SO(4)-SYMMETRIC SOLUTIONS OF MINKOWSKIAN YANG-MILLS FIELD EQUATIONS

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We construct all solutions to the SU(2) Yang-Mills field equations in Minkowski space that are invariant under an SO(4) subgroup of the conformal group. They are real, regular and have finite energy and action. A connection with the instanton solution is pointed out.

The similarity between the four dimensional SU(2) Yang-Mills field and the two dimensional O(3) nonlinear σ-model [3] with regard to pseudoparticles [4] suggests that the Minkowskian classical Yang-Mills field theory is completely integrable, too. However, so far we know of only one non-trivial, real finite energy solution to the Yang-Mills field equations in Minkowski space. This one solution was found (in a complex form) by De Alfaro et al. [1] and was later shown to be gauge equivalent to a real solution by Cervero et al. [2].

In this letter we exploit the invariance of the Yang-Mills field equations under an SO(4)-subgroup of the Minkowskian conformal group. First we rewrite the field equations in a manifestly SO(4)-covariant way. Then, upon mixing isospin with one of the SO(4)-spins, we obtain all SO(4)-symmetric solutions. They are real, regular, non-abelian and have finite energy and action.

We consider an SU(2) Yang-Mills field $A_\mu = A^a_\mu \sigma^a/2\mathrm{i}$ in Minkowski space*. The action is

$$S = -\frac{1}{4g^2} \int \mathrm{d}^4x\, F^a_{\mu\nu} F^{a\mu\nu} \tag{1}$$

where

$$F_{\mu\nu} = F^a_{\mu\nu} \frac{\sigma^a}{2\mathrm{i}} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{2}$$

Correspondingly, the field equations are

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0. \tag{3}$$

* Greek indices $\mu, \nu, \rho, \ldots$ run from 0 to 3 and are subject to the metric $g_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$. Greek indices $\alpha, \beta, \gamma \ldots$ resp. latin indices $a, b, \ldots k, l \ldots$ take on values from 1 to 4 resp. 1 to 3. Repeated indices are always summed over.

They are conformally invariant, i.e. if A_μ is a solution and $x^\mu \to x'^\mu$ is a conformal mapping in Minkowski space then

$$A'_\mu(x) = A_\nu(x'(x))\, \partial x'^\nu / \partial x^\mu$$

solves the eqs. (3), too.

The conformal group in Minkowski space is isomorphic to O(4,2). The action of an element $\Lambda \in \mathrm{O}(4,2)$ on a point x is conveniently written down by using projective coordinates for x, viz. we identify x with a ray in the cone [5]

$$C_{4,2} = \{\xi^A, A = 0, \ldots, 5 | (\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - (\xi^4)^2 + (\xi^5)^2 = 0\} \tag{5}$$

$$x^\mu = \xi^\mu/(\xi^4 + \xi^5).$$

The conformal mapping $x^\mu \to x'^\mu$ corresponding to Λ is then given by:

$$x'^\mu = \xi'^\mu/(\xi'^4 + \xi'^5); \qquad \xi'^A = \Lambda^A{}_B\, \xi^B.$$

Let us now focus on the SO(4) subgroup of O(4,2), which acts on ξ^α, $\alpha = 1, \ldots 4$, and leaves ξ^0 and ξ^5 fixed. An SO(4) adapted parametrization of the rays in $C_{4,2}$ is:

$$\xi = r(\sin\tau, n^1, n^2, n^3, n^4, \cos\tau). \tag{6}$$

This amounts to a parametrization of space-time points according to:

$$x^0 = \frac{\sin\tau}{\cos\tau + n^4}; \qquad x^k = \frac{n^k}{\cos\tau + n^4}. \tag{7}$$

The space $\widetilde{\mathrm{M}} = \{(\tau, n) | \tau \in \mathbf{R}, n \in S^3\}$ will be called "superworld". The mapping (7) is one-to-one on the subspace $\mathrm{M} = \{(\tau, n) |\ |\tau| < \pi; (\cos\tau + n^4) > 0\}$. Therefore, we may identify Minkowski space with M.

The superworld $\widetilde{\mathrm{M}}$ has previously been used in conformal quantum field theory. Its outstanding features are firstly that the universal covering of the conformal group acts on it in a differentiable manner and secondly that it allows for a conformally invariant causal ordering of its points. For details, see ref. [6].

In order to rewrite the field equations (3) in (τ, n)-coordinates it is convenient to introduce new vector potentials B_μ:

$$A_\mu \, \mathrm{d}x^\mu = B_\mu \, \omega^\mu \quad \text{where} \quad \omega^0 = \mathrm{d}\tau, \omega^k = \eta^k_{\alpha\beta} n^\alpha \, \mathrm{d}n^\beta \tag{8}$$

$\eta^k_{\alpha\beta}$ denote 't Hooft's ([7], appendix) SO(4) – covariant η-symbols, i.e.:

$$\eta^k_{\alpha\beta} = -\eta^k_{\beta\alpha}, \quad \eta^k_{l4} = \delta^k_l, \quad \eta^k_{lj} = \epsilon_{klj}. \tag{9}$$

Thus, under the conformal transformations

$$\tau' = \tau; \quad n'^\alpha = \Lambda^\alpha{}_\beta n^\beta, \quad \Lambda \in \mathrm{SO}(4)$$

the B-field transforms as: (10)

$$B'_0(\tau, n) = B_0(\tau, \Lambda n); \quad B'_k(\tau, n) = B_l(\tau, \Lambda n) R(\Lambda)^l{}_k$$

$R(\Lambda)$ is a three dimensional orthogonal representation of SO(4):

$$R(\Lambda)^k{}_l \, \eta^l_{\alpha\beta} = \eta^k_{\gamma\delta} \, \Lambda^\gamma{}_\alpha \, \Lambda^\delta{}_\beta. \tag{11}$$

Let us also define a new field tensor $G_{\mu\nu}$ by

$$F_{\mu\nu} \, \mathrm{d}x^\mu \times \mathrm{d}x^\nu = G_{\mu\nu} \, \omega^\mu \times \omega^\nu; \quad G_{\mu\nu} = -G_{\nu\mu}.$$

In terms of the vector potential this becomes:

$$\begin{aligned} G_{0k} &= \nabla_0 B_k - \nabla_k B_0 + [B_0, B_k] \\ G_{lk} &= \nabla_l B_k - \nabla_k B_l + 2\,\epsilon_{lkj} B_j + [B_l, B_k] \end{aligned} \tag{13}$$

where we have introduced covariant derivatives:

$$\nabla_0 = \partial/\partial\tau, \quad \nabla_k = \frac{1}{2}\eta^k_{\alpha\beta}\left(n^\alpha \frac{\partial}{\partial n^\beta} - n^\beta \frac{\partial}{\partial n^\alpha}\right). \tag{14}$$

They are dual to the differential forms ω^μ, i.e. for any function $f(\tau, n)$ we have:

$$\mathrm{d}f = \omega^\mu \, \nabla_\mu f.$$

Note also that ∇_k and ∇_l do not commute:

$$[\nabla_k, \nabla_l] = -2\,\epsilon_{klj} \, \nabla_j.$$

From eq. (1) we now obtain the action in the new language:

$$S = -\frac{1}{2g^2} \int_{-\pi}^{\pi} \mathrm{d}\tau \int \mathrm{d}^4 n$$

$$\times \, \delta(1 - n^\alpha n^\alpha) \, \theta(\cos\tau + n^4) \, G^a_{\mu\nu} \, G^{a\mu\nu}. \tag{15}$$

As expected, the integration runs over the subspace M of the superworld $\widetilde{\mathrm{M}}$, i.e. over a region of *finite* volume. Hence, any regular field B_μ on $\widetilde{\mathrm{M}}$ has finite action.

Finally, the action principle yields the field equations:

$$\begin{aligned} &\nabla_l G_{l0} + [B_l, G_{l0}] = 0 \\ &\nabla_0 G_{0k} + [B_0, G_{0k}] - \nabla_l G_{lk} - [B_l, G_{lk}] \\ &\quad + \epsilon_{klj} G_{lj} = 0. \end{aligned} \tag{16}$$

We now seek SO(4)-symmetric solutions to the field equations (16). Thus, we are concentrating on potentials B^a_μ satisfying:

$$\begin{aligned} B^a_0(\tau, \Lambda n) &= \pi(\Lambda)^a{}_b \, B^b_0(\tau, n) \\ B^a_k(\tau, \Lambda n) &= \pi(\Lambda)^a{}_b \, B^b_l(\tau, n) \, R(\Lambda^{-1})^l{}_k \end{aligned} \tag{17}$$

for all $\Lambda \in \mathrm{SO}(4)$. Here, $\pi(\Lambda)$ denotes some three dimensional orthogonal representation of SO(4). A careful analysis shows that all such fields are gauge equivalent to

$$B^a_0 = 0, \quad B^a_k(\tau, n) = q(\tau) \, \delta^a_k, \quad \pi(\Lambda) = R(\Lambda) \tag{18}$$

where q is an arbitrary (real) function of τ.

Let us insert the ansatz (18) into the field equations:

$$G^a_{0k} = \dot{q}\ \delta^a_k;\quad G^a_{kl} = q(q+2)\ \epsilon_{akl} \tag{19}$$

$$\ddot{q} + 2q(q+1)(q+2) = 0 \tag{20}$$

(the dots denote derivation with respect to τ). This is the equation of motion of a particle moving in the double well potential $V(q) = \frac{1}{2}q^2(q+2)^2$. There are two types of solutions of eq. (20): when the "energy"

$$\epsilon = \tfrac{1}{2}\{\dot{q}^2 + q^2(q+2)^2\} \tag{21}$$

is smaller than the bump of $V(q)$ at $q = -1$, we find

$$q = -1 \pm (1+\sqrt{2\epsilon})^{1/2}\ \mathrm{dn}\{(1+\sqrt{2\epsilon})^{1/2}(\tau-\tau_0); k_1\}$$

$$k_1^2 = 2\sqrt{2\epsilon}/(1+\sqrt{2\epsilon});\qquad \epsilon \leqslant \tfrac{1}{2} \tag{22}$$

whereas when $\epsilon > \frac{1}{2}$ the solution is:

$$q = -1 + (1+\sqrt{2\epsilon})^{1/2}\ \mathrm{cn}\{(8\epsilon)^{1/4}\ (\tau-\tau_0); k_2\} \tag{23}$$

$$k_2^2 = (1+\sqrt{2\epsilon})/2\sqrt{2\epsilon};\qquad \epsilon > \tfrac{1}{2}$$

(dn and cn denote Jacobian elliptic functions, [8 §8.14]). There is also a τ-independent, but unstable solution, namely:

$$q = \text{constant} = -1. \tag{24}$$

When translated back to x-coordinates one recovers the solution of De Alfaro et al. in a real gauge.

All the solutions above are regular functions on $\widetilde{\mathrm{M}}$. Their restriction to M therefore provides a set of regular, finite action solutions of the original equations (3). A simple expression for the energy-momentum tensor

$$\theta_{\mu\nu} = \frac{1}{g^2}\{-F^a_{\mu\lambda}F^{a\lambda}_{\nu} + \tfrac{1}{4}g_{\mu\nu}F^a_{\rho\sigma}F^{a\rho\sigma}\} \tag{25}$$

is obtained at $x^0 = 0$:

$$\theta_{\mu\nu} = 0 \text{ for } \mu \neq \nu;\ \theta_{00} = \frac{3\epsilon}{g^2}\frac{16}{(1+x^2)^4},\qquad \theta_{kl} = \delta_{kl}\frac{\epsilon}{g^2}\frac{16}{(1+x^2)^4}. \tag{26}$$

Hence, the energy of these solutions is finite, too. For the sake of completeness we finally reproduce the solutions in their x-coordinate form:

$$A^a_0 = 4q(\tau(x))\gamma^2x^0x^a;\quad \gamma = [(1+x^\mu x_\mu)^2 + 4x^2]^{-1/2} \tag{27}$$

$$A^a_k = -4q(\tau(x))\gamma^2\{\tfrac{1}{2}(1+x^\mu x_\mu)\delta^a_k + \epsilon_{akj}x^j + x^ax^k\}$$

where $q(\tau)$ is one of the functions (22), (23) or (24) and $\tau = \tau(x)$ must be calculated from

$$\sin\tau = 2x^0\gamma;\qquad \cos\tau = (1-x^\mu x_\mu)\gamma;\qquad |\tau| < \pi.$$

There is a simple connection between the mechanics of SO(4)-symmetric fields described here and the one (anti-) instanton solutions in euclidean space. In terms of (τ, n)-coordinates the euclidean equations of motion are obtained by performing the following substitutions:

$$\sigma = \mathrm{i}\tau,\quad B^{\mathrm{E}}_0 = (-\mathrm{i})B_0,\quad B^{\mathrm{E}}_k = B_k,\qquad G^{\mathrm{E}}_{0k} = (-\mathrm{i})\,G_{0k},\quad G^{\mathrm{E}}_{lk} = G_{lk}. \tag{28}$$

The variable σ ranges from $-\infty$ to $+\infty$, i.e. the euclidean action is:

$$S^{\mathrm{E}} = \frac{1}{2g^2}\int_{-\infty}^{\infty}\mathrm{d}\sigma\ \int \mathrm{d}^4n\ \delta(1-n^\alpha n^\alpha)\ G^{a\mathrm{E}}_{\mu\nu}G^{a\mathrm{E}}_{\mu\nu}. \tag{29}$$

For SO(4) symmetric fields (eq. (18)) this becomes

$$S^{\mathrm{E}} = \frac{6\pi^2}{g^2}\int \mathrm{d}\sigma\{\tfrac{1}{2}(\dot{q}^{\mathrm{E}})^2 + V(q^{\mathrm{E}})\} \tag{30}$$

and the field equations reduce to

$$\ddot{q}^{\mathrm{E}} - 2q^{\mathrm{E}}(q^{\mathrm{E}}+1)(q^{\mathrm{E}}+2) = 0.$$

The potential $V(q)$ has two minima, one for vanishing fields and one at $q = -2$, i.e. for

$$B_k = u^{-1}\nabla_k u,\quad u = n^4 - \mathrm{i}n^k\sigma^k \tag{32}$$

which is a pure gauge. The mapping $n \to u(n)$ has winding number one [9]. In quantum mechanics tunneling takes place between the two separated vacua. In leading order of $\hbar$ the amplitude for this process is given by the contribution to the euclidean path integral of those histories which interpolate between the vacua in question and that minimize the action (30) [9, 10]. These configurations are:

$$q^{\mathrm{E}}(\sigma) = -2[1 + e^{\pm 2(\sigma - \sigma_0)}]^{-1}. \tag{33}$$

In euclidean x-space these solutions turn out to be precisely the one (anti-) instanton solutions (in a singular gauge).

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