Nuclear Physics B120 (1977) 113–124

ELECTRON-POSITRON SCALING IN BOUND STATE MODELS

M. BÖHM
Physikalisches Institut der Universität, Würzburg

M. KRAMMER
Deutsches Elektronen-Synchrotron DESY, Hamburg

Received 13 May 1976

We investigate scaling assuming a generalized vector meson dominance picture. The vector mesons are described as relativistic quark-antiquark bound states by a Bethe-Salpeter equation which yields the mass spectrum and the coupling to e^+e^- pairs. We discuss the spin structure and find that scaling can occur only for a γ_μ type amplitude. We solve the BS equation using a generalized WKB approximation and find scaling, independent of the detailed shape of the interaction. This means that scaling in e^+e^- annihilation does not select a particular "confinement potential". The scaling constant depends on the current renormalization constant and on the details of the relativistic spin structure.

1. Introduction

The data on e^+e^--annihilation into hadrons [1] indicate that the ratio $R \equiv \sigma_{tot}(e^+e^- \to \text{hadrons})/\sigma_{tot}(e^+e^- \to \mu^+\mu^-)$ becomes again a constant, once the new resonance region between J/ψ (3.1) and 4.5 GeV is passed (fig. 1). One is wont to think of the scaling constant R as the measure of the sum of the squared charges of the fundamental spin-$\frac{1}{2}$ constituents [2]. The simplest way of exemplifying this is the parton model. Asymptotically free field theoretical models predict a logarithmic approach down to $R = \Sigma Q_i^2$. The experimental value $R \approx 5.5$ is not explained by these models if there are four quark flavours only ($R = 3\frac{1}{3}$).

The excitation of a series of intermediate vector mesons, on the other hand, can lead to scaling if the masses and leptonic decay widths satisfy locally the condition [3]

$$R = \sum_{\substack{V=\rho^0,\omega;\\ \phi;J/\psi\text{,type}}} \frac{9\pi\, M_V \Gamma_{V\to e^+e^-}}{\alpha^2 \Delta M_V^2}\,. \tag{1}$$

It is interesting that phenomenological investigations of the ρ, ω, ϕ or the J/ψ threshold region [4,5] tend to give a higher value for R than the counting of quark char-

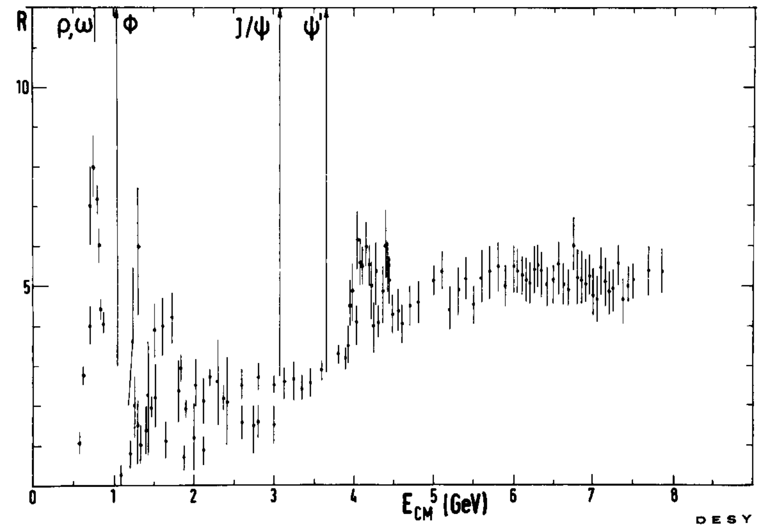

Fig. 1. $R \equiv \sigma_{tot}(e^+e^- \to \text{hadrons})/\sigma_{tot}^{QED}(e^+e^- \to \mu^+\mu^-)$. Data points were taken from ref. [1].

ges. E.g., from the J/ψ leptonic width and the $J/\psi - \psi'$ mass difference follows $\Delta R \approx 2.0$, which figure has to be seen in contrast to the number $\frac{4}{3}$.

Dynamical quark-antiquark bound state models yield a spectrum of vector mesons and their leptonic decay widths. Special models of this type show scaling behaviour [6,7]. Thus the question arises whether or not scaling is a general feature of bound state models. For non-relativistic Schrödinger type models it has been shown that scaling occurs independent of the shape of a confinement potential regular at the origin, provided the eigenvalue E is linearly related to the mass squared M^2 of the bound state [8].

In this paper we shall investigate the problem of scaling in relativistic, field theoretic bound state models, formulated with the help of the fermion-antifermion Bethe-Salpeter equation. The paper is organized as follows: Sects. 2 and 3 contain formulas related to e^+e^- processes and to the general structure of the BS amplitudes of vector mesons plus their photon coupling. A short review of the dynamical model for the strong binding of heavy quarks is given in sect. 4. Scaling behaviour can be expected only if the leading component of the vector meson BS amplitude is of the γ_μ type. For this class of models the radial equation is derived. Sect. 5 treats in detail the solution of this radial equation by a generalized WKB method. A relation between the wave function at the origin and the mass spectrum is found which leads to scaling, independent of the spatial shape of the interaction. In sect. 6 we compare the scaling constant of these dynamical models with the scaling constant of the parton model.

2. Matrix elements, widths and cross sections

The cross section for producing a pair of free, pointlike quarks q_i with masses m_i and charges Q_i is given by [9]

$$\frac{d\sigma}{d\cos\theta}(e^+e^- \to q_i\bar{q}_i) = \frac{Q_i^2\alpha^2\pi}{2s}\sqrt{1-\frac{4m_i^2}{s}}\left[1+\frac{4m_i^2}{s}+\left(1-\frac{4m_i^2}{s}\right)\cos^2\theta\right], \quad (2a)$$

$$s = E_{CM}^2, \quad m_e^2 \approx 0\,,$$

$$\sigma(e^+e^- \to q_i\bar{q}_i) = \frac{Q_i^2\alpha^2 4\pi}{3s}\sqrt{1-\frac{4m_i^2}{s}}\left(1+\frac{2m_i^2}{s}\right) \xrightarrow[s \gg 4m_i^2]{} \frac{Q_i^2\alpha^2 4\pi}{3s}\,, \quad 2b)$$

and

$$R \equiv \frac{\Sigma_i\,\sigma(e^+e^- \to q_i\bar{q}_i)}{\sigma(e^+e^- \to \mu^+\mu^-)} \xrightarrow[s \gg 4m_i^2]{} \sum_i Q_i^2\,. \quad (3)$$

The vector meson photon coupling is defined as

$$\left\langle 0 \left| j_\mu^{em}(0) \right| \begin{matrix} V & 1^{--} \\ P & s_3 \end{matrix} \right\rangle = (2\pi)^{-3/2}\, g_V \epsilon_\mu^{s_3}(P)\,. \quad (4)$$

It determines the leptonic width

$$\Gamma_{V\to\ell^+\ell^-} = \frac{4\pi\alpha^2}{3}\frac{g_V^2}{M_V^3}\sqrt{1-\frac{4m_\ell^2}{M_V^2}}\left(1+\frac{2m_\ell^2}{M_V^2}\right), \quad (5)$$

$$\xrightarrow[M_V^2 \gg 4m_\ell^2]{} \frac{4\pi\alpha^2}{3}\frac{g_V^2}{M_V^3}$$

and the total e^+e^- annihilation cross section

$$\sigma(e^+e^- \to V \to \text{all}) = \frac{16\pi^2\alpha^2}{s^2}\, g_V^2\, \frac{M_V\Gamma_V^{tot}}{(s-M_V^2)^2 + M_V^2\Gamma_V^{tot\,2}}\,. \quad (6)$$

For a narrow resonance we have

$$\int_{res} \sigma(e^+e^- \to V \to \text{all})\, ds = \frac{16\pi^3\alpha^2 g_V^2}{M_V^4} = 12\pi^2\,\frac{\Gamma_{V\to e^+e^-}}{M_V}\,, \quad (7)$$

whereas for a broad one the leptonic width can be determined from the peak cross section

$$\sigma_{peak}(e^+e^- \to V \to \text{all}) = \frac{12\pi}{M_V^2}\,\frac{\Gamma_{V\to e^+e^-}}{\Gamma_V^{tot}}\,. \quad (8)$$

3. Bound state models

In field theoretic bound state models the quark structure of a meson with mass M and spin-parity j^{Π} is described by Bethe-Salpeter amplitudes

$$\chi(q, P) = (2\pi)^{3/2} \int \mathrm{d}^4x \, \mathrm{e}^{iqx} \langle 0|T\psi(\tfrac{1}{2}x)\, \overline{\psi}(-\tfrac{1}{2}x)|{}^{M}_{P}{}^{j^{\Pi}}_{s_3}\rangle , \tag{9}$$

where $\psi(x)$ denotes the renormalized quark field. From Π, C, T and Lorentz invariance follows the most general structure for vector mesons

$$\chi^{\mathrm{V}}(q, P, s_3) = \{\not\epsilon\chi_1 + \not\epsilon\not P\chi_2 + [\not q, \not\epsilon\not P]_+\chi_3 + \epsilon\not q\chi_4 + \epsilon q\not q\chi_5$$
$$+ qP[\not\epsilon, \not q]\chi_6 + \epsilon q[\not P, \not q]\chi_7 + \epsilon q\, qP\not P\chi_8\}\, \mathrm{V} , \qquad \epsilon_\mu = \epsilon_\mu^{s_3}(P) . \tag{10}$$

The invariant functions χ_i depend on q^2 and $(qP)^2$ only; V denotes the internal sym metry part of the amplitude.

Expressing the electromagnetic current by the quark fields

$$j_\mu^{\mathrm{em}}(x) = Z\overline{\psi}(x)\gamma_\mu Q\psi(x) , \qquad Q = \text{quark charge matrix} , \tag{11}$$

we obtain for the photon vector meson couplings

$$g_{\mathrm{V}}\epsilon_\mu = \mathrm{Tr}\, Z\gamma_\mu Q(2\pi)^{-4} \int \mathrm{d}^4q\, \chi^{\mathrm{V}}(q, P) ,$$

or in configuration space

$$g_{\mathrm{V}}\epsilon_\mu = \mathrm{Tr}\, Z\gamma_\mu Q\chi^{\mathrm{V}}(x = 0, P) . \tag{12}$$

Because of the Dirac trace, $\chi_2, \chi_4, \chi_6, \chi_7$ cannot contribute. If we go to the rest system, $P_\mu = (M, \mathbf{0})$, $\epsilon_\mu^{s_3} = (0, \delta_{s_3,i})$, then we see that χ_3 and χ_8 give no contribution either and we find

$$g_{\mathrm{V}} = 4Z\langle Q_{\mathrm{V}}\rangle(2\pi)^{-4} \int \mathrm{d}^4q\, \{\chi_1(q_0^2, \boldsymbol{q}^2) + \tfrac{1}{3}\boldsymbol{q}^2\chi_5(q_0^2, \boldsymbol{q}^2)\} ,$$
$$\langle Q_{\mathrm{V}}\rangle \equiv \mathrm{Tr}\,(QV) . \tag{13a}$$

In configuration space this formula reads

$$g_{\mathrm{V}} = 4Z\langle Q_{\mathrm{V}}\rangle\{\chi_1(x = 0) - \tfrac{1}{3}(\Delta\chi_5)(x = 0)\} . \tag{13b}$$

Only the $\ell = 0$ part of the amplitude contributes to the photon coupling.

We would like to point out here that in any quark model the relation

$$\sum_{\mathrm{V}} \langle Q_{\mathrm{V}}\rangle^2 = \sum_i Q_i^2 \tag{14}$$

holds. A comparison of the ratio R in the parton model and in bound state models thus involves dynamical quantities only.

4. Dynamical Bethe-Salpeter equation

The Bethe-Salpeter amplitudes, eq. (9), satisfy the bound state BS equation [10]

$$S_F^{-1}\left(\tfrac{1}{2}P+q\right)\chi(q,P)\,\bar{S}_F^{-1}\left(\tfrac{1}{2}P-q\right)=i\int d^4q'\,\mathcal{K}(\lambda;q,q',P)\chi(q',P)\,, \tag{15}$$

and are normalized according to

$$(2\pi)^{-4}\,i\,\mathrm{Tr}\int d^4q\,\bar{\chi}_{r'}(q,P)\,S_F^{-1}\left(\tfrac{1}{2}P+q\right)\chi_r(q,P)\,\bar{S}_F^{-1}\left(\tfrac{1}{2}P-q\right)=\lambda\left.\frac{\partial M^2}{\partial\lambda}\right|_{M_r^2}\delta_{rr'}\,. \tag{16}$$

In order to arrive at a tractable model we had to develop dynamical ideas [11]

(a) Heavy quarks, $m_q \gg 1$ GeV. When investigating scaling in e^+e^- annihilation we, therefore, have the situation that $s = M^2_{\text{meson}}$ is large on a hadronic scale but still small compared to the quark production threshold $4m^2$:

$$1\ \text{GeV}^2 \ll s = M^2_{\text{meson}} \ll 4m^2\,.$$

(b) Free propagators, $S^{-1}(P_1) = \gamma P_1 - m$.

(c) Convolution type, energy independent kernels,

$$\mathcal{K}(\lambda;q,q',P)=\sum_{i=S,V,T,A,P} K_i(\lambda;q-q')\,\mathcal{P}_i\,,$$

with the projectors $\mathcal{P}_i\Gamma_j = \Gamma_j\delta_{ij}$ (no summation),

$$\Gamma_j = (1,\gamma_\mu,\sigma_{\mu\nu},\gamma_5\gamma_\mu,\gamma_5)\,.$$

(d) Wick rotation in the q_0 plane, $q_0 \to iq_4$ (leading to $O^{\Pi,C}(4)$ symmetry of the bound state mass $M = 0$ equation). With these assumptions, eq. (15) takes in the CMS, $P = (M, \mathbf{0})$, the form

$$(\not{q} - im + \tfrac{1}{2}iM\gamma_4)\chi(q,M)\,(\not{q} - im - \tfrac{1}{2}iM\gamma_4) = \int d^4q'\,\mathcal{K}(\lambda, q-q')\chi(q,M)$$

in momentum space or

$$(-i\not{\partial} - im + \tfrac{1}{2}iM\gamma_4)\,\chi(x,M)(-i\overleftarrow{\not{\partial}} - im - \tfrac{1}{2}iM\gamma_4) = \mathcal{K}(\lambda,x)\,\chi(x,M)\,, \tag{17}$$

in configuration space.

As is well known, the $M = 0$ equation decomposes into three sectors:

(S,	V)	(T,	A)	(P)
1,	γ_μ	$\sigma_{\mu\nu}$,	$\gamma_5\gamma_\mu$	γ_5

which will decompose further if one makes full use of the $O^{\Pi,C}(4)$ symmetry [10].

The amplitudes with the Dirac structures (γ_i, σ_{4i}) and $(\gamma_5\gamma_4, \gamma_5)$ are good candidates for the leading terms in the BS amplitudes of the quark spin triplet and quark spin singlet mesons, respectively. As in each case there should be one state only, we have to look for only one linear combination of γ_i and σ_{4i}, $\gamma_5\gamma_4$ and γ_5, resp., in

Table 1

	$\gamma_5\gamma_4$	γ_5
γ_i	V ⊕ A	V ⊕ P
σ_{4i}	T ⊕ A	T ⊕ P

which the interaction is strongly attractive. Since the V and the T, the A and the P sector are not coupled in the strong binding ($M = 0$) limit, we must have V or T and A or P. This leads us to four possible models (table 1).

If the leading component of the BS amplitudes for the quark spin triplet mesons is of the tensor type, the vector mesons do not couple to the electromagnetic current in leading order ($g_V \sim 1/m_q$). Thus we cannot expect a scaling behaviour for $\sigma_{tot}(e^+e^-)$. We therefore concentrate our discussion on the models of the V-type (first row in table 1).

Subsequently we indicate an elimination procedure which reduces eq. (17) to a simple radial equation [12]. We start with the $M = 0$ equation which we write

$$-\not{\partial}\chi(x)\overleftarrow{\not{\partial}} - m^2\chi(x) - m\{\not{\partial}, \chi(x)\} = \sum_i K_i(\lambda, x)\, \mathcal{P}_i\chi(x)\,. \tag{18}$$

We put $\chi = \chi_0 + \chi_1$, because there occur at most two Dirac amplitudes in one sector, and obtain

$$\begin{aligned}&(-\not{\partial}\chi\overleftarrow{\not{\partial}} - m^2 - K_0)\chi_0 - m\{\not{\partial}, \chi_1\} = 0,\\ &(-\not{\partial}\chi\overleftarrow{\not{\partial}} - m^2 - K_1)\chi_1 - m\{\not{\partial}, \chi_0\} = 0\,.\end{aligned} \tag{18'}$$

The interaction potential in configuration space should be smooth, allowing an expan sion around the origin

$$K_i(R) = \alpha_i + \beta_i R + \gamma_i R^2 + \ldots\,. \tag{19}$$

Since we want χ_0 to be the large component, K_0 has to be strongly attractive in order to compensate the quark mass term ($\alpha \approx -m^2$), whereas K_1 is not attractive. We assume that K_1 is repulsive with the strength $\alpha_1 \approx +m^2$. Under these conditions we can get χ_1 algebraically:

$$\chi = \chi_0 + \chi_1 \approx \chi_0 - \frac{1}{2m}\{\not{\partial}, \chi_0\}\,, \tag{20}$$

and we obtain for χ_0 the dynamical equation

$$(\Box_x - m^2 - K_0(R))\chi_0(x) = 0\,. \tag{21}$$

The BS equation for the general case, $M \neq 0$, can be discussed along similar lines,

Table 2

	ξ triplets	ξ singlets
$\chi_0^{\text{triplets}} \sim \gamma_i$	$=\dfrac{3m^2 - \alpha_T}{m^2 + \alpha_T}$	$=\dfrac{3m^2 - \alpha_P}{m^2 + \alpha_P}$
$\chi_0^{\text{singlets}} \sim \gamma_5\gamma_4$	$= 1$	$\alpha_P >> - m^2$
$\chi_0^{\text{triplets}} \sim \gamma_i$	$=\dfrac{3m^2 - \alpha_T}{m^2 + \alpha_T}$	$=\dfrac{3m^2 - \alpha_A}{m^2 + \alpha_A}$
$\chi_0^{\text{singlets}} \sim \gamma_5$	$\alpha_T >> - m^2$	$\alpha_A >> -m^2$

by rewriting eq. (17) in the form

$$[-\partial\!\!\!/ \times \overleftarrow{\partial\!\!\!/} - m^2 - m \, \{\partial\!\!\!/, x\} - \mathcal{K} + \tfrac{1}{4}M^2\gamma_4 \times \gamma_4 + \tfrac{1}{2}Mm \, [\gamma_4, x] + \tfrac{1}{2}M(\gamma_4 \times \overleftarrow{\partial\!\!\!/} - \partial\!\!\!/ \times \gamma_4)] \, \chi(x,M) = 0. \tag{17'}$$

With the ansatz $\chi = \chi_0 + \chi_1 + \chi_{br}$, where χ_1 is already given by eq. (20), one obtains [12]

$$\chi = \chi_0 - \frac{1}{2m}\{\partial\!\!\!/, \chi_0\} + \frac{m}{2(m^2 + \alpha_{br})} [P\!\!\!/, \chi_0] + O\left(\frac{1}{m^2}\right), \tag{22}$$

and the dynamical equation

$$(\Box_x - m^2 - K_0(R) + \tfrac{1}{4}\xi M^2)\chi_0(x) \approx 0, \quad \xi \equiv \frac{3m^2 - \alpha_{br}}{m^2 + \alpha_{br}} . \tag{23}$$

In eq. (22) we made use of the requirement that, with respect to spin structure, there occurs only one strongly bound state solution χ_0. This allows the algebraic determination of χ_{br} because of $\alpha_{br} >> - m^2$. The expression for the parameter ξ has been obtained by making use of the fact that the amplitudes $\chi_0 \sim \gamma_i, \gamma_5\gamma_4$ or γ_i, γ_5 are eigenfunctions of $\gamma_4 \times \gamma_4$ with the eigenvalue -1. Table 2 contains the parameters ξ in the V $\oplus$ A and the V $\oplus$ P model.
From eq. (23) we see that the mass spectrum M^2 is not only determined by the shape of the "potential" K_0 but also by ξ, i.e. by the detailed spin structure of the interaction.

5. Asymptotic solution of the BS radial equation (generalized WKB method)

In the last section we have derived the dynamical equation

$$(\Box_x - m^2 + \tfrac{1}{4}\xi M^2 - K_0(R))\chi_0(x) = 0 \tag{23}$$

for the leading component $\chi_0(x)$ of the BS amplitude. For the orbital part of $\chi_0(x)$ we make the ansatz

$$\chi_0^{\mathrm{orb}}(x) = \mathcal{Y}_{nlm}(\hat{x})\ \frac{u_n(R)}{R^{3/2}}\ , \tag{24}$$

where the $\mathcal{Y}_{nlm}(\hat{x})$ are the SO(4) spherical functions, and we obtain the radial equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}R^2} - \frac{(n+\frac{1}{2})(n+\frac{3}{2})}{R^2} - m^2 + \tfrac{1}{4}\xi M^2 - K_0(R)\right) u_n(R) = 0\ . \tag{25}$$

Since we are interested in those solutions $\chi_0(x)$ which do not vanish at $x = 0$ (see eq. (12)), we only have to consider the $n = 0$ case

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}R^2} - \frac{3}{4R^2} + \mu^2 - K_0(R)\right) u_0(R) = 0\ , \qquad \mu^2 = \tfrac{1}{4}\xi M^2 - m^2\ . \tag{26}$$

We need those solutions which correspond to high excitations, because we want to investigate the behaviour of $\sigma^{\mathrm{tot}}(\mathrm{e}^+\mathrm{e}^- \to \text{hadrons})$ for large values of the energy squared s. This suggests the use of asymptotic methods, i.e. a suitably modified WKB approximation [13].

The solution of the free equation: $(\mathrm{d}^2/\mathrm{d}R^2 - 3/4R^2 + \mu^2)\,\hat{u}(R) = 0$ with the correct threshold behaviour is $\hat{u}(R) = \sqrt{\mu R}\cdot J_1(\mu R)$, which leads to the ansatz

$$u_0(R) = H(R)\sqrt{S(R)}\,J_1(S(R))\ . \tag{27}$$

Insertion into eq. (26) and use of the derivative relations for the Bessel functions gives the two equations

$$HS'^2 - (\mu^2 - K_0(R))\,H = H'' + \tfrac{3}{4}H\left(\frac{S'^2}{S^2} - \frac{1}{R^2}\right)\ , \tag{28a}$$

$$2H'S' + HS'' = 0\ . \tag{28b}$$

Eq. (28b) is solved by $H = \mathrm{const}/\sqrt{S'}$. Eq. (a) is solved approximately by $S(R) = \pm \int^R \sqrt{\mu^2 - K_0(\rho)}\,\mathrm{d}\rho$, neglecting the right-hand side (in analogy to the standard WKB procedure). The amplitudes $\chi_0^{\mathrm{orb}}(x)$ for $n = 0$ are therefore approximately given by

$$\chi_0^{\mathrm{orb}}(x) = \frac{N}{\sqrt{2\pi^2}}\,\frac{\left[\int_0^R \sqrt{\mu^2 - K_0(\rho)}\,\mathrm{d}\rho\right]^{1/2}}{R^{3/2}(\mu^2 - K_0(R))^{1/4}}\ J_1\left(\int_0^R \sqrt{\mu^2 - K_0(\rho)}\,\mathrm{d}\rho\right)\ . \tag{29}$$

For $0 < 1/M \leqslant R < R_0$ (= classical turning point) we can replace the Bessel function by its asymptotic expression which leads to

$$\chi_0^{\mathrm{orb}}(x) = \frac{N}{\sqrt{\pi^3}} \frac{\sin\left(\int_0^R \sqrt{\mu^2 - K_0(\rho)}\, \mathrm{d}\rho - \frac{1}{4}\pi\right)}{R^{3/2}(\mu^2 - K_0(R))^{1/4}}\,, \qquad \frac{1}{M} \leqslant R < R_0\,. \tag{30}$$

As in the ordinary WKB method we have the quantum condition

$$\int_0^{R_0} \sqrt{\mu^2 - K_0(\rho)}\, \mathrm{d}\rho = \pi(r + \gamma); \qquad \mu^2 = \tfrac{1}{4}\xi M^2 - m^2\,, \tag{31}$$

which yields the mass spectrum M_r^2.

The constant N is determined from the BS normalization, eq. (16). It reads, in leading order of the quark mass m and for the Wick rotated configuration space amplitudes $\chi_0(x)$,

$$m^2 \mathrm{Tr}^{\mathrm{Dirac}} \int \mathrm{d}^4x\, \bar{\chi}_{0,r'}(x)\, \chi_{0,r}(x) = \lambda \left(\frac{\partial P^2}{\partial \lambda}\right)_{P^2 = M_r^2} \delta_{rr'}\,. \tag{32}$$

With $\bar{\chi}(x) = \gamma_4 \chi^*(-x_4, \boldsymbol{x})\gamma_4 = (-1)^{n-l}\gamma_4\chi^*(x_4, \boldsymbol{x})\gamma_4$, because of $\gamma_4\chi^*\gamma_4 = -\chi^*$ for the physical amplitudes (Table 1) and with the coupling constant $\lambda \approx \alpha_0 \approx -m^2$, we obtain

$$4 \int \mathrm{d}^4x\, \chi_{0,r'}^{*\,\mathrm{orb}}(x)\, \chi_{0,r}^{\mathrm{orb}}(x) = \frac{4}{\xi}\delta_{rr'}\,. \tag{33}$$

We evaluate this integral with the asymptotic solution given in eq. (30), thereby replacing $\sin^2$ by $\frac{1}{2}$ in the same approximation. This gives the normalized orbital part of the BS amplitude

$$\chi_0^{\mathrm{orb}}(x) = \frac{1}{\sqrt{2\pi\xi}} \frac{1}{\left(\int_0^{R_0} \frac{\mathrm{d}\rho}{(\mu^2 - K_0(\rho))^{1/2}}\right)^{1/2}} \frac{\left(\int_0^R \sqrt{\mu^2 - K_0(\rho)}\, \mathrm{d}\rho\right)^{1/2}}{R^{3/2}(\mu^2 - K_0(R))^{1/4}}$$

$$\times J_1\left(\int_0^R \sqrt{\mu^2 - K_0(\rho)}\, \mathrm{d}\rho\right). \tag{29'}$$

Because of $J_1(x) \xrightarrow[x\to 0]{} \frac{1}{2}x$, the amplitude at the origin has the value

$$\chi_0^{\mathrm{orb}}(x=0) = \frac{1}{2}\frac{1}{\sqrt{2\pi\xi}} \left[\frac{\mu^2 - K_0(0)}{\int_0^{R_0} \frac{\mathrm{d}\rho}{(\mu^2 - K_0(\rho))^{1/2}}}\right]^{1/2}, \qquad \mu^2 = \tfrac{1}{4}\xi M^2 - m^2\,. \tag{34}$$

By differentiating the quantum condition (31) with respect to M^2 one obtains

$$\frac{\xi}{8\pi}\frac{\mathrm{d}M^2}{\mathrm{d}r} = \frac{1}{\displaystyle\int_0^{R_0} \frac{\mathrm{d}\rho}{(\mu^2 - K_0(\rho))^{1/2}}} \tag{35}$$

Inserting this expression into eq. (34) and observing that for strong binding: $\mu^2 - K_0(0) \approx \frac{1}{4}\xi M^2$, we obtain a relation between the amplitudes at $x = 0$ and the mass spectrum, into which the kernel $K_0(\rho)$ does not enter explicitly,

$$|\chi_{0,r}^{\mathrm{orb}}(x=0)|^2 = \frac{\xi}{4(8\pi)^2} M_r^2 \frac{\mathrm{d}M_r^2}{\mathrm{d}r} . \tag{36}$$

With the help of eqs. (13b) and (5) we relate $|\chi_{0r}^{\mathrm{orb}}(x=0)|^2$ to the leptonic decay width of the rth vector meson, obtaining

$$\Gamma_{\mathrm{V}_r \to \mathrm{e}^+\mathrm{e}^-} = \frac{4\pi\alpha^2}{3} Z^2 \langle Q_{\mathrm{V}}\rangle^2 \frac{\xi}{16\pi^2} \frac{\mathrm{d}M_{\mathrm{V}_r}^2/\mathrm{d}r}{M_{\mathrm{V}_r}} \tag{37}$$

When we use this expression in eq. (1) we find that the ratio R is independent of the excitation level

$$\frac{\sigma_{\mathrm{tot}}(\mathrm{e}^+\mathrm{e}^- \to \mathrm{hadrons})}{\sigma_{\mathrm{tot}}(\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-)} \equiv R = \frac{3\xi}{4} Z^2 \sum_{\mathrm{V}} \langle Q_{\mathrm{V}}\rangle^2 \tag{38}$$

Thus, for high excitations, we obtain scaling, and the scaling constant R is independent of the shape of the smooth, potential type interaction.

Before giving a more detailed discussion of this result in the next section, we would like to add a few remarks about "potentials" which are singular at zero distance. If the kernel $K_0(R)$ of the dynamical equation (23) has a $1/R$ singularity at the origin but is confining at large distances, then, by using the so-called Langer method for the asymptotic solutions of differential equations [13], we obtain again eq. (36). However, we want to point out that the eq. (23) was obtained by an algebraic elimination of the "small" components of the BS amplitude. This elimination remains valid at least for the case that the projection of the interaction on these small components is nonsingular. Thus, with this restriction, scaling remains preserved for an interaction containing a $1/R$ singularity.

6. Discussion

We have shown that in relativistic bound state models with heavy quarks the total $\mathrm{e}^+\mathrm{e}^-$ annihilation into hadrons has scaling behaviour once the amplitudes of the vector meson bound states have a γ_μ Dirac structure like the electromagnetic quark current, and that the scaling constant is independent of the detailed shape of the potential-type interaction. Thus scaling is a rather general property of bound state models *. Turning the argument around, from scaling in e^+e^- annihilation one cannot select a particular "confinement" potential. In view of scaling being one of the main features of models with free, or asymptotically free, quarks (partons), we want to compare the two pictures. For this we choose the scaling constant R which in the asymptotically free quark model is given by $R^{\text{free}} = \Sigma_i Q_i^2$, whereas in the bound state models we found $R^{\text{bound}} = \frac{3}{4}\xi Z^2 \Sigma_{\text{V}} \langle Q_{\text{V}} \rangle^2$. Because of the equality of the sum over quarks and the sum over vector mesons, eq. (14), the comparison actually involves the reduced quantity $R^{\text{bound}}/R^{\text{free}} = \frac{3}{4}\xi Z^2$ only. We observe that in bound state models the scaling constant R^{bound} depends, apart from the flavours and colours of the quarks, on several factors:

(a) The current renormalization constant Z, $Z \leqslant 1$.

(b) The dynamical factor $\frac{3}{4}\xi$ which is determined by the detailed spin structure of the interaction kernel, reflecting the fact that in a bound state model one can excite the off-shell degrees of freedom, in contrast to the parton model. In our dynamically favoured model [11], where the strong forces binding the heavy quarks get saturated in the mesonic vertices, $\xi = 1$, in a model with the Fierz symmetric pseudoscalar-vector-scalar kernel, $\xi = 3$ [12].

(c) An energy dependence of the interaction of the form $K = K(R) + \text{const.}\, M^2$ which we did not consider here, would also enter the scaling constant.

(d) Thus far we have tacitly assumed the zero-width approximation for the vector mesons. In reality the total widths are finite, and actually they have to be rather large in order to yield the smooth cross section. A description where the total width of the resonances is proportional to their mass will not destroy scaling in our model, but from unitarity arguments we expect that the scaling constant becomes smaller [14].

In conclusion, scaling behaviour in e^+e^- annihilation seems to be an almost unavoidable property of bound state models. At present there is an indication for scaling above $s = (5\ \text{GeV})^2$ with $R^{\text{exp}} \approx 5.5$. This number is bigger than the values from the charm or Han-Nambu quark model. If, however, the experimental scaling function will turn out to approach a parton model value, then this would require a subtle interplay between current renormalization, spin structure and finite width effects of the intermediate vector mesons in the bound state picture.

One of us (M.B.) would like to express his gratitude to Professors H. Schopper and G. Weber for the hospitality during several visits at DESY.

* Scaling occurs also in non-relativistic bound state models if the Schrödinger eigenvalue is linearly related to the squared bound state mass M^2 [8]. This is so if the potential is confining and not more singular than $1/R$.

References

[1] Proc. 4th Int. Symp. on electron and photon interactions at high energies (Daresbury Nuclear Physics Lab., 1969) p. 213, 227;
Proc. 6th Int. Symp. on electron and photon interactions at high energies, Bonn, 1973 (North-Holland, Amsterdam, 1974) p. 1;
Proc. 1975 Int. Symp. on lepton and photon interactions at high energies (Stanford, 1975) p. 5.
[2] J.D. Bjorken, Proc. 6th Int. Symp. on electron and photon interactions at high energies (North-Holland, Amsterdam, 1974) p. 25.
[3] A. Bramon, E. Etim and M. Greco, Phys. Letters 41B (1972) 609.
[4] J.J. Sakurai, Phys. Letters 46B (1973) 207.
[5] D. Schildknecht and F. Steiner, Phys. Letters 56B (1975) 36.
[6] M. Böhm, H. Joos and M. Krammer, Acta Phys. Austriaca 38 (1973) 123.
[7] G. Preparata and N.S. Craigie, Ref. TH. 2038 CERN (1975);
G. Preparata, Lecture at Int. Summer Institute in theoretical particle physics, Hamburg, 1975.
[8] M. Krammer and P. Leal Ferreira, IFT preprint 213/75.
[9] M. Gourdin, Hadronic interactions of electrons and photons, eds. J. Cumming and H. Osborn (Academic Press, London, 1971).
[10] N. Nakanishi, Progr. Theor. Phys. Suppl. 43 (1969) 1.
[11] M. Böhm, H. Joos and M. Krammer, Ref. TH 1949-CERN (1974).
[12] M. Böhm, H. Joos and M. Krammer, Recent developments in mathematical physics, ed. P. Urban (Springer, Wien and New York, 1973) p. 3;
M. Krammer, Acta Phys. Austr. 40 (1974) 187.
[13] S. Flügge, Practical quantum mechanics (Springer, Berlin, 1971).
[14] F.M. Renard, Phys. Letters 49B (1974) 477.