Dilations and interaction

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As a consequence of the geometrical features of dilations massless particles do not interact in a local, dilationally invariant quantum theory. This result also holds in models in which dilations are only a symmetry of the S matrix.

1. INTRODUCTION AND MAIN RESULTS

The conventional argument showing that massless particles do not interact in a local, dilationally invariant quantum theory is in the simplest case the following one (see, e.g., Ref. 1): suppose ϕ is a scalar Wightman field transforming under dilations according to

$$D(\lambda)\phi(\lambda^{-1}x)D(\lambda)^{-1}=\lambda^{d}\cdot\phi(x). \tag{1}$$

If ϕ has a nonvanishing matrix element between the vacuum and a massless one-particle state, d can only be one. Then ϕ has canonical dimension and this implies that it is a free field. This reasoning is quite correct. However, since the argument depends upon the existence of a field ϕ with the special properties mentioned above, the conclusion appears to us to be rather premature. First, there is no physical reason to rule out *ab initio* all models in which the basic fields do not transform like a finite-dimensional representation under dilations. And secondly, even if the fields transform in this way, it could happen that they do not interpolate between the vacuum and the massless one-particle states. In general one should only expect that suitable polynomials in the fields have this property. It is the aim of the present note to close these apparent loopholes. Using only the geometrical features of dilations and the basic properties of local field theory, we give a fairly general argument confirming the above no-go theorem.

The setting used for the analysis may be sketched as follows: We deal with an irreducible field algebra $\mathfrak{F}$ of bounded operators acting on a Hilbert space $\mathcal{H}$. $\mathfrak{F}$ is generated by a net $\mathcal{O}\to\mathfrak{F}(\mathcal{O})$ of local algebras attached to the regions $\mathcal{O}$ of Minkowski space. We may forego here a formal specification of the usual structural assumptions on the theory like locality, covariance, spectrum condition and uniqueness of the vacuum. (For a detailed discussion see, for example, Ref. 2). In addition to these familiar properties we require that there be a continuous, unitary representation $\lambda\to D(\lambda)$ of the multiplicative group of the positive reals in $\mathcal{H}$. The operators $D(\lambda)$, the dilations, satisfy

$$D(\lambda)U(x)=U(\lambda x)D(\lambda)\quad\text{and}\quad D(\lambda)U(\Lambda)=U(\Lambda)D(\lambda), \tag{2a}$$

where $x=(x_0,\mathbf{x})\to U(x)$ are the translations and $\Lambda\to U(\Lambda)$ the Lorentz transformations. Moreover, the dilations $D(\lambda)$ induce automorphisms of the field algebra $\mathfrak{F}$ with appropriate geometric properties:

$$D(\lambda)\mathfrak{F}(\mathcal{O})D(\lambda)^{-1}=\mathfrak{F}(\lambda\cdot\mathcal{O}). \tag{2b}$$

These rather general assumptions suffice to prove the following statement:

If there exist massless particles in the model, (i.e., a family of subspaces $\mathcal{H}_1^{(i)}\subset\mathcal{H}$ on which the unitaries $U(x)$, $U(\Lambda)$ act like an irreducible representation of the Poincaré group with mass zero and helicity s_i), then the S matrix for these particles is trivial.

Our interest in this problem arose in discussions with Haag on supersymmetric field theories. In a recent article Haag, Lopuszanski, and Sohnius have analyzed the structure of all possible supersymmetries of the S matrix.[3] They found out that in a pure S-matrix formalism there is essentially only one way of a complete fusion between internal and geometrical symmetries, including dilations. Since such a structure looks very promising from the point of view of physics, one may ask whether it can be embedded into a conventional field theoretical setting. As a consequence of our analysis the answer to this question is negative: If the theory is to describe collisions of massless particles and if dilations are to be a proper, unbroken symmetry, one has to abandon some of the usual field theoretical assumptions. At present it is unclear how the assumptions have to be modified and we refrain from speculations. However, we want to emphasize that even in a modified scheme the local observables (the currents, etc.) should have a structure similar to that of $\mathfrak{F}$ given above. What may then be learned from our analysis is that massless particles in the vacuum sector of the observable algebra do not interact. It is therefore unlikely that particles like the photon and the η'-meson (both of which carry the charge quantum numbers of the vacuum) can be incorporated into such a scheme. This apparently restricts the possible range of application of these models to weak interaction physics.

2. THE PROOF

The central idea of the proof is very simple: we derive an asymptotic expansion for the function $\lambda\to D(\lambda)AD(\lambda)^{-1}$ at $\lambda=0$, where A is a suitable local operator taken from $\mathfrak{F}$. It turns out that

$$D(\lambda)AD(\lambda)^{-1}=(\Omega,A\Omega)\cdot 1+\lambda\cdot\phi+o(\lambda), \tag{3}$$

where this expansion is understood in the sense of operator valued distributions; Ω denotes the vector representing the vacuum and ϕ is some local field. Now the crucial point is that if ϕ is not zero, it creates a massless particle from the vacuum. It then follows from

Huyghens' principle (i.e., the timelike commutation relations between local and asymptotic fields given in Ref. 4) that the S matrix of this particle can only be trivial.

Unfortunately, there are models in which, for kinematical reasons, all local operators A give rise to a vanishing ϕ. However, this defect can be cured by a slight modification of the above expansion: dilating and boosting the operator A simultaneously, one arrives at an expression similar to (3), but with a nontrivial ϕ. To abbreviate the argument, we confine our attention to models involving only one type of massless particles with helicity $s=0$. But we shall give a brief outline of how to proceed in more complicated situations.

Now let A be any operator from $\mathfrak{F}$ which is localized in a bounded region $\mathcal{O}\subset \mathbb{R}^4$. We regularize A according to

$$A_\varphi = \int dt\, \varphi(t)\, U(t) A U(t)^{-1}, \tag{4}$$

where $t \to U(t)$ are the time translations. $\varphi(t)$ is a test function with compact support which has a Fourier transform $\tilde{\varphi}(\omega)$ with a twofold zero at $\omega=0$. The smoothed operator A_φ is still local and we get the following bound on its two-point function:

Lemma 1: Let $\Delta \to E(\Delta)$ be the spectral projections of the mass operator $M=(P^2)^{1/2}$ where $\Delta \subsetneq \mathbb{R}^+$ is any Borel set of mass values. Then

$$|(A_\varphi\Omega, E(\Delta)U(\mathbf{x})A_\varphi\Omega)| \leqslant c\cdot(1+|\mathbf{x}|^4)^{-1} \cdot\{\|E(\Delta)A\Omega\|^2+\|E(\Delta)A^*\Omega\|^2\}$$

where the constant c depends neither on $\mathbf{x}$ nor on Δ.

Proof: Using the methods of the Jost–Lehmann–Dyson representation, one can show that the function

$$h_\Delta(x) = (A\Omega, E(\Delta)U(x)A\Omega) - (A^*\Omega, E(\Delta)U(-x)A^*\Omega)$$

vanishes in the spacelike complement of some bounded region $\mathcal{O}_1$ which depends only on the localization region $\mathcal{O}$ of A (see, e.g., Ref. 5, Lemma 6.2). Now, if one puts $\psi(t)=\int ds\, \overline{\varphi}(s)\varphi(s+t)$, one gets, owing to the spectrum condition,

$$\begin{aligned}(A_\varphi\Omega, &E(\Delta)U(\mathbf{x})A_\varphi\Omega)\\ &= \int dt\, \psi(t)(A\Omega, E(\Delta)U(t,\mathbf{x})A\Omega)\\ &= \int dt\, \psi^+(t)(A\Omega, E(\Delta)U(t,\mathbf{x})A\Omega)\\ &= \int dt\psi^+(t)\{(A\Omega, E(\Delta)U(t,\mathbf{x})A\Omega)\\ &\quad - (A^*\Omega, E(\Delta)U(-t,-\mathbf{x})A^*\Omega)\}\\ &= \int dt\, \psi^+(t)h_\Delta(t,\mathbf{x}),\end{aligned}$$

where

$$\begin{aligned}\psi^+(t) &= (2\pi)^{-1/2}\int_0^\infty d\omega\tilde{\psi}(\omega)\exp(-i\omega t)\\ &= \int_0^\infty d\omega\, |\tilde{\varphi}(\omega)|^2\exp(-i\omega t).\end{aligned}$$

Since $|\tilde{\varphi}(\omega)|^2$ is a test function with a fourfold zero at $\omega=0$, it is easy to verify that $\psi^+(t)$ is continuous and $|\psi^+(t)|\leqslant c\cdot(1+|t|^5)^{-1}$. Taking the support properties of $h_\Delta(x)$ into account, one arrives at

$$\begin{aligned}|(A_\varphi\Omega, &E(\Delta)U(\mathbf{x})A_\varphi\Omega)|\\ &\leqslant \int_{|t|\geqslant|\mathbf{x}|-R} dt\,|\psi^+(t)|\cdot|h_\Delta(t,\mathbf{x})|\\ &\leqslant c\cdot\int_{|t|\geqslant|\mathbf{x}|-R} dt(1+|t|^5)^{-1}\cdot\{\|E(\Delta)A\Omega\|^2+\|E(\Delta)A^*\Omega\|^2\},\end{aligned}$$

where R is some length which depends only on A. From this inequality the statement of the lemma follows at once. ∎

We take now the operator A_φ and carry out the following manipulations: First we dilate it, then we boost it in a fixed direction, and finally we smear it in the two remaining spatial directions. For the boosts we take those in the x_1 direction:

$$K_\lambda = \begin{pmatrix}\frac{1}{2}(\lambda+\lambda^{-1}) & \frac{1}{2}(\lambda-\lambda^{-1}) & & \\ \frac{1}{2}(\lambda-\lambda^{-1}) & \frac{1}{2}(\lambda+\lambda^{-1}) & & \\ & & 1 & \\ & & & 1\end{pmatrix}, \quad \lambda>0. \tag{5}$$

Then if $\mathbf{x}^\perp=(0,x_2,x_3)$ denotes the projection of $\mathbf{x}$ onto the (x_2,x_3)-plane and if $d^2x^\perp = dx_2\,dx_3$, we set

$$B_\lambda = \lambda^{-1}\cdot\int d^2x^\perp f(\mathbf{x}^\perp)U(\mathbf{x}^\perp)U(K_\lambda)D(\lambda)A_\varphi \times D(\lambda)^{-1}U(K_\lambda)^{-1}U(\mathbf{x}^\perp)^{-1}, \tag{6}$$

where $f(\mathbf{x}^\perp)$ is any test function with compact support. To begin with, we examine the localization properties of B_λ: Since A_φ is localized in some bounded region $\mathcal{O}$, it follows from (6) that B_λ is localized in $\{\lambda\cdot K_\lambda\mathcal{O}+\operatorname{supp} f\}$. Now $\lim_{\lambda\to 0}\lambda\cdot K_\lambda = P$, where P is the projection onto the ray $(a,-a,0,0)$, $a\in\mathbb{R}$. Therefore, the operators B_λ are, for sufficiently small λ, localized in a fixed bounded region $\mathcal{O}_1$. The next step is to show that the sequence $B_\lambda\Omega$ converges to a (possibly zero) one-particle state in the limit of small λ:

Proposition 2: Let B_λ be the operator defined in relation (6). Then the weak limit w-$\lim_{\lambda\to 0}B_\lambda\Omega$ exists and is an element of $\mathcal{H}_1$.

Proof: The proof of this assertion is based on Lemma 1. Since $U(K_\lambda)$ commutes with $U(\mathbf{x}^\perp)$ and $E(\Delta)$, we may write

$$\|E(\Delta)B_\lambda\Omega\|^2 = \lambda^{-2}\cdot\int d^2x^\perp\int d^2y^\perp\bar{f}(\mathbf{x}^\perp)f(\mathbf{y}^\perp) \times(A_\varphi\Omega, E(\lambda\Delta)U(\lambda^{-1}[\mathbf{y}^\perp-\mathbf{x}^\perp])A_\varphi\Omega),$$

where we have made use of relation (2a). If we set $g(\mathbf{x}^\perp)=\int d^2y^\perp\bar{f}(\mathbf{y}^\perp)f(\mathbf{x}^\perp+\mathbf{y}^\perp)$, we get, using Lemma 1,

$$\begin{aligned}\|E(\Delta)&B_\lambda\Omega\|^2\\ &= \lambda^{-2}\cdot\int d^2x^\perp g(\mathbf{x}^\perp)(A_\varphi\Omega, E(\lambda\Delta)U(\lambda^{-1}\mathbf{x}^\perp)A_\varphi\Omega)\\ &\leqslant \sup_{\mathbf{y}^\perp}|g(\mathbf{y}^\perp)|\cdot c\int d^2x^\perp(1+|\mathbf{x}^\perp|^4)^{-1}\\ &\quad\cdot\{\|E(\lambda\Delta)A\Omega\|^2+\|E(\lambda\Delta)A^*\Omega\|^2\}.\end{aligned}$$

Putting $\Delta=\mathbb{R}^+$, it follows that the sequence $B_\lambda\Omega$ is uniformly bounded in λ. Putting $\Delta_c=[a,b]$, where $0<a\leqslant b<\infty$, it follows that $\lim_{\lambda\to 0}\|E(\Delta_c)B_\lambda\Omega\|=0$ because the continuity properties of the spectral resolution imply $\lim_{\lambda\to 0}\|E(\lambda\Delta_c)\Phi\| = 0$ for every vector $\Phi \in \mathcal{H}$. Thus the sequence $B_\lambda\Omega$ converges weakly to zero on the orthogonal complement of the one-particle space $\mathcal{H}_1$. It remains to establish its convergence on $\mathcal{H}_1$. Since we are dealing with only one type of massless particles with helicity $s = 0$, we may identify the one-particle states $\Psi \in \mathcal{H}_1$ with their momentum space wavefunctions $\Psi(\mathbf{p})$ in $L^2(\mathbf{R}^3, d^3p/2|\mathbf{p}|)$. The dilations and Poincaré transformations act on these functions as follows:

$$(D(\lambda)\Psi)(\mathbf{p}) = \lambda \cdot \Psi(\lambda\mathbf{p}),$$
$$(U(t, \mathbf{x})\Psi)(\mathbf{p}) = \exp[i(t|\mathbf{p}| - \mathbf{x}\mathbf{p})]\Psi(\mathbf{p}), \tag{7a}$$

and

$$(U(\Lambda)\Psi)(\mathbf{p}) = \Psi(\Lambda^{-1} \circ \mathbf{p}), \tag{7b}$$

where $\Lambda^{-1} \circ \mathbf{p}$ denotes the spatial components of the 4-vector $\Lambda^{-1}(|\mathbf{p}|, \mathbf{p})$. What is crucial now is that the wavefunction $(A_\varphi\Omega)(\mathbf{p})$ of the one-particle state $E(\{0\})A_\varphi\Omega$ is continuous in $\mathbf{p}$ if A_φ is the operator defined in relation (4). To verify this, we fix a set $\mathcal{L}$ of Lorentz transformations Λ which are close to the identity I, e.g., $\mathcal{L} = \{\Lambda : \|\Lambda - I\| \leq \frac{1}{2}\}$. Since A is local it is obvious that all operators $(U(\Lambda)^{-1}AU(\Lambda) - A)$, $\Lambda \in \mathcal{L}$, are localized in a bounded region $\mathcal{O}$ of configuration space. Therefore, we get the estimate, using relation (7) and Lemma 1,

$$\begin{aligned}
&(\pi/|\mathbf{p}|)|\tilde{\varphi}(|\mathbf{p}|)|^2 \cdot |(A\Omega)(\Lambda \circ \mathbf{p}) - (A\Omega)(\mathbf{p})|^2 \\
&= (1/2|\mathbf{p}|) \cdot ((U(\Lambda)^{-1}AU(\Lambda) - A)_\varphi\Omega)(\mathbf{p})|^2 \\
&= (2\pi)^{-3} \cdot \int d^3x \exp(i\mathbf{x}\mathbf{p}) \cdot ((U(\Lambda)^{-1}AU(\Lambda) - A)_\varphi\Omega, \\
&\quad E(\{0\})U(\mathbf{x})(U(\Lambda)^{-1}AU(\Lambda) - A)_\varphi\Omega) \\
&\leq c \cdot \{\|(U(\Lambda) - 1)E(\{0\})A\Omega\|^2 + \|(U(\Lambda) - 1)E(\{0\})A^*\Omega\|^2\},
\end{aligned}$$

and this inequality holds for all $\Lambda \in \mathcal{L}$ and $\mathbf{p} \in \mathbf{R}^3$. Since we may take for $\tilde{\varphi}$ a test function which has a zero only at the origin, it is evident that $\lim_{\Lambda\to I}(A\Omega)(\Lambda \circ \mathbf{p}) = (A\Omega)(\mathbf{p})$ for $\mathbf{p} \neq 0$. But this shows that $(A\Omega)(\mathbf{p})$ and therefore also $(A_\varphi\Omega)(\mathbf{p}) = (2\pi)^{1/2}\tilde{\varphi}(|\mathbf{p}|)(A\Omega)(\mathbf{p})$ are continuous at $\mathbf{p} \neq 0$ because for every sequence $\mathbf{p}_n$ converging to $\mathbf{p}$ we can specify a sequence of Lorentz transformations Λ_n such that, for sufficiently large n, $\Lambda_n \circ \mathbf{p} = \mathbf{p}_n$ and $\lim_n \Lambda_n = I$. In order to establish the continuity of $(A_\varphi\Omega)(\mathbf{p})$ at $\mathbf{p} = 0$, we estimate

$$\begin{aligned}
&(1/2|\mathbf{p}|)|(A_\varphi\Omega)(\mathbf{p})|^2 \\
&\quad = (2\pi)^{-3} \int d^3x \exp(i\mathbf{x}\mathbf{p}) \cdot (A_\varphi\Omega, E(\{0\})U(\mathbf{x})A_\varphi\Omega) \leq c.
\end{aligned}$$

This bound holds uniformly for all $\mathbf{p} \in \mathbf{R}^3$ and implies $\lim_{\mathbf{p}\to 0}(A_\varphi\Omega)(\mathbf{p}) = 0$. Now we are almost finished: Using relation (7), we get for the wavefunction $(B_\lambda\Omega)(\mathbf{p})$ of the one-particle state $E(\{0\})B_\lambda\Omega$

$$(B_\lambda\Omega)(\mathbf{p}) = 2\pi \cdot \tilde{f}(\mathbf{p}^\perp) \cdot (A_\varphi\Omega)(\lambda \cdot K_\lambda^{-1} \circ \mathbf{p}),$$

where

$$\tilde{f}(\mathbf{p}^\perp) = (2\pi)^{-1} \int d^2x^\perp \exp(-i\mathbf{x}^\perp\mathbf{p}^\perp)f(\mathbf{x}^\perp) \quad \text{with } \mathbf{p}^\perp = (0, p_2, p_3).$$

An easy calculation shows that $\lim_{\lambda\to 0} \lambda \cdot K_\lambda^{-1} \circ \mathbf{p} = \frac{1}{2}(|\mathbf{p}| + p_1)\mathbf{e}_1$, where $\mathbf{e}_1 = (1, 0, 0)$. Taking into account the continuity of $\mathbf{p} \to (A_\varphi\Omega)(\mathbf{p})$, we get

$$\lim_{\lambda\to 0}(B_\lambda\Omega)(\mathbf{p}) = 2\pi \cdot \tilde{f}(\mathbf{p}^\perp) \cdot (A_\varphi\Omega)(\tfrac{1}{2}[|\mathbf{p}| + p_1]\mathbf{e}_1). \tag{8}$$

It then follows from the bounded convergence theorem that the limit $\lim_{\lambda\to 0} \int (d^3p/2|\mathbf{p}|)\Psi(\mathbf{p})(B_\lambda\Omega)(\mathbf{p})$ exists for all test functions $\Psi(\mathbf{p})$ with compact support. These functions are dense in $L^2(\mathbf{R}^3, d^3p/2|\mathbf{p}|)$ and since the vectors $B_\lambda\Omega$ are uniformly bounded in λ we conclude that the weak limit $w\text{-}\lim_{\lambda\to 0}E(\{0\})B_\lambda\Omega$ exists. This finishes the proof of the statement. ∎

Remark: Using the above proposition and the localization properties of the operators B_λ, one can show that $\lim_{\lambda\to 0}B_\lambda$ also exists on a dense set of vectors in $\mathcal{H}$.

The wavefunction of the one-particle state w-$\lim_{\lambda\to 0} B_\lambda\Omega$ is given by the right hand side of equation (8). It is therefore easy to specify a local operator A for which this vector is nontrivial: Pick, for example, a one-particle state $\Phi \in \mathcal{H}_1$ which is invariant under spatial rotations $R \to U(R)$. Since $\mathfrak{F}$ is irreducible, there exists a local operator $A_1 \in \mathfrak{F}$ such that the matrix element $(\Phi, A_1\Omega)$ is not zero. The operator $A = \int d\mu(R)U(R) \times A_1 U(R)^{-1}$, where $d\mu(R)$ is the Haar measure on the group of rotations, then has the desired property. If one takes A_1 Hermitian and the functions φ, f real, one can even arrange for the approximating operators B_λ to be Hermitian.

In the remainder of this section we shall show that the existence of an operator sequence B_λ with properties mentioned above implies that the massless particles do not scatter. The argument is based on results recently derived in Ref 4 in the context of collision theory for massless particles. We recapitulate the main facts briefly: As in the massive case, there are collision states

$$\Phi_1 \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \Phi_n \text{ and } \Phi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \Phi_n$$

in $\mathcal{H}$ corresponding to incoming and outgoing configurations $\Phi_1, \ldots, \Phi_n \in \mathcal{H}_1$ of massless particles. These vectors have the familiar Fock structure known from a free theory. They can be generated from the vacuum Ω with the aid of asymptotic fields A^{in} and A^{out}. The bounded functions of the fields which are localized in a region $\mathcal{O}$ constitute the local asymptotic field algebras $\mathfrak{F}^{\text{in}}(\mathcal{O})$ and $\mathfrak{F}^{\text{out}}(\mathcal{O})$ respectively. They have commutation relations with the basic fields which may be interpreted as the field theoretical version of Huyghens' principle: If $\mathcal{O}$ is any bounded region and if $\mathcal{O}_+$, $\mathcal{O}_-$ are two regions which have a positive and negative timelike distance from $\mathcal{O}$, then

$$[F, F_+^{\text{in}}] = 0 \quad \text{and} \quad [F, F_-^{\text{out}}] = 0 \tag{9}$$

for arbitrary $F \in \mathfrak{F}(\mathcal{O})$, $F_+^{\text{in}} \in \mathfrak{F}^{\text{in}}(\mathcal{O}_+)$, and $F_-^{\text{out}} \in \mathfrak{F}^{\text{out}}(\mathcal{O}_-)$ This relation is the key to the proof of the following statement.

Proposition 3: If there exists a bounded region $\mathcal{O} \subset \mathbb{R}^4$ and a sequence of Hermitian operators $B_\lambda \in \mathfrak{F}(\mathcal{O})$ which converges weakly on the vacuum to some nonzero vector in $\mathcal{H}_1$, then the collision states

$$\Phi_1 \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \Phi_n \text{ and } \Phi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \Phi_n$$

coincide for arbitrary configurations $\Phi_1, \ldots, \Phi_n \in \mathcal{H}_1$. Consequently, the S matrix is trivial.

Proof: We define $B_\lambda^L = U(L)B_\lambda U(L)^{-1}$, where $L=(\Lambda, x)$ is an arbitrary Poincaré transformation and $U(L) = U(x)U(\Lambda)$ is the corresponding unitary in $\mathcal{H}$. Since B_λ converges weakly on the vacuum to some nontrivial one-particle state $\Phi \in \mathcal{H}_1$ we get $w\text{-}\lim_{\lambda\to 0} B_\lambda^L\Omega = U(L)\Phi = \Phi_L$. These vectors form a total set in $\mathcal{H}_1$ because the Poincaré transformations are irreducibly represented in $\mathcal{H}_1$. Now the operators B_λ^L are localized in the region $L\mathcal{O}$. Using relation (9), we get therefore

$$(F_+^{\text{in}}\Phi_L, F_-^{\text{out}}\Omega) = \lim_{\lambda\to 0}(F_+^{\text{in}}B_\lambda^L\Omega, F_-^{\text{out}}\Omega)$$
$$= \lim_{\lambda\to 0}(F_+^{\text{in}}\Omega, F_-^{\text{out}}B_\lambda^L\Omega) = (F_+^{\text{in}}\Omega, F_-^{\text{out}}\Phi_L), \quad (10)$$

provided $F_+^{\text{in}} \in \mathfrak{F}^{\text{in}}(L\mathcal{O}_+)$ and $F_-^{\text{out}} \in \mathfrak{F}^{\text{out}}(L\mathcal{O}_-)$. Since the operators A^{in} and A^{out} are free fields, it is straightforward to verify that the bounded operators F_+^{in} and F_-^{out} in this relation may be replaced by products of smeared field operators $A_1^{\text{in}}, \ldots, A_m^{\text{in}}$ and $A_{m+1}^{\text{out}}, \ldots, A_n^{\text{out}}$, which are localized in $L\mathcal{O}_+$ and $L\mathcal{O}_-$ respectively. Thus we arrive at

$$(A_1^{\text{in}}\cdots A_m^{\text{in}}\Phi_L, A_{m+1}^{\text{out}}\cdots A_n^{\text{out}}\Omega) = (A_1^{\text{in}}\cdots A_m^{\text{in}}\Omega, A_{m+1}^{\text{out}}\cdots A_n^{\text{out}}\Phi_L). \quad (11)$$

Now we can prove the proposition by induction. For a one-particle state there is nothing to show, so let us assume that

$$\Phi_1 \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \Phi_m = \Phi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \Phi_m$$

for arbitrary configurations $\Phi_1, \ldots, \Phi_m \in \mathcal{H}_1$. This implies in particular that $A_1^{\text{in}}\cdots A_m^{\text{in}}\Omega = A_1^{\text{out}}\cdots A_m^{\text{out}}\Omega$ and, using relation (11), we get

$$(A_1^{\text{in}}\cdots A_m^{\text{in}}\Phi_L, A_{m+1}^{\text{out}}\cdots A_n^{\text{out}}\Omega)$$
$$= (A_1^{\text{in}}\cdots A_m^{\text{in}}\Omega, A_{m+1}^{\text{out}}\cdots A_n^{\text{out}}\Phi_L)$$
$$= (A_1^{\text{out}}\cdots A_m^{\text{out}}\Omega, A_{m+1}^{\text{out}}\cdots A_n^{\text{out}}\Phi_L)$$
$$= (A_1^{\text{out}}\cdots A_m^{\text{out}}\Phi_L, A_{m+1}^{\text{out}}\cdots A_n^{\text{out}}\Omega),$$

where the last equality sign follows from an explicit calculation of the scalar products. If we set $\Phi_1 = A_1^{\text{in}}\Omega, \ldots, \Phi_n = A_n^{\text{out}}\Omega$, we can reexpress this equation in terms of the collision states,

$$(\Phi_1 \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \Phi_m \overset{\text{in}}{\times} \Phi_L, \Phi_{m+1} \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \Phi_n)$$
$$= (\Phi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \Phi_m \overset{\text{out}}{\times} \Phi_L, \Phi_{m+1} \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \Phi_n),$$

provided $A_1^{\text{in}}, \ldots, A_n^{\text{out}}$ are operators with the special localization properties mentioned above. However, keeping in mind that the vectors Φ_L form a total set in $\mathcal{H}_1$, one can extend this equation by continuity to arbitrary configurations $\Phi_1, \ldots, \Phi_n, \Phi_L \in \mathcal{H}_1$[4] and it is then obvious that

$$\Phi_1 \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} \Phi_{m+1} = \Phi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \Phi_{m+1}. \qquad \blacksquare$$

Combining the two propositions it follows that the massless particles in $\mathcal{H}_1$ do not interact if the dilations are a true symmetry. We have established this result only for one type of massless particles with helicity $s=0$. In the presence of a family of one-particle spaces $\mathcal{H}_1^{(i)} \subset \mathcal{H}$ on which the unitaries $U(x)$, $U(\Lambda)$ act like an irreducible representation of the Poincaré group with mass zero and helicity s_i, the main modifications are in the second part of the proof of Proposition 2: For vectors $\Psi \in \mathcal{H}_1^{(k)}$ relation (7b) changes according to

$$(U(\Lambda)\Psi)_k(\mathbf{p}) = \exp[is_k\alpha(\Lambda, \mathbf{p})](\Psi)_k(\Lambda^{-1}\circ\mathbf{p}),$$

where the index k refers to the space $\mathcal{H}_1^{(k)}$. The functions $\alpha(\Lambda, \mathbf{p})$ are the Wigner phases.[6] They are not completely fixed by the structural relations imposed by the Lorentz group. As a matter of fact we may choose a convention such that the functions $\alpha(\Lambda, \mathbf{p})$ are simultaneously continuous in Λ and $\mathbf{p}$ except at $\mathbf{p}=0$; moreover, we may require that $\alpha(K_\lambda, \mathbf{p}) = 0$, where K_λ are the boosts in the x_1 direction introduced in relation (5). It is then easy to verify that the functions $(B_\lambda\Omega)_k(\mathbf{p})$ are continuous and that the analog of relation (8) holds. The proof of Proposition 3 carries over almost literally, and we may therefore omit the details.

Finally we want to point out a further generalization of our main result. In an asymptotically complete theory of massless particles there always exist two representations $D^{\text{in}}(\lambda)$ and $D^{\text{out}}(\lambda)$ of the group of dilations which act on the asymptotic fields A^{in} and A^{out}, respectively, as in a free field theory. Their commutation relations with the translations $U(x)$ and Lorentz transformations $U(\Lambda)$ are again given by (2a). However, they do not, in general, act on the basic fields according to relation (2b). In order that the dilations are an asymptotically visible symmetry, it would be sufficient to require

$$D^{\text{in}}(\lambda) = D^{\text{out}}(\lambda) = D(\lambda) \quad (12)$$

and relation (2b) could be dropped. But this assumption still implies that the S matrix is trivial! To verify this, one has only to realize that Propositions 2 and 3 still hold in this case with obvious modifications. The proof of Proposition 2 depends on the clustering properties of the vacuum and relation (2a) and therefore applies. Of course, the operators B_λ are in general not local. However, relation (10) which was crucial for the proof of Proposition 3 can still be established. This follows simply from the fact that the asymptotic nets $\mathcal{O} \to \mathfrak{F}^{\text{in}}(\mathcal{O})$ and $\mathcal{O} \to \mathfrak{F}^{\text{out}}(\mathcal{O})$ transform under the dilations $D(\lambda) = D^{\text{in}}(\lambda) = D^{\text{out}}(\lambda)$ according to relation (2b). Hence, if, for example, $A \in \mathfrak{F}(\mathcal{O})$, where $\mathcal{O}$ is any bounded region which contains the origin and if $F_+^{\text{in}} \in \mathfrak{F}^{\text{in}}(\mathcal{O}_+)$, where $\mathcal{O}_+$ has a positive timelike separation from $\mathcal{O}$, one gets for $\lambda < 1$

$$[D(\lambda)AD(\lambda)^{-1}, F_+^{\text{in}}]$$
$$= D(\lambda)[A, D(\lambda^{-1})F_+^{\text{in}}D(\lambda^{-1})^{-1}]D(\lambda)^{-1} = 0$$

by Huyghens' principle. A similar relation holds for $F_-^{\text{out}} \in \mathfrak{F}^{\text{out}}(\mathcal{O}_-)$. It is then easy to verify that the operators B_λ commute for small λ with the operators in $\mathfrak{F}^{\text{in}}(\mathcal{O}_+)$ and $\mathfrak{F}^{\text{out}}(\mathcal{O}_-)$ where the regions $\mathcal{O}_+$, $\mathcal{O}_-$ depend only on the localization properties of f and A_φ. The rest of the argument can then be carried over.

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