Nuclear Physics B118 (1977) 445–457

DUALITY IN QUANTUM FIELD THEORY

G. MACK
II. Institut für Theoretische Physik der Universität Hamburg

Received 19 March 1976
(Revised 27 September 1976)

We postulate a convergent version of operator product expansions on the vacuum and explore some of their consequences. They lead to structures much reminiscent of dual resonance models.

1. Introduction

The study of models has often paved the way for the discovery of general laws in physics.

In the present paper we formulate a thesis about local quantum field theory (QFT) and explore some of its consequences. If it is true then QFT has a structure much reminiscent of dual resonance models [1,2]. Our thesis is abstracted from conformal invariant quantum field theory [3, 7].

According to Wilson [4] the product of two local fields $\phi^i(x)$, $\phi^j(y)$ should admit an asymptotic expansion at short distances of the form

$$\phi^i(\tfrac{1}{2}x)\,\phi^j(-\tfrac{1}{2}x)\Omega = \sum_k C^{ijk}(x)\,\phi^k(0)\Omega\,. \tag{1.1}$$

Herein ϕ^k are local fields, and C^{ijk} are singular c-number functions. In a scale invariant theory, they are homogeneous functions of x. The expansion is presumably valid on all states Ω in the field theoretic domain $\mathcal{D}$ which is created out of the vacuum by polynomials in smeared field operators. We shall however only consider the special case

$$\Omega = \text{vacuum}\,.$$

Studies in perturbation theory [5] indicate that expansion (1.1) is then valid as an asymptotic expansion to arbitrary accuracy for matrix elements $(\psi, \phi^i(x)\,\phi^j(y)\Omega)$, ψ in $\mathcal{D}$.

Among the fields ϕ^k in expansion (1.1) there are derivatives of other local fields. In general there appears $\partial^\mu\phi$ etc. together with any non-derivative field ϕ.

In a conformal invariant QFT the coefficients C^{ijk} in (1.1) are much constrained by summetry. From the work of Ferrara, Gatto, Grillo and Parisi [6] one knows that the terms involving non-derivative fields determine all the others. Using this, the terms involving derivatives of one and the same nonderivative local field can be formally summed. In ref. [7] we proved that the resulting expansions are not only asymptotic but strongly convergent. More precisely one has [7]:

Theorem. Consider conformal invariant quantum field theory (in four space-time dimensions) and suppose that vacuum expansions (1.1) are valid as asymptotic expansions in homogeneous functions of x to arbitrary accuracy for $(\psi, \phi^i(\frac{1}{2}x)\, \phi^i(-\frac{1}{2}x)\Omega)$, ψ in $\mathcal{D}$. Then $\phi^i\phi^j\Omega$ admits a *convergent* expansion,

$$\phi^i(x)\, \phi^j(y)\Omega = \sum_k \int \mathrm{d}z\, \phi^k(z)\Omega\, B^{kij}(z;xy)\,. \tag{1.2}$$

The B^{kij} are generalized c-number functions. Summation is over non-derivative fields ϕ^k only (including the unit operator) and integration is over Minkowski space. Convergence is strong convergence in Hilbert space after smearing with test functions $f(xy)$.

In conformal invariant QFT, nonderivative fields can be recognized by their conformal transformation law [16]. In general QFT we shall speak of a complete set of non-derivative fields ϕ^k if every local field in the theory (including composite ones) can be written as a sum of such ϕ^k and their derivatives.

Example [18]: In a theory of one massless scalar free field $\phi(x)$, one can expand $\phi^*(x)\, \phi(y)\, \Omega$ in terms of $\phi^k(z)\Omega$, with fields ϕ^k that are components of tensor fields (normal products): $:\phi^* \overleftrightarrow{D}_{\mu_1 \ldots \mu_n} \phi:(x)$, where $\overleftrightarrow{D}_{\ldots}$ are some differential operators.

We will now state:

Thesis. A convergent expansion of the form (1.2) is also valid in realistic theories with mass and without conformal symmetry, for arbitrary local fields ϕ^i, ϕ^j.

It will suffice for our applications to assume convergence of $(\psi, \phi^i(x)\, \phi^j(y)\Omega)$ for all states ψ in the field theoretic domain $\mathcal{D}$, after smearing in x, y.

We shall show in the next section how the kernels $B^{kij}(z;xy)$ are determined from Wightman two- and three-point functions.

One does not expect convergence of expansions (1.2) on states Ω other than the vacuum because otherwise one would be dealing with Lie field theories, and they run into trouble with no-go-theorems in more than two dimensions [19, 9].

At present we are only able to give some plausibility arguments (at best) in favor of our thesis. They are

(i) Its validity in conformal invariant QFT.

(ii) According to Schroer, Swieca and Völkel it is also true in massive free field theory [10].

(iii) It looks like a plausible sharpening of the Reeh-Schlieder theorem [11] of axiomatic QFT.

Conformal invariant QFT is special from the point of view of axioms in that it does not satisfy the asymptotic condition. However, massive free field theory does. Also, the asymptotic condition is a condition on the behavior of e.g. Wightman functions at large distances. Our expansions are supposed to be valid in the distribution theoretic sense, i.e. after smearing with test functions. Test functions fall off fast at large distances, and the precise form of the asymptotic behavior there should therefore not be crucial.

The Reeh-Schlieder theorem will be discussed in sect. 6.

In sects. 3, 4, 5 we will discuss the possible significance of our thesis for constructive quantum field theory. It allows to write down expansions of arbitrary n-point Wightman functions in terms of two- and three-point functions of all local fields in the theory (including composite ones). Given their convergence, Lorentz invariance, spectrum condition and positivity are automatic, and locality is satisfied if in addition the 4-point functions satisfy a certain crossing symmetry.

Expansions can be written down not only for Wightman functions, but also for absorptive parts of (off mass shell) scattering amplitudes. It is hoped that they may provide a convenient frame work for phenomenological theories also. Some remarks on this are found in sect. 7. In particular, if it were consistent to make narrow resonance approximations in such expansions, the result would look much like a dual resonance model.

2. The kernels B^{kij}

Let us introduce Wightman two- and three-point functions

$$(\Omega, \phi^i(x)\, \phi^j(y)\Omega) = \Delta^{ij}(x-y) = \int_{\text{sptr.}} dp\; e^{-ip(x-y)}\widetilde{\Delta}^{ij}(p)\,, \tag{2.1}$$

$$(\Omega, \phi^k(z)\, \phi^i(x)\, \phi^j(y)\Omega) = W^{kij}(zxy) = \int_{\text{sptr.}} dp\; e^{-ipz}\, \widetilde{W}^{kij}(p;xy)\,. \tag{2.2}$$

Integration is over the energy-momentum spectrum of the theory, in it $p^0 \geqslant |\boldsymbol{p}|$. We introduce also

$$\widetilde{B}^{kij}(p;xy) = \int dz\; e^{ipz} B^{kij}(z;xy)\,. \tag{2.3}$$

In the following we shall assume that all our fields are hermitian.

Let us take the scalar product of the states in (1.2) with $\widetilde{\phi}^l(p)\Omega = \int dz\; e^{-ipz}\phi^l(z)\Omega$. In view of definitions (2.1) – (2.3) the result reads

$$\widetilde{W}^{lij}(p;xy) = \sum_k \widetilde{\Delta}^{lk}(p)\,\widetilde{B}^{kij}(p;xy) \qquad (p \in \text{sptr}). \tag{2.4}$$

Because of the spectrum condition for $\widetilde{\Delta}$ and $\widetilde{W}$, both sides of this equation vanish automatically if $p \notin$ sptr. Because of the spectral properties of states $\phi^k\Omega$ in (1.2), the value of the kernels $\widetilde{B}^{kij}(p;xy)$ for p outside the spectrum is physically irrelevant; therefore the kernels $B^{kij}(z;xy)$ are not unique. They are however determined by eq. (2.4) to the extent that they are physically relevant. This will become clear later on. It means that we may use for kernels B any solution of (2.4).

It is of course implicit in our thesis that eqs. (2.4) must have a solution. But that is expected also on axiomatic grounds: Consider fixed $p \in$ sptr. and sequences $z = \{z^k\}$ of complex numbers such that $z\Delta(p)\bar{z} = \Sigma_{k,l} z^k \widetilde{\Delta}^{kl}(p) z^l < \infty$. By axiomatic positivity, the two-point matrix is positive semi-definite, $\Delta(p) \geqslant 0$ and diagonalizable. Moreover, if $\bar{z}\,\Delta(p) z = 0$ then z belongs to the zero eigenspace of $\widetilde{\Delta}(p)$ and it follows that also $\Sigma z^k \widetilde{\phi}^k(p)\Omega = 0$ hence $zW \equiv \Sigma_k z^k W^{kij}(p;xy) = 0$ for all x, y, i.e. W is orthogonal to the zero eigenspace. Since $\widetilde{\Delta}(p) \geqslant 0$ there is a pseudo-inverse $\widetilde{\Delta}_*(p)^{-1}$ such that $\widetilde{\Delta}(p)\,\widetilde{\Delta}_*(p)^{-1}\zeta = \zeta$ for ζ orthogonal to the zero eigenspace of $\Delta(p)$. $B = \widetilde{\Delta}_*(p)^{-1}$ W is a formal solution of (2.4).

3. Expansion of Wightman functions

Consider finite sequences of test functions $f = f_0, f_1^{i_1}(x_1) \ldots f_N^{i_N \ldots i_1}(x_N \ldots x_1)$. According to the principles of local QFT, states

$$\psi(f) = \sum_n \int \mathrm{d}x_n \ldots \mathrm{d}x_1\, f_n^{i_n \ldots i_1}(x_n \ldots x_1)\, \psi^{i_n \ldots i_1}(x_n \ldots x_1)\,, \tag{3.1}$$

with

$$\psi^{i_n \ldots i_1}(x_n \ldots x_1) = \phi^{i_n}(x_n) \ldots \phi^{i_2}(x_2)\, \phi^{i_1}(x_1)\,\Omega\,, \tag{3.2}$$

form a dense subset $\mathcal{D}$ of the Hilbert space $\mathcal{H}$ of physical states. In conventional Lagrangian field theory, polynomials in the *fundamental* fields of the theory alone create a dense set of states out of the vacuum also. But even in theories without fundamental fields (e.g. no-cutoff-limit of gauge theories with quark- and colour confinement) the weaker statement above is still expected to be correct. All fields ϕ^k would then be composite in a sense and on an equal footing.

The Wightman functions are

$$\begin{aligned} W^{i_n \ldots i_1}(x_n \ldots x_1) &= (\Omega, \phi^{i_n}(x_n) \ldots \phi^{i_1}(x_1)\,\Omega) \\ &= (\psi^{i_{m+1} \ldots i_n}(x_{m+1} \ldots x_n), \psi^{i_m \ldots i_1}(x_m \ldots x_1))\,, \end{aligned} \tag{3.3}$$

independent of m, $0 \leqslant m \leqslant n$.

We can expand the "state" $\phi^{i_2}(x_2)\,\phi^{i_1}(x_1)\Omega$ in (3.2) according to our thesis to obtain

$$\psi^{i_n \ldots i_1}(x_n \ldots x_1) = \sum_j \int \mathrm{d}z\, \phi^{i_n}(x_n) \ldots \phi^{i_3}(x_3)\, \phi^j(z)\Omega\, B^{j i_2 i_1}(z; x_2 x_1)\,.$$

The expansion process may now be repeated. Next one expands $\phi^{i_3}(x_3)\,\phi^j(z)\Omega$, and so on. As a final result one obtains

$$\psi^{i_n\dots i_1}(x_n \dots x_1) = \sum_k \int \mathrm{d}z\ \phi^k(z)\Omega\, B^{ki_n\dots i_1}(z; x_n \dots x_1)\ , \tag{3.4}$$

with

$$B^{k_n i_n \dots i_1}(z_n; x_n \dots x_1) = \sum_{k_{n-1}} \dots \sum_{k_2}\ \mathrm{d}z_{n-1} \dots \mathrm{d}z_2\ B^{k_n i_n k_{n-1}}(z_n; x_n z_{n-1})$$

$$\dots B^{k_3 i_3 k_2}(z_3; x_3 z_2)\, B^{k_2 i_2 i_1}(z_2; x_2 x_1)\ . \tag{3.5}$$

This is an expansion in terms of states $\phi^k(z)\Omega$ which are obtained by applying only the first power of some field to the vacuum. Thus, states $\phi^k(f)\Omega = \int \mathrm{d}z\, f(z)\,\phi^k(z)\Omega$ (f test functions) span the Hilbert space of physical states.

Let us return briefly to the remark after (2.4). The kernels $B^{kij}(z; xy)$ are determined to the extent that expansion (1.2) must correctly reproduce $(\psi, \phi^i(x)\phi^j(y)\Omega)$ for arbitrary states ψ. Since one only needs to consider states $\psi = \phi^k(f)\Omega$ by what has just been said, it suffices then to satisfy eq. (2.4) for the 3-point functions.

Next we turn to expansions for the Wightman functions. We insert expansion (3.5) for the states into definition (3.3) of the Wightman functions. After Fourier transformation in variables z we obtain

$$W^{i_n \dots i_1}(x_n \dots x_1) = \sum_{k,l} \int_{\text{sptr.}} \mathrm{d}p\ \widetilde{\bar{B}}^{k_{m+1}\dots i_n}(p; x_{m+1} \dots x_n)\ \widetilde{\Delta}^{kl}(p)$$

$$\times\ B^{li_m \dots i_1}(p; x_m \dots x_1)\ , \tag{3.6}$$

independent of m, $0 \leqslant m \leqslant n$.

This formula is valid also for $m = 0{,}1$ and $n-1, n$ if we interpret (cp. (3.4))

$$B^{kl}(p; x) = \mathrm{e}^{ipx}\delta_{kl}\ ,\qquad B^i(p) = \delta_{i0}\ ,\qquad \phi^0 = \mathbb{1}\quad \text{(unit operator)}\ .$$

In the expansions (3.5) and (3.6) there are terms coming from the unit operator. The corresponding 2- and 3-point functions are

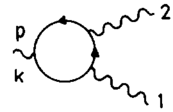
$$\widetilde{\Delta}^{0j}(p) = \delta(p)\,\langle\phi^j\rangle\ ,\qquad \widetilde{B}^{0ij}(p; xy) = \Delta^{ij}(x-y)\ . \tag{3.7}$$

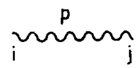
One may assume that the vacuum expectation value $\langle\phi^j\rangle$ vanishes for all fields ϕ^j except the unit operator $\phi^0 = 1$. Eq. (3.5) can be rewritten in the equivalent form

$$\widetilde{B}^{k_n i_n \dots i_1}(p_n; x_n \dots x_1) = \sum_{k_{n-1}} \dots \sum_{k_1} \int \dots \int \prod_{r=2}^{n-1} \left((2\pi)^{-4}\ \mathrm{d}p_r \mathrm{d}z_r\ \mathrm{e}^{-ip_r z_r}\right)$$

$$\times\ \widetilde{B}^{k_n i_n k_{n-1}}(p_n; x_n z_{n-1}) \dots \widetilde{B}^{k_3 i_3 k_2}(p_3; x_3 z_2)\, \widetilde{B}^{k_2 i_2 i_1}(p_2; x_2 x_1)\ . \tag{3.5$'$}$$

Let us introduce a graphical notation

$\tilde{B}^{k i_2 i_1}(p; x_2 x_1) =$ 

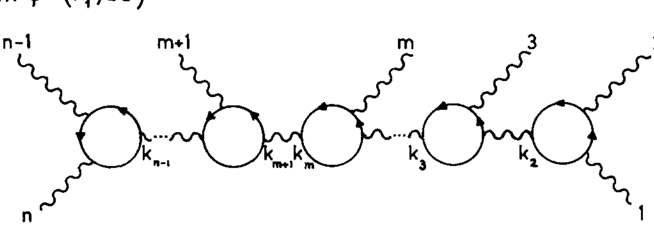
; 

$= \tilde{\Delta}^{ij}(p)$

Reversing all arrows will mean complex conjugation; omitting an arrow will mean that its direction is immaterial.

In this language, expansion (3.6) with (3.5′) reads

$(\Omega, \phi^{i_n}(x_n) \dots \phi^{i_1}(x_1)\Omega) =$ (3.6′)

$= \sum_{k_{n-1}} \dots \sum_{k_2}$ 

(3.6′)

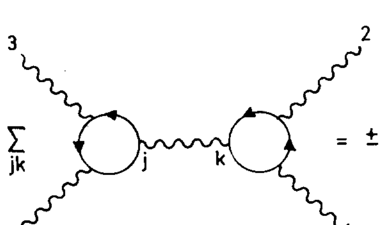
Integration over all internal momenta etc. is understood, summation is over field labels $k_2 \dots k_{n-1}$, and m is arbitrary.

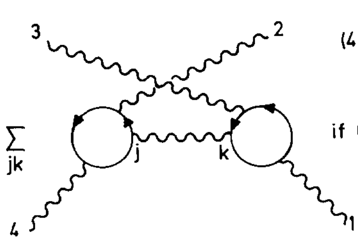
4. Constraint on the 4-point function from locality

Consider 4-point Wightman functions. Because of locality

$$(\Omega, \phi^{i_4}(x_4)\, \phi^{i_3}(x_3)\, \phi^{i_2}(x_2)\, \phi^{i_1}(x_1)\Omega)$$
$$= \pm\, (\Omega, \phi^{i_4}(x_4)\, \phi^{i_2}(x_2)\, \phi^{i_3}(x_3)\, \phi^{i_1}(x_1)\Omega) \qquad \text{if } (x_2 - x_3)^2 < 0\,, \qquad (4.0)$$

with − if both ϕ^{i_2} and ϕ^{i_3} are Fermi fields, and + otherwise. Upon inserting expansion (3.6′) on both sides, this becomes a crossing relation

$\sum_{jk}$ 

$= \pm \sum_{jk}$ 

(4.1)

if $(x_2 - x_3)^2 < 0$

(4.1)

5. The quantum field theory axioms

Let us first review the axiomatic properties of Wightman two- and three-point functions. (We assume throughout that all fields are hermitian). They are [12]

(i) Lorentz invariance. (5.1)

(ii) Positivity and spectrum condition of the 2-point matrix $\widetilde{\Delta}^{ij}(p) = 0$ for $p \notin \bar{V}_+$ i.e. outside forward lightcone; $\widetilde{\Delta}^{ij}(p) = \widetilde{\Delta}^{ji}(p)$ and $\Sigma_{i,j}\bar{z}^i\widetilde{\Delta}^{ij}(p)z^j \geqslant 0$ for all * sequences of complex numbers $\{z^i\}$ and all p.

(iii) The 3-point functions W^{kij} resp. Fourier transforms $\widetilde{W}^{kij}$ satisfy (a) hermiticity condition, $W^{kij}(zxy) = \bar{W}^{jik}(yxz)$; (b) spectrum condition $\widetilde{W}^{kij}(p;xy) = 0$ for $p \notin \bar{V}_+$; (c) locality $\widetilde{W}^{i_3i_2i_1}(x_3x_2x_1) = W^{i_3i_2i_1}(x_3x_1x_2)$ if $(x_1 - x_2)^2 < 0$, etc. Locality of 2-point functions has not been displayed since it is automatic consequence of (i) and (ii). Also omitted are distribution theoretic axioms (temperedness), they should be added according to taste. The 3-point functions $\widetilde{W}^{kij}(p;x_1x_2)$ must be measured in p after smearing in x_1, x_2, with support properties described at the end of sect. 2.

Expansions (3.6) could be useful in constructive quantum field theory. This is suggested by the following:

Theorem. Suppose that one can find a set of 2-point functions $\widetilde{\Delta}^{ij}(p)$ and amputated 3-point functions $\widetilde{B}^{kij}(p;xy)$ such that (i) axiomatic properties (5.1 i, ii, iii) are true for the 2- and 3-point functions when W^{kij} are defined by eqs. (2.4); (ii) expansions (3.5), (3.6) converge; (iii) crossing relations (4.1) are fulfilled. Then the Wightman functions defined by expansion (3.6) with (3.5) (or 3.6')) are independent of the choice of $m(0 \leqslant m \leqslant n)$ and satisfy the usual postulates of local quantum field theory: Lorentz invariance, spectrum condition, positivity and locality.

If a theory can be constructed in this way we call it dual, because crossing relation (4.1) is reminiscent of dual resonance models. In this language, our thesis postulates duality of local QFT.

Let us verify the assertion of theorem. First we show that the r.h.s. of eq. (3.6) is independent of m. This only requires to reshuffle the amputations on the 3-point functions. They can be reshuffled with the help of the following identity

$$\sum_k \int \mathrm{d}y\, \mathrm{e}^{-ipy}\, \widetilde{\Delta}^{lk}(p)\, \widetilde{B}^{kij}(p;xy)$$

$$= \sum_k \int \mathrm{d}y'\, \mathrm{e}^{ip'y'}\, \widetilde{\bar{B}}^{kil}(p;xy)\, \widetilde{\Delta}^{kj}(p)\,. \tag{5.2}$$

This identity follows from hermiticity condition (5.1 iii) and hermiticity of the ma-

* $\lim_{N\to\infty}\Sigma_{i,j \leqslant N}\bar{z}^i\widetilde{\Delta}^{ij}(p)\, z^j$ is allowed to converge to $+\infty$ for infinite sequences.

trix $\widetilde{\Delta}^{lk}(p)$. They imply that in (5.2),

$$\text{l.h.s.} = (2\pi)^{-4} \int \mathrm{d}y\, \mathrm{d}y'\, \mathrm{e}^{-i(py-p'y')}\, W^{lij}(y'\, xy)$$

$$= (2\pi)^{-4} \int \mathrm{d}y\, \mathrm{d}y'\, \mathrm{e}^{-i(py-p'y')}\, \overline{W}^{jil}(yxy') = \text{r.h.s}\,.$$

Inserting (5.2) into (3.6) with (3.5′), we see that the r.h.s. of (3.6) remains unchanged when $m-1$ is substituted for m.

Next we turn to the axioms.

Lorentz invariance is obvious. The *spectrum condition* is also obvious because m can be chosen arbitrary in (3.6′) and the propagator $\widetilde{\Delta}^{k_{m+1}k_m}(p)$ which joins the bubbles with legs m and $m+1$ can only transmit positive energy according to (5.1 ii).

Positivity. Let $f_0, f_1^{i_1}(x_1), \ldots f_N^{i_N \ldots i_1}(x_N \ldots x_1)$ be an arbitrary finite sequence of test functions. Define

$$z^j(p) = \sum_n \sum_{\{i\}} \int \mathrm{d}x_n \ldots \mathrm{d}x_1\, B^{ji_n \ldots i_1}(p; x_n \ldots x_1)\, f_n^{i_n \ldots i_1}(x_n \ldots x_1)\,.$$

Then, using (3.6)

$$\sum_{r,s} \sum_{\{i\}\{j\}} \int \mathrm{d}x_r \ldots \mathrm{d}x_1\, \mathrm{d}x_1' \ldots \mathrm{d}x_s'\, \overline{f}^{j_1 \ldots j_s}(x_1' \ldots x_s')\, W^{j_1 \ldots j_s i_1 \ldots i_r}(x_1' \ldots x_s'\, x_r \ldots x_1)$$

$$\times f^{i_r \ldots i_1}(x_r \ldots x_1) = \sum_{k,l} \int \mathrm{d}p\, \overline{z}^k(p)\, \widetilde{\Delta}^{kl}(p)\, z^l(p) \geqslant 0\,,$$

as required by axiomatic positivity [12].

Locality requires that

$$\pm W^{i_N \ldots i_{r+1} i_r \ldots i_1}(x_N \ldots x_{r+1} x_r \ldots x_1) = W^{i_N \ldots i_r i_{r+1} \ldots i_1}(x_N \ldots x_r x_{r+1} \ldots x_1)$$

$$\text{if } (x_r - x_{r+1})^2 < 0 \qquad (r = 1 \ldots n-1)\,. \tag{5.3}$$

We distinguish two cases. If $r = 1$ or $r = n-1$, validity of (5.3) follows from locality property (5.1 iii) of the 3-point functions, $\widetilde{W}^{kij}(p; xy) = \pm \widetilde{W}^{kji}(p; yx)$ for $(x-y)^2 < 0$, which carry over to the kernels $\widetilde{B}^{kij}(p; xy)$ by (2.4). If $r = 2 \ldots n-2$, we put $m = r$ in expansion (3.6′). Validity of (5.3) follows then by inserting (4.1) for the center piece of the diagram (3.6′).

In conclusion we see that, given convergence of expansions (3.6), verification of locality only involves inspection of the 4-point functions to make sure that they are local in the sense of (4.0).

Remark: Since the individual terms in the expansion (3.6) do not satisfy locality, they need not share the full axiomatic x-space analyticity of Wightman functions: They will still be analytic in the extended tube but not in the permuted extended tube. After summing up, the unphysical singularities must cancel if crossing relation

(4.1) holds. This point has been stressed by Ferrara et al. in the context of conformal invariant QFT [13].

6. The Reeh-Schlieder theorem

For simplicity of writing suppose that we are given a field $\phi(x)$ that is irreducible so that states of the form $\psi(f) = \Sigma \int dx_n \dots dx_1 \, f_n(x_n \dots x_1)\, \phi(x_n) \dots \phi(x_1)\Omega$ form a dense subspace of the Hilbert space of physical states. The Reeh-Schlieder theorem asserts [11, 12] that the same is then still true if one allows only test functions f whose support is constrained in all arguments to an arbitrarily small open subset O of Minkowski space M^4, *viz*, $f_n(x_n \dots x_1) = 0$ unless $x_j \in O$ for all j. It has often been said that this is a surprising result.

Let us introduce new variables

$$x_j = x + \xi_j \quad \text{with } \sum \xi_j = 0\,, \qquad x = \frac{1}{n}\sum x_j \qquad (j = 1 \dots n)\,.$$

Given an arbitrarily small neighborhood O' of zero in M^4, there exists an open set $O \subset M^4$ such that $x_j \in O$ for all j implies $\xi_j \in O'$. Test functions $f(x_n \dots x_1) = f'(x, \xi_{n-1} \dots \xi_1)$ with compact support can be approximated by sums of products $\Sigma_r h_r(x) g_r(\xi_{n-1} \dots \xi_1)$. No harm is done if we drop the constraints on the support of f_n in the c.m. variable x. Thus it is still true that states of the form

$$\sum \int dx\, h_r(x) \int d\xi_{n-1} \dots d\xi_1 \, g_r(\xi_{n-1} \dots \xi_1) \phi(x + \xi_1) \dots \phi(x + \xi_n)\Omega\,,$$

with

$$g_r(\xi_{n-1} \dots \xi_1) = 0 \quad \text{unless all } \xi_j \in O' \; (j = 1 \dots n; \; \sum_1^n \xi_j = 0)\,. \tag{6.1}$$

span the Hilbert space of physical states for arbitrarily small open neighborhood O' of zero.

Let us now remember how composite fields are related to $\phi(x)$. Wilson expansions for products of fields $\phi(x + \xi_n) \dots \phi(x + \xi_1)$ at short distances [4,5] give at the same time formulae for all composite fields $\phi^k(x)$. Usually one uses point splitting, one may write just as well

$$\phi^k(x) = \lim_{\nu \to \infty} \int_{O'_\nu} \dots \int_{O'_\nu} d\xi_{n-1} \dots d\xi_1 \; h_\nu\,(\xi_{n-1} \dots \xi_1)\, \phi(x + \xi_n) \dots \phi(x + \xi_1)\,, \tag{6.2}$$

where h_ν is a suitable sequence of test functions with support constrained to $\xi_j \in O'_\nu$ $(j = 1 \dots n)$, and the open set O'_ν shrinks to zero as $\nu \to \infty$.

As we have discussed earlier, our thesis says that states of the form $\int dx\, h(x)\, \phi^k(x)\Omega$ span the Hilbert space of physical states. Comparing (6.2) with (6.1) we see that this means roughly that the corollary (6.1) of the Reeh-Schlieder theorem continues to hold when we shrink the support in the relative coordinates to the point zero.

7. Phenomenology

More than a decade ago, the bootstrap idea emerged in analytic S-matrix theory [14]. It says for instance that the ρ is bound together from π's by forces that come largely from the exchange of ρ's again (or of its Regge trajectory). Veneziano duality is a further development of such ideas. In S-matrix language, crossed-channel exchanges produce direct channel poles. Crossing relations (5), (8) may be looked upon as a QFT pendant of this. Let us then try to make a connection as good as we can.

Consider the elastic scattering of two identical scalar particles which are their own antiparticles

$$1 + 2 \rightarrow 3 + 4 \,. \tag{7.1}$$

Consider the off mass shell scattering amplitude $A(p_1 \dots p_4) = A(stu; p_1^2 \dots p_4^2)$, $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $s + t + u = \Sigma p_i^2$. Its absorptive part in the channel (12) is [15]

$$\mathrm{Abs}_{12} A(p_1 \dots p_4)\,(2\pi)^4\, \delta\left(\sum p_i\right) = \frac{1}{16\pi^3}\int \mathrm{d}x_1 \dots \mathrm{d}x_4 \exp\left(-i \sum p_j x_j\right)$$
$$\times\, (\Omega, R'\{\phi(x_3)\phi(x_4)\}\, R'\{\phi(x_1)\,\phi(x_2)\}\Omega) \,, \tag{7.2}$$

with

$$R'\{\phi(x)\phi(y)\} = -iK_x K_y \theta(x-y)[\phi(x), \phi(y)] \,, \qquad K_x = \Box_x + m^2 \,,$$

i.e. R' is a retarded product. The *partially* retarded function appearing here can also be expanded as in (3.6), Define

$$C^j(p; xy) = -iK_x K_y \theta(x-y)\{\widetilde{B}^j(p; xy) - \widetilde{B}^j(p; yx)\} \,.$$

Then we can expand

$$(\Omega, R'\{\phi(x_3)\,\phi(x_4)\}\, R'\{\phi(x_1)\phi(x_2)\}\Omega) = \sum_{ij} \int_{\mathrm{sptr.}} \mathrm{d}p\, \bar{C}^i(p; x_3 x_4)\, \widetilde{\Delta}^{ij}(p)$$
$$\times\, C^j(p; x_1 x_2) \,. \tag{7.3}$$

This can be inserted in (7.2). Convergence is still in the distribution theoretic sense, i.e. after smearing with test functions. This means essentially that one must sum first before going to sharp masses on the mass shell $p_i^2 = m^2$. To extract information from crossing relations one must determine the absorptive part in the crossed channel $1 + 3 \to 2 + 4$ from $\mathrm{Abs}_{12}A(p_1 \dots p_4)$. In the QFT context we did this by use of axiomatic analyticity in x space, the Wightman functions are determined by their values for relatively space-like arguments, where (4.1) holds. In analytic S-matrix theory one uses p-space analyticity instead. One would first have to recover the scattering amplitude from its absorptive part by dispersion relations – hopefully they converge without subtraction for some range of t. Then one would have to analytically continue in s, t to the crossed channel assuming e.g. Mandelstam analyticity. Finally one could then take the absorptive part in the crossed channel and write down its expansion analog to (7.3).

In conclusion, it it were not for a questionable interchange of limits (expanding and going to the mass shell) we would get a relation between the expansions of the on-shell absorptive parts of scattering amplitudes in crossed channels at the cost of having to use analyticity in p space in place of x space. If one is willing to make the further hypothesis (approximation) that the 2-point function $\widetilde{\Delta}^{ij}(p)$ can be written as a sum of δ functions supported at physical particle masses then the result would look much like a dual resonance model.

Let us add few remarks on Regge trajectories and fundamental fields. Consider temporarily QFT in an arbitrary not necessarily integer number D of space-time dimensions. In models, local fields come in families. We call them towers. For instance, in conformal invariant ϕ^3 theory in $D = 6 + \epsilon$ dimensions [16] the fundamental field ϕ has anomalous dimension $d = \frac{1}{2}D - 1 + \Delta$ with anomalous part $\Delta = \frac{1}{18}\epsilon + \dots$. Then there is a tower of traceless symmetric tensor fields $O_{\alpha_1 \dots \alpha_2}$ of even rank $s = 2, 4, \dots$ with dimension $d_s = D - 2 + s + \sigma_s$ whose anomalous part

$$\sigma_s = \left[\frac{1}{9} - \frac{1}{3(s+2)(s+1)}\right]\epsilon + \dots , \qquad \sigma_s \to 2\Delta \qquad \text{as } s \to \infty .$$

They are composite fields quadratic in ϕ. The dimensions of the component fields become additive in the limit $s \to \infty$, so we have asymptotically straight trajectories in dimension. One would think that there is also a scalar field – call it ϕ^2 – with dimension $d_0 = D - 2 + \sigma_0$. But such a field does actually not exist, it simply does not appear in operator product expansions of $\phi(x)\phi(0)$ or in expansions (1.2). [We see here the ghost of the field equations, remember that we count only non-derivative fields, they transform differently.] Instead the "shaddow" of the missing point on the trajectory appears as the fundamental field with dimension $d = D - d_0$. To order ϵ this relation between dimensions may be checked from the explicit formulae given above. Because of the normal product algorithm of renormalized perturbation theory [5] one would expect that towers of fields exist whether or not there is conformal symmetry. Moreover, one will speculate that in the real world there is a connection between towers of fields and Regge trajectories – the fields of suitable rank would serve as interpolating fields for particles on the Regge trajectories [17].

We have seen above a model with a fundamental field. It is not a member of a tower but appears instead as a shadow of a missing member in the tower. Typically it has a dimension $d < \frac{1}{2}D$. In the present frame work one could however very well imagine theories without any fundamental fields.

If crossing relations (4.1) have any solutions such that expansions (3.6) converge, they will usually also have solutions which do not involve any fundamental field. This is not the whole issue though. Take for instance $\theta_{\mu\nu}$, the stress tensor. Everybody believes in that local field. Consider vacuum expectation values of products of stress tensors. Their expansions (3.6) will only involve fields that are singlets under all exact internal symmetries, and fundamental fields will usually not appear in them at all. Nevertheless we have at hand a perfectly respectable Wightman QFT and the crossing relation (4.0) will be satisfied. We call this the rudimentary theory. Its main short-coming is that it will usually not furnish interpolating fields for all particles. Nevertheless one should see these particles in stress-tensor correlation functions, because particles could in principle be detected in a laboratory by the gravitational forces which they exert.

The question of fundamental fields and quanta associated with them is thus a rather subtle one. Nevertheless it may be useful to remark that one could in principle start the bootstrap – i.e. try to solve crossing relations (4.1) – without knowing whether there are fundamental fields, or even what the internal symmetry group is, by considering the rudimentary theory first.

The rudimentary theory will fix an algebra of observables (measurements based on gravitational forces). According to ideas of Doplicher, Haag and Roberts in algebraic QFT this algebra should in turn fix the complete theory uniquely, including internal symmetries [18].

The author is much indebted to R. Haag, J. Jersak, M. Lüscher, B. Schroer and K. Symanzik for helpful and stimulating discussions.

References

[1] G. Veneziano, Elementary processes at high energy (Academic press, New York, 1971).
[2] P.H. Frampton, Dual resonance models (Benjamin, Reading, 1974).
[3] Abdus Salam and G. Mack, Ann. of Phys. 53 (1969) 174;
G. Mack, J. de Phys. 34 (1973) 99; Renormalization and invariance in quantum field theory ed. E.R. Caianello (Plenum Press, New York, 1974); Lecture notes in physics 37 (eds. H. Rollnik and K. Dietz (Springer, Heidelberg, 1975);
V.K. Dobrev, V.B. Petkova, S.G. Petrova and I.T. Todorov, Dynamical derivation of operator product expansions in Euclidean conformal quantum field theory. IAS preprint, Princeton (July 1975), Phys. Rev. D, to be published;
V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova and I.T. Todorov, Harmonic analysis on the n-dimensional Lorentz group and its application to conformal quantum field theory (to be published in Lecture notes in physics);
A.A. Migdal, 4-dimensional soluble models of conformal field theory, Landau institute preprint, Chernogolovka (1972);
A.M. Polyakov, ZhETF 66 (1974) 23; JETP (Sov. Phys.) 39 (1974) 10;
M. Lüscher and G. Mack, Comm. Math. Phys. 41 (1975) 203.

[4] K.G. Wilson, Phys. Rev. 179 (1969) 1499.
[5] W. Zimmermann, Comm. Math. Phys. 15 (1969) 208; Lectures on elementary particles and quantum field theory, Brandeis University, vol. 1 (MIT press, 1970);
J.M. Löwenstein, Phys. Rev. D4 (1971) 2281.
[6] S. Ferrara, R. Gatto and A.F. Grillo, Springer tracts in modern physics 67 (Springer, Heidelberg, 1973) and references therein; Ann. of Phys. 76 (1973) 116;
S. Ferrara, R. Gatto, A.F. Grillo and G. Parisi, Nucl. Phys. B49 (1972) 77; Nuovo Cimento Letters (1972) 115.
[7] G. Mack, Convergence of operator product expansions on the vacuum in conformal invariant quantum field theory, DESY 76/30 (June 1976) and to be published in Comm. Math. Phys.;
M. Lüscher, Comm. Math. Phys., in press;
W. Rühl and B.C. Yunn, Comm. Math. Phys., in press.
[8] B. Schroer, J.A. Swieca and A.H. Völkel, Phys. Rev. D11 (1975) 1509.
[9] K. Baumann, Comm. Math. Phys. 47 (1976) 69.
[10] B. Schroer, private communication.
[11] H. Reeh and S. Schlieder, Nuovo Cimento 22 (1961) 1051.
[12] R.F. Streater and A.S. Wightman, *PCT*, spin statistics, and all that (Benjamin, New York, 1964).
[13] S. Ferrara, A.F. Grillo and R. Gatto, Nuovo Cimento 26A (1975) 226.
[14] G.F. Chew and S. Mandelstam, Nuovo Cimento 19 (1961) 752;
G.F. Chew, The analytic *S*-matrix (Benjamin, New York, 1966).
[15] H. Lehmann, Nuovo Cimento 10 (1958) 579.
[16] G. Mack, Lecture notes in physics 17, eds. W. Rühl and V.A. Vancura (Springer, Heidelberg, 1973).
[17] J.M. Cornwall and R. Jackiw, Phys. Rev. D4 (1971) 367.
[18] S. Doplicher, R. Haag and J.E. Roberts, Comm. Math. Phys. 13 (1969) 1; 35 (1974) 49.
[19] J.T. Lopuszanski, Phys. Letters 8 (1964) 85;
O.W. Greenberg, Ann. of Phys. 16 (1961) 158.