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Construction of a Selfadjoint, Strictly Positive Transfer Matrix for Euclidean Lattice Gauge Theories

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Abstract. It is shown that physical positivity holds in Wilson's lattice gauge theories, i.e. transition probabilities between gauge invariant states are non-negative and the quantum mechanical Hamiltonian has real eigenvalues only.

I. Introduction

Ever since lattice gauge theories were proposed by Wilson [1] there was the question, wether the scheme will indeed yield an acceptable quantum field theory in the continuum limit. One of the required properties that does not obviously hold in the lattice theory is physical positivity[1]. In this paper we are going to explicitly construct the quantum mechanical space of states for euclidean lattice gauge theories. We will also derive a formula for the transfer matrix, i.e. the operator e^{-aH}, where H is the q.m. Hamiltonian and a is the lattice spacing.

Euclidean lattice gauge theories are defined as follows (for details, the reader is refered to Wilson's papers). We consider a cubic, four dimensional lattice, whose points will be labelled by four integer numbers $n=(n_0, n_1, n_2, n_3)$, $|n_0| \leqq M$, $|n_i| \leqq L$ ($i=1,2,3$), thus giving a total of $(2M+1)(2L+1)^3$ sites. At each lattice point n there is attached a classical Dirac spinor ψ_n (the quark field) whose entries are elements of a Grassmann algebra (cp. Appendix). The gauge field $U(n, \mu)$ ($\mu=0,1,2,3$) sits on the links between the lattice sites. It is an element of the gauge group G, which is taken to be $SU(N)$. Correspondingly, the quark field ψ_n carries a colour index α, $\alpha=1,\ldots,N$. To keep the reasoning as transparent as possible, we will assume that there are no flavour degrees of freedom. Our results are however true for the more general case aswell.

The dynamics of euclidean quark and gluon fields can be expressed in terms of their correlation functions (euclidean expectations, Schwinger functions):

$$\langle \varphi_1 \ldots \varphi_m \rangle = Z^{-1} \int \mathscr{D}U \mathscr{D}\psi \mathscr{D}\bar{\psi} \varphi_1 \ldots \varphi_m e^{A} . \tag{1}$$

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[1] Osterwalder and Seiler have announced a result concerning this question [5]

The φ_i's are any of the fields ψ_n, $\bar{\psi}_n$, $U(n,\mu)$, $U^+(n,\mu)$. The quantity A is the action [1]:

$$A=\sum_n\left\{-\bar{\psi}_n\psi_n+K\sum_{\mu=0}^{3}[\bar{\psi}_n(1+\gamma_\mu)U(n,\mu)\psi_{n+\hat{\mu}}+\bar{\psi}_{n+\hat{\mu}}(1-\gamma_\mu)U^+(n,\mu)\psi_n]\right.$$
$$\left.+\tfrac{1}{2}g_0^{-2}\sum_{\substack{\mu,\nu\\ \mu\neq\nu}}\mathrm{Tr}[U(n,\mu)U(n+\hat{\mu},\nu)U^+(n+\hat{\nu},\mu)U^+(n,\nu)]\right\} \tag{2}$$

where K is a dimensionless parameter related to the quark mass and g_0 is the bare quark-gluon coupling constant. Finally, Z is the partition function:

$$Z=\int\mathscr{D}U\mathscr{D}\psi\mathscr{D}\bar{\psi}e^A \tag{3}$$

and $\mathscr{D}U$, $\mathscr{D}\psi$, $\mathscr{D}\bar{\psi}$ denote the product measures $\prod_{n,\mu}dU(n,\mu)$ (dU is the invariant measure on the gauge group G), $\prod_{n,\alpha}d\psi_{n,\alpha}$ and $\prod_{n,\alpha}d\bar{\psi}_{n,\alpha}$ respectively. The normalizations have been choosen such that $\int_G dU=\int d\psi\psi=\int d\bar{\psi}\bar{\psi}=1$ [2].

Letting M and L tend to infinity we obtain the infinite volume lattice gauge theory. Hopefully, this theory has a critical point at $K=1/8$ and $g_0=0$. The continuum gauge theory can then be thought of as a certain limit of the lattice theory for large distances (in lattice units) keeping K and g_0 near their critical values.

Physical positivity can be expressed in terms of the euclidean expectation values defined above. This is the celebrated Osterwalder-Schrader positivity condition [2]. It states that if $\mathcal{O}$ is any polynomial of positive time ($n_0\geqq 1$) fields ψ, $\bar{\psi}$, U, and U^+, we should find

$$\langle\theta(\mathcal{O}^+)\mathcal{O}\rangle\geqq 0\,. \tag{4}$$

Here, θ denotes euclidean time reflection and $\mathcal{O}^+$ is the complex conjugate of $\mathcal{O}$ (e.g. $\bar{\psi}^+=\gamma_0\psi$). It will be proven in Section II that the expectations (1) indeed satisfy this positivity condition provided $0<K<1/6$. The Hilbert space $\mathscr{H}$ of physical states can then be constructed in a standard fashion. As a linear space $\mathscr{H}$ is just the set $\mathscr{E}_+$ of all gauge invariant polynomials $\mathcal{O}$ of positive time fields $\bar{\psi}$, ψ, U and U^+. The inner product in $\mathscr{H}$ is taken to be:

$$(\mathcal{O}_1,\mathcal{O}_2)=\langle\theta(\mathcal{O}_1^+)\mathcal{O}_2\rangle;\qquad \mathcal{O}_1,\mathcal{O}_2\in\mathscr{E}_+ \tag{5}$$

(division by the subspace of all null vectors and completion are understood). The q.m. Hamiltonian can now be found by identifying e^{-aH} with the operator T that shifts the elements of $\mathscr{E}_+$ by one lattice unit in the positive euclidean time direction. In order that this procedure yields an acceptable Hamiltonian we must make sure that $T>0$. For a continuum field theory this requirement is automatically fulfilled whenever (4) holds. In the lattice case an additional argument is needed.

The construction of the space of physical states of a euclidean field theory via the Osterwalder-Schrader positivity property of the euclidean expectations is

[2] Boundary conditions will be specified in Section II

generally applicable. However, for a lattice theory there is a more natural procedure. Having a momentum aswell as a volume cutoff, it should be possible to set up a canonical, hamiltonian formalism for equal time fields, such that the euclidean expectations become vacuum expectation values of appropriately translated time zero fields. Positivity (4) then holds automatically and $\mathscr{H}$ is equal to the time zero quantum mechanical Hilbert space.

At the cost of fixing a convenient gauge, one would also like to have a space of states $\hat{\mathscr{H}} \supset \mathscr{H}$ where the fundamental fields can act. In the euclidean framework the gauge can be fixed (or partly fixed) by modifying the definition (1) of the expectation values as follows:

$$\langle \varphi_1 \dots \varphi_m \rangle_g = Z_g^{-1} \int \mathscr{D}U \mathscr{D}\psi \mathscr{D}\bar{\psi} \varphi_1 \dots \varphi_m e^A g(U). \tag{6}$$

The function g is called the gauge fixing function. In order that the physics is not affected by the introduction of g, we must require that

$$\langle \mathcal{O} \rangle_g = \langle \mathcal{O} \rangle \tag{7}$$

for all gauge invariant polynomials $\mathcal{O}$ of the fundamental fields.

If positivity (4) is now true for all combinations $\mathcal{O}$ of positive time fields, we can construct $\hat{\mathscr{H}}$ in the same way as $\mathscr{H}$.

An example of an admissable gauge fixing function is[3]:

$$g(U) = \prod_n \delta(U(n,0) - 1) \tag{8}$$

i.e. this is the gauge, where the time component of the gauge vector field has been set equal to zero.

In Section II we are going to use this gauge, establish a canonical scheme for equal time fundamental fields ψ_n, $\bar{\psi}_n$, $U(n,j)$ $(j=1,2,3)$, give an expression for the transfer matrix T in terms of these operators and finally prove that the corresponding Schwinger functions are equal to the euclidean expectations (6).

II. Construction of the Quark-Gluon Quantum Mechanics[4]

A. *Definition of the Time Zero Hilbert Space* $\hat{\mathscr{H}}$

In this paragraph we consider a three dimensional lattice with vertices $\boldsymbol{n} = (n_1, n_2, n_3)$, $n_i \in \mathbb{Z}$, $|n_i| \leq L$. At each site $\boldsymbol{n}$ there is an operator Dirac spinor $\hat{\psi}_{\boldsymbol{n}}$. With the links between the lattice points we associate an $N \times N$-matrix $\hat{U}(\boldsymbol{n},j)_{\alpha\beta}$ $(j=1,2,3)$ of gauge field operators. We assume periodic boundary conditions, i.e. there are also gauge fields on the links connecting boundary points of the lattice.

The time zero fields $\hat{\psi}_{\boldsymbol{n}}$, $\hat{U}(\boldsymbol{n},j)$ act in a Hilbert space $\hat{\mathscr{H}}$ that is the tensor product of a pure fermion space of states $\hat{\mathscr{H}}_F$ and a pure gauge field Hilbert space $\hat{\mathscr{H}}_G$.

$\hat{\mathscr{H}}_G$ is easily described. It is just the space of all square integrable, complex valued functions $f(U(\boldsymbol{n},j))$, i.e. since there are $3(2L+1)^3$ links we have

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[3] In case cyclic boundary conditions are choosen for the gauge field, Equation (7) is true only in the infinite volume limit, provided the color symmetry is not spontaneously broken (cp. Section II)

[4] Quark-Gluon quantum mechanics on a (space-) lattice has been considered by Kogut and Susskind [9]

$\hat{\mathscr{H}}_G = [L^2(G)]^{3\cdot(2L+1)^3}$. The operator $\hat{U}(\boldsymbol{n},j)_{\alpha\beta}$ acts as a multiplication operator on the wave functions $f(U)$:

$$[\hat{U}(\boldsymbol{n},j)_{\alpha\beta} f](U) = U(\boldsymbol{n},j)_{\alpha\beta} \cdot f(U). \tag{9}$$

We will henceforth always work in the representation for vectors of $\hat{\mathscr{H}}_G$ described here and thus agree to write $U(\boldsymbol{n},j)_{\alpha\beta}$ instead of $\hat{U}(\boldsymbol{n},j)_{\alpha\beta}$ respectively $U^+(\boldsymbol{n},j)_{\alpha\beta}$ for the operator which multiplies $f(U)$ with $U^+(\boldsymbol{n},j)_{\alpha\beta} = U^{-1}(\boldsymbol{n},j)_{\alpha\beta}$.

Sufficiently well behaved operators A in $\hat{\mathscr{H}}_G$ can be represented with the help of an integral kernel $K_A(U, U')$:

$$(Af)(U) = \int \prod_{\boldsymbol{n},j} dU'(\boldsymbol{n},j) K_A(U, U') f(U'). \tag{10}$$

This will be the case for the transfer matrix to be studied later.

The fermion Hilbert space $\hat{\mathscr{H}}_F$ is the Fock space built from an operator spinor field $\hat{\chi}_{\boldsymbol{n}}$ that satisfies canonical anticommutation relations[5]:

$$\{\hat{\chi}_{\boldsymbol{n}\alpha}, \hat{\chi}^+_{\boldsymbol{m}\beta}\} = \delta_{\boldsymbol{nm}}\delta_{\alpha\beta}; \qquad \{\hat{\chi}_{\boldsymbol{n}\alpha}, \hat{\chi}_{\boldsymbol{m}\beta}\} = \{\hat{\chi}^+_{\boldsymbol{n}\alpha}, \hat{\chi}^+_{\boldsymbol{m}\beta}\} = 0. \tag{11}$$

The field $\hat{\psi}_{\boldsymbol{n}}$ acts in $\hat{\mathscr{H}} = \hat{\mathscr{H}}_F \otimes \hat{\mathscr{H}}_G$. It does *not* have a canonical anticommutator, but

$$\{\hat{\psi}_{\boldsymbol{n}\alpha}, \hat{\psi}^+_{\boldsymbol{m}\beta}\} = (B^{-1})_{\boldsymbol{n}\alpha,\boldsymbol{m}\beta}; \qquad \{\hat{\psi}_{\boldsymbol{n}\alpha}, \hat{\psi}_{\boldsymbol{m}\beta}\} = \{\hat{\psi}^+_{\boldsymbol{n}\alpha}, \hat{\psi}^+_{\boldsymbol{m}\beta}\} = 0. \tag{12}$$

The matrix $B_{\boldsymbol{n}\alpha,\boldsymbol{m}\beta}$ depends on the gauge field and is given by

$$B_{\boldsymbol{n}\alpha,\boldsymbol{m}\beta} = \delta_{\boldsymbol{nm}}\delta_{\alpha\beta} - K \sum_{j=1,2,3} (U(\boldsymbol{n},j)_{\alpha\beta}\delta_{\boldsymbol{n}+\hat{j},\boldsymbol{m}} + U^+(\boldsymbol{m},j)_{\alpha\beta}\delta_{\boldsymbol{m}+\hat{j},\boldsymbol{n}}). \tag{13}$$

It is an easy exercise to prove that $B_{\boldsymbol{n}\alpha,\boldsymbol{m}\beta}$ is hermitian and strictly positive for any configuration of gauge fields, provided $|K| < 1/6$. In fact, $B \geqq 1 - 6|K|$. Thus B^{-1} exists and (12) is well defined. The restriction $|K| < 1/6$ already occurs in the free field case [3] and is unessential, because K is near 1/8 in the continuum limit.

An explicit representation of $\hat{\psi}$ in terms of the canonical field $\hat{\chi}$ is:

$$\hat{\psi}_{\boldsymbol{n}\alpha} = \sum_{\boldsymbol{m}\beta} (B^{-1/2})_{\boldsymbol{n}\alpha,\boldsymbol{m}\beta}\hat{\chi}_{\boldsymbol{m}\beta}; \qquad \hat{\psi}^+_{\boldsymbol{n}\alpha} = \sum_{\boldsymbol{m}\beta} \hat{\chi}^+_{\boldsymbol{m}\beta}(B^{-1/2})_{\boldsymbol{m}\beta,\boldsymbol{n}\alpha}. \tag{14}$$

Note that B does not act on Lorentz indices.

One can perform gauge transformations in $\hat{\mathscr{H}}$, too. They correspond to time independent gauge changes in the euclidean framework (such gauge transformations survive, when the gauge fixing function (8) is introduced).

For any field $V_{\boldsymbol{n}}$ of $SU(N)$-matrices there is a unitary operator $R(V)$ such that

$$\begin{aligned} &[R(V)f](U(\boldsymbol{n},j)) = f(V_{\boldsymbol{n}}^{-1} U(\boldsymbol{n},j) V_{\boldsymbol{n}+\hat{j}}) \quad \text{for all} \quad f \in \hat{\mathscr{H}}_G \\ &R(V)\hat{\chi}_{\boldsymbol{n}} R(V)^{-1} = V_{\boldsymbol{n}}^{-1} \cdot \hat{\chi}_{\boldsymbol{n}}. \end{aligned} \tag{15}$$

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[5] α is a shorthand for colour *and* Lorentz indices. Of course, when we write $U(\boldsymbol{n},j)_{\alpha\beta}$ resp. $(\gamma_j)_{\alpha\beta}$ there are only colour resp. Lorentz indices involved. The γ-matrices used are euclidean ones, i.e.

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}; \ \gamma_\mu^+ = \gamma_\mu.$$

An explicit representation is given in Equation (32)

By its definition B is gauge covariant. Therefore, $\hat{\psi}$ transforms as $\hat{\chi}$ under gauge transformations. An operator $\mathcal{O}$ is called gauge invariant, if it commutes with $R(V)$.

For later convenience let us define a normal ordering prescription for operators $\hat{\varphi}$ that are polynomials in $\hat{\psi}$, $\hat{\psi}^+$. We set

$$N[\hat{\varphi}] = \text{the same as } \hat{\varphi}\text{, but the fermi operators reordered such that all the fields } (1+\gamma_0)\hat{\psi},\ \hat{\psi}^+(1-\gamma_0) \text{ stand to the left of all operators } (1-\gamma_0)\hat{\psi},\ \hat{\psi}^+(1+\gamma_0). \tag{16}$$

For example, $N[\hat{\bar{\psi}}\hat{\psi}] = \frac{1}{2}(\hat{\bar{\psi}}(1-\gamma_0)\hat{\psi} - \hat{\psi}^T(1+\gamma_0)\hat{\bar{\psi}}^T)$ [6].

It's clear that to any classical composite field φ involving equal time ψ's, $\bar{\psi}$'s, U's and U^+'s we can uniquely associate an operator in $\hat{\mathscr{H}}$ by replacing the classical variables by the corresponding time zero operators and performing the normal ordering.

B. *Definition of the Transfer Matrix* $\hat{T}$

We will now give an explicit expression for the transfer matrix $\hat{T}$ as an operator acting in $\hat{\mathscr{H}}$. In the next paragraph it will be verified that $\hat{T}$ is indeed the transfer matrix of the euclidean theory discussed in Section I.

With respect to the gauge field $\hat{T}$ is an integral operator. Its kernel $K_T(U, U')$ is however an operator in the fermion Hilbert space $\hat{\mathscr{H}}_F$. It has the following structure:

$$K_T(U, U') = T_F^+(U) \cdot T_G^+(U) \cdot S(U, U') \cdot T_G(U') \cdot T_F(U'). \tag{17}$$

T_F is the only part of $\hat{T}$ that depends on fermion operators:

$$T_F(U) = \det(2KB)^{1/4} \exp(\hat{\bar{\chi}} \cdot \tfrac{1}{2}(1-\gamma_0)C\hat{\chi}) \exp(-\hat{\bar{\chi}} \cdot M\hat{\chi}). \tag{18}$$

The matrix $C_{\boldsymbol{n}\alpha,\boldsymbol{m}\beta}$ is similar to $B_{\boldsymbol{n}\alpha,\boldsymbol{m}\beta}$ (see footnote 5):

$$C_{\boldsymbol{n}\alpha,\boldsymbol{m}\beta} = \frac{1}{2} \sum_{j=1,2,3} \{U(\boldsymbol{n},j)_{\alpha\beta}(\gamma_j)_{\alpha\beta}\delta_{\boldsymbol{n}+\hat{j},\boldsymbol{m}} - U^+(\boldsymbol{m},j)_{\alpha\beta}(\gamma_j)_{\alpha\beta}\delta_{\boldsymbol{m}+\hat{j},\boldsymbol{n}}\}. \tag{19}$$

C is a skew-hermitian matrix independently of the gauge field configuration. The matrix M is equal to $1/2 \ln B/2K$ and hence hermitian (we assume $0<K<1/6$). In Equation (18) summation over all indices $\boldsymbol{n}\alpha$ resp. $\boldsymbol{m}\beta$ is understood.

The remaining contributions to $K_T(U, U')$ account for the plaquette terms in Equation (2):

$$T_G(U) = \exp(2g_0)^{-2} \sum_{\boldsymbol{n}} \sum_{i \neq j = 1,2,3} \mathrm{Tr}\{U(\boldsymbol{n}, i)U(\boldsymbol{n}+\hat{\imath}, j)U^+(\boldsymbol{n}+\hat{\jmath}, i)U^+(\boldsymbol{n}, j)\}, \tag{20}$$

$$S(U, U') = \exp \tfrac{1}{2} g_0^{-2} \sum_{\boldsymbol{n}} \sum_{j=1,2,3} \{\mathrm{Tr}[U(\boldsymbol{n},j)U'^+(\boldsymbol{n},j)] + \mathrm{Tr}[U'(\boldsymbol{n},j)U^+(\boldsymbol{n},j)]\}. \tag{21}$$

The relevant properties of the transfer matrix thus defined are summarized by the following theorem.

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[6] $\hat{\psi}^T$ is the transpose of $\hat{\psi}$ and the sign in the second term is due to fermi statistics

Proposition 1. a) *$\hat{T}$ is a selfadjoint, bounded operator in $\hat{\mathscr{H}}$.*

b) *It is gauge invariant under the restricted class of gauge transformations discussed in the preceding paragraph.*

c) *It is strictly positive, i.e. all its eigenvalues are larger than zero.*

The only not obviously true statement made here is the claim that $\hat{T}$ is strictly positive. This property of $\hat{T}$ allows one to define a Hamiltonian H by setting: $H = -1/a \ln \hat{T}$ (a is the lattice constant). H is bounded and has real eigenvalues only.

To prove the proposition, we first observe that it suffices to verify that $\langle X|\hat{T}|X\rangle > 0$ for all nonvanishing vectors $|X\rangle \in \hat{\mathscr{H}}$. But, from Equation (17) and the fact that T_G and T_F are bounded and invertible, we see that it is enough to show strict positivity for the integral operator S that acts in $\hat{\mathscr{H}}_G$. Our task is furthermore simplified by noting that S is a product of identical operators, one for each link. Thus we are left to prove that

$$\int dU dU' f^*(U) \exp\tfrac{1}{2} g_0^{-2}\{\mathrm{Tr}(U^{-1}U') + \mathrm{Tr}(U^{-1}U')^+\} f(U') > 0 \tag{22}$$

for all square integrable nonvanishing functions f on the gauge group $G = SU(N)$. This is easily done. For, the kernel in (22) can be expanded in a Fourier series on the group G:

$$\exp\tfrac{1}{2} g_0^{-2}\{\mathrm{Tr}\, V + \mathrm{Tr}\, V^+\} = \sum_{\nu \in \hat{G}} c_\nu \chi^{(\nu)}(V) \qquad (V = U^{-1}U'). \tag{23}$$

$\hat{G}$ is the set of all irreducible representations of G and $\chi^{(\nu)}$ is the character of the representation $\nu \in \hat{G}$. In order that (22) holds, it is necessary and sufficient that the numbers c_ν are all positive.

To calculate the c_ν's we expand the left hand side of Equation (23):

$$\exp\tfrac{1}{2} g_0^{-2}\{\mathrm{Tr}\, V + \mathrm{Tr}\, V^+\} = \sum_{n,m=0}^{\infty} a_{nm} (\mathrm{Tr}\, V)^n (\mathrm{Tr}\, V^+)^m, \qquad a_{nm} > 0\,.$$

Now, $(\mathrm{Tr}\, V)^n (\mathrm{Tr}\, V^+)^m$ is nothing else than the trace of the tensor product representation of $SU(N)$ that is composed of n quark and m antiquark representations. Reducing out the tensor product mentaly, we find

$$(\mathrm{Tr}\, V)^n (\mathrm{Tr}\, V^+)^m = \sum_{\nu \in \hat{G}} c_\nu(n,m) \chi^{(\nu)}(V)$$

where $c_\nu(n,m)$ is just the number of times the irreducible representation ν occurs in the tensor product of n quark and m antiquark representations. Hence,

$$c_\nu = \sum_{n,m=0}^{\infty} a_{nm} c_\nu(n,m)$$

is nonnegative. It must be positive, because *all* irreducible representations of $SU(N)$ can be obtained by reducing out tensor products of quark representations [4].

C. Reconstruction of the Euclidean Expectation Values

Given any polynomial $\hat{\phi}$ of the time zero fundamental fields we define its time translate $\hat{\phi}_t$ by

$$\hat{\phi}_t = \hat{T}^t \hat{\phi} \hat{T}^{-t} \qquad t = -M, -M+1, \ldots, M-1, M\,. \tag{24}$$

The Schwinger functions of a set $\hat{\phi}_{1,t_1}, \ldots, \hat{\phi}_{k,t_k}$ of such operators with $t_1 < \ldots < t_k$ are [2]

$$S(\hat{\phi}_{1,t_1} \ldots \hat{\phi}_{k,t_k}) = \mathscr{Z}^{-1} \operatorname{Tr}\{\hat{T}^{2M+1} \hat{\phi}_{1,t_1} \ldots \hat{\phi}_{k,t_k}\}$$
$$= \mathscr{Z}^{-1} \operatorname{Tr}\{\hat{T}^{t_1+M} \hat{\phi}_1 \hat{T}^{t_2-t_1} \hat{\phi}_2 \ldots \hat{\phi}_k \hat{T}^{M+1-t_k}\} \tag{25}$$

where $\mathscr{Z} = \operatorname{Tr} \hat{T}^{2M+1}$.

Letting M tend to infinity we get

$$S(\hat{\phi}_{1,t_1} \ldots \hat{\phi}_{k,t_k}) = (\operatorname{Tr} P_0)^{-1} \operatorname{Tr}\{P_0 \hat{\phi}_{1,t_1} \ldots \hat{\phi}_{k,t_k}\} . \tag{26}$$

P_0 is the projector on the lowest energy subspace. If the largest eigenvalue of $\hat{T}$ is not degenerate, (26) becomes

$$S(\hat{\phi}_{1,t_1} \ldots \hat{\phi}_{k,t_k}) = \langle 0 | \hat{\phi}_{1,t_1} \ldots \hat{\phi}_{k,t_k} | 0 \rangle . \tag{27}$$

$|0\rangle$ being the physical vacuum state. I do not know wether this happens for lattice gauge theories. If not there could be spontaneous colour symmetry breaking.

We wish to establish equality of the euclidean expectations defined in Section I and the corresponding Schwinger functions considered here. To this end we must specify the boundary conditions to be used in Equations (1) and (2). In the space directions of the lattice we assume periodic boundary conditions as was done for the time zero fields. As for the time direction, we choose periodic boundary conditions for the gauge field and anticyclic boundary conditions for the quark field:

$$\psi_{M+1,\boldsymbol{n}} = -\psi_{-M,\boldsymbol{n}} ; \qquad \psi_{-M-1,\boldsymbol{n}} = -\psi_{M,\boldsymbol{n}} . \tag{28}$$

For $M \to \infty$ these boundary conditions are no worse than e.g. periodic boundary conditions, provided the vacuum is unique[7].

We are now ready to formulate and prove the reconstruction theorem:

Proposition 2. *Let $\phi_1, \ldots, \phi_k$ be a set of polynomials of the classical fields ψ, $\bar{\psi}$, U, and U^+ at equal times $t_1 < \ldots < t_k$ respectively. Assuming boundary conditions as defined above, we have*

$$\langle \phi_1 \ldots \phi_k \rangle_g = S(N[\hat{\phi}_1]_{t_1} \ldots N[\hat{\phi}_k]_{t_k}) \tag{29}$$

where g is the gauge fixing function (8) *and $N[\hat{\phi}]$ is the normally ordered operator $\hat{\phi}$* (cp. § A)).

Of course, this theorem proves Osterwalder-Schrader positivity (4) of the euclidean expectations $\langle \ldots \rangle_g$. For $M \to \infty$ one readily verifies that the Hilbert spaces $\hat{\mathscr{H}}$ defined in Sections I resp. II A) can be identified naturally. Moreover, the space $\mathscr{H}$ of gauge invariant states is equal to the subspace of $\hat{\mathscr{H}}$ that consists of the vectors which are invariant under the restricted class of gauge transformations mentioned in § A).

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[7] I do not consider the question of boundary conditions in euclidean lattice gauge theories as settled (physically). For, we know that there are topological sectors for the continuum gauge field [6]. Such sectors are easily missed, when choosing wrong boundary conditions for the finite volume theory. As an example for a similar situation we may take the lattice massive Thirring model, where a delicate dependence of the particle spectrum on boundary conditions has been observed [7]

To prove Proposition 2 we have to show that

$$\mathrm{Tr}\{\hat{T}^{t_1+M}N[\hat{\phi}_1]\hat{T}^{t_2-t_1}N[\hat{\phi}_2]\dots N[\hat{\phi}_k]\hat{T}^{M+1-t_k}\}$$
$$=\int \mathscr{D}U\mathscr{D}\psi\mathscr{D}\bar{\psi}\,\phi_1\dots\phi_k e^A g(U)\,. \tag{30}$$

The idea is, to use a representation of $\hat{T}$ in terms of Grassmann variables as explained in the appendix. The integrations appearing on the right hand side of Equation (30) stand for the summations (sum over all intermediate states) needed when multiplying operators. The U-integrations are already there because the gauge field part of $\hat{T}$ is an integral operator.

To explicitly write down the Grassmann form of $\hat{T}$ it is convenient to use the following variables:

$$\hat{\chi}_n=\begin{pmatrix}\hat{x}_{1,n}\\ \hat{x}_{2,n}\\ \hat{y}^+_{1,n}\\ \hat{y}^+_{2,n}\end{pmatrix};\qquad \hat{\chi}^+=(\hat{x}^+_{1,n},\hat{x}^+_{2,n},\hat{y}_{1,n},\hat{y}_{2,n}) \tag{31}$$

(the colour index has been omitted). The two-spinors $\hat{x}_n$ and $\hat{y}_n$ are canonical operators and play the role the operators $\hat{a}_l$ do in the appendix.

Choosing the representation

$$\gamma_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix};\qquad \gamma_j=i\begin{pmatrix}0&\sigma_j\\-\sigma_j&0\end{pmatrix}\qquad(\sigma_j:\text{Pauli matrices}) \tag{32}$$

we can rewrite $T_F(U)$ [Eq. (18)] as

$$T_F(U)=\det(2KB)^{1/4}\exp(\hat{y}\cdot c\hat{x})\exp(-\hat{x}^+M\hat{x}+\hat{y}M\hat{y}^+)$$

where, according to (19) and (32), we have set $C=\begin{pmatrix}0&c\\-c&0\end{pmatrix}$. Using canonical anticommutation relations and $\exp M=(2K)^{-1/2}B^{1/2}$ we find

$$T_F(U)=\det(2KB)^{1/4}\exp(-\hat{x}^+M\hat{x}+\hat{y}M\hat{y}^+)\exp(2K\hat{y}B^{-1/2}cB^{-1/2}\hat{x})\,.$$

Hence, by (A8), (A9), and (A10), the Grassmann equivalent of $T_F(U)^+T_F(U')$ is

$$[T_F^+(U)T_F(U')](x^+,y^+;x,y)=\det(B\cdot B')^{1/2}\exp(2Kx^+B^{-1/2}cB^{-1/2}y^+)$$
$$\cdot\exp(2Kx^+B^{-1/2}B'^{-1/2}x-2KyB'^{-1/2}B^{-1/2}y^+)\exp(2KyB'^{-1/2}c'B'^{-1/2}x)\,. \tag{33}$$

With the help of this formula and Equation (A6) we can rewrite the left hand side of Equation (30). The normal products $N[\hat{\phi}_j]$ thereby translate into the classical form ϕ_j. This is so because the order of fermi operators in $N[\hat{\phi}]$ has been choosen such that all fields $\hat{x}$, $\hat{y}$ are the to left of all operators $\hat{x}^+$, $\hat{y}^+$. Thus (A10) is applicable.

At each fixed time there will be an integration over x, x^+ and y, y^+. In these integrals we make the substitution

$$\psi=B^{-1/2}\begin{pmatrix}x_1\\x_2\\y_1^+\\y_2^+\end{pmatrix};\qquad \bar{\psi}=(x_1^+,x_2^+,y_1,y_2)\gamma_0B^{-1/2}\,.$$

The Jacobian of this transformation is killed by the factors $\det B^{1/2}$ resp. $\det B'^{1/2}$ in Equation (33).

Using these rules, collecting all contributions patiently, finally yields Equation (30) [the anticyclic boundaty conditions for the fermi field match with the trace formula for Grassmann integral operators, cp. (A7)].

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Appendix. Some Useful Formulas Involving Grassmann Variables [8]

We consider a system of fermion operators $\hat{a}_1, \hat{a}_1^+ ;\dots ; \hat{a}_n, \hat{a}_n^+$ satisfying canonical anti commutation relations:

$$\{\hat{a}_k, \hat{a}_l^+\} = \delta_{kl} ; \{\hat{a}_k, \hat{a}_l\} = \{\hat{a}_k^+, \hat{a}_l^+\} = 0 .$$

The corresponding Fock space F is spanned by the vectors

$$|k_1 \dots k_j\rangle = \hat{a}_{k_1}^+ \dots \hat{a}_{k_j}^+|0\rangle ; \hat{a}_k|0\rangle = 0 \quad \text{for all} \quad k ; \| |0\rangle \| = 1 .$$

We are now going to establish an isomorphism between F and a subalgebra $\mathscr{G}^+$ of a Grassmann algebra $\mathscr{G}$. $\mathscr{G}$ is taken to be generated by the totally anticommuting objects $a_1, a_1^+ ;\dots ; a_n, a_n^+$. $\mathscr{G}^+$ is the subalgebra generated by the elements $a_1^+, \dots, a_n^+$ alone. The isomorphism mentioned above is defined by mapping the vectors $|k_1, \dots, k_j\rangle$ onto the elements $a_{k_1}^+ \dots a_{k_j}^+$ of $\mathscr{G}^+$. The image of a general vector $|X\rangle \in F$ under this mapping will be denoted by $X(a^+)$.

The scalar product in F can be expressed in terms of Grassmann variables:

$$\langle X|Y\rangle = \int da_n^+ da_n \dots da_1^+ da_1 e^{-\sum_{j=1}^{n} a_j^+ a_j} (X(a^+))^+ Y(a^+) . \tag{A1}$$

The integral over Grassmann variables is defined as follows. First, we note that the integrand is always a polynomial in the generators a_k and a_k^+. Hence, it suffices to specify the integral for monomials. We choose

$$\int da_n^+ da_n \dots da_1^+ da_1 a_1 a_1^+ \dots a_n a_n^+ = 1 .$$

The integral of other monomials vanishes.

Linear changes of variables can be done as for ordinary integrals. Thus, if $b_k = \sum_l A_{kl} a_l$, $b_k^+ = \sum_l A_{kl}^* a_l^+$ one finds

$$\int da_n^+ \dots da_1 f(a, a^+) = |\det A|^2 \int db_n^+ \dots db_1 f(A^{-1} b, A^{*-1} b^+) . \tag{A2}$$

Linear operators in F have a representation in terms of Grassmann variables too. To any operator A in F we associate an element $A(a^+, a)$ of $\mathscr{G}$ by the following rule: if

$$\langle k_1, \dots, k_j|A|l_1 \dots l_i\rangle = \langle 0|\hat{a}_{k_j} \dots \hat{a}_{k_1} A \hat{a}_{l_1}^+ \dots \hat{a}_{l_i}^+|0\rangle = A(k_1 \dots k_j|l_1 \dots l_i) \tag{A3}$$

we set

$$A(a^+, a) = \sum_{\substack{\{k_1 \dots k_j\} \\ \{l_1 \dots l_i\}}} (1/j!i!)\, a_{k_1}^+ \dots a_{k_j}^+ A(k_1 \dots k_j|l_1 \dots l_i) a_{l_i} \dots a_{l_1} . \tag{A4}$$

This definition has been choosen such that

$$(AX)(a^+) = \int db_n^+ \dots db_1 \, e^{-\sum_{j=1}^{n} b_j^+ b_j} A(a^+, b) X(b^+), \tag{A5}$$

i.e. $A(a^+, a)$ is the integral kernel representing the operator A.

For the product of two operators A and B we obtain

$$(A \cdot B)(a^+, a) = \int db_n^+ \dots db_1 \, e^{-\sum_{j=1}^{n} b_j^+ b_j} A(a^+, b) B(b^+, a) \tag{A6}$$

and the trace of A is given by:

$$\mathrm{Tr} A = \int da_n^+ \dots da_1 \, e^{-\sum_{j=1}^{n} a_j^+ a_j} A(a^+, -a). \tag{A7}$$

It is the somewhat unexpected sign appearing in this formula that forces one to assume anticyclic boundary conditions in the time direction for the fermi field ψ (cp. Sec. II).

We finally collect some formulas for special operators A.

$$A = e^{\sum_{k,l} a_k^+ A_{kl} a_l} \Rightarrow A(a^+, a) = e^{\sum_{k,l} a_k^+ (e^A)_{kl} a_l} \tag{A8}$$

$$A = e^{\sum_{k,l} A_{kl} a_l} \cdot e^{\sum_{k,l} a_k^+ \Xi_{kl} a_l} \Rightarrow A(a^+, a) = e^{\sum_{k,l} a_k^+ (e^A \cdot e^\Xi)_{kl} a_l} \tag{A9}$$

If $B = B(\hat{a}^+)$ and $C = C(\hat{a})$ are operators that depend on $\hat{a}^+$ resp. $\hat{a}$ only, then we have for any A

$$(B \cdot A \cdot C)(a^+, a) = B(a^+) A(a^+, a) C(a). \tag{A10}$$

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