K_2 - \sinh K_2 \cos q)^2 + \exp(-2K_1^*)\,(\sinh K_2 \sin q)^2$$

$$B_q = \exp(2K_1^*)\,(\sinh K_2 \sin q)^2 + \exp(-2K_1^*)\,(\cosh K_2 + \sinh K_2 \cos q)^2$$

$$C_q = (2\sinh K_2 \sin q)\,(\sinh 2K_1^* \sinh K_2 \cos q - \cosh 2K_1^* \cosh K_2)$$

and

$$\cosh \epsilon_q = \cosh 2K_2 \cosh 2K_1^* - \sinh 2K_2 \sinh 2K_1^* \cos q\ , \tag{3.7}$$

where the notation is from ref. [5].

The original classical formula (2.3) for the σ-correlation functions now becomes a "quantum-like" formula in the Hilbert-space of the transfer matrix (if $i_1 \cdots i_n$ are indices in a row):

$$\langle \sigma_{i_1} \cdots \sigma_{i_n} \rangle = \langle \psi_0 | \sigma_{i_1}^x \cdots \sigma_{i_n}^x | \psi_0 \rangle \tag{3.8}$$

$$|\psi_0\rangle = \text{highest eigenvalue of transfer matrix} \tag{3.9a}$$

and

$$\sigma_m^x = C_m^x \mu_m\,, \qquad C_m^y = \mu_{m+1}\, \mathrm{i}\, C_m^y \tag{3.9b}$$

with

$$\mu_m = \exp\left\{ -\mathrm{i}\pi \sum_{m+1}^{M} C_i^- C_i^+ \right\}.$$

μ_m is the "disorder" variable of Kadanoff [7]. Using mathematical techniques of Wu [8] one may obtain a very transparent derivation of the following Kadanoff–Ceva formula [9]:

Statement (Kadanoff and Ceva). At $T = T_c$ for large distances the correlation function within a row are given by the formula:

$$\langle \prod_{i=1}^{N} D_{\gamma_i}(\tau_i) \rangle = \begin{cases} 0, & \text{if } \Gamma = 0 \\ \prod\limits_{1 \leqslant i < j \leqslant N} [c(\tau_j - \tau_i)]^{\gamma_i \gamma_j p_i p_j}, & \text{if } \Gamma \neq 0\,. \end{cases} \tag{3.10}$$

Here

$$\tau_1 < \tau_2 \cdots < \tau_N\,, \qquad \gamma_i = \pm\tfrac{1}{2}\,, \qquad D_{1/2} \equiv \sigma, \qquad D_{-1/2} \equiv \mu\,.$$

The p_i are determined recursively by:

$$p_i = (-1)^{2\Gamma_i}, \qquad \Gamma_{i+1} = \Gamma_i + (-1)^{2\Gamma_i}\gamma_{i+1}$$

where

$$\Gamma_1 = \gamma_1 \quad \text{and} \quad \Gamma = \Gamma_N\,.$$

The main step in the proof is the realization that the special case: K_1 (vertical coupling) $\to 0$, $K_2 \to \infty$ dominates for long distances in a row [7].

The special formula for the fermion two point functions

$$\langle C^x(\tau_1)\, C^x(\tau_2)\rangle = 0 = \langle \mathrm{i}\, C^y(\tau_1)\, \mathrm{i}\, C^y(\tau_2)\rangle \quad ,$$
$$\langle C^x(\tau_i)\, \mathrm{i}\, C^y(\tau_j)\rangle = \frac{1}{\pi(\tau_j - \tau_i)} \tag{3.11}$$

and the statement that the higher point functions of fermions are just products of two point functions is a side result of the same considerations which lead to the Kadanoff–Ceva formula. We want to interpret now the critical long distance correlation functions as *the correlation functions of operators in a scale invariant field theory*. In this theory one may form composite fields for example:

$$E(\tau) = \lim_{\tau' \to \tau} (\tau' - \tau)^{-3/4}\, \sigma(\tau)\, \sigma(\tau') \,. \tag{3.12}$$

Kadanoff has postulated two-dimensional transformation properties (which cannot be rigorously derived by investigations on one line only, the best one can do is give certain consistency checks) for the operators in the scale invariant field theory. For example C^x and $\mathrm{i}\, C^y$ are supposed to be linear combinations of a two dimensional spinor-field, whereas σ, μ and E are scalars. This field theory is an Euclidean field theory. An Euclidean field theory under certain circumstances may be viewed as the Euclidean continuation of a relativistic quantum field theory [10].

We will in the following indicate the explicit construction of a relativistic field theory which leads to the Kadanoff algebra [11]. Consider the following $D = 2$ relativistic Majorana (self-conjugate) two component spinor field (ℓ, r = left resp. right):

$$\psi_1(u) = \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{\mathrm{i}pu}\, a_\ell^+(p)\, \mathrm{d}p + \text{h.c.}$$
$$\psi_2(v) = \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{\mathrm{i}pv}\, a_\mathrm{r}^+(p)\, \mathrm{d}p - \text{h.c.} \tag{3.13}$$

with $u = t + x$, $v = t - x$.

The antihermiticity of ψ_2 is a result of the following representation of γ-matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

For $T = T_\mathrm{c}$, $q \approx 0$ we may view our transfer matrix:

$$V = \mathrm{e}^{-H}, \qquad H = \int \mathrm{d}q\, |q|\, (\xi_q^+ \xi_q - \tfrac{1}{2}) \tag{3.14}$$

in terms of a relativistic Majorana Hamiltonian:

$$H = \tfrac{1}{2}\mathrm{i} \int \mathrm{d}x\, \bar{\psi}\, \gamma^0 \partial_0 \psi \,. \tag{3.15}$$

So we expect this Majorana field to play a crucial role in the construction of scale invariant operators.

We identify the C's in the following way with ψ:

$$C^x(\tau) \sim \psi_1(t) + \psi_2(t), \qquad \mathrm{i}\,C^y(\tau) \sim \psi_2(t) - \psi_1(t)\,. \tag{3.16}$$

Here the tilde $\sim$ means equality after continuing to relativistic operators to imaginary time: $\tau = \mathrm{i}t$.

The energy density in terms of the Majorana fields is represented by the local operator:

$$E(\tau) = \tfrac{1}{2} C^x\, \mathrm{i}\, C^y \sim \psi_1(t)\, \psi_2(t) = E(t)\,. \tag{3.17}$$

For the bilocals our lattice formulas go over into the following line integrals [11]:

$$\tau < \tau': \qquad \mu(\tau)\,\mu(\tau') \sim Z \exp\left\{-\mathrm{i}\pi \int_t^{t'} E(t'')\,\mathrm{d}t''\right\} = \mu(t)\,\mu(t') \tag{3.18}$$

$$\sigma(\tau)\,\sigma(\tau') \sim \lim_{\substack{\epsilon\to 0\\ \epsilon'\to 0}} (\epsilon)^{1/2} (\epsilon')^{1/2} (\psi_2(t-\epsilon) - \psi_1(t - \mathrm{i}\epsilon))\, \mu(t)\, \mu(t')\, (\psi_1(t'+\epsilon') + \psi_2(t'+\epsilon'))$$

$$\sigma(\tau)\,\mu(\tau') = \lim_{\epsilon\to 0} \epsilon^{1/2} (\psi_2(t-\epsilon) - \psi_1(t-\epsilon))\, \mu(t)\, \mu(t')$$

$$\mu(\tau)\,\sigma(\tau') = \lim_{\epsilon'\to 0} \epsilon^{1/2}\, \mu(t)\, \mu(t')\, (\psi_1(t'+\epsilon') + \psi_2(t'+\epsilon'))\,.$$

The Z is a renormalization factor due to infinities coming the endpoints if one takes functions (for example the exponential) of a line integral. The problem of this renormalization and of computation of line integrals is significantly simplified by the existence of "Bosons on a time-like line".

Theorem [11]: The operator $E(t)$ can be written as

$$E(t) = E^{(+)}(t) + \text{h.c.} \tag{3.19}$$

with

$$E^{(+)}(t) = \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{\mathrm{i}p^+} C^+(p)\,\mathrm{d}p$$

and:

$$[C(p), C(p')] = 0 = [C^+(p), C^+(p')]\,,$$

$$[C(p), C^+(p')] = p\,\delta(p-p')\,\theta(p)\,.$$

The proof of this theorem follows from the definition of E leading to:

$$C(p) = \int \mathrm{d}k\,\theta(k)\,[a_\mathrm{r}(p-k)\,a_\ell(h)\,\theta(p-k) - a_\ell^+(k)\,a_\mathrm{r}(k+p)\,\theta(p+k) - a_\mathrm{r}^+(k)\,a_\ell(k+p)\,\theta(k+p)]\,. \tag{3.20}$$

The rest is straightforward computation.

The line integrals can be easily computed with the Bose potentials

$$E(t) = \partial_t \Phi(t), \qquad \Phi(t) = \frac{\mathrm{i}}{2\pi} \int_0^\infty \mathrm{e}^{-\mathrm{i}pt} \frac{C(p)}{p}\,\mathrm{d}p + \text{h.c.} \tag{3.21}$$

For the ordered bilocal we write:

$$\mu(t)\,\mu(t') = :\mathrm{e}^{\mathrm{i}\pi\Phi(t)}:\ :\mathrm{e}^{-\mathrm{i}\pi\Phi(t')}: \tag{3.22}$$

with the Z having done its duty after obtaining a well defined finite operator.

Clearly the $2n$-point function of ordered μ's on a line computed from (3.22) by Wick-contractions will be exactly that of the Kadanoff–Ceva formula (3.10).

With the help of the commutation relations [11]:

$$[\psi_1(t) + \psi_2(t), \Phi^{\pm}(t')] = \frac{\mp 1}{4\pi i} \log[-(t-t')^2_{\mp}] (\psi_1(t) + \psi_2(t)) \tag{3.23}$$

$$(t-t')^2_{\mp} \equiv (t - t' \mp i\epsilon)^2$$

and the corresponding commutation relations for the difference $\psi_2 - \psi_1$ with $\Phi^{\pm}$ which follows from (3.23) by taking the hermitean adjoint one may now Wick order the fermions relative to the boson exponentials. In this way we may establish the Kadanoff–Ceva formula as a special result of our bilocal line-integrals of μ and σ (3.18). The important commutation relation (3.23) follows if we interprete the infrared infinite $\Phi(t)$ with the help of a particular infrared regularization. An alternative more elegant possibility is to realize that the commutator of $\psi_1 + \psi_2$ with potential differences i.e. $\Phi(t_1') - \Phi(t_2')$ is infrared finite and its knowledge suffices to do all Wick-contractions necessary for the computation of ordered correlation functions of σ and μ's on a time-line.

Note finally that all even local functions of σ, μ are objects whose correlation functions one can compute for an arbitrary configuration. Only for correlation functions involving odd polynomials of σ and μ our computational technique force us to restrict ourself to a time-line [28].

The main result of our investigation is that the scale invariant Lenz–Ising-model is most easily described in terms of a relativistic Majorana field. This field is complete in the Hilbert-space, but its relation to σ is nonlocal involving line integrals. Only even polynomials of σ have a simple local relation to the Majorana field.

Let us now briefly indicate the extension of our methods to the Baxter-model [12] which consists of two superimposed Lenz–Ising-models coupled by an energy–energy coupling [13]:

$$\mathcal{H}_{\text{int}} = -8g\, E_1(x)\, E_2(x)\, . \tag{3.24}$$

In the scale invariant limit for each Lenz–Ising-model the relativistic interaction operator takes on the form

$$\mathcal{L}_{\text{int}} \sim \psi_1^{\text{I}} \psi_2^{\text{I}} \psi_1^{\text{II}} \psi_2^{\text{II}}\, . \tag{3.25}$$

Because of Fermi-statistics one may add terms as

$$:\psi_1^{\text{I}} \psi_1^{\text{II}} \psi_1^{\text{I}} \psi_1^{\text{II}}: \; = 0 = \; :\psi_2^{\text{I}} \psi_2^{\text{II}} \psi_2^{\text{I}} \psi_2^{\text{II}}:$$

in order to obtain a O_2-invariant form of the interaction. With the help of Dirac combinations

$$\psi = \frac{1}{\sqrt{2}} (\psi^{\text{I}} + i\psi^{\text{II}}) \tag{3.26}$$

we obtain the $U(1) \times U(1)$ invariant Lagrangian

$$\mathcal{L}_{\text{int}} = 2g :\bar{\psi}\gamma_\mu \psi\, \bar{\psi}\gamma^\mu \psi: \tag{3.27}$$

Note that all 4-fermion couplings involving only one Dirac-field are identical (Fierz-identities).

Our assumption that one may construct the scale invariant limit of the Baxter-model directly from the scale invariant versions of the superimposed Lenz–Ising-models has led us the *Thirring-model* as the *scale invariant limit* of the *Baxter-model.*

It is well known that the $U(1) \times U(1)$ Thirring interaction (3.27) maintains scale invariance to all orders, only the dimensions change from their canonical value to a g-dependent value. The concrete form of this relation will depend on the parametrization in terms of the coupling constant we use. For example in Klaiber's parametrization [14] we obtain:

$$\begin{aligned} &\dim \psi = \tfrac{1}{2} + 4g^2/\pi^2 \\ &\dim \bar{\psi}\psi = 1 + \frac{8g^2}{\pi^2} - \frac{4g}{\pi}\sqrt{1 + \frac{4g^2}{\pi^2}} \\ &\dim \psi^{tr}\gamma_0\gamma^5\psi = 1 + \frac{8g^2}{\pi^2} + \frac{4g}{\pi}\sqrt{1 + \frac{4g^2}{\pi^2}} \ . \end{aligned} \tag{3.28}$$

Other parametrizations as the Johnson [15] or the Sommerfield [16] parametrization agree with (3.28) in first order in g but deviate in higher orders. The lattice approach, which is very complicated, leads to Baxter's parametrization which again agrees to lowest order.

An important intrinsic (parameter independent) feature of this model is that if the change of coupling leads to an increase in $\dim \psi$, it automatically decreases $\dim \bar{\psi}\psi$ (zero is limiting value). This limit is called the strong coupling limit of the model. In terms of the original Lenz–Ising variables we have

$$\bar{\psi}\psi = E^{\mathrm{I}} + E^{\mathrm{II}} \tag{3.29a}$$

$$\tfrac{1}{2}(\psi^{tr}\gamma_0\gamma^5\psi + \mathrm{h.c.}) = E^{\mathrm{I}} - E^{\mathrm{II}} \ . \tag{3.29b}$$

Therefore we obtain from the addition of a mass perturbation $m\bar{\psi}\psi$:

$$\begin{aligned} &\text{Index for correlation length: } \nu = (2 - \dim \bar{\psi}\psi)^{-1} = 1 - 4g/\pi + \cdots , \\ &\text{Index for specific heat:} \qquad \alpha = \frac{2(1 - \dim \bar{\psi}\psi)}{2 - \dim \bar{\psi}\psi} = \frac{8g}{\pi} + \cdots . \end{aligned} \tag{3.30}$$

Now consider the bilocals. The simplest one is

$$t < t', \qquad \mu(t)\,\mu(t') = Z \exp\left\{-\mathrm{i}\pi \int_t^{t'} E(t'')\,\mathrm{d}t''\right\} . \tag{3.31}$$

In order to see the interpretation of the operator, let us switch off the interaction. Due to (3.29a) we have:

$$\mu(t)\,\mu(t') = (\mu(t)\,\mu(t'))^{\mathrm{I}}\,(\mu(t)\,\mu(t'))^{\mathrm{II}} \ . \tag{3.32}$$

This square of the bilocals, which after switching on the interaction does not factorize, is related to the electric polarisability of the Baxter-model (actually this statement holds for the closely related $\sigma\sigma$-bilocal).

However in contrast to the decoupled case, the line integrals over the "dressed" operator $E(x)$ have a power scaling and therefore our construction of the scale invariant Baxter-model would lead to a breakdown of power scaling for the bilocals. A closer examination of this situation suggest that for $g > 0$ the correlation functions of the bilocals may develop an exponential scaling of the type:

$$\langle \mu(t)\,\mu(t')\rangle \sim \exp(-\kappa/2)\,(t'-t)^{2-2\dim\bar\psi\psi}\,, \qquad \kappa^{-1} = (2\dim\bar\psi\psi - 1)(2-2\dim\bar\psi\psi)\,. \tag{3.33}$$

This would be the mathematical expression for a *phase transition of infinite order* on the level of correlation functions.

The behaviour of the Baxter-model in the *neighbourhood of the critical temperature* is in our language described by the *massive Thirring-model*. The massive Thirring-model contains bound states of the soliton–antisoliton type [17] as will be discussed in the next section.

4. Sine-Gordon equation and superrenormalizable perturbations on Thirring-like models

Consider a massive two dimensional quantum field theory with an abelian charge structure. The conserved vector current j_μ can always be affiliated with a curl-free axial-current:

$$j_{\mu 5} = \epsilon_{\mu\nu} j^\nu, \qquad \operatorname{curl} j_{\mu 5} = 0\,. \tag{4.1}$$

In the classical version of the theory this situation would immediately lead to a pseudo-scalar potential:

$$j_{\mu 5} = \frac{1}{\sqrt{\pi}}\,\partial_\mu \phi\,. \tag{4.2}$$

The quantized theory will also give a potential, however its construction is much more delicate [18].

The following definition turns out to be helpful:

Definition: A state $B\,|0\rangle$ is called quasilocal if it is of the form:

$$B = \sum_0^N \int f(x_1\cdots x_n) \prod_{i=1}^n A_i(x_i)\,|0\rangle\, \mathrm{d}^2x_1 \cdots \mathrm{d}^2x_n \tag{4.3}$$

where the A_i are any local fields (including spinor fields) of the theory and the f's are fast decreasing test functions.

First we define the ϕ on the vacuum state:

$$\frac{1}{\sqrt{\pi}}\,\phi(x)\,|0\rangle = \frac{1}{\partial_\mu\partial^\mu}\,\partial^\kappa j_{\kappa 5}(x)\,|0\rangle\,. \tag{4.4}$$

The division by the differential operator is meant in momentum space; it causes no problem because the vacuum component of the right-hand side is zero and above the vacuum there is a mass gap according to our assumption. It is easy to see that even though the division destroys locality, the state is still quasilocal (after smearing with a fast decreasing test function). The state transforms as a pseudoscalar. Classically we would define a *pseudoscalar* field as:

$$\frac{1}{\sqrt{\pi}}\,\phi_{\mathrm{c}\ell}(x) = \tfrac{1}{2}\int_{-\infty}^{\infty} \epsilon(s)\, j_{\mu 5}^{\mathrm{c}\ell}(x - se)\, e^\mu\, \mathrm{d}s\,, \qquad e^\mu = \text{space-like unit vector}\,. \tag{4.5}$$

The independence on e^μ follows upon using the vanishing of the curl (4.1). The antisymmetrical definition (4.5) guarantees the pseudo-scalarity. This classical consideration together with (4.4) suggests to define the quantum operator on the dense set of quasilocal states as:

$$\frac{1}{\sqrt{\pi}}\,\phi(x)B|0\rangle = \tfrac{1}{2}\int_{-\infty}^{\infty} \epsilon(s)\,[j_{\mu 5}(x-se)\,e^\mu, B]\,|0\rangle + B\,\frac{1}{\partial_\mu\partial^\mu}\,\partial^\kappa j_{\mu 5}(x)\,|0\rangle\,. \tag{4.6}$$

The commutator poses no convergence problems. The so constructed ϕ in addition to pseudo-scalarity (and independence of e) has the following desirable properties [18] which we state without proof:

1. It is local relative to itself.
2. With the other basic fields A_i of the theory it leads to the following spacelike commutator:

$$\frac{1}{\sqrt{\pi}}\,[\phi(x), A_i(y)] = \tfrac{1}{2}\epsilon(x'-y')\,[Q, A_i(y)] \qquad \text{for } (x-y)^2 < 0\,. \tag{4.7}$$

The field ϕ is pseudo-scalar and has negative C-parity. By sacrificing the pseudo-scalarity in the higher sectors one may obtain with

$$\hat{\phi} = \phi + \tfrac{1}{2}\sqrt{\pi}\,Q\,.$$

2′. $$\frac{1}{\sqrt{\pi}}\,[\hat{\phi}(x), A_i(y)] = \theta(x'-y')\,[Q, A_i(y)]\,. \tag{4.8}$$

Let us now assume the dynamics is such that the current for a fixed time is complete in each charge sector. Equivalently we may say that $\hat{\phi}$ and $\partial\hat{\phi}/\partial t$ form a complete set. In a local covariant theory we should then expect the validity of a hyperbolic equation of motion:

$$\partial_\mu\partial^\mu\hat{\phi} = F(\hat{\phi}, \partial_\kappa\hat{\phi}\,\partial^\kappa\hat{\phi})\,. \tag{4.9}$$

If we demand this equation of motion in each sector, and then consistency with 2′ will lead to the periodicity [19]

$$F(\hat{\phi} - \sqrt{\pi}\,Q, \partial_\mu\hat{\phi}\,\partial^\mu\hat{\phi}) = F(\hat{\phi}, \partial_\mu\hat{\phi}\,\partial^\mu\hat{\phi})\,. \tag{4.10}$$

If in addition we insist in the absence of derivative coupling we arrive at:

$$F(\phi + \sqrt{\pi}\,n) = F(\phi), \qquad n = \pm 1, \pm 2, \ldots \tag{4.11}$$

where we have simplified our notation by omitting the hat on ϕ. Clearly $F = C\sin 2\sqrt{\pi}\phi$ and higher Fourier-components (the pseudoscalarity in the vacuum-sector demands F to be antisymmetric!) are the only possibilities. Such an interaction is superrenormalizable (this is equivalent to a finite Schwinger-term in the currents) and by introducing a renormalized field with $\phi_R = \beta\phi$ with

$$[\phi_R(x), \dot{\phi}_R(y)] = i\delta(x-y) \tag{4.12}$$

obtain the sine-Gordon equation in the usual form (omitting again subscript R):

$$\partial_\mu\partial^\mu\phi = C : \sin\frac{2\sqrt{\pi}}{\beta}\phi :\,. \tag{4.13}$$

The double dot is the Wick-ordering with respect to the interacting vacuum i.e.

$$:\phi(x)^2: = \lim_{y\to x} \{\phi(x)\,\phi(y) - \langle\phi(x)\,\phi(y)\rangle\}$$
$$:\phi^3(x): = \lim_{y,z\to x} \{\phi(x)\,\phi(y)\,\phi(z) - 3\langle\phi(y)\,\phi(z)\rangle\,\phi(x)\} \tag{4.14}$$

etc.

The most likely candidates for models fulfilling these requirements are the massive Thirring-model and certain abelian generalizations, since their zero mass – respectively short distance – limits lead to currents which are known to fulfill the completeness criteria in each sector [20]. Indeed for the massive Thirring-model Coleman [21] was able to establish a direct equivalence between certain quantities in that model and corresponding quantities in the properly parameter – adjusted sine-Gordon equation.

Our version [22] of this equivalence is the following. Consider first the massless Thirring-model:

$$\mathcal{L} = \tfrac{1}{2}\mathrm{i}\bar\psi\gamma^\mu\partial_\mu\psi - g\bar\psi\gamma_\mu\psi\,\bar\psi\gamma^\mu\psi\,. \tag{4.15}$$

The solution is usually written in the form

$$\psi(x) = \exp\{\mathrm{i}\chi^{(+)}(x)\}\,\psi_0(x)\,\exp\{\mathrm{i}\chi^-(x)\}$$
$$\chi = \alpha j(x) + \gamma^5\beta\tilde{j}(x) \tag{4.16}$$

where j and $\tilde{j}$ are related to the free spinor current by:

$$j_\mu = \frac{1}{\sqrt{\pi}}\partial_\mu j, \qquad \tilde{j}_\mu = \frac{1}{\sqrt{\pi}}\partial_\mu\tilde{j}\,. \tag{4.17}$$

Their two point functions and commutators with ψ_0 have been infrared-regularized [23], the j and $\tilde{j}$ behave as two free scalar massless fields and their relative commutator is the space-like antisymmetric function. From the canonical behaviour of j and $\tilde{j}$ as well as their commutation relations with the spinors the following formula for the dimension and the spin of ψ follows by straightforward computation

$$d_\psi = \tfrac{1}{2} + \frac{\alpha^2 - 2\sqrt{\pi}\alpha + \beta^2 - 2\sqrt{\pi}\beta}{4\pi}\,, \qquad s_\psi = \tfrac{1}{2} + \frac{\alpha\beta - \alpha\sqrt{\pi} - \beta\sqrt{\pi}}{2\pi}\,. \tag{4.18}$$

For fixed s_ψ the remaining parameter may be related to the coupling constant g (4.15), however since in this model there is no mass shell, we cannot work with the usual "natural" definition of the coupling constant. The definition would depend on the way one defines the current from the ψ's by a space-time limiting procedure and there are almost as many coupling constant parametrizations of the Thirring-model as articles on that model. Common to all parametrizations is the statement that $g \to +\infty$, i.e. "strong coupling" means:

$$\dim\psi \to \infty \qquad \text{and} \qquad \dim\bar\psi\psi \to 0\,. \tag{4.19}$$

The only reason why we sometimes prefer the Klaiber parametrization [14] is that the connection between $\dim\psi$ and g is non-singular on the real g-axis.

One comment on continuous spin. The Lorentz-group being abelian in two-dimensional space-time allows for the covariant transformation property

$$U(\Lambda)\,\psi(x)\,U^{+}(\Lambda)=\begin{pmatrix} e^{-s\chi} & 0 \\ 0 & e^{s\chi}\end{pmatrix}\psi(\Lambda x)\,. \tag{4.20}$$

The spin in the Lorentz-sense complies with the space-like commutation properties of these quantities [14]

$$\psi_1(x)\,\psi_1(y)+\exp\{is\epsilon(x'-y')\}\,\psi_1(y)\,\psi_1(x)=0, \qquad (x-y)^2<0\ , \text{ etc.} \tag{4.21}$$

The significance on the level of particle states (which in massless case do not exist) has not yet been investigated. Note that for $s=\frac{1}{2}N$, N integer, the ψ would be called in the usual language a *nonlocal field*.

It is convenient to "bosonize" [22] the free fermion field by applying first a Klein transformation:

$$\psi_0 \to \exp\{\tfrac{1}{2}i\pi(Q+\tilde{Q})\}\,\psi_0 \tag{4.22}$$

which will lead to commutation between the two different irreducible Lorentz-components, and then to write this field (no new notation introduced) as

$$\psi_0=\frac{1}{\sqrt{2\pi}}:\exp\{i\sqrt{\pi}j+i\sqrt{\pi}\gamma^5\tilde{j}\}:\sigma\,. \tag{4.23}$$

Here σ is the Thirring field for $d_\psi=0=s_\psi$ i.e. a constant field which commutes [14] with j and $\tilde{j}$. It is constant unitary operator which can be diagonalized in terms of angles [24]:

$$|0\rangle=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}|\theta_1\theta_2\rangle\,d\theta_1\,d\theta_2 \tag{4.24}$$

with

$$\sigma_i^{+}|\theta_1\theta_2\rangle=\exp(i\theta_i)\,|\theta_1\theta_2\rangle\,.$$

We would omit this field in (4.23) if we would infer the uniqueness of the vacuum i.e. the cluster property. The latter property applied to the 4-point function of two ψ's and two ψ^{+}'s would i.e. yield

$$\langle\psi_0\psi_0\rangle=0 \tag{4.25}$$

for consistency reasons. The θ_i's and their subsequent averaging (4.24), i.e.:

$$\langle\prod_1^N\psi(x_i)\prod_1^N\psi^{+}(y_i)\rangle=\frac{1}{(2\pi)^2}\int d\theta_1\,d\theta_2\,\langle\prod\psi(x_i)\,\prod\psi^{+}(y_i)\rangle_{\theta_1\theta_2} \tag{4.26}$$

would build in our bose formalism these consistency relations (the Q and $\tilde{Q}$ selection rules of the fermion language) in a "foolproof" way.

The next step is to bring the symmetric energy momentum tensor, obtained by applying Noether formalism to the Thirring-model, into the Sugawara form:

$$\theta_{\mu\nu}=\pi(:\tilde{j}_\mu\tilde{j}_\nu-\tfrac{1}{2}g_{\mu\nu}:\tilde{j}_\kappa\tilde{j}^\kappa:)\,. \tag{4.27}$$

The easiest way is by short distance limiting procedure on the completely bosonized Thirring field

$$\psi = \frac{1}{\sqrt{2\pi}} : \exp\{i\hat{\alpha} j + i\hat{\beta}\gamma^5 \tilde{j}\} : \sigma \tag{4.28}$$

with

$$d_\psi = \frac{\hat{\alpha}^2 + \hat{\beta}^2}{4\pi}, \qquad s_\psi = \frac{\hat{\alpha}\hat{\beta}}{2\pi} .$$

But even without knowing anything about "bosonization" we may obtain (4.27) by performing a short distance limiting procedure directly on the linear spinor-part of $\theta_{\mu\nu}$ [25]. The bosonization formula (4.28) will be only used in an essential way if we rewrite the mass perturbation on the Thirring-model:

$$m_0 N[\psi_1^+ \psi_2] + \text{h.c.}$$

where the normal product N is defined by a space-time limiting procedure in the Thirring-model. Using the Thirring-model as an interaction picture for the massive case, we obtain with the help of (4.28) the bosonized form of the mass term, which after diagonalization of σ depends on θ. In terms of the Hamiltonian density $\mathcal{H} = \theta_{00}$ we get the expression:

$$\mathcal{H}_\theta = \tfrac{1}{2}(\partial_t \phi)^2 + \tfrac{1}{2}(\partial_x \phi)^2 + \frac{m_0}{2\pi} : \cos(2\hat{\beta}\phi + \theta) : , \tag{4.29}$$

$$\theta = \theta_1 - \theta_2, \qquad \phi = \tilde{j} .$$

The mass perturbation in the Thirring interaction picture as well as the equivalent sine-Gordon Hamiltonian (4.29) have a perturbation series with divergencies coming from intermediate integrations over large distances and hence need a spatial cut-off in order to be infrared-finite.

In ref. [26] this infrared cut-off (i.e. a space-box) has been seriously considered. We take as Coleman a formal point of view since at the end the non-perturbative sine-Gordon problem is supposed to be the infrared cure for the infrared disease of the mass perturbations on the Thirring-model. Let us consider expectation values of ϕ differences:

$$\langle \prod_1^N (\phi(x_i) - \phi(y_i))\rangle = \frac{1}{4\pi^2} \int d\theta_1\, d\theta_2 \langle \prod_1^N (\phi(x_i) - \phi(y_i))\rangle_{\mathcal{H}_\theta} . \tag{4.30}$$

In m_0-perturbation theory there is no difference between a Thirring computation and a computation with $\mathcal{H}_\theta$. Imagine now that in the summed up perturbation series we perform the field translation:

$$\phi \to \phi - \frac{\theta_2 - \theta_1}{2\hat{\beta}} .$$

In this way we completely eliminate the θ-dependence of the Hamiltonian and we are not generating an explicit dependence in the potential differences. The cluster property is maintained and the θ_i averaging is rendered manifestly superfluous. By cluster arguments (taking some coordinates "behind the moon") we may get rid of potential differences. The relation to the Thirring currents is precisely that given by Coleman [21].

We may go one step further and consider spinor-bilocals i.e.:

$$:\psi_{\substack{1\\2}}(x)\,\psi^+_{\substack{1\\2}}(y): = \frac{1}{2\pi}:\exp\left[i\hat{\alpha}\int_0^1 \epsilon_{\mu\nu}\partial^\nu\phi(x-s(y-x))\,e^\mu\,ds \mp i\hat{\beta}(\phi(y)-\phi(x))\right]:\sigma_{\substack{1\\2}}\sigma^+_{\substack{1\\2}}\,,$$
$$:\psi_1(x)\,\psi^+_2(y): = \frac{1}{2\pi}:\exp\left[i\hat{\alpha}\int_0^1 \epsilon_{\mu\nu}\partial^\nu\phi(x-s(y-x))\,e^\mu\,ds + i\hat{\beta}(\phi(y)+\phi(x))\right]:\sigma_1\sigma^+_2\,; \tag{4.31}$$

e = unit vector in direction $y-x$.

The reader may convince himself that the explicit θ_i factors in the σ_i together with the θ-dependence coming from the shift leads to a complete cancellation.

In this approach, the two point functions F

$$\psi_i(x)\,\psi^+_k(y) = F_{ik}(x-y):\psi_i(x)\,\psi^+_k(y): \tag{4.32}$$

are not computed directly, they have to be reconstructed by applying cluster arguments from the higher expectation values of the ordered bilocals. Note that operators as $\psi\psi$ which do not commute with the charge involve line integrals to infinity. In the massless Thirring-model such infinite line integrals decrease with an inverse (noninteger) power of the line-length. The corresponding line integrals over the non-perturbative sine-Gordon field ϕ are expected to decrease exponentially in the line distance leading in this way to a vanishing of expectation values as $\psi\psi$. It would be desirable to establish rigorously those properties by fermion reconstruction starting from the bilocals (4.31). Our results are as yet incomplete and do not merit further discussion. On this rather formal level of our consideration we have not yet seen any reason why the reconstructed ψ-fields should anticommute for space-like distances. This problem may be traced back to the problem of whether the massive Thirring field has to have $s = \frac{1}{2}$. The mass perturbation: $N[\bar{\psi}\psi]$ in

$$\mathcal{L} = \mathcal{L}_{\text{Thirring}} - m_0 N[\bar{\psi}\psi] \tag{4.33}$$

is a scalar field for any spin s_ψ. Due to the nonlocal nature of continuous spin we should not expect a local field equation for ψ. The ordered bilocals (4.31) satisfy however locality properties (in the massless version of the model) consistent with their formal bilocal nature, and have a good chance of satisfying bilocal massive equation of motions. This would suggest the tentative conclusion that the solitons of the sine-Gordon equation may have any spin, a problem which clearly merits further investigation.

One remark on fermion mass-renormalizations. For dim $\bar{\psi}\psi < 1$, the infrared-regularized perturbation theory indicates that in addition to the Thirring normal ordering resp. the Wick-ordering on the sine-Gordon level there are no ultraviolet divergencies which require the introduction of a mass-counterterm. For $1 <$ dim $\bar{\psi}\psi < 2$ we do not have a clear ultraviolet picture yet.

Let us finally discuss an abelian generalization of the Thirring-model which has played a role in solid state physics as a physically interesting generalization of the Tomonaga–Luttinger-model [27].

Consider a doublet Thirring field:

$$\psi(x,s), \qquad s = 1,2\,. \tag{4.34}$$

Let $\mathcal{L}_{\text{Th.}}$ be the Lagrangian for the two decoupled massless Thirring fields with identical interaction strength and add the interaction:

$$\mathcal{L}_{\text{int}} = -\lambda N \psi_1^+(x,1)\,\psi_2(x,1)\,\psi_2^+(x,2)\,\psi_1(x,2) + \text{h.c.} \tag{4.35}$$

Written in terms of two-dimensional γ-matrices this interaction is a superposition of scalar and pseudo-scalar quadrilinear terms.

The short distance limiting procedure on the energy-momentum tensor again leads to the Sugawara form for the Thirring part. "Bosonization", as explained before, yields

$$\mathcal{H}_\theta = \sum (\tfrac{1}{2}:(\partial_0\phi_s)^2: + \tfrac{1}{2}:(\partial_1\phi_s)^2:) + \frac{\lambda}{(2\pi)^2}\cos[2\hat{\beta}(\Phi_1 - \Phi_2) + \theta] \tag{4.36}$$

with $\theta = \theta_2(2) - \theta_1(2) + \theta_1(1) - \theta_2(1)$.

Introducing the linear combinations (α and β_0 suitably determined)

$$\Phi = \frac{\Phi_1 + \Phi_2}{\sqrt{2}}, \qquad \sigma = \frac{\Phi_1 - \Phi_2}{\sqrt{2}} \tag{4.37}$$

we obtain with

$$\mathcal{H}_\theta = \tfrac{1}{2}:(\partial_0\phi)^2: + \tfrac{1}{2}:(\partial_1\phi)^2: + \tfrac{1}{2}:(\partial_0\sigma)^2: + \tfrac{1}{2}:(\partial_1\sigma)^2: + \frac{\lambda}{(2\pi)^2}\cos(2\hat{\beta}_\sigma\sqrt{2}\sigma + \theta) \tag{4.38}$$

a decoupled Hamiltonian.

The σ-part is of the sine-Gordon type and if we take over the conjecture of Dashen, Neveu and Hasslacher [17] that the WKB method is exact for the sine-Gordon equation we obtain a gap in the σ excitation spectrum (twice the soliton mass) and depending on the size of β_σ a number of bound states inside this gap. Small coupling β_σ i.e. small dimension of the perturbing operator (i.e. big Thirring couplings resp. field dimension) are favorable for the formation of bound states, and there is the usual critical upper limit from which on the perturbation becomes nonrenormalizable and the sine-Gordon theory looses its "ground state bottom".

A proof of the Luther–Emery gap conjecture using the more conventional formulation of the Tomonaga–Luttinger model has been given elsewhere [27].

Note added in proof: We have meanwhile been able to give a direct proof of the sine-Gordon equation for the potential ϕ without using any zero-mass intermediate steps.

Acknowledgement

I am deeply indebted to my colleagues J.L. Gervais and A. Neveu for inviting me to this interesting conference. This gave me the chance to have some interesting discussions with S. Mandelstam, C. Itzykson, E. Brezin and J.B. Zuber.

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