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Operator Product Expansions on the Vacuum in Conformal Quantum Field Theory in Two Spacetime Dimensions*

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Abstract. Let $\varphi_1(x)$ and $\varphi_2(y)$ be two local fields in a conformal quantum field theory (CQFT) in two dimensional spacetime. It is then shown that the vector-valued distribution $\varphi_1(x)\varphi_2(y)|0\rangle$ is a boundary value of a vectorvalued holomorphic function which is defined on a large conformally invariant domain. By group theoretical arguments alone it is proved that $\varphi_1(x)\varphi_2(y)|0\rangle$ can be expanded into conformal partial waves. These have all the properties of a global version of Wilson's operatór product expansions when applied to the vacuum state $|0\rangle$. Finally, the corresponding calculations are carried out more explicitly in the Thirring model. Here, a complete set of local conformally covariant fields is found, which is closed under vacuum expansion of any two it its elements (a vacuum expansion is an operator product expansion applied to the vacuum).

I. Introduction

Some time ago partial wave expansions of the euclidean Greensfunctions (i.e. the Schwingerfunctions) of a CQFT have been established [1]. These expansions are useful to solve the nonlinear dynamical integralequations and also help to study the implications of locality. However, when one tries to express Osterwalder-Schrader-positivity (i.e. the euclidean counterpart of ordinary Wightman-positivity) in terms of the conformal partial waves, a complicated process of analytic continuation in the expansion parameters is needed [2]. In fact, one performs something like an inverse Sommerfeld-Watson-transform. The resulting discrete expansion is then termwise positive. Moreover the series looks exactly like a globally valid form of an operator product expansion applied to the vacuum. The above mentioned manipulations with the euclidean partial waves can only be done under suitable technical assumptions. For instance, to prove the validity of the inverse Sommerfeld-Walson-transform one must make sure that the partial waves have appropriate asymptotic properties in the expansion parameters.

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Therefore, it is natural to try to obtain the discrete expansions directly from an analysis of the Wightmandistributions rather than making the detour via the euclidean formalism. It is the aim of this paper to carry out such a program [3].

Globally valid operator product expansions in CQFT are also interesting from the following point of view. A CQFT is in general not a particle theory. Therefore, there is a priori no natural language to describe such a theory. In case there are "sufficiently" many local, conformally covariant fields, they can be looked at as the fundamental entities of a new language. The interelation between them (i.e. the dynamics) is then expressed by the operator product expansions. As will be discussed later, the Thirring model exhibits such a structure.
Considering CQFT'ies as valuable models for more realistic QFT'ies, one can try to translate the above picture to the general case. Such a proposal has recently been advanced by Mack [4].

The restriction of the present work to two dimensional CQFT'ies needs some justification:

a) The kinematic complexity grows rapidly by going from two to four spacetime dimensions.

b) The results obtained hold presumably also in the case of four dimensions. In fact all the deeper mathematical tools are also available for this case. Moreover, the discrete expansions emerging from the euclidean method are equally valid for any spacetime dimension.

c) The only not completely trivial, soluble models live in two dimensions.
The paper is organized as follows. For the readers convenience and to fix notations some wellknown facts concerning the conformal group in two dimensions are collected in Section II. The definition of a CQFT is also included here. Section III deals with the problem of describing the minimal conformally invariant analyticity domain for two point vectors $\varphi_1(x)\varphi_2(y)|0\rangle$. Then, in Section IV, the tensorproduct of two holomorphic, irreducible, unitary representations of the universal covering $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ of the group $\mathrm{SL}(2,\mathbb{R})$ is decomposed into its irreducible parts. The result is applied (Section V) to the vectors $\varphi_1(x)\varphi_2(y)|0\rangle$ yielding their vacuum expansion. In the last section the Thirring model is analysed; thereby the dimensions and spins of a complete set of local fields are determined.

II. The Conformal Group in Two Spacetime Dimensions

2.1. *Some Definitions*

To exhibit the action of the conformal group on a point (x^0, x^1) of two dimensional Minkowskispace M it is convenient to introduce lightcone variables, namely:

$$x_+ = x^0 + x^1 ; x_- = x^0 - x^1 . \tag{2.1}$$

The conformal group C is then defined to be

$$\mathrm{SO}_0(2,2)/Z_2 \cong \mathrm{Sl}(2,\mathbb{R})/Z_2 \times \mathrm{Sl}(2,\mathbb{R})/Z_2 . \tag{2.2}$$

In fact the group $\mathrm{Sl}(2,\mathbb{R})\times\mathrm{Sl}(2,\mathbb{R})$ acts on $(x_+,x_-)\in M$ as follows:

$$g=g_+\times g_-;\qquad g_\pm=\begin{pmatrix}\sigma_\pm & \tau_\pm\\ \xi_\pm & \eta_\pm\end{pmatrix}\in\mathrm{Sl}(2,\mathbb{R}),\tag{2.3}$$

$$g(x_+,x_-)\doteq\left(\frac{\sigma_+x_++\tau_+}{\xi_+x_++\eta_+};\quad\frac{\sigma_-x_-+\tau_-}{\xi_-x_-+\eta_-}\right).$$

Therefore, the study of the conformal group in two spacetime dimensions boils down to the investigation of the group $\mathrm{Sl}(2,\mathbb{R})$.

The transformation law (2.3) is not well defined, since $\xi x+\eta$ may vanish. However, this problem can be solved by compactifying Minkowskispace, i.e. adding points at infinity (e.g. [5, 6]).

The group $\mathrm{Sl}(2,\mathbb{R})$ is a simple, threedimensional Liegroup. Its Liealgebra $\mathfrak{sl}(2,\mathbb{R})$ can be represented by all real, traceless 2×2-matrices. I will use the following basis for $\mathfrak{sl}(2,\mathbb{R})$:

$$H\doteq\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix};\quad D\doteq\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix};\quad P\doteq\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.\tag{2.4}$$

For $g=\begin{pmatrix}\sigma & \tau\\ \xi & \eta\end{pmatrix}\in\mathrm{Sl}(2,\mathbb{R})$ and $x\in\mathbb{R}$, set

$$g(x)\doteq(\sigma x+\tau)/(\xi x+\eta)\tag{2.5}$$

(one should compactify $\mathbb{R}$).

The generators D and P generate dilatations and translations of x, respectively:

$$e^{\varrho D}=\begin{pmatrix}e^{\varrho} & 0\\ 0 & e^{-\varrho}\end{pmatrix}\doteq a;\qquad e^{\tau P}=\begin{pmatrix}1 & \tau\\ 0 & 1\end{pmatrix}\doteq n.\tag{2.6}$$

The generator H generates a maximal compact subgroup of $\mathrm{Sl}(2,\mathbb{R})$:

$$e^{\psi H}=\begin{pmatrix}\cos\psi & \sin\psi\\ -\sin\psi & \cos\psi\end{pmatrix}\doteq k.\tag{2.7}$$

Every element g of $\mathrm{Sl}(2,\mathbb{R})$ can be decomposed uniquely and in a differentiable manner into $k\cdot a\cdot n$ (the Iwasawa decomposition). Therefore, as manifolds, one has the equality:

$$\mathrm{Sl}(2,\mathbb{R})\cong S^1\times\mathbb{R}_+\times\mathbb{R};\qquad g=\begin{pmatrix}\cos\psi & \sin\psi\\ -\sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}\sigma & \tau\\ 0 & \sigma^{-1}\end{pmatrix}.\tag{2.8}$$

Here, S^1 is the unit circle, $\mathbb{R}_+=\{\sigma\in\mathbb{R}\,|\,\sigma>0\}$.

Because of the factor S^1, these manifolds are not simply connected. Unrolling S^1 yields the universal covering $\widetilde{\mathrm{Sl}(2,\mathbb{R})}$ of $\mathrm{Sl}(2,\mathbb{R})$:

$$\widetilde{\mathrm{Sl}(2,\mathbb{R})}\cong\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}.\tag{2.9}$$

The canonical projection $\pi:\widetilde{\mathrm{Sl}(2,\mathbb{R})}\to\mathrm{Sl}(2,\mathbb{R})$ is given by

$$\pi(\psi,\sigma,\tau)=\begin{pmatrix}\cos\psi & \sin\psi\\ -\sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}\sigma & \tau\\ 0 & \sigma^{-1}\end{pmatrix}.\tag{2.10}$$

π is one to one on the open set

$$O=\{(\psi,\sigma,\tau)\in\widetilde{\mathrm{Sl}(2,\mathbb{R})}\,|\,|\psi|<\pi\}. \tag{2.11}$$

The group multiplication law on $\widetilde{\mathrm{Sl}(2,\mathbb{R})}$ could be written down in terms of the coordinates (ψ,σ,τ). I will however never use this complicated formula. It suffices to know that

$$\left.\begin{array}{l}\pi[g_1\cdot g_2]=\pi(g_1)\cdot\pi(g_2)\\ (0,1,0)\cdot g=g\end{array}\right\}\quad\text{for all}\quad g_1,g_2,g\in\widetilde{\mathrm{Sl}(2,\mathbb{R})}. \tag{2.12}$$

For example, one has

$$(\psi,1,0)\cdot(\psi',1,0)=(\psi+\psi',1,0).$$

The center $\mathfrak{Z}$ of $\widetilde{\mathrm{Sl}(2,\mathbb{R})}$ is generated by just one element, namely

$$\begin{aligned}&z\doteq(\pi,1,0)\,;\qquad \pi(z)=\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\\ &\mathfrak{Z}=\{(n\cdot\pi,1,0)\,|\,n\in\mathbb{Z}\}.\end{aligned} \tag{2.13}$$

2.2. *The Irreducible, Analytic Representations of* $\widetilde{\mathrm{Sl}(2,\mathbb{R})}$ *(*[8, 9]*)*

A nontrivial, unitary representation of a Lie group $\tilde{G}$ is called analytic, if some of its nonzero generators is represented by $\sqrt{-1}$ times a positive selfadjoint operator (for a general treatment of such representations, see [7]). For $\tilde{G}=\widetilde{\mathrm{Sl}(2,\mathbb{R})}$ all the irreducible analytic representations are known explicitly. They are induced representations on the homogeneous space $\tilde{G}/\tilde{K}$, where $\tilde{K}$ is the one parameter subgroup of $\tilde{G}$ generated by H (2.4). It turns out that $\tilde{G}/\tilde{K}$ is isomorphic to the upper half plane Π:

$$\tilde{G}/\tilde{K}\cong\Pi\doteq\{w\in\mathbb{C}\,|\,\mathrm{Im}\,w>0\}. \tag{2.14}$$

$\tilde{G}$ acts on Π as follows:

$$\begin{aligned}&g\in\tilde{G}\,;\qquad \pi(g)=\begin{pmatrix}\sigma&\tau\\ \xi&\eta\end{pmatrix}\in\mathrm{Sl}(2,\mathbb{R}),\\ &w\in\Pi\Rightarrow g(w)\doteq(\sigma w+\tau)/(\xi w+\eta).\end{aligned} \tag{2.15}$$

Note that (2.15) is well defined since $\xi w+\eta\neq0$ for all $w\in\Pi$.

The analytic representations of $\tilde{G}$ are carried by the following two types of function spaces:

For $n>0$ define $H_n(H_n^*)$ to be the linear space of all holomorphic (anti-holomorphic) functions F on Π with the additional properties:

a) F has a C^∞-boundary value for $\mathrm{Im}\,w\searrow0$.

b) For $\mathrm{Im}\,w\geqq0$ and $|w|\to\infty$ the following expansions are valid:

$$\begin{aligned}&F(w)\sim(1-iw)^{-n}(a_0+a_1w^{-1}+a_2w^{-2}+\ldots)\qquad(F\in H_n)\\ &F(w)\sim(1+iw^*)^{-n}(a_0+a_1w^{-1}+a_2w^{-2}+\ldots)\quad(F\in H_n^*)\end{aligned} \tag{2.16}$$

(w^* denotes the complex conjugate of w).

Of course, $F \in H_n$ iff $F^* \in H_n^*$. From now on I will restrict the discussion to H_n only.

The action of $\tilde{G}$ on H_n is first specified for $g \in O$, the open neighbourhood (2.11) of 1. Set $\pi(g) = \begin{pmatrix} \sigma & \tau \\ \xi & \eta \end{pmatrix}$ and define:

$$(T_n(g)F)(w) = (-\xi w + \sigma)^{-n} F(g^{-1}(w)), \tag{2.17}$$
$$|\arg(-\xi w + \sigma)| < \pi \qquad (g \in O).$$

If g_1, g_2 and $g_1 \cdot g_2$ are contained in O, the multiplication law is satisfied:

$$T_n(g_1) \cdot T_n(g_2) = T_n(g_1 \cdot g_2). \tag{2.18}$$

Due to the fact that $\tilde{G}$ is simply connected, $T_n(\cdot)$ may be extended uniquely to all of $\tilde{G}$ such that (2.18) holds. Thus for $g = g_1 \cdot g_2 \cdot \ldots \cdot g_k$, $g_j \in O$, one defines:

$$T_n(g) = T_n(g_1) \cdot T_n(g_2) \cdot \ldots \cdot T_n(g_k).$$

For instance, one obtaines in this way:

$$[T_n((k\pi, 1, 0))F](w) = e^{-ik\pi \cdot n} F(w) \qquad (k \in \mathbb{Z}). \tag{2.19}$$

Hence $T_n(z) = e^{-i\pi n}$ (see (2.13)).

An invariant scalarproduct on H_n is given by

$$(F_1, F_2)_n \doteq (n-1)\pi^{-1} \int_{\Pi} |dw| F_1(w)^* (\operatorname{Im} w)^{n-2} F_2(w), \tag{2.20}$$
$$(F_1, F_2 \in H_n;\ |dw| \doteq dx \cdot dy \quad \text{for} \quad w = x + iy \in \Pi).$$

This integral converges absolutely for $n > 1$ and can be analytically continued down to all $n > 0$ by means of a suitable chosen orthogonal basis [8]. By completion of H_n with respect to $(\cdot, \cdot)_n$ one obtaines a Hilbert space $\mathscr{H}_n$. The operators T_n extend by continuity to all of $\mathscr{H}_n$ yielding a unitary, irreducible, analytic representation of $\widetilde{\mathrm{Sl}(2, \mathbb{R})}$. These representations will be referred to as the holomorphic irreducible representations of $\tilde{G}$, in contrast to the antiholomorphic representations obtained by starting from H_n^* instead of H_n (in fact, as can be seen from (2.20), the representations on $\mathscr{H}_n$ and $\mathscr{H}_n^*$ are dual with respect to each other).

Note that the holomorphic irreducible representations of $\widetilde{\mathrm{Sl}(2, \mathbb{R})}$ are all irreducible representations $U(\cdot)$ such that $(1/i)U(H) \leq 0$, H defined by (2.4).

Any function $F \in H_n$ has the Fourier representation:

$$F(w) = \int_0^\infty dp e^{ipw} \tilde{F}(p); \quad \tilde{F}(p) = p^{n-1} \cdot g(p) \tag{2.21}$$
$$g(p) \in \mathscr{S}(\bar{\mathbb{R}}_+).$$

$(\varphi(\bar{\mathbb{R}}_+) = \{g|_{p \geq 0} | g \in \mathscr{S}\}; \mathscr{S}$ is the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}$). Conversely, any function $F(w)$ which is representable in the form (2.21) belongs to H_n. In terms of $\tilde{F}(p)$ the scalar product (2.20) becomes:

$$(F_1, F_2)_n = 2^{2-n} \Gamma(n) \int_0^\infty dp p^{1-n} \tilde{F}_1^*(p) \cdot \tilde{F}_2(p). \tag{2.22}$$

From this, one easily observes, that

$$G_n(w,w') \doteq [2^{2-n}\Gamma(n)]^{-1} \int_0^\infty dp p^{n-1} e^{ip(w'-w^*)} \tag{2.23}$$

has the reproducing property:

$$(G_n(w,\cdot),F)_n = F(w) \quad \text{for all} \quad F \in \mathscr{H}_n . \tag{2.24}$$

The explicit form of $G_n(w,w')$ is:

$$G_n(w,w') = 2^{n-2} e^{-i(\pi/2)n} (w^* - w')^{-n} \tag{2.25}$$

$$(w, w' \in \Pi ; |\arg(w^* - w')| < \pi) .$$

2.3. *Formulation of a General* CQFT *in Two Spacetime Dimensions*

There are two points to take care of: first, in two dimensional spacetime there is no natural concept of spin because there is no rotation group. Secondly, the implementation of the conformal group cannot be done in a canonical manner since a conformal transformation may carry points to infinity and moreover convert spacelike point pairs into timelike ones (see however [5]).

Concerning spin, I will be as conservative as possible: carrying over the transformation laws of spinning multicomponent fields to two dimensions, one realizes that it is possible to chose a basis in index space, such that the Lorentz transformations act diagonally, viz.

$$U(\Lambda)\psi_\alpha(x)U(\Lambda)^{-1} = e^{s_\alpha \chi}\psi_\alpha(\Lambda x) ; \qquad |s_\alpha| = s = \text{spin} . \tag{2.26}$$

Here, $\Lambda^\mu_\nu = \begin{pmatrix} \mathrm{ch}\,\chi & \mathrm{sh}\,\chi \\ \mathrm{sh}\,\chi & \mathrm{ch}\,\chi \end{pmatrix}$ is a boost. For instance, if $j_\mu(x)$ is a vectorfield, one defines:

$$j_+ = j_0 + j_1, j_- = j_0 - j_1$$

and (2.26) reads:

$$U(\Lambda)j_\pm(x)U(\Lambda)^{-1} = e^{\pm\chi} j_\pm(\Lambda x) , \quad \text{i.e.} \quad s_+ = 1, s_- = -1 .$$

Thus, given any field $\varphi(x)$ such that

$$U(\Lambda)\varphi(x)U(\Lambda)^{-1} = e^{s\cdot\chi}\varphi(\Lambda x) . \tag{2.27}$$

I will call s the spin of φ and restrict the allowed values of s by hand to be 0, $\pm 1/2$, $\pm 1, \ldots$.

A general CQFT in two dimensional space is now defined as follows: First of all it should be a fieldtheory satisfying Wightman's axioms [10] excluding, of course, the requirement of asymptotic completeness. Then, it is assumed that the conformal group $\tilde{C} \doteq \widetilde{\mathrm{Sl}(2,\mathbb{R})} \times \widetilde{\mathrm{Sl}(2,\mathbb{R})}$ is unitarily represented by operators $U(g)$, $g \in \tilde{C}$. If g is an element of the Poincaré group, $U(g)$ should coincide with the corresponding operator given by the underlying Wightman theory.

The action of $U(\cdot)$ on the fields is complicated. I will not make the assumption, that there is a conformally invariant, dense domain of definition for the fields, but require that the infinitesimal generators of $U(\cdot)$ and the fields have a common domain of definition D left invariant by them (and containing the vacuum $|0\rangle$). This domain hence contains the dense, linear space D_0 of all vectors which are built by applying a polynomial of smeared fields to the vacuumstate.

It is now postulated that the fields have the infinitesimal transformation law (valid on D) corresponding to the formal expression

$$U(g)\varphi(x)U(g)^{-1}=(\xi_+x_++\eta_+)^{-n_+}(\xi_-x_-+\eta_-)^{-n_-}\varphi(g(x)) \tag{2.28}$$

$$g=g_+\times g_-;\qquad \pi(g_\pm)=\begin{pmatrix}\sigma_\pm & \tau_\pm\\ \xi_\pm & \eta_\pm\end{pmatrix},$$

n_+ and n_- are the conformal quantum numbers of φ.

Formula (2.28) is then also globally valid on D_0, if $\xi_+=\xi_-=0$, $\eta_+,\eta_->0$. Specifically, for Lorentz boosts one has

$$g=(0,e^{\chi/2},0)\times(0,e^{-\chi/2},0)$$

$$U(g)\varphi(x)U(g)^{-1}=e^{\chi\frac{1}{2}(n_+-n_-)}\varphi(\Lambda x)$$

implying, that the spin of φ is equal to $1/2(n_+-n_-)$.

For dilatations $g=(0,\sqrt{\lambda},0)\times(0,\sqrt{\lambda},0)$ the transformation law yields:

$$U(g)\varphi(x)U(g)^{-1}=\lambda^{\frac{1}{2}(n_++n_-)}\varphi(\lambda x)=\lambda^d\varphi(\lambda x).$$

The number d is called the dimension of φ.

Thus

$$\begin{aligned} d&=\tfrac{1}{2}(n_++n_-)\\ s&=\tfrac{1}{2}(n_+-n_-). \end{aligned} \tag{2.29}$$

The two point function of any two local fields is determined up to a normalization constant by conformal invariance. It vanishes if the spins and dimensions of the two fields are not the same.

The result is:

$$\langle 0|\varphi_1(x)\varphi_2(y)|0\rangle=N(x_+-y_+-i\varepsilon)^{-n_+}(x_--y_--i\varepsilon)^{-n_-}. \tag{2.30}$$

Positivity requires: $n_+\geqq 0$, $n_-\geqq 0$, i.e. $d\geqq|s|$.

In case, say, $n_+=0$, the fields φ_1 and φ_2 do not depend on x_+.

In two dimensional QFT's the spectrum condition can be written in a factorized form. Let $\mathbb{P}_+$ and $\mathbb{P}_-$ be defined through

$$e^{i\mathbb{P}_+a}=U((0,1,a)\times(0,1,0));\qquad e^{i\mathbb{P}_-a}=U((0,1,0)\times(0,1,a)).$$

Then, the spectrum condition is equivalent to the statement:

$$\mathbb{P}_+\geqq 0;\qquad \mathbb{P}_-\geqq 0. \tag{2.31}$$

This implies, that the two unitary representations of $\widetilde{\mathrm{Sl}(2,\mathbb{R})}$ associated with $U(\cdot)$ are both analytic representations [7].

III. The Analytic Continuation of $\varphi_1(x)\varphi_2(y)|0\rangle$

The analyticity of "vectors" $\varphi_1(x)\varphi_2(y)|0\rangle$ in the variables x and y is a consequence of the spectrum condition (2.31) and of the conformal transformation law (2.28). It has been shown generally [7] that if the spectrumcondition holds and $|\psi\rangle$ is a vector in the Hilbert space $\mathscr{H}$ of physical states, then $U(g)|\psi\rangle$ is, as a function of $g\in\tilde{C}$, a boundary value of a holomorphic function $T(g)|\psi\rangle$, g running through a sixdimensional complex manifold $\tilde{S}_{\tilde{C}}$. Moreover, this manifold carries also a semigroup structure and $T(g)$ actually denotes a holomorphic, contractive ($\|T(g)\|\leqq 1$) representation of $\tilde{S}_{\tilde{C}}$.

When the operators $T(g)$, $g\in\tilde{S}_{\tilde{C}}$, are applied to the states $\varphi_1(x)\varphi_2(y)|0\rangle$ one gets a vectorvalued function of g for fixed x, y. However, because of (2.28) some of the variables specifying g are actually redundant, leaving just four independent complex parameters. These describe a conformally invariant two point analyticity domain for $\varphi_1(x)\varphi_2(y)|0\rangle$.

So far the general idea of what is now going to be done in great detail. Since the problem factorizes completely into x_+- and x_--variables, it is possible to "forget" about the presence of, say, x_-. Therefore, I will henceforth (in this section) argue as if spacetime were one dimensional and the conformal group were just $\tilde{G}=\widetilde{\mathrm{Sl}(2,\mathbb{R})}$ (i.e. the first factor of $\tilde{C}$).

3.1. *Summary of Properties of the Holomorphic Semigroup $\tilde{S}$ Belonging to $\tilde{G}$*

Let me first describe a holomorphic semigroup S which has the group $G=\mathrm{Sl}(2,\mathbb{R})$ at its boundary. G itself is a real form of the complex group $G_c=\mathrm{Sl}(2,\mathbb{C})$. As is wellknown, the Riemannian sphere S^2 is a homogeneous space for G_c, namely for $z\in\mathbb{C}\subset S^2$, we have:

$$g(z)=(\alpha z+\beta)/(\gamma z+\delta)\,; \qquad g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathrm{Sl}(2,\mathbb{C})\,. \tag{3.1}$$

Under the action of G the manifold S^2 splits into three pieces, each of them being a homogeneous space of G. These pieces are:

i) the upper halfplane Π,

ii) the real axis including the point at infinity,

iii) the lower halfplane Π^*.

Two complex semigroups are now defined as follows:

$$S^0=\{g\in G_c\,|\,g(\Pi)\text{ and its closure are contained in }\Pi\}\,, \tag{3.2}$$

$$S=S^0\cup G\,. \tag{3.3}$$

S^0 is an open submanifold of G_c and G belongs to its boundary. A holomorphic parametrization of S^0 is achieved by means of a Bruhat decomposition for G_c: any element $s\in S^0$ can be written uniquely in the form

$$s=\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sigma & 0\\ 0 & \sigma^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ \xi & 1\end{pmatrix}; \qquad (z,\sigma,\xi)\in\mathbb{C}^3,\ \sigma\neq 0,\ \mathrm{Im}\,\xi<0 \tag{3.4}$$

$$\mathrm{Im}\,z\cdot\mathrm{Im}\,\xi<-(\mathrm{Im}\,\sigma)^2\,.$$

Another useful parametrization can be obtained using an open, G-invariant cone V contained in the Liealgebra of G:

$$V \doteq \{X \in \mathrm{sl}(2, \mathbb{R}) \mid X = \lambda g H g^{-1}, \lambda > 0, g \in G\} \tag{3.5}$$

(H is defined by (2.4)). More explicitly, V consists precisely of those real matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ with $\beta > 0$, $\gamma < 0$, $\alpha^2 + \beta \cdot \gamma < 0$. As manifolds, the following equality holds:

$$S^0 \cong G \times V. \tag{3.6}$$

The corresponding diffeomorphism is given by:

$$s = g \cdot e^{iX}; \quad s \in S^0, \quad g \in G, \quad X \in V. \tag{3.7}$$

From (3.6) it is clear that S^0 is not simply connected since G is not. Taking the universal covering

$$\tilde{S}^0 \cong \tilde{G} \times V \tag{3.8}$$

of S^0 and lifting the semigroup structure yields a new holomorphic semigroup. By adding points of $\tilde{G}$ in a continuous fashion [7] one obtains a semigroup

$$\tilde{S} \cong \tilde{G} \times (V \cup \{0\}). \tag{3.9}$$

The above mentioned theorem, which is the key to the construction of the conformal analyticity domain of $\varphi_1(x)\varphi_2(y)|0\rangle$, now reads:

Theorem 3.1. [7]. *Suppose $U(\cdot)$ is a unitary (continuous) representation of $\tilde{G}$ in a Hilbert space $\mathscr{H}$, such that $(1/i)U(H) \geqq 0$ ($U(H)$ is defined by $e^{tU(H)} = U(e^{t \cdot H})$). Then there exists a representation $T(\cdot)$ of $\tilde{S}$ satisfying:*

(i) $\|T(s)\| \leqq 1$; $T(s_1) \cdot T(s_2) = T(s_1 \cdot s_2)$.

(ii) $T(s) = U(s)$ *for* $s \in \tilde{G}$.

(iii) *for any* $|\psi\rangle \in \mathscr{H}$, *the vectorvalued function* $T(s)|\psi\rangle$ *of* $s \in \tilde{S}$ *is continuous and holomorphic when restricted to* $\tilde{S}^0$.

Briefly, $T(\cdot)$ is the (unique) analytic continuation of $U(\cdot)$.

3.2. *The Geometry of the Conformal Two Point Forward Tube*

The one point forward tube in one "spacetime" dimension is the upper half plane Π. Indeed, due to the spectrum condition (2.31) one can analytically continue $\varphi(x)|0\rangle$ to a vectorvalued holomorphic function $|w\rangle$, $w \in \Pi$, such that $\lim_{\mathrm{Im}\, w \searrow 0} |w\rangle = \varphi(x)|0\rangle$, $x = \mathrm{Re}\, w$ [11] (here, the presence of x_- has not been noticed and x is identified with x_+ for simplicity; compare the remark at the beginning of this section).

Now $T(s)$, $s \in \tilde{S}$, acts on $|w\rangle$ as follows:

$$T(s)|w\rangle = (\gamma w + \delta)^{-n} |s(w)\rangle; \quad \pi(s) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in S. \tag{3.10}$$

π denotes the canonical projection of $\tilde{S}$ onto S. This projection reduces to (2.10) for elements $s \in \tilde{G}$, thus excluding a notational ambiguity. Observe that by definition of S, $s(w) \in \Pi$. The phase of $\gamma w + \delta$ is determined by requiring it to lie in the interval $(-\pi, \pi)$ as s approaches unity and by imposing the validity of the multiplication law for $T(\cdot)$. The number $n \geqq 0$ is the conformal quantum number n_+ of φ see (2.28).

The same argument does not apply to the two point vector $\varphi_1(x_1)\varphi_2(x_2)|0\rangle$. In fact, using the spectrum condition alone yields a vectorvalued, holomorphic function $|w_1, w_2\rangle$, $0 < \operatorname{Im} w_1 < \operatorname{Im} w_2$, such that

$$\lim_{0 < \operatorname{Im} w_1 < \operatorname{Im} w_2 \searrow 0} |w_1, w_2\rangle = \varphi_1(x_1)\varphi_2(x_2)|0\rangle, \; x_1 = \operatorname{Re} w_1, \; x_2 = \operatorname{Re} w_2 .$$

The set $\{(w_1, w_2) \in \Pi \times \Pi | 0 < \operatorname{Im} w_1 < \operatorname{Im} w_2\}$ is however not invariant under the action of $\tilde{G}$, and, a fortiori, of $\tilde{S}$.

A first guess of how a conformally invariant twopoint analyticity domain could look like is the following set:

$$\Pi \mathring{\times} \Pi = \{(w_1, w_2) \in \Pi \times \Pi | w_1 \neq w_2\} . \tag{3.11}$$

The center of $\tilde{G}$ acts trivially on $\Pi \mathring{\times} \Pi$, i.e. $z(w_1, w_2) = (z(w_1), z(w_2)) = (w_1, w_2)$ for all $(w_1, w_2) \in \Pi \mathring{\times} \Pi$.

Now, this would imply

$$U(z)\varphi_1(x)\varphi_2(y)|0\rangle = \text{phase factor} \times \varphi_1(x)\varphi_2(y)|0\rangle \tag{3.12}$$

a formula, which will not hold generally [8].

To avoid (3.12) one must have some two point manifold where z does not act trivially. To get an idea, what is needed, let $(z_1, z_2) \doteq (i/2, i)$ a special point in the forward tube and $\gamma(t) = (t, 1, 0)$, $0 \leqq t \leqq \pi$ a curve in $\tilde{G}$. Then

$$\pi(\gamma(t)) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}; \quad \gamma(0) = 1 , \quad \gamma(\pi) = z$$

$$\gamma(t)(z_2) = z_2 \quad \text{for all} \quad t .$$

As shown in the figure, $\gamma(t)(z_1)$ walks around z_2 as t increases from 0 to π.

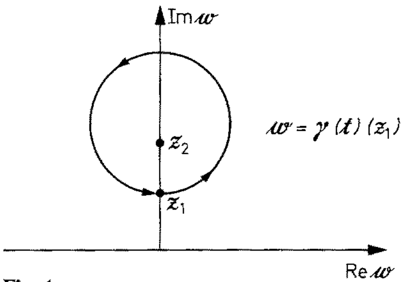

Fig. 1

The requirement that $\gamma(\pi)(z_1, z_2) = z(z_1, z_2) \neq (z_1, z_2)$ implies therefore, that, while w_1 is running around w_2, another Riemannian sheet of the domain of holomorphy of $|w_1, w_2\rangle$ is reached. This suggests the consideration of the complex manifold

$$\widetilde{\Pi \mathring{\times} \Pi} = \text{universal covering of } \Pi \mathring{\times} \Pi . \tag{3.13}$$

Let $p:\widetilde{\Pi \mathring{\times} \Pi} \to \Pi \mathring{\times} \Pi$ the natural projection. Because $\tilde{G}$ and $\tilde{S}$ both can act on $\Pi \mathring{\times} \Pi$ and are simply connected, they act on $\widetilde{\Pi \mathring{\times} \Pi}$ aswell. This action "commutes" with p:

$$p[s(\omega)] = s[p(\omega)] \quad \text{for all} \quad s \in \tilde{S}, \omega \in \widetilde{\Pi \mathring{\times} \Pi}. \tag{3.14}$$

Points of $\widetilde{\Pi \mathring{\times} \Pi}$ can be described by their projection on $\Pi \mathring{\times} \Pi$ and a sheet label: $\omega = (w_1, w_2, n)$; $(w_1, w_2) \in \Pi \mathring{\times} \Pi$, $n \in \mathbb{Z}$. Two consecutive sheets of $\widetilde{\Pi \mathring{\times} \Pi}$ are glued together along the surface: $\mathrm{Re}\, w_1 = \mathrm{Re}\, w_2$, $0 < \mathrm{Im}\, w_2 < \mathrm{Im}\, w_1$. This can be done in such a way, that

$$z(w_1, w_2, n) = (w_1, w_2, n+1); \ z \text{ as in (2.13)} \tag{3.15}$$

i.e. the center of $\tilde{G}$ does no longer act trivially, but maps one sheet of $\widetilde{\Pi \mathring{\times} \Pi}$ onto another.

$\widetilde{\Pi \mathring{\times} \Pi}$ is not a homogeneous space of $\tilde{G}$. In fact there are infinitely many orbits described by the following Lemma.

Lemma 3.2. *For* $\omega \in \widetilde{\Pi \mathring{\times} \Pi}$ *define:* $\theta(\omega) = \mathrm{Im}\, w_1 \cdot \mathrm{Im}\, w_2 / |w_1 - w_2|^2$ *where* $(w_1, w_2) = p(\omega)$. *The orbits of* $\tilde{G}$ *in* $\widetilde{\Pi \mathring{\times} \Pi}$ *are then precisely the subsets of* $\widetilde{\Pi \mathring{\times} \Pi}$ *with a constant value of* θ.

Proof. See Appendix A.

On the other hand, the semigroup $\tilde{S}^0$ acts on $\widetilde{\Pi \mathring{\times} \Pi}$ as if it were a "homogeneous space":

Lemma 3.3. *Define* θ *as above. The orbit of a point* $\omega \in \widetilde{\Pi \mathring{\times} \Pi}$ *under the action of* $\tilde{S}^0$ *consists precisely of all* $\omega' \in \widetilde{\Pi \mathring{\times} \Pi}$ *having* $\theta(\omega') > \theta(\omega)$.

This lemma is also proved in Appendix A.

The orbit $O(\omega)$ of $\omega \in \widetilde{\Pi \mathring{\times} \Pi}$ under the action of $\tilde{S}^0$ is thus an open subset of $\widetilde{\Pi \mathring{\times} \Pi}$ and moreover $O(\omega) \nearrow \widetilde{\Pi \mathring{\times} \Pi}$ as $\theta(\omega) \searrow 0$.

3.3. *Carrying out the Continuation of* $\varphi_1(x)\varphi_2(y)|0\rangle$

Let φ_1 and φ_2 two local fields in a CQFT having conformal quantum numbers $n_1^{\pm}$ and $n_2^{\pm}$ respectively.

To formulate the main result of this section, the x_--variables are not ignored. For the proof of the theorem below they will not be noticed.

First, one can analytically continue $\varphi_1(x_1^+, x_1^-)\varphi_2(x_2^+, x_2^-)|0\rangle$ to the relativistic forward tube. Thus, there are vectorvalued, holomorphic functions $|w_1^+, w_2^+; w_1^-, w_2^-\rangle$, $0 < \mathrm{Im}\, w_1^+ < \mathrm{Im}\, w_2^+$, $0 < \mathrm{Im}\, w_1^- < \mathrm{Im}\, w_2^-$, such that

$$\lim_{\substack{0 < \mathrm{Im}\, w_1^+ < \mathrm{Im}\, w_2^+ \searrow 0 \\ 0 < \mathrm{Im}\, w_1^- < \mathrm{Im}\, w_2^- \searrow 0 \\ \mathrm{Re}\, w_1^{\pm} = x_1^{\pm};\ \mathrm{Re}\, w_2^{\pm} = x_2^{\pm}}} |w_1^+, w_2^+; w_1^-, w_2^-\rangle = \varphi_1(x_1^+, x_1^-)\varphi_2(x_2^+, x_2^-)|0\rangle \tag{3.16}$$

in the distribution sense.

Theorem 3.4. *There are vectorvalued, holomorphic functions* $|\omega_+, \omega_-\rangle$, $\omega_{\pm} \in \widetilde{\Pi \mathring{\times} \Pi}$ *such that:*

a) $|\omega_+, \omega_-\rangle$ *analytically continues* $|w_1^+, w_2^+; w_1^-, w_2^-\rangle$, *i.e. for* $\omega_+ = (w_1^+, w_2^+, 0)$ *and* $\omega_- = (w_1^-, w_2^-, 0)$ *we have:* $|\omega_+, \omega_-\rangle = |w_1^+, w_2^+; w_1^-, w_2^-\rangle$.

b) *if* $s = s_+ \times s_- \in \tilde{S} \times \tilde{S}$ *and* $\pi(s_\pm) = \begin{pmatrix} \alpha_\pm & \beta_\pm \\ \gamma_\pm & \delta_\pm \end{pmatrix}$ *then:*

$$T(s)|\omega_+, \omega_-\rangle = (\gamma_+ w_1^+ + \delta_+)^{-n_1^+} (\gamma_+ w_2^+ + \delta_+)^{-n_2^+} \times (\gamma_- w_1^- + \delta_-)^{-n_1^-} (\gamma_- w_2^- + \delta_-)^{-n_2^-} |s_+(\omega_+), s_-(\omega_-)\rangle . \tag{3.17}$$

Here $(w_1^\pm, w_2^\pm) = p(\omega_\pm)$ and the phases of the multipliers $(\gamma_+ w_1^+ + \delta_+)^{-n_1^+}$ etc. are determined as in the case of the transformation law (3.10) of the one point vectors.

Proof. Reducing the theorem to plus-variables only, one starts with a holomorphic function $|w_1, w_2\rangle$, $0 < \operatorname{Im} w_1 < \operatorname{Im} w_2$, and looks for vectors $|\omega\rangle$, $\omega \in \widetilde{\Pi \mathring{\times} \Pi}$, such that

$$T(s)|\omega\rangle = (\gamma w_1 + \delta)^{-n_1} (\gamma w_2 + \delta)^{-n_2} |s(\omega)\rangle \tag{3.18}$$

for all $s \in \tilde{S}$, $\pi(s) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sl}(2, \mathbb{C})$. Moreover, if $\omega = (w_1, w_2, 0)$, then $|\omega\rangle = |w_1, w_2\rangle$.

This suggests, that one should define $|\omega\rangle$ through

$$|\omega\rangle = (\gamma w_1 + \delta)^{n_1} (\gamma w_2 + \delta)^{n_2} T(s) |w_1, w_2\rangle \tag{3.19}$$

whenever $0 < \operatorname{Im} w_1 < \operatorname{Im} w_2$ and $s((w_1, w_2, 0)) = \omega$, $s \in \tilde{S}$. However, there are many triples s, w_1, w_2 such that $s((w_1, w_2, 0) = \omega$. This is described more precisely in the following Lemma:

Lemma 3.5. *For* $\omega, \omega_0 \in \widetilde{\Pi \mathring{\times} \Pi}$ *define:*

$$\mu(\omega|\omega_0) \doteq \{s \in \tilde{S}^0 | s(\omega_0) = \omega\} . \tag{3.20}$$

Then, $\mu(\omega|\omega_0)$ *is a closed, connected, holomorphic submanifold of* $\tilde{S}^0$.

Proof. See Appendix B.

Next, have a closer look at the multipliers $(\gamma w + \delta)^n$ which appear in (3.19). I claim, that the map $\lambda_n : \Pi \times \tilde{S}^0 \to \mathbb{C}$, $\lambda_n(w, s) = (\gamma w + \delta)^n$ (phases defined as in (3.10)) is holomorphic. Indeed, the map $\varrho : \Pi \times S^0 \to \mathbb{C}$, $\varrho(w, s) \doteq (\gamma w + \delta)$ vanishes nowhere by definition of S^0 and is hence a holomorphic map of $\Pi \times S^0$ into $\mathbb{C} \setminus \{0\}$. Let $\mathbb{C}^*$ the universal covering of $\mathbb{C} \setminus \{0\}$, i.e. $\mathbb{C}^*$ is the Riemann-surface of the logarithm. By the monodromy principle [12], ϱ can be uniquely lifted to a mapping $\tilde{\varrho} : \Pi \times \tilde{S}^0 \to \mathbb{C}^*$ such that $\arg[\tilde{\varrho}(w, s)] \in (-\pi, \pi)$ as s approaches unity. Now, powers are defined as holomorphic functions on $\mathbb{C}^*$. Thus, for any n, $\tilde{\varrho}^n$ is a well defined holomorphic mapping of $\Pi \times \tilde{S}^0 \to \mathbb{C}$. Clearly $\tilde{\varrho}^n \cdot \tilde{\varrho}^{-n} = 1$. It remains to show that $\tilde{\varrho}^n = \lambda_n$. To this end, consider the transformation law (3.10). For fixed w and s near enough to 1, one can replace $(\gamma w + \delta)^{-n}$ by $\tilde{\varrho}(w, s)^{-n}$ there. By uniqueness of analytic continuation the "new" equation holds for all $s \in \tilde{S}$ and therefore $\tilde{\varrho}(w, s)^{-n} = \lambda_{-n}(w, s)$ for all w, s.

The multipliers $\lambda_n(w, s) = (\gamma w + \delta)^n$ satisfy the following algebraic identity:

$$\lambda_n(w, s_1 \cdot s_2) = \lambda_n(s_2(w), s_1) \cdot \lambda_n(w, s_2) ; \qquad w \in \Pi, s_1, s_2 \in \tilde{S} . \tag{3.21}$$

For this relation holds when s_1 and s_2 are near 1 and extends by uniqueness of analytic continuation to all of $\tilde{S}$.

For $s \in \tilde{S}^0$ and $(w_1, w_2) \in \Pi \times \Pi$, $0 < \operatorname{Im} w_1 < \operatorname{Im} w_2$, we now define:

$$|s, w_1, w_2\rangle \doteq \lambda_{n_1}(w_1, s)\lambda_{n_2}(w_2, s)T(s)|w_1, w_2\rangle . \tag{3.22}$$

By the above this is a holomorphic vectorvalued function. From (3.17) we expect that it depends only on the combination $\omega = s(w_1, w_2, 0) \in \widetilde{\Pi \mathring{\times} \Pi}$ of the variables involved. In order to prove this statement we need another lemma.

Lemma. *For any element X of the Liealgebra of $\tilde{G}$ denote by $U(X)$ the infinitesimal generator of $U(\operatorname{expt} X)$. Then $|w_1, w_2\rangle$ lies in the domain of definition of $U(X)$ and there are differential operators*

$$L_X = Q(X, w_1) + P(X, w_1)\partial/\partial w_1 + Q(X, w_2) + P(X, w_2)\partial/\partial w_2 .$$

$Q(X, w)$, $P(X, w)$: polynomials of w, linearly dependent on X such that

$$U(X)|w_1, w_2\rangle = L_X|w_1, w_2\rangle . \tag{3.23}$$

Proof. $|w_1, w_2\rangle$ is the Fourier-Laplace transform of $\varphi_1(x_1)\varphi_2(x_2)|0\rangle$ and moreover there is a testfunction $f_{w_1 w_2}(x_1, x_2) \in \mathscr{S}(\mathbb{R}^2 \times \mathbb{R}^2)$ such that

$$|w_1, w_2\rangle = \int dx_1 dx_2 f_{w_1 w_2}(x_1, x_2)\varphi_1(x_1)\varphi_2(x_2)|0\rangle .$$

Therefore $|w_1, w_2\rangle$ is an element of the domain of $U(X)$. By infinitesimal conformal covariance (2.28) and elementary properties of Fourier-Laplace transforms we have

$$\begin{aligned} U(X)|w_1, w_2\rangle &= \int dx_1 dx_2 f_{w_1 w_2}(x_1 x_2) L_X \varphi_1(x_1)\varphi_2(x_2)|0\rangle \\ &= L_X|w_1, w_2\rangle \quad \square \text{ (Lemma)}. \end{aligned}$$

The uniqueness theorem for solving the problem

$$(d/dt)|\psi, t\rangle = U(X)|\psi, t\rangle ; \qquad |\psi, 0\rangle = |\psi\rangle$$

now implies that

$$T(\exp X)|w_1, w_2\rangle = \lambda_{-n_1}(w_1, \exp X)\lambda_{-n_2}(w_2, \exp X)|\exp X w_1, \exp X w_2\rangle \tag{3.24}$$

for all X such that $\operatorname{Im}(\operatorname{expt} X w_1) < \operatorname{Im}(\operatorname{expt} X w_2) (\forall t, 0 \leqq t \leqq 1)$. Hence by (3.21) we have

$$\begin{aligned} |s \cdot \exp X, w_1, w_2\rangle &= \lambda_{n_1}(w_1, s \cdot \exp X)\lambda_{n_2}(w_2, s \cdot \exp X)T(s)T(\exp X)|w_1, w_2\rangle \\ &= |s, \exp X w_1, \exp X w_2\rangle . \end{aligned} \tag{3.25}$$

The holomorphy of $|s, w_1, w_2\rangle$ allows us to extend this relation to all *complex* X lying in some open neighborhood N_{s, w_1, w_2} of zero.

Let $\omega \in \widetilde{\Pi \mathring{\times} \Pi}$ and $\omega_0 = (w_1, w_2, 0) \in \widetilde{\Pi \mathring{\times} \Pi}$ such that $0 < \operatorname{Im} w_1 < \operatorname{Im} w_2$ and $\Theta(\omega_0) < \Theta(\omega)$ (such ω_0 always exists). By Lemma 3.3 there is some $s \in \tilde{S}^0$ with $\omega = s(\omega_0)$. Since $\mu(\omega|\omega_0)$ is a connected closed submanifold of $\tilde{S}^0$ (Lemma 3.5) (3.25) implies that $|s, w_1, w_2\rangle$ depends only on ω but not on the particular tripel s, w_1, w_2 chosen. Hence we may unambiguously define

$$|\omega\rangle \doteq |s, w_1, w_2\rangle .$$

Obviously $|\omega\rangle$ analytically continues $|w_1, w_2\rangle$ and the transformation law (3.17) follows from Equations (3.21) and (3.22). $\square$

Though the semigroup formalism is a quite complicated tool it gives deeper insight into the question of where the whole analyticity comes from. It also yields a good guess what the conformal n-point forward tube should look like. A point of this set is formally given by:

$$\omega = (w_1, \ldots, w_n); \quad \exists s_1, \ldots, s_n \in \tilde{S}^0$$

such that

$$w_1 = s_1(0); w_2 = (s_1 \cdot s_2)(0); \ldots; w_n = (s_1 \cdot s_2 \cdot \ldots \cdot s_n)(0).$$

More precisely, this is a subset of the universal covering of

$$\{(w_1, \ldots, w_n) \in \Pi \times \ldots \times \Pi \mid w_i \neq w_j \quad \text{for all} \quad i \neq j\}$$

corresponding to a complicated ordering.

IV. Decomposition of the Tensorproduct of Two Holomorphic, Irreducible Representations of $\widetilde{\mathrm{Sl}(2, \mathbb{R})}$ into Irreducible Subspaces

The problem of decomposing a tensorproduct of irreducible representations of $\widetilde{\mathrm{Sl}(2, \mathbb{R})}$ has been discussed recently by Rühl and Yunn [13]. For the case of two holomorphic representations the formulas and proofs involved simplify considerably and moreover, can be given a very elegant form. I will therefore not refer to the work mentioned above, but derive the harmonic expansion newly.

In view of Theorem 3.4 it seems to be more promising to do harmonic analysis on the space $\widetilde{\Pi \mathring{\times} \Pi}$ instead of $\Pi \times \Pi$. However, as will be shown in Section V, the group theoretical machinery developped in the present section will be sufficient to expand the vectors $|\omega_+, \omega_-\rangle$ in conformal partial waves.

Let me first define the tensor product of two irreducible, holomorphic representations $(T_{n_1}, \mathscr{H}_{n_1})$, $(T_{n_2}, \mathscr{H}_{n_2})$, $n_1, n_2 > 0$, of $\tilde{G}$. The appropriate function space is denoted by $H_{n_1} \hat{\otimes} H_{n_2}$. Its elements are functions $F(w_1, w_2)$, $\mathrm{Im}\, w_{1,2} > 0$, satisfying:

a) F is holomorphic and has a C^∞-extension to all of

$$\{(w_1, w_2) \in \mathbb{C} \times \mathbb{C} \mid \mathrm{Im}\, w_1 \geqq 0, \mathrm{Im}\, w_2 \geqq 0\}.$$

b) the same as in a) is true for

$$w_1^{-n_1} F(-1/w_1, w_2); w_2^{-n_2} F(w_1, -1/w_2); w_1^{-n_1} w_2^{-n_2} F(-1/w_1, -1/w_2). \tag{4.1}$$

Equation (4.1) implies various asymptotic expansions for $F(w_1, w_2)$ as $|w_1| \to \infty$ and/or $|w_2| \to \infty$.

$\tilde{G}$ acts on $H_{n_1} \hat{\otimes} H_{n_2}$ as follows:

$$[T_{n_1 \times n_2}(g) F](w_1, w_2) = (-\xi w_1 + \sigma)^{-n_1} (-\xi w_2 + \sigma)^{-n_2} F(g^{-1}(w_1), g^{-1}(w_2)), \tag{4.2}$$

$$\pi(g) = \begin{pmatrix} \sigma & \tau \\ \xi & \eta \end{pmatrix} \in \mathrm{Sl}(2, \mathbb{R}); |\arg(-\xi w_{1,2} + \sigma)| < \pi.$$

This formula is valid for $g \in O$ (see (2.11)). Since $\tilde{G}$ is simply connected, $T_{n_1 \times n_2}$ extends uniquely to a representation of $\tilde{G}$.

The completion of $H_{n_1} \hat{\otimes} H_{n_2}$ with respect to the invariant scalarproduct

$$(F_1, F_2)_{n_1 \times n_2} = (n_1 - 1)(n_2 - 1)\pi^{-2} \int_{\Pi \times \Pi} |dw_1| \cdot |dw_2| F_1^*(w_1, w_2)(\operatorname{Im} w_1)^{n_1 - 2} \cdot (\operatorname{Im} w_2)^{n_2 - 2} F_2(w_1, w_2) \tag{4.3}$$

yields a Hilbertspace $\mathscr{H}_{n_1} \hat{\otimes} \mathscr{H}_{n_2}$. The operators $T_{n_1 \times n_2}(g)$ extend to all of $\mathscr{H}_{n_1} \hat{\otimes} \mathscr{H}_{n_2}$ forming a unitary, analytic representation of $\tilde{G}$.

An analytic representation of a Liegroup can always be decomposed into irreducible analytic representations [7]. This suggests the consideration of the following Clebsch-Gordon-kernels:

$$K^k_{n_1 n_2}(w_1, w_2 | w) = i^k (w_1 - w_2)^k (w_1 - w^*)^{-n_1 - k} (w_2 - w^*)^{-n_2 - k} \tag{4.4}$$

$$(w_1, w_2, w \in \Pi;\ k = 0, 1, 2, \dots).$$

For fixed w, $K^k_{n_1 n_2}(w_1, w_2 | w) \in H_{n_1} \hat{\otimes} H_{n_2}$ and for fixed w_1, w_2 $K^k_{n_1 n_2}(w_1, w_2 | w)^* \in H_{n_1 + n_2 + 2k}$. Therefore, the scalarproduct $(K^k_{n_1 n_2}(\cdot, \cdot | w), K^{k'}_{n_1 n_2}(\cdot, \cdot | w'))_{n_1 \times n_2}$ is well defined and we have the orthogonality relation:

$$(K^k_{n_1 n_2}(\cdot, \cdot | w), K^{k'}_{n_1 n_2}(\cdot, \cdot | w'))_{n_1 \times n_2} = C_k \delta_{k,k'} G^*_{n_1 + n_2 + 2k}(w, w'), \tag{4.5}$$

$$C_k = 4^{3 - n_1 - n_2 - k} k! \Gamma(n_1) \Gamma(n_2) \Gamma(n_1 + n_2 + 2k - 1) / \Gamma(n_1 + k) \Gamma(n_2 + k) \cdot \Gamma(n_1 + n_2 + k - 1).$$

Proof. $K^k_{n_1 n_2}$ has the following Fourier representation:

$$K^k_{n_1 n_2}(w_1, w_2 | w) = \{\Gamma(n_1 + k) \Gamma(n_2 + k)\}^{-1} e^{-i\pi/2(n_1 + n_2)} \cdot \int_0^\infty dp_1 dp_2 e^{ip_1(w_1 - w^*)} e^{ip_2(w_2 - w^*)} (\partial/\partial p_1 - \partial/\partial p_2)^k [p_1^{n_1 + k - 1} p_2^{n_2 + k - 1}].$$

Performing the substitution

$$p_1 = y(1 - x);\ p_2 = y(1 + x);\ 0 < y < \infty,\ -1 < x < 1$$

yields:

$$K^k_{n_1 n_2}(w_1, w_2 | w) = M_k \int_0^\infty dy \int_{-1}^1 dx\, e^{iy(1 - x)(w_1 - w^*)} e^{iy(1 + x)(w_2 - w^*)} \cdot y^{n_1 + n_2 + k - 1} (1 - x)^{n_1 - 1} (1 + x)^{n_2 - 1} P^k_{n_1 n_2}(x), \tag{4.6}$$

$$M_k \doteq 2^{k+1} k! e^{-i\pi/2(n_1 + n_2)} / \Gamma(n_1 + k) \Gamma(n_2 + k). \tag{4.7}$$

$P^k_{n_1 n_2}(x)$ is a Jacobi polynomial:

$$P^k_{n_1 n_2}(x) = (-1)^k / (2^k k!)(1 - x)^{1 - n_1} (1 + x)^{1 - n_2} (d^k / dx^k) \cdot [(1 - x)^{n_1 + k - 1} (1 + x)^{n_2 + k - 1}]. \tag{4.8}$$

Now formulas (2.22), (2.23) and the orthogonality relation

$$\int_{-1}^1 dx (1 - x)^{n_1 - 1} (1 + x)^{n_2 - 1} P^k_{n_1 n_2}(x) P^l_{n_1 n_2}(x) = h_k \delta_{k,l} \tag{4.9}$$

$$h_k = 2^{n_1 + n_2 - 1} \Gamma(n_1 + k) \Gamma(n_2 + k) / (n_1 + n_2 + 2k - 1) k! \Gamma(n_1 + n_2 + k - 1)$$

for the Jacobi polynomials imply (4.5). □

For any function $F(w_1, w_2) \in H_{n_1} \hat{\otimes} H_{n_2}$ define its k'th Fourier component by

$$F_k(w) = C_k^{-1}(K_{n_1n_2}^k(\cdot,\cdot|w), F)_{n_1 \times n_2}. \tag{4.10}$$

It will be shown later that $F_k(w) \in H_{n_1+n_2+2k}$.

Clearly, the kernel (4.4) is covariant, i.e.

$$[T_{n_1+n_2+2k}(g)F_k](w) = [T_{n_1 \times n_2}(g)F]_k(w) \quad \text{for all} \quad g \in \tilde{G}. \tag{4.11}$$

Moreover, since the Jacobi polynomials $P_{n_1n_2}^k$ are a complete orthogonal basis in the Hilbertspace of all measurable functions $f(x)$, $x \in [-1, 1]$ with finite norm

$$\|f\|^2 = \int_{-1}^{1} dx(1-x)^{n_1-1}(1+x)^{n_2-1}|f(x)|^2$$

it is easily proved (Appendix C) from (4.6), (4.10) that

$$F(w_1, w_2) = \sum_{k=0}^{\infty} (n_1+n_2+2k-1)\pi^{-1} \int_{\Pi} |dw| K_{n_1n_2}^k(w_1, w_2|w) \cdot (\operatorname{Im} w)^{n_1+n_2+2k-2} F_k(w). \tag{4.12}$$

This sum converges pointwise and also in the $\mathscr{H}_{n_1} \hat{\otimes} \mathscr{H}_{n_2}$-norm:

$$\|F\|_{n_1 \times n_2}^2 = \sum_{k=0}^{\infty} C_k \|F_k\|_{n_1+n_2+2k}^2 \qquad C_k \text{ as in (4.5)}. \tag{4.13}$$

Briefly, the decomposition

$$\mathscr{H}_{n_1} \hat{\otimes} \mathscr{H}_{n_2} = \bigoplus_{k=0}^{\infty} \mathscr{H}_{n_1+n_2+2k} \tag{4.14}$$

holds.

There is a very useful alternative way to express the orthogonality relation (4.5). This is done by means of a set of homogeneous polynomials

$$Q_{n_1n_2}^k(x_1, x_2) \doteq x_1^{1-n_1} x_2^{1-n_2} (\partial/\partial x_1 - \partial/\partial x_2)^k [x_1^{n_1+k-1} x_2^{n_2+k-1}]. \tag{4.15}$$

We have

$$\begin{aligned} Q_{n_1n_2}^k(\lambda x_1, \lambda x_2) &= \lambda^k Q_{n_1n_2}^k(x_1, x_2) \\ Q_{n_1n_2}^k(1-x, 1+x) &= 2^k \cdot k! \cdot P_{n_1n_2}^k(x) \end{aligned} \tag{4.16}$$

and

Lemma. *Let* $n_1, n_2 \in \mathbb{R}$, $k, k' = 0, 1, 2, \ldots$ *and* $F(w_1, w_2)$ *a function holomorphic on* $\Pi \times \Pi$[1]. *Then:*

a) $$\begin{aligned} &Q_{n_1n_2}^k(\partial/\partial w_1, \partial/\partial w_2) K_{n_1n_2}^{k'}(w_1, w_2|w)|_{w_1=w_2=w'} = N_k \delta_{k,k'} G_{n_1+n_2+2k}^*(w', w), \\ &N_k = (k! \Gamma(n_1+n_2+2k-1)/2^{n_1+n_2+2k-2} \Gamma(n_1+n_2+k-1)) e^{-i\pi(n_1+n_2-k)/2}. \end{aligned} \tag{4.17}$$

b) *Define* $f_k(w) = Q_{n_1n_2}^k(\partial/\partial w_1, \partial/\partial w_2) F(w_1, w_2)|_{w_1=w_2=w}$.

[1] $K_{n_1n_2}^k$, G_n, $T_n(g)$, $T_{n_1 \times n_2}(g)$ $(g \in \tilde{G})$ can be defined for arbitrary $n_1, n_2, n \in \mathbb{R}$ and holomorphic $F(w_1, w_2)$ resp. $F(w)$

For $g \in \tilde{G}$ *one has*

$$[T_{n_1+n_2+2k}(g)f_k](w) = Q^k_{n_1n_2}(\partial/\partial w_1, \partial/\partial w_2)\{[T_{n_1\times n_2}(g)F](w_1, w_2)\}|_{w_1=w_2=w}. \tag{4.18}$$

If $F \in H_{n_1} \hat{\otimes} H_{n_2}$ *we can calculate its Fourier components through:*

$$F_k(w) = N_k^{-1} Q^k_{n_1n_2}(\partial/\partial w_1, \partial/\partial w_2)F(w_1, w_2)|_{w_1=w_2=w}. \tag{4.19}$$

From this formula and (4.11) *it is easily seen that* $F_k \in H_{n_1+n_2+2k}$ *as promised earlier.*

Proof. The Equations (4.17) and (4.18) are analytically dependent on n_1, n_2. It is therefore sufficient to prove them for $n_1, n_2 > 0$. In this case, the representation (4.6) of $K^k_{n_1n_2}$ is valid and Equation (4.17) then follows from (4.16) and the orthogonality of the Jacobipolynomials $P^k_{n_1n_2}$.

To prove Equation (4.18) assume first that $F \in H_{n_1} \hat{\otimes} H_{n_2}$. Using the Fourier representation for F one easily establishes

$$(K^k_{n_1n_2}(\cdot, \cdot|w), F)_{n_1\times n_2} = C_k N_k^{-1} Q^k(\partial/\partial w_1, \partial/\partial w_2)F(w_1, w_2)|_{w_1=w_2=w}.$$

Covariance of the Clebsch-Gordon-kernels then implies (4.18). If F is arbitrary, one writes each side of Equation (4.18) in the form

$$\sum_{j,l=0}^{k} a_{j,l}(\partial^{j+l}/\partial w_1^j \partial w_2^l)F(w_1, w_2)|_{w_1=w_2=g^{-1}(w)}.$$

Since it is possible to find a function $L(w_1, w_2) \in H_{n_1} \hat{\otimes} H_{n_2}$ having prescribed derivatives $(\partial^{j+l}/\partial w_1^j \partial w_2^l)\, L(w_1, w_2)|_{w_1=w_2=w'}(0 \leqq j, l \leqq k)$ at a particular point $w' \in \Pi$ the equality of the coefficients $a_{j,l}$ on both sides of (4.18) follows. □

V. The Conformal Partial Wave Expansion of the Vectors $\varphi_1(x)\varphi_2(y)|0\rangle$

A straight forward application of the results obtained in the preceding section to the vectors $|\omega_+, \omega_-\rangle$ (Theorem 3.4) is not possible, because these vectors are defined only for $\omega_\pm \in \widetilde{\Pi \mathring{\times} \Pi}$ rather than $\omega_\pm \in \Pi \times \Pi$. However, the different sheets of $\widetilde{\Pi \mathring{\times} \Pi}$ can be reached from one special sheet by acting with central elements of $\tilde{G}$. Let z_+ resp. z_- the generators (2.13) of the center of the first resp. second factor of $\tilde{C}$. Define:

$$Z_+ \doteq U(z_+ \times 1);\, Z_- = U(1 \times z_-). \tag{5.1}$$

According to Theorem (3.4) we have:

$$\begin{aligned} &Z_+|\omega_+, \omega_-\rangle = e^{i\pi(n_1^+ + n_2^+)}|z_+(\omega_+), \omega_-\rangle, \\ &z_+(w_1^+, w_2^+, n) = (w_1^+, w_2^+, n+1)\,(\omega_+ = (w_1^+, w_2^+, n)). \end{aligned} \tag{5.2}$$

I will now make a simplifying assumption, namely that the spectrum of the unitary operators $Z_\pm$ are purely discrete, i.e.

$$Z_\pm = \sum_{k \in I_\pm} \lambda_k^\pm E(\lambda_k^\pm) \qquad \text{(strong convergence; } I_\pm \subset \mathbb{N}).$$

Here, $\lambda_k^\pm$ runs through all eigenvalues of $Z_\pm$ and $E(\lambda_k^\pm)$ are the corresponding (ordinary) projection operators in the Hilbertspace $\mathscr{H}$ of physical states.

This assumption holds automatically if there is a set $\mathscr{F}$ of (composite) conformally covariant fields such that the vectors $\phi(x)|0\rangle$, $\phi\in\mathscr{F}$, span $\mathscr{H}$. Since this is the situation envisioned when doing operator product expansions this is a natural assumption. Another important point is that by using more complicated mathematical notions (i.e. the general spectral theorem and "vectorvalued measures") one could well do without Equation (5.3). Since $Z_\pm$ are unitary, we have

$$\lambda_k^\pm = e^{i\pi\mu_k^\pm}; \quad 0 \leqq \mu_k^\pm < 2. \tag{5.4}$$

Our procedure will be to carry out a harmonic analysis on the center of $\tilde{C}$ first, and then use the results of Section IV.

The main result of this section is now formulated in the following theorem:

Theorem 5.1. *Let φ_1 and φ_2 be two local fields in a CQFT in two space time dimensions and $n_1^\pm$ resp. $n_2^\pm$ their conformal quantum numbers. Denote by $|\omega_+,\omega_-\rangle$ the analytic continuation of $\varphi_1(x_1)\varphi_2(x_2)|0\rangle$ as explained in Theorem 3.4. Assume furthermore, that the spectrums of Z_+ and Z_- are discrete, i.e. Equation (5.3) holds. Then there are vectorvalued, holomorphic functions $|w_+,w_-,k\rangle$, $w_\pm\in\Pi$, $k=3,4,5,\ldots$ such that*

a) $U(g_+\times g_-)|w_+,w_-,k\rangle = (\xi_+w_++\eta_+)^{-n_k^+}(\xi_-w_-+\eta_-)^{-n_k^-}|g_+(w_+),$
$g_-(w_-),k\rangle\,.$ (5.5)

Here, $n_k^\pm \geqq 0$ and $\pi(g_\pm)=\begin{pmatrix}\sigma_\pm & \tau_\pm\\ \xi_\pm & \eta_\pm\end{pmatrix}$. The phases of the multipliers are the same as in (2.17).

b) *$|\omega_+,\omega_-\rangle$ can be expanded into an orthogonal sum as follows:*

$$\begin{aligned}|\omega_+,\omega_-\rangle = \sum_{k=3}^{\infty}(n_k^+-1)(n_k^--1)\pi^{-2}\int_{\Pi\times\Pi}&|dw_+||dw_-|C_{n_1^+n_2^+}^{n_k^+}(\omega_+|w_+)\\ &\cdot C_{n_1^-n_2^-}^{n_k^-}(\omega_-|w_-)\\ &\cdot(\operatorname{Im}w_+)^{n_k^+-2}(\operatorname{Im}w_-)^{n_k^--2}|w_+,w_-,k\rangle\,.\end{aligned} \tag{5.6}$$

The kernels $C_{n_1n_2}^n(\omega|w)$ are first defined for $\omega=(w_1,w_2,0)$

$$\begin{aligned}C_{n_1n_2}^n(\omega|w) &\doteq [i(w_1-w_2)]^{-\delta_3}[w_1-w^*]^{-\delta_2}[w_2-w^*]^{-\delta_1}\\ &\delta_1\doteq\tfrac{1}{2}(n-n_1+n_2);\ \delta_2\doteq\tfrac{1}{2}(n-n_2+n_1);\ \delta_3=\tfrac{1}{2}(n_1+n_2-n)\\ &|\arg[i(w_1-w_2)]|<\pi\end{aligned} \tag{5.7}$$

and otherwise through analytic continuation. In case, say $n_k^+=0$ $|w_+,w_-,k\rangle$ does not depend on w_+. This can happen only if $n_1^+=n_2^+$ and the corresponding contribution to Equation (5.6) should be read as follows: $(n_1^+=n_2^+=n^+)$

$$[i(w_1^+-w_2^+)]^{-n^+}(n_k^--1)\pi^{-1}\int_{\Pi}|dw_-|C_{n_1^-n_2^-}^{n_k^-}(\omega_-|w_-)(\operatorname{Im}w_-)^{n_k^--2}|w_+,w_-,k\rangle. \tag{5.8}$$

In case $n_k^+=n_k^-=0$ (which can happen only if $n_1^+=n_2^+=n^+$, $n_1^-=n_2^-\doteq n^-$), the contribution to (5.6) reduces to:

$$\varrho[i(w_1^+-w_2^+)]^{-n^+}[i(w_1^--w_2^-)]^{-n^-}|0\rangle; \quad \varrho\in\mathbb{C}. \tag{5.9}$$

Comments. a) Equation (5.5) says, that $U(g)$, $g \in \tilde{C}$, acts irreducibly in the subspace $\mathscr{H}_k$ of $\mathscr{H}$ spanned by $|w_+, w_-, k\rangle$. If $k \neq l$, the spaces $\mathscr{H}_k$ and $\mathscr{H}_l$ are orthogonal.

b) Equation (5.6) arises from the fact, that the subspace $\mathscr{H}_{\varphi_1\varphi_2}$ of $\mathscr{H}$ spanned by the vectors $|\omega_+, \omega_-\rangle$, is the unitary direct sum of the $\mathscr{H}_k$'s:

$$\mathscr{H}_{\varphi_1\varphi_2} = \bigoplus_{k=3}^{\infty} \mathscr{H}_k \, .$$

c) Remark a) can be expressed in formulas as follows:

$$\langle w_+, w_-, k | w'_+, w'_-, l \rangle = \delta_{k,l} a_k G_{n_k^+}(w_+, w'_+) G_{n_k^-}(w_-, w'_-) \, . \tag{5.10}$$

The numbers $a_k > 0$ carry a piece of the dynamical information contained in $\varphi_1(x)\varphi_2(y)|0\rangle$. The same is true for the set $\{(n_k^+, n_k^-) | k = 3, 4, \ldots\}$ of conformal quantum numbers appearing in the expansion (5.6). This set will be called the conformal spectrum of the operator product $\varphi_1(x)\varphi_2(y)$.

The anomalous parts of the quantum numbers (n_k^+, n_k^-) are related to the eigenvalues of (Z_+, Z_-). In fact we have

$$n_k^+ \equiv \mu_j^+ (\mathrm{mod}\, 2) \, ; \qquad n_k^- \equiv \mu_l^- (\mathrm{mod}\, 2)$$

for some j, l depending on k. The vector $|w_+, w_-, k\rangle$ is then an element of $E(\lambda_j^+) E(\lambda_l^-) \mathscr{H}$.

d) The conformal spectrum of $\varphi_1(x)\varphi_2(y)$ is restricted by locality, namely the "spins" $s_k = \frac{1}{2}(n_k^+ - n_k^-)$ take on values only from $\{0, \pm 1, \pm 2, \ldots\}$ or $\{\pm \frac{1}{2}, \pm \frac{3}{2}, \ldots\}$ depending on the spins of φ_1 and φ_2.

e) From (5.10) it is easily seen that the integrals appearing in (5.6) are well-defined (for say $n_k^+ \leqq 1$, they need a special treatment: see the remark after (2.20)). For fixed ω_+, ω_- the sum (5.6) converges in the norm of the underlying Hilbert-space $\mathscr{H}$. Looking at each term in this series as a tempered distribution (for $\mathrm{Im}\, w_{1,2}^{\mp} \searrow 0$), smearing with some test function $f \in \varphi(\mathbb{R}^4)$ and summing up, yields $\int dx_1 dx_2 f(x_1, x_2) \varphi_1(x_1) \varphi_2(x_2) |0\rangle$.

f) The above theorem relies purely on "conformal kinematics". Without having some dynamical information about the model considered, there seems to be no way to prove the existence of local, conformally covariant fields ϕ_k, such that

$$\lim_{\mathrm{Im}\, w_\pm \searrow 0} |w_+, w_-, k\rangle = \phi_k(x_+, x_-)|0\rangle \, ; \qquad \mathrm{Re}\, w_\pm = x_\pm \, .$$

g) The conformal spectrum of $\varphi_1(x)\varphi_2(y)$ and the numbers a_l in (5.10) can be calculated from the four point function

$$\langle 0 | \varphi_2^+(x_1) \varphi_1^+(x_2) \varphi_1(x_3) \varphi_2(x_4) | 0 \rangle \, .$$

How this can be done, will become clear during the proof of the theorem.

h) When the expansion (5.6) is inserted into the fourpoint function, one obtains a series that looks exactly like the discrete expansion, which is extracted from the (euclidean) partialwave expansion of the fourpoint Schwinger function by means of an "inverse Sommerfeld-Watson transform" [2]. That the amplitudes a_l (5.10) and the quantum numbers $n_k^\pm$ are the same in both expansions, has been checked in the Thirringmodel [14].

Proof of 5.1. Let $E(\lambda)$ one of the projectors appearing in the spectral decomposition of (Z_+, Z_-), i.e.

$$E(\lambda) = E(\lambda_k^+)\cdot E(\lambda_l^-) = E(\lambda_l^-)\cdot E(\lambda_k^+) \tag{5.11}$$

for some k, l. Because Z_+ and Z_- commute with all $U(g)$, $g \in \tilde{C}$, we have

$$U(g)E(\lambda)U(g)^{-1} = E(\lambda), \quad g \in \tilde{C}. \tag{5.12}$$

Now set $(\lambda_k^+, \lambda_l^-) = (\lambda^+, \lambda^-) = (e^{i\pi\mu^+}, e^{i\pi\mu^-})$ and

$$|\omega_+, \omega_-; \lambda\rangle \doteq E(\lambda)|\omega_+, \omega_-\rangle. \tag{5.13}$$

Since $E(\lambda)$ is a bounded operator $|\omega_+, \omega_-; \lambda\rangle$ is a holomorphic, vectorvalued function. Moreover, from (3.17) and (5.12):

$$U(g_+ \times g_-)|\omega_+, \omega_-; \lambda\rangle = (\xi_+ w_1^+ + \eta_+)^{-n_1^+}(\xi_+ w_2^+ + \eta_+)^{-n_2^+} \cdot (\xi_- w_1^- + \eta_-)^{-n_1^-}(\xi_- w_2^- + \eta_-)^{-n_2^-}|g_+(\omega_+), g_-(\omega_-); \lambda\rangle \tag{5.14}$$

$$\pi(g_\pm) = \begin{pmatrix} \sigma_\pm & \tau_\pm \\ \xi_\pm & \eta_\pm \end{pmatrix}; \quad p(\omega_\pm) = (w_1^\pm, w_2^\pm).$$

Especially for $g_+ = z_+$, $g_- = 1$:

$$U(z_+ \times 1)|\omega_+, \omega_-; \lambda\rangle = e^{i\pi(n_1^+ + n_2^+)}|z_+(\omega_+), \omega_-; \lambda\rangle.$$

On the other hand by definition of $E(\lambda)$:

$$U(z_+ \times 1)|\omega_+, \omega_-; \lambda\rangle = Z_+|\omega_+, \omega_-; \lambda\rangle = e^{i\pi\mu^+}|\omega_+, \omega_-; \lambda\rangle.$$

Hence:

$$\begin{aligned} |z_+(\omega_+), \omega_-; \lambda\rangle &= e^{i\pi(\mu^+ - n_1^+ - n_2^+)}|\omega_+, \omega_-; \lambda\rangle, \\ |\omega_+, z_-(\omega_-); \lambda\rangle &= e^{i\pi(\mu^- - n_1^- - n_2^-)}|\omega_+, \omega_-; \lambda\rangle. \end{aligned} \tag{5.15}$$

One can even get rid of the phases $e^{i\pi(\mu^\pm - n_1^\pm - n_2^\pm)}$ by multiplying $|\omega_+, \omega_-; \lambda\rangle$ with an appropriate factor. Define $\alpha_\nu(\omega)$, $\omega \in \widetilde{\Pi \mathring{\times} \Pi}$, first for $\omega = (w_1, w_2, 0)$:

$$\alpha_\nu(\omega) = [i(w_1 - w_2)]^{-\nu}, \ |\arg[i(w_1 - w_2)]| < \pi \tag{5.16}$$

and for general ω by analytic continuation. Then:

$$\alpha_\nu(z(\omega)) = e^{-2\pi i\nu}\alpha_\nu(\omega). \tag{5.17}$$

Now renormalize $|\omega_+, \omega_-; \lambda\rangle$ as follows:

$$|\omega_+, \omega_-; \lambda\rangle' \doteq \alpha_{\nu^+}(\omega_+)\alpha_{\nu^-}(\omega_-)|\omega_+, \omega_-; \lambda\rangle$$
$$\nu^\pm \doteq \tfrac{1}{2}(\mu^\pm - n_1^\pm - n_2^\pm).$$

Because

$$|z_+(\omega_+), \omega_-; \lambda\rangle' = |\omega_+, z_-(\omega_-); \lambda\rangle' = |\omega_+, \omega_-; \lambda\rangle'$$

it is possible to define

$$|w_1^+, w_2^+; w_1^-, w_2^-; \lambda\rangle \doteq |\omega_+, \omega_-; \lambda\rangle'; \quad p(\omega_\pm) = (w_1^\pm, w_2^\pm)$$

for all pairs $(w_1^\pm, w_2^\pm)\in\Pi \mathring{\times} \Pi$. By construction, this new vectorvalued function is holomorphic and moreover:

$$U(g_+\times g_-)|w_1^+, w_2^+; w_1^-, w_2^-; \lambda\rangle = (\xi_+ w_1^+ + \eta_+)^{-m_1^+}(\xi_+ w_2^+ + \eta_+)^{-m_2^+} \tag{5.18}$$
$$(\xi_- w_1^- + \eta_-)^{-m_1^-}(\xi_- w_2^- + \eta_-)^{-m_2^-}|g_+(w_1^+), g_+(w_2^+); g_-(w_1^-), g_-(w_2^-); \lambda\rangle$$
$$m_1^\pm \doteq \tfrac{1}{2}(\mu^\pm + n_1^\pm - n_2^\pm); m_2^\pm \doteq \tfrac{1}{2}(\mu^\pm - n_1^\pm + n_2^\pm).$$

Lemma. $|w_1^+, w_2^+; w_1^-, w_2^-; \lambda\rangle$ *can be extended to a holomorphic vectorvalued function on* $(\Pi\times\Pi)\times(\Pi\times\Pi)$.

Proof. Since $\varphi_1(x)\varphi_2(y)|0\rangle$ is a tempered, vectorvalued distribution and $\|U(g)\| = 1 (g\in\tilde{C})$ it follows that there exists some $k_\pm\in\mathbb{Z}$, $k_\pm \geqq 0$ such that

$$\|(w_1^+ - w_2^+)^{k^+}(w_1^- - w_2^-)^{k-}|w_1^+, w_2^+; w_1^-, w_2^-; \lambda\rangle\|$$

remains bounded as $(w_1^\pm, w_2^\pm)$, $w_1^\pm \neq w_2^\pm$, varies through any compact subset of $\Pi\times\Pi$. We can assume that $k_\pm$ are the smallest numbers satisfying this requirement. Like in the case of an isolated singularity of a holomorphic function $f(z)$ of one complex variable, it can be shown that $(w_1^+ - w_2^+)^{k_+}(w_1^- - w_2^-)^{k_-}|w_1^+, w_2^+; w_1^-, w_2^-; \lambda\rangle$ can be extended to a holomorphic function on all of $(\Pi\times\Pi)^2$.

Especially

$$\lim_{\substack{w_1^+\to w^+\\ w_2^+\to w^+}} (w_1^+ - w_2^+)^{k_+}(w_1^- - w_2^-)^{k_-}|w_1^+, w_2^+; w_1^-, w_2^-; \lambda\rangle; \qquad w^+\in\Pi$$

exists and defines a holomorphic, vectorvalued function $|w^+; w_1^-, w_2^-\rangle$. From covariance (5.18) of $|w_1^+, w_2^+; w_1^-, w_2^-; \lambda\rangle$ we have:

$$\langle w^+; w_1^-, w_2^-|w^{+\prime}; w_1^{-\prime}, w_2^{-\prime}\rangle = G_{\mu^+ - 2k_+}(w^+, w^{+\prime})F(w_1^-, w_2^-, w_1^{-\prime}, w_2^{-\prime}).$$

Positivity now implies, that $\mu^+ - 2k_+ \geqq 0$ or $F\equiv 0$, i.e. by minimality of $k^\pm$, we have: $k^\pm = 0$. □ (Lemma)

Now assume that $\mu^\pm > 0$ (for $\mu^+ = 0$ and/or $\mu^- = 0$ one has to take care of the degenerate cases (5.8) and (5.9); since no serious difficulties are encountered here, I will not discuss these cases further). Recall the polynomials Q^k_{m,m_2} defined by (4.15). They are now of great use: set

$$|w_+, w_-; k, l\rangle \doteq N_k^{-1}Q^k_{m_1^+ m_2^+}(\partial/\partial w_1^+, \partial/\partial w_2^+)N_l^{-1}Q^l_{m_1^- m_2^-}(\partial/\partial w_1^-, \partial/\partial w_2^-)$$
$$\cdot|w_1^+ w_2^+; w_1^- w_2^-\rangle|_{\substack{w_1^+ = w_2^+ = w^+\\ w_1^- = w_2^- = w^-}}$$

N_k, N_1 as in (4.17) with $n_{1,2} = m_{1,2}^+$ resp. $n_{1,2} = m_{1,2}^-$. (5.19)

(the index λ has been omitted). These are vectorvalued holomorphic functions for $w_\pm\in\Pi$ and they transform as:

$$U(g_+\times g_-)|w_+, w_-; k, l\rangle = (\xi_+ w_+ + \eta_+)^{-\mu^+ - 2k}(\xi_- w_- + \eta_-)^{-\mu^- - 2l}$$
$$\cdot|g_+(w_+), g_-(w_-); k, l\rangle.$$

Therefore, for fixed k, l the vectors $|w_+, w_-; k, l\rangle$ span a subspace $\mathscr{H}_{k,l}$ of $\mathscr{H}$ where the conformal group $\tilde{C}$ acts irreducibly. Let $\mathbb{P}_{k,l}$ the projector on $\mathscr{H}_{k,l}$. It is a simple matter to show that

$$\mathbb{P}_{k,l}|w_1^+, w_2^+; w_1^-, w_2^-\rangle = (\mu^+ + 2k - 1)(\mu^- + 2l - 1)\pi^{-2} \int_{\Pi \times \Pi} |dw_+||dw_-| \,. \tag{5.20}$$
$$K^k_{m_1^+ m_2^+}(w_1^+, w_2^+|w_+) K^l_{m_1^- m_2^-}(w_1^-, w_2^-|w_-)(\operatorname{Im} w_+)^{\mu^+ + 2k - 2}(\operatorname{Im} w_-)^{\mu^- + 2l - 2}$$
$$|w_+, w_-; k, l\rangle \,.$$

Because the $\mathbb{P}_{k,l}$'s are mutually orthogonal projectors, the sum

$$\sum_{k,l=0}^{\infty} \mathbb{P}_{k,l} \doteq \mathbb{P}$$

is strongly convergent and $\mathbb{P}$ is again a projector. Hence

$$\mathbb{P}|w_1^+, w_2^+; w_1^-, w_2^-\rangle = \sum_{k,l=0}^{\infty} \mathbb{P}_{k,l}|w_1^+, w_2^+; w_1^-, w_2^-\rangle \tag{5.21}$$

is a vectorvalued holomorphic function as well.

By construction:

$$|w_+, w_-; k, l\rangle = N_k^{-1} Q^k_{m_1^+ m_2^+}(\partial/\partial w_1^+, \partial/\partial w_2^+) N_l^{-1} Q^l_{m_1^- m_2^-}(\partial/\partial w_1^-, \partial/\partial w_2^-)$$
$$\mathbb{P}|w_1^+, w_2^+; w_1^-, w_2^-\rangle|_{\substack{w_1^+ = w_2^+ = w_+ \\ w_1^- = w_2^- = w_-}} \,.$$

Analyticity and the following elementary lemma now imply that

$$|w_1^+, w_2^+; w_1^-, w_2^-\rangle = \mathbb{P}|w_1^+, w_2^+; w_1^-, w_2^-\rangle \,. \tag{5.22}$$

Lemma 5.2. *Any polynomial $Y(x_1, x_2)$ of two variables is a finite linear combination of the polynomials*

$$(x_1 + x_2)^j Q^k_{m_1 m_2}(x_1, x_2); \qquad k, j = 0, 1, 2, \ldots$$
$$(m_1, m_2 \in \mathbb{R} \text{ fixed}; \qquad m_1 + m_2 \neq 0, -1, -2, \ldots) \,.$$

Putting together formulae (5.22), (5.21), (5.20) yields the conformal partial wave expansion for $|w_1^+, w_2^+; w_1^-, w_2^-; \lambda\rangle$. Taking into account the definition of this vector, summing over λ and rearranging some factors finally yields (5.6) and thereby proves the theorem. □

VI. Vacuum Expansions in the Thirring Model [3]

6.1. *Definition of the Model*

Notations and standard results concerning the Thirring model (e.g. [15, 16]) will be taken over from Ref. [16]. To make the present discussion sufficiently self contained, the basic structure of the model is briefly exhibited.

The theory is conveniently formulated in terms of two fields: a current j^μ and a spinor field ψ with two components ψ_1, ψ_2. The current j^μ and its axial brother $\tilde{j}_\mu = \varepsilon_{\mu\nu} j^\nu (\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \varepsilon_{10} = +1)$ are both conserved:

$$\partial_\mu j^\mu = \partial_\mu \tilde{j}^\mu = 0 \,. \tag{6.1}$$

This implies $\Box j^{\mu}=0$, i.e. j^{μ} is a free field. However the operator representation of j^{μ} is not of the Fock type. In fact, the Hilbertspace of the model carries a reducible representation of the canonical commutation relations:

$$[j_{\mu}(x^0,x^1),j_{\nu}(y^0,y^1)]_{x^0=y^0}=c\cdot i^{-1}\varepsilon_{\mu\nu}\delta'(x^1-y^1)\,. \tag{6.2}$$

$c>0$ is a normalization constant. Because of (6.1) the charges

$$Q_{\pm}\doteq\int dx^1 j_{\pm}(x)\,; \qquad j_{\pm}(x)\doteq j_0(x)\pm j_1(x) \tag{6.3}$$

do not depend on x^0. From (6.2) we have: $[Q_+,Q_-]=0$.

The (simultaneous) spectrum of $Q_{\pm}$ is given by:

$$(Q_+,Q_-)=(m_1(a+\bar{a})+m_2(a-\bar{a});m_1(a-\bar{a})+m_2(a+\bar{a})) \tag{6.4}$$
$$m_1,m_2=0,\pm1,\pm2,\ldots\,.$$

The real numbers a and $\bar{a}$ parametrize the model (like coupling constants do). The representation of (6.2) is now specified as follows:

First, the Hilbertspace $\mathscr{H}$ splits into charge sectors:

$$\mathscr{H}=\bigoplus_{(q_+,q_-)}\mathscr{H}_{q_+,q_-} \tag{6.5}$$
$$Q_{\pm}|\chi\rangle=q_{\pm}|\chi\rangle \quad \text{for} \quad |\chi\rangle\in\mathscr{H}_{q^+,q^-}\,.$$

Each charge sector $\mathscr{H}_{q+q-}$ carries an irreducible representation of (6.2) or equivalently of:

$$[j_+(x_+),j_-(y_-)]=0;[j_{\pm}(x_{\pm}),j_{\pm}(y_{\pm})]=2ic\delta'(x_{\pm}-y_{\pm}) \tag{6.6}$$

(due to (6.1), $j_{\pm}$ depends only upon $x_{\pm}$ respectively).

This representation is characterized by:

"If $I_{\pm}(x_{\pm})$ are two $\mathbb{R}$-number functions, such that

a) $I_{\pm}(x_{\pm})$ and $x_{\pm}^{-2}I_{\pm}(x_{\pm}^{-1})$ are C^{∞},

b) $\int dx_{\pm}I_{\pm}(x_{\pm})=q_{\pm}$,

then the currents

$$j_{\pm}^{I_{\pm}}(x_{\pm})\doteq j_{\pm}(x_{\pm})-I_{\pm}(x_{\pm})$$

are of Focktype"[2].

The spinor field $\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ now "intertwines" the representations of the current in different charge sectors:

$$\begin{aligned}[][j_+(x_+),\psi(y_+,y_-)]&=-(a+\bar{a}\gamma_5)\psi(y_+,y_-)\delta(x_+-y_+)\\ [j_-(x_-),\psi(y_+,y_-)]&=-(a-\bar{a}\gamma_5)\psi(y_+,y_-)\delta(x_--y_-)\end{aligned} \tag{6.7}$$

where $\gamma_5=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Accordingly, the ψ-field is charged:

$$[Q_{\pm},\psi_1]=-(a\pm\bar{a})\psi_1\,;[Q_{\pm},\psi_2]=-(a\mp\bar{a})\psi_2\,. \tag{6.8}$$

[2] In the charge sector $\mathscr{H}_{0,0}$, the representation of j^{μ} is itself of Focktype; the vacuum in $\mathscr{H}_{0,0}$ will be denoted by $|0\rangle$

It has been shown that there exists up to normalization one and only one spinor field ψ satisfying (6.7) [16].

This defines the Thirring model.

The spin $s_1(s_2)$ of $\psi_1(\psi_2)$ turns out to be

$$s_1 = -s_2 = a \cdot \bar{a}/2\pi c\,. \tag{6.9}$$

In case the parameters a, $\bar{a}$ are restricted such that $s_1 = \pm 1/2, \pm 1, \ldots$ the Thirring model becomes an ordinary Wightman quantum field theory [16]. I will assume throughout this section that $s_1 = \pm 1/2, \pm 1, \ldots$.

6.2. *The Implementation of the Conformal Group $\tilde{C}$*

The Wightman distributions for ψ, ψ^+ can be calculated explicitly. They are conformally covariant under infinitesimal transformations (2.28), when ψ is given a dimension (2.29) d,

$$d = (a^2 + \bar{a}^2)/4\pi c\,. \tag{6.10}$$

According to a general theorem [5], there exists a unitary representation $U(\cdot)$ of the conformal group $\tilde{C}$ acting on the field ψ as described in § 2.3.

The current j^μ transforms simply:

$$U(g_+ \times g_-) j_\pm(x_\pm) U(g_+ \times g_-)^{-1} = (\xi_\pm x_\pm + \eta_\pm)^{-2} j_\pm(g_\pm(x_\pm))\,. \tag{6.11}$$

$$\pi(g_\pm) = \begin{pmatrix} \sigma_\pm & \tau_\pm \\ \xi_\pm & \eta_\pm \end{pmatrix} \in \mathrm{Sl}(2, \mathbb{R})\,.$$

This implies that the charges $Q_\pm$ are conformally invariant. To make use of Theorem 5.1, it is necessary to calculate the operators $Z_\pm$ (5.1). Equation (6.11) is valid in each charge sector $\mathscr{H}_{q_+q_-}$ separately. Since the current is represented irreducibly in $\mathscr{H}_{q_+,q_-}$, $U(g_+ \times g_-)$ is determined by Equation (6.11) up to a phase. There is a unique choice of phases, such that the operators $U(g)$, $g \in \tilde{C}$, satisfy the multiplication law. The outcome is that: [17]

$$Z_\pm = e^{iQ_\pm^2/4c}\,; \qquad c \text{ as in } (6.2)\,. \tag{6.12}$$

Briefly: due to the fact that there are non-Fock-representations of the current, the conformal group C is forced to unroll itself to become $\tilde{C}$. Thereby, the center of $\tilde{C}$ will be represented as shown in (6.12).

When formula (6.12) is applied to the twopoint function (2.30) of a covariant field ϕ having spin s, dimension d and charges q_+, q_-, an interesting relation emerges:

$$\begin{aligned} d &\equiv (q_+^2 + q_-^2)/8\pi c \quad (\mathrm{mod}\, 1) \\ s &\equiv (q_+^2 - q_-^2)/8\pi c \quad (\mathrm{mod}\, 1) \end{aligned} \tag{6.13}$$

i.e. spin and dimension of ϕ are functions of its charges modulo integer numbers. From (6.4) it follows that $s = 0, \pm\frac{1}{2}, \pm 1, \ldots$.

6.3. *Construction of a Complete Set of Local, Covariant Fields*

Now we are well prepared to apply Theorem 5.1. From (6.12) it follows that $Z_\pm$ has indeed a purly discrete spectrum. Let φ_1 and φ_2 two conformally covariant fields with conformal quantum numbers $n^{\pm}_{1,2}$ and charges $q^{\pm}_{1,2}$ respectively. The vector $\varphi_1(x)\varphi_2(y)|0\rangle$ is then already an eigenvector of $Z_{\pm}$:

$$Z_{\pm}\varphi_1(x_1)\varphi_2(x_2)|0\rangle = e^{i[q_1^{\pm}+q_2^{\pm}]^2/4c}\varphi_1(x_1)\varphi_2(x_2)|0\rangle . \tag{6.14}$$

Thus, to construct its partial waves, the projection (5.13) is superfluous. Following the program given in the proof of Theorem 5.1, one has to multiply $\varphi_1(x_1)\varphi_2(x_2)|0\rangle$ with a factor $[i(x_1^+ - x_2^+ - i\varepsilon)]^{-\nu^+}[i(x_1^- - x_2^- - i\varepsilon)]^{-\nu^-}$ where

$$\begin{aligned} &\nu^{\pm} = \tfrac{1}{2}(\mu^{\pm} - n_1^{\pm} - n_2^{\pm}) \\ &0 \leqq \mu^{\pm} < 2;\ \mu^{\pm} \equiv [q_1^{\pm} + q_2^{\pm}]^2/4\pi c \qquad (\mathrm{mod}\, 2)\,. \end{aligned} \tag{6.15}$$

Thereby the singularity of $\varphi_1(x_1)\varphi_2(x_2)|0\rangle$ as $x_1 \to x_2$ is killed. We may then apply the differential operators

$$Q^k_{m_1^+ m_2^+}(\partial/\partial x_1^+, \partial/\partial x_2^+)Q^l_{m_1^- m_2^-}(\partial/\partial x_1^-, \partial/\partial x_2^-) \tag{6.16}$$

and evaluate the new function at $x_1 = x_2 = x$ (rigorously speaking, one should do everything in the conformal forward tube, coming back to Minkowskispace at the end of the calculation). By this procedure, one obtains vectorvalued distributions $|x; k, l\rangle$ transforming irreducibly under the conformal group $\tilde{C}$.

If, e.g., φ_1 and φ_2 are any of the fields $\psi_1, \psi_1^+, \psi_2, \psi_2^+$ one easily shows from the explicit form of the Wightman distributions (Appendix D) that

$$\begin{aligned} \phi_{k,l}(x) \doteq\ & Q^k_{m_1^+ m_2^+}(\partial/\partial x_1^+, \partial/\partial x_2^+)Q^l_{m_1^- m_2^-}(\partial/\partial x_1^-, \partial x_2^-) \\ & \cdot [i(x_1^+ - x_2^+)]^{-\nu^+}[i(x_1^- - x_2^-)]^{-\nu^-}\varphi_1(x_1)\varphi_2(x_2)|_{x_1 = x_2 = x} \end{aligned} \tag{6.17}$$

exists as an operator valued distribution. Of course,

$$|x; k, l\rangle = \phi_{k,l}(x)|0\rangle \tag{6.18}$$

and $\phi_{k,l}$ is a covariant, local field with conformal quantum numbers

$$n^+_{k,l} = \mu^+ + 2k$$

$$n^-_{k,l} = \mu^- + 2l$$

and charges

$$q^{\pm}_{k,l} = q_1^{\pm} + q_2^{\pm}\,.$$

Wightman distributions involving ψ and $\phi_{k,l}$ fields have the same general structure as the Wightman distributions of ψ fields only (i.e. they are sums of products of powers of difference variables $(x_{\pm} - y_{\pm})$ [15]). We can therefore iterate the above procedure, e.g. by taking $\varphi_1 = \psi_1$ and $\varphi_2 = \varphi_{k,l}$. In this way one arrives at an infinite set $\mathscr{F}$ of new fields. $\mathscr{F}$ has the following properties:

I. The fields contained in $\mathscr{F}$ satisfy the axioms of a *CQFT*. Especially, any two fields $\phi_1, \phi_2 \in \mathscr{F}$ are relatively local:

$$[\phi_1(x), \phi_2(y)]_{\pm} = 0 \quad \text{for} \quad (x-y)^2 < 0\,. \tag{6.19}$$

II. Each $\phi \in \mathscr{F}$ carries charges q_+, q_-, i.e.

$$[Q_{\pm}, \phi] = q_{\pm}\phi\,.$$

Charges, spin and dimension of ϕ are related through Equation (6.13).

III. The states $\int d^2x f(x)\phi(x)|0\rangle$, $f\in\mathscr{S}$, $\phi\in\mathscr{F}$ span the Hilbertspace $\mathscr{H}$ of physical states.

IV. The partial wave expansion (vacuumexpansion) (5.6) of vectors $\varphi_1(x)\varphi_2(y)|0\rangle$, $\varphi_{1,2}\in\mathscr{F}$ can be done in terms of vectors $\phi(x)|0\rangle$, $\phi\in\mathscr{F}$. More precisely, there are fields $\phi_k\in\mathscr{F}$, such that the vectors $|w_+, w_-, k\rangle$ described in Theorem 5.1 are determined by $\phi_k(x)|0\rangle$:

$$\lim_{\mathrm{Im}\, w_\pm \searrow 0} |w_+, w_-, k\rangle = N\phi_k(x_+, x_-)|0\rangle\,; \mathrm{Re}\, w_\pm = x_\pm, N\in\mathbb{C}\,.$$

The question arises of whether this remarkable algebraic structure might hold more generally. The only serious point apart from some regularity problems seems to be locality (6.19). Locality implies some crossing relations between partial wave amplitudes [1, 2]. To answer the question above presumably forces one to analyse the crossing conditions, a task which is by no means simple.

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Appendix A. Proof of Lemmas 3.2/3

It is a trivial calculation to prove that $\Theta(g(\omega))=\Theta(\omega)$ for all $g\in\tilde{G}$. For $s\in\tilde{S}^0$ we have: $\Theta(s(\omega))>\Theta(\omega)$. Indeed, this inequality is easily proved for $s=\exp i\cdot t\cdot H$, $t>0$. Equations (3.5) and (3.7) then imply its validity for general $s\in\tilde{S}^0$.

Now assume that $\omega, \omega'\in\widetilde{\Pi\mathring{\times}\Pi}$ and that $\Theta(\omega)=\Theta(\omega')$. We have to show that there is some $\tilde{g}\in\tilde{G}$ such that $\tilde{g}(\omega)=\omega'$. Set $p(\omega)=(w_1, w_2)$, $p(\omega')=(w_1', w_2')$. Since the center of $\tilde{G}$ acts transitively on the sheets of $\widetilde{\Pi\mathring{\times}\Pi}$ the problem can be solved when an element g of $G=\mathrm{Sl}(2,\mathbb{R})$ can be found, such that $g(w_1, w_2)=(w_1', w_2')$. First there are translations/dilations (2.6) g_1 and g_1' with:

$$g_1(w_1, w_2)=(i, z_2);\, g_1'(w_1', w_2')=(i, z_2')\,.$$

Applying suitable transformations from k (2.7) one obtains:

$$\left.\begin{aligned}(g_2\cdot g_1)(w_1, w_2)&=(i, \lambda i)\\ (g_2', g_1')(w_1', w_2')&=(i, \lambda' i)\end{aligned}\right\}\lambda, \lambda'>1\,. \tag{A1}$$

Therefore $\Theta(\omega)=\lambda/(\lambda-1)^2$, $\Theta(\omega')=\lambda'/(\lambda'-1)^2$ and hence $\lambda=\lambda'$. Setting $g=g_1'^{-1}g_2'^{-1}\cdot g_2\cdot g_1$ we have: $g(w_1, w_2)=(w_1', w_2')$ as required.

To prove Lemma 3.3 we may dublicate the proof of Lemma 3.2 up to Equation (A1). From $\Theta(\omega')>\Theta(\omega)$ it follows that $\lambda'<\lambda$. But in this case we can find some $t>0$ such that $e^{itH}(i, \lambda i)=(i, \lambda' i)$. Therefore $s(w_1, w_2)=(w_1', w_2')$ with $s=g_1'^{-1}\cdot g_2'^{-1}\cdot e^{itH}g_2\cdot g_1\in S^0$.

Appendix B. Proof of Lemma 3.5

Let $\hat{\mu}\doteq\pi(\mu(\omega|\omega_0))$ and $p(\omega)=(w_1, w_2)$, $p(\omega_0)=(w_1^0, w_2^0)$. Obviously

$$\hat{\mu}=\{s\in S^0|s(w_1^0, w_2^0)=(w_1, w_2)\}\,.$$

Now, $S^2\mathring{\times}S^2=\{(z_1, z_2)\in S^2\times S^2|z_1\neq z_2\}$ is a homogeneous space of $\mathrm{Sl}(2,\mathbb{C})$ (S^2 is the Riemannian sphere). Let L the little group of the point $(w_1^0, w_2^0)\in S^2\mathring{\times}S^2$

and s_0 a particular element of $\hat{\mu}$. Clearly then

$$\hat{\mu}=S^0 \cap s_0 \cdot L.$$

Therefore, $\hat{\mu}$ is a closed, holomorphic submanifold of S^0. This property of $\hat{\mu}$ carries over to $\mu(\omega|\omega_0)$, i.e. $\mu(\omega|\omega_0)$ is a closed, holomorphic submanifold of $\tilde{S}^0$.

If $\hat{\mu}$ were connected, $\mu(\omega|\omega_0)$ would be too. Indeed, given two elements s_1, s_2 of $\mu(\omega|\omega_0)$ there is a curve $\gamma(t)\subset\hat{\mu}$, $\gamma(0)=\pi(s_1)$, $\gamma(1)=\pi(s_2)$. This curve can be lifted uniquely to a curve $\Gamma(t)$ in $\tilde{S}^0$ such that $\Gamma(0)=s_1$. By continuity $\Gamma(t)\subset\mu(\omega|\omega_0)$. Moreover, $\pi(\Gamma(1))=\pi(s_2)$ hence $\Gamma(1)=\mathfrak{z}\cdot s_2$, $\mathfrak{z}$ a central element of $\tilde{G}$. But $\Gamma(1)(\omega_0)=s_2(\omega_0)=\omega$. Thus $\mathfrak{z}=1$ and $\Gamma(t)$ is therefore a curve connecting s_1 and s_2.

Thus it remains to show that $\hat{\mu}$ is connected. As in Appendix A we may assume that

$$(w_1^0, w_2^0)=(i\,\mathrm{tgh}\,t, i\,\mathrm{ctgh}\,t)=b_t(0,\infty); \quad t>0.$$

b_t denotes the matrix $\begin{pmatrix} \mathrm{ch}\,t & i\,\mathrm{sh}\,t \\ -i\,\mathrm{sh}\,t & \mathrm{ch}\,t \end{pmatrix}\in S^0$. The little group of (w_1^0, w_2^0) is given by

$$L=\{g\in \mathrm{Sl}(2.\,\mathbb{C})|g=b_t\begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1}\end{pmatrix}b_t^{-1}; \quad \sigma\in\mathbb{C}, \sigma\neq 0\}.$$

It is then a simple geometric task to verify that the set (s a fixed element from $S^0\cdot b_t$)

$$\{\sigma\in\mathbb{C}|s\cdot\begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1}\end{pmatrix}\in S^0\cdot b_t\}$$

is connected. Therefore $\hat{\mu}$ is connected too.

Appendix C. Proof of the Plancherel Formula (4.13)

Let $F(w_1, w_2)\in H_{n_1}\hat{\otimes} H_{n_2}$. Define $F_k(w)$ by (4.10). Using (4.6) and

$$F(w_1, w_2)=\int_0^\infty dp_1 dp_2 e^{ip_1w_1}e^{ip_2w_2}\tilde{F}(p_1, p_2)$$

we have:

$$\begin{aligned}F_k(w)&=(2^{5-n_1-n_2+k}k!\Gamma(n_1)\Gamma(n_2)/C_k\Gamma(n_1+k)\Gamma(n_2+k))e^{i\pi(n_1+n_2)/2}\\ &\quad\cdot\int_0^\infty dy\int_{-1}^1 dx e^{2iyw}y^{k+1}P^k_{n_1n_2}(x)\tilde{F}(y(1-x), y(1+x))\\ &=\int_0^\infty dp e^{ipw}\tilde{F}_k(p)\end{aligned}$$

where

$$\tilde{F}_k(p)=(e^{i\pi/2(n_1+n_2)}\Gamma(n_1+n_2+k-1)/2^{3-n_1-n_2-2k}\Gamma(n_1+n_2+2k-1))$$
$$p^{k+1}\cdot\int_{-1}^1 dx P^k_{n_1n_2}(x)\tilde{F}(p(1-x)/2, p(1+x)/2).$$

Therefore:

$$\|F_k\|^2_{n_1+n_2+2k}=(\Gamma(n_1+n_2+2k)(\Gamma(n_1+n_2+k-1))^2/2^{4-n_1-n_2-2k}$$
$$\cdot(\Gamma(n_1+n_2+2k-1))^2)$$
$$\cdot\int_0^\infty dp p^{3-n_1-n_2}$$
$$\cdot\int_{-1}^{1} dx dx' \tilde{F}(p(1-x)/2, p(1+x)/2)^* P^k_{n_1n_2}(x)P^k_{n_1n_2}(x')$$
$$\cdot\tilde{F}(p(1-x')/2, p(1+x')/2)\,.$$

$|\tilde{F}_k(p)|^2$ is measureable and nonnegative. Applying the monotone convergence theorem ([12], 13.8.1) we may interchange summation and p-integration in $\sum_{k=0}^{\infty} C_k\|F_k\|^2_{n_1+n_2+2k}$.

This yields:

$$\sum_{k=0}^{\infty} C_k\|F_k\|^2_{n_1+n_2+2k}=\int_0^\infty dp p^{3-n_1-n_2}\sum_{k=0}^{\infty} 2\Gamma(n_1)\Gamma(n_2)/h_k$$
$$\cdot\int_{-1}^{1} dx dx' \tilde{F}(p(1-x)/2, p(1+x)/2)^* P^k_{n_1n_2}(x)P^k_{n_1n_2}(x')\tilde{F}(p(1-x')/2, p(1+x')/2)$$

(h_k is defined in (4.9)). $F(w_1, w_2)$ is an element of $H_{n_1}\hat{\otimes} H_{n_2}$. For almost all $p\geqq 0$ the function

$$f_p(x)=(1-x)^{1-n_1}(1+x)^{1-n_2}\tilde{F}(p(1-x)/2, p(1+x)/2)$$

is therefore square integrable with respect to the measure $dx(1-x)^{n_1-1}\cdot(1+x)^{n_2-1}$ on $[-1, 1]$. Hence, the completeness relation for the Jacobipolynomials applies:

$$\sum_{k=0}^{\infty} C_k\|F_k\|^2_{n_1+n_2+2k}=2\Gamma(n_1)\Gamma(n_2)\int_0^\infty dp p^{3-n_1-n_2}$$
$$\cdot\int_{-1}^{1} dx(1-x)^{n_1-1}(1+x)^{n_2-1}|f_p(x)|^2\,.$$

Substituting $p_1=p(1-x)/2$, $p_2=p(1+x)/2$ yields the Plancherel formula (4.13).

Note that (4.13) implies

$$(G, F)_{n_1\times n_2}=\sum_{k=0}^{\infty}(G_k, F_k)_{n_1+n_2+2k} \qquad \text{(absolute convergence)}$$

for all $F, G\in\mathscr{H}_{n_1}\hat{\otimes}\mathscr{H}_{n_2}$. Especially when choosing G to be the reproducing kernel $G_{n_1}(w_1, w_1')\cdot G_{n_2}(w_2, w_2')$ formula (4.12) emerges. This series therefore converges pointlike.

Appendix D. The General Form of the Wightman Distributions in the Thirringmodel [15]

For any two points x, y of Minkowski space M and real numbers n_+, n_- define:

$$\Delta_{n_+n_-}(x, y)\doteq(x_+-y_+-i\varepsilon)^{-n_+}(x_--y_--i\varepsilon)^{-n_-}\,.$$

This is a tempered distribution on $M\times M$.

If $\varphi_1, \varphi_2, \ldots, \varphi_n$ are any of the fields $\psi_1, \psi_1^+, \psi_2, \psi_2^+$ in the Thirringmodel we have

$$\langle 0|\varphi_1(x_1)\ldots\varphi_n(x_n)|0\rangle = N\prod_{i<j}\Delta_{n_+^{ij}\, n_-^{ij}}(x_i, x_j)\,; \qquad N\in\mathbb{C}\,. \tag{D1}$$

This is also a tempered distribution, since it is a boundary value of a holomorphic function defined in the forward tube. From locality it follows that for any permutation π of $(1, \ldots, n)$

$$\langle 0|\varphi_{\pi(1)}(x_{\pi(1)})\ldots\varphi_{\pi(n)}(x_{\pi(n)})|0\rangle = \pm N\prod_{i<j}\Delta_{m_+^{ij}\, m_-^{ij}}(x_{\pi(i)}, x_{\pi(j)}) \tag{D2}$$

where $m_\pm^{ij} = n_\pm^{\pi(i),\pi(j)}$ $(n_\pm^{ij} = n_\pm^{ji})$. The singularity of $\varphi_1(x_1)\varphi_2(x_2)$ as $x_1 \to x_2$ is thus independent of where this product is placed in the n-point distribution. Especially, when taking a permutation π with $\pi(n-1)=1$, $\pi(n)=2$ we see that the operator product cannot be more singular than the vector $\varphi_1(x_1)\varphi_2(x_2)|0\rangle$ i.e. $-\nu^\pm - n_\pm^{12} = 0, 1, 2, \ldots$. Inserting the definition (6.17) of $\phi_{k,l}$ into Equations (D2) yields the Wightman distributions of fields $\phi_{k,l}, \psi_1, \psi_1^+, \psi_2, \psi_2^+$. They are sums of distributions of the type (D1). Each summand is again local in the sense that its permuted form (like (D2)) shows up also in the permuted Wightman distribution. The argument to prove the regularity of $\varphi_1(x_1)\varphi_2(x_2)\cdot[i(x_1^+ - x_2^+ - i\varepsilon)]^{-\nu^+}[i(x_1^- - x_2^- - i\varepsilon)]^{-\nu^-}$ as $x_1 \to x_2$ can thus also be applied to products of $\phi_{k,l}$'s, etc.

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17. This result has also been established by W. Rühl by inspection of the Wightman distributions, see [6]

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Fig. 1