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SCATTERING OF MASSLESS LUMPS AND NON-LOCAL CHARGES IN THE TWO-DIMENSIONAL CLASSICAL NON-LINEAR σ-MODEL

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Finite-energy solutions of the field equations of the non-linear σ-model are shown to decay asymptotically into massless lumps. By means of a linear eigenvalue problem connected with the field equations we then find an infinite set of dynamical conserved charges. They, however, do not provide sufficient information to decode the complicated scattering of lumps.

1. Introduction

Recently there has been much interest in the two-dimensional non-linear σ-model [1] as a simulator for the four-dimensional pure Yang-Mills fields, both on the classical and on the quantum level. The classical field equations of the non-linear σ-model in Minkowski space have been shown to be related to partial differential equations solvable by the inverse scattering method [2]. However, these results did not yield a clear understanding of the classical model in physical terms due to the complicated form of the local charges which were derived with the help of cumbersome "normal coordinates" in Minkowski space.

In this paper we give an account of the classical field equations without relying on normal coordinates. We first analyze energy-momentum conservation and show that a generic solution decomposes into a set of massless lumps for large times. The problem of how to (exactly) linearize the field equations is attacked in sect. 3. Although we are not able to find a transformation to action and angle variables, our results are good enough to provide an infinite set of conserved charges (sect. 4). They are non-local but nevertheless satisfy an (non-Abelian) addition law.

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2. Basic facts about the non-linear σ-model

The two-dimensional O(n) σ-model describes the motion of a string of n-dimensional classical spins $q^a(t,x)$, $a = 1, ..., n$ of unit length: $q^a q^a = 1$ ★. The action is

$$S = \tfrac{1}{2}\int d^2x\ \partial_\mu q^a\ \partial^\mu q^a\ , \tag{1}$$

giving rise to the field equations

$$\partial_\mu \partial^\mu q^a + (\partial_\mu q^b\ \partial^\mu q^b) q^a = 0\ , \qquad q^b q^b = 1\ . \tag{2}$$

The invariance of the action under dilatations implies that the energy-momentum tensor $\Theta_{\mu\nu}$ is traceless:

$$\Theta_{\mu\nu} = \partial_\mu q^a\ \partial_\nu q^a - \tfrac{1}{2} g_{\mu\nu}\ \partial_\lambda q^a\ \partial^\lambda q^a\ , \tag{3}$$

$$\Theta^\mu{}_\mu = 0\ , \qquad \Theta_{\mu\nu} = \Theta_{\nu\mu}\ . \tag{4}$$

Therefore, $\Theta_{\mu\nu}$ has only two independent components

$$\Theta_\xi = \tfrac{1}{2}(\Theta_{00} + \Theta_{01})\ , \qquad \Theta_\eta = \tfrac{1}{2}(\Theta_{00} - \Theta_{01})\ , \tag{5}$$

representing the density of energy flowing from the right to the left and from the left to the right, respectively. Energy-momentum conservation says that

$$\frac{\partial}{\partial\eta}\Theta_\xi = 0\ , \qquad \frac{\partial}{\partial\xi}\Theta_\eta = 0\ , \tag{6}$$

where we have introduced light-cone coordinates defined by

$$\xi = \tfrac{1}{2}(x^0 + x^1)\ , \qquad \eta = \tfrac{1}{2}(x^0 - x^1)\ . \tag{7}$$

Hence, Θ_ξ depends on ξ only. In other words, the energy flowing from right to left runs with exactly the speed of light and does not dissipate.

The meaning of energy-momentum conservation is made most transparent when considering the following situation. Assume that at $t = 0$ all the energy-momentum is concentrated on a compact interval $x \in [\alpha, \beta]$, i.e. $(\partial_\mu q)(t = 0, x) = 0$ outside $[\alpha, \beta]$ (see fig. 1).

Eq. (6) now implies that $\Theta_{\mu\nu}$ is supported as indicated in fig. 1. We thus see that after some time has elapsed the spin string has separated into two lumps moving away from each other with the velocity of light. Let us look more closely at the lump running to the right. There,

$$\Theta_\xi = \frac{\partial}{\partial\xi} q^a \frac{\partial}{\partial\xi} q^a = 0$$

★ Notation: Greek indices $\mu, \nu, ...$ run from 0 to 1, Latin ones $a, b, ...$ from 1 to n. The metric is $g_{00} = -g_{11} = 1$ and the summation convention is implied.

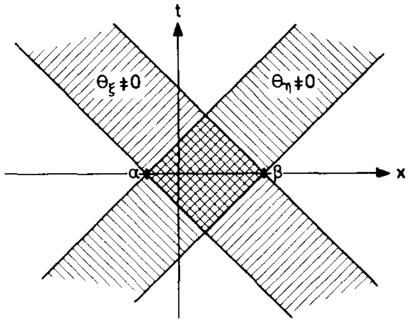

Fig. 1. The support of $\Theta_{\mu\nu}$ when $(\partial_\mu q)(0, x) = 0$ for $x \notin [\alpha, \beta]$.

and hence

$$\frac{\partial}{\partial \xi} q^a = 0 \, .$$

Therefore, as t increases, this spin string is just shifted to the right but does not change its shape.

To sum up, we have found that the two-dimensional σ-model describes massless lumps which scatter. These lumps are characterized by $(\partial/\partial\xi)q^a = 0$ and $(\partial/\partial\eta)q^a = 0$ if they move to the right and to the left, respectively. When two lumps scatter they will deform each other. These deformations reflect the non-trivial dynamics of the model.

Note finally that the lumps discussed here fit into Coleman's [3] definition of a classical lump. However, our lumps are of an essentially kinematic nature and are not a manifestation of an attractive force between elementary waves.

3. Derivation of a family of linear eigenvalue problems associated with the equations of motion

Let us first consider the case $n = 3$. We start from the following set of compatible equations [2]:

$$\frac{\partial}{\partial \xi} \mathcal{O}^{(\gamma)} = (1 - \gamma^{-1})\left(q \times \frac{\partial}{\partial \xi} q\right)^a I^a \mathcal{O}^{(\gamma)} \, ,$$

$$\frac{\partial}{\partial \eta} \mathcal{O}^{(\gamma)} = (1 - \gamma)\left(q \times \frac{\partial}{\partial \eta} q\right)^a I^a \mathcal{O}^{(\gamma)} \, ,$$

$$\mathcal{O}^{(\gamma)\mathrm{T}} \cdot \mathcal{O}^{(\gamma)} = \mathcal{O}^{(\gamma)} \cdot \mathcal{O}^{(\gamma)\mathrm{T}} = \mathbb{1} \tag{8}$$

Here, the symbol γ denotes a complex constant parameter different from zero. I^a, $a = 1, 2, 3$ stand for the antihermitian infinitesimal generators of the rotation group

O(3) ★. $\mathcal{O}^{(\gamma)\mathrm{T}}$ is identical with the rotation matrix $\mathcal{R}^{(\gamma)}(\cdot, q)$ of ref. [2].

For the spin-$\frac{1}{2}$ representatives $U^{(\gamma)}$ of the space-time dependent rotations $\mathcal{O}^{(\gamma)}$, eqs. (8) become

$$\frac{\partial}{\partial\xi} U^{(\gamma)} = (1-\gamma^{-1})\left(q \times \frac{\partial}{\partial\xi} q\right)^a \frac{\sigma^a}{2i} U^{(\gamma)} ,$$

$$\frac{\partial}{\partial\eta} U^{(\gamma)} = (1-\gamma)\left(q \times \frac{\partial}{\partial\eta} q\right)^a \frac{\sigma^a}{2i} U^{(\gamma)} ,$$

$$U^{(\gamma)} \cdot U^{(\gamma^*)+} = U^{(\gamma^*)+} \cdot U^{(\gamma)} = \mathbb{1} , \tag{9}$$

where σ^a, $a = 1, 2, 3$ denote the three Pauli matrices. The compatibility of these linear equations for two values of the eigenvalue parameter γ different from one (and the initial data for the field vector q) imply the equations of motion (2) and *vice versa*. Hence, the above set of equations constitutes the desired family of linear eigenvalue problems closely related to the equations of motion.

In space and time coordinates (x, t), eqs. (9) read

$$\frac{\partial}{\partial t} U = \frac{w}{1-w^2} \{(q_x \times q)^a - w(\dot{q} \times q)^a\} i\sigma^a U , \tag{10a}$$

$$\frac{\partial}{\partial x} U = \frac{w}{1-w^2} \{(\dot{q} \times q)^a - w(q_x \times q)^a\} i\sigma^a U , \tag{10b}$$

where (for later convenience) we have set

$$w = \frac{\gamma - 1}{\gamma + 1} , \qquad \dot{q} = \frac{\partial}{\partial t} q , \qquad q_x = \frac{\partial}{\partial x} q ,$$

and the dependence of U on γ and on w has been suppressed.

In the following, we shall restrict our attention to situations for which

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \{(\dot{q}^a \dot{q}^a)^{1/2} + (q_x^a q_x^a)^{1/2}\} < \infty . \tag{11}$$

If this condition is met at one time, then, due to eq. (6), it holds at all times. It implies the time independence of $U(t, \pm\infty)$ and $q(t, \pm\infty) \equiv q(\pm\infty)$. We may therefore normalize $U(t, -\infty)$ to $\mathbf{1}$, the 2×2 unit matrix. Having thus fixed the integration constants in eq. (10), $U^{(\gamma)}(t, x)$ becomes a uniquely defined functional of q. It is then easy to prove that

$$\det U^{(\gamma)}(t, x) = 1 ,$$

$$U^{(-\gamma)}(t, x) = \not{q}(t, x)\, U^{(\gamma)}(t, x)\, \not{q}(-\infty) ,$$

★ The symbols T, +, *, the latter ones to be used later, indicate transposition, hermitian conjugation and complex conjugation, respectively.

$$U^{(1)}(t,x) = \mathbb{1}\,, \qquad U^{(-1)}(t,x) = \not{q}(t,x)\,\not{q}(-\infty)\,, \tag{12}$$

where $\not{q} = q^a\sigma^a$. The last relation allows the reconstruction of q, the solution vector of the equations of motion, from the special unitary matrix $U^{(-1)}$ (and $q(-\infty)$).

From the matrices $U^{(\gamma)}$ we can obtain two families of new solutions of the field equations (2) by simple algebraic manipulations. The first of these families is defined by [2]

$$q^{(\gamma)} = \mathcal{O}^{(\gamma)\mathrm{T}} \cdot q\,, \tag{13}$$

i.e.

$$\not{q}^{(\gamma)} = (U^{(\gamma)}U_0^{(\gamma)})^{-1}\,\not{q}\,(U^{(\gamma)}U_0^{(\gamma)})\,, \tag{14}$$

where $U_0^{(\gamma)}$ is a constant SU(2) matrix.

The second family consists of the Bäcklund transforms $q^{(\gamma)'}$ of $q^{(\gamma)}$. Up to an integration constant (which can be fixed by specifying $q^{(\gamma)'}(t, -\infty) \equiv q^{(\gamma)'}(-\infty)$, $q^{(\gamma)'}(-\infty) \perp q^{(\gamma)}(-\infty)$), $q^{(\gamma)'}$ is determined by [2]

$$\frac{\partial}{\partial\xi}(q^{(\gamma)'} + q^{(\gamma)}) \parallel (q^{(\gamma)'} - q^{(\gamma)})\,,$$

$$\frac{\partial}{\partial\eta}(q^{(\gamma)'} - q^{(\gamma)}) \parallel (q^{(\gamma)'} + q^{(\gamma)})\,,$$

$$(q^{(\gamma)'a}q^{(\gamma)a}) = 0\,, \qquad (q^{(\gamma)'a}q^{(\gamma)'a}) = 1\,. \tag{15}$$

$q^{(\gamma)'}$ can then be constructed from $U^{(\gamma)}$ and $U^{(\pm i\gamma)}$:

$$\not{q}^{(\gamma)'} = W^{(\gamma)}\not{p}^{(\gamma)}(-\infty)\,W^{(\gamma)-1}\,, \tag{16}$$

with

$$\not{p}^{(\gamma)}(-\infty) = U_0^{(\gamma)}\not{q}^{(\gamma)'}(-\infty)\,U_0^{(\gamma)-1}\,,$$

$$W^{(\gamma)} = (U^{(\gamma)}U_0^{(\gamma)})^{-1}\,V^{(\gamma)}\,,$$

$$V^{(\gamma)} = \tfrac{1}{2}(U^{(i\gamma)} + U^{(-i\gamma)})\,\not{p}^{(\gamma)}(-\infty) - \frac{1}{2i}\,(U^{(i\gamma)} - U^{(-i\gamma)})\,\not{q}(-\infty)\,. \tag{17}$$

Thus we see that the eigenvalue problems (9) provide a linearization of the Bäcklund transformation (15) *.

Let us now turn to the general case $n \geqslant 3$. As before, we start from the set of compatible equations

$$\frac{\partial}{\partial\xi}\,\mathcal{O}^{(\gamma)} = (1 - \gamma^{-1})\,q^a\,\frac{\partial}{\partial\xi}\,q^b\,\Gamma^{ab}\mathcal{O}^{(\gamma)}\,,$$

* For the expert, we note that eq. (9) could have been obtained from the Bäcklund transformation (17) *via* a pair of matrix Riccati equations.

$$\frac{\partial}{\partial \eta} \mathcal{O}^{(\gamma)} = (1-\gamma)\, q^a \frac{\partial}{\partial \eta} q^b I^{ab} \mathcal{O}^{(\gamma)} ,$$

$$\mathcal{O}^{(\gamma)\mathrm{T}} \mathcal{O}^{(\gamma)} = \mathcal{O}^{(\gamma)} \mathcal{O}^{(\gamma)\mathrm{T}} = \mathbb{1} , \qquad (18)$$

where $I^{ab} = -I^{ba}$ $(a, b = 1, ..., n)$ denote the infinitesimal generators of the group $\mathrm{O}(n)$ for rotations in the (a, b) plane. Again, $\mathcal{O}^{(\gamma)\mathrm{T}}$ coincides with the rotation matrix $\mathcal{R}^{(\gamma)}(\cdot, q)$ of ref. [2].

Let Γ^a, $a = 1, ..., n$, stand for the lowest-dimensional matrix representation of the basis elements of the Clifford algebra [4]

$$\{\Gamma^a, \Gamma^b\}_+ = 2\delta^{ab} , \qquad (19)$$

and let the symbol [,] denote the commutator. The Lie algebra with basis $J^{ab} = -\frac{1}{4}[\Gamma^a, \Gamma^b]$ is a representation of the Lie algebra of the group $\mathrm{O}(n)$. The corresponding representatives $U^{(\gamma)}$ of the space-time dependent rotations $\mathcal{O}^{(\gamma)}$ satisfy the following equations:

$$\frac{\partial}{\partial \xi} U^{(\gamma)} = (1-\gamma^{-1})\, q^a \frac{\partial}{\partial \xi} q^b J^{ab} U^{(\gamma)} ,$$

$$\frac{\partial}{\partial \eta} U^{(\gamma)} = (1-\gamma)\, q^a \frac{\partial}{\partial \eta} q^b J^{ab} U^{(\gamma)} ,$$

$$U^{(\gamma^*)+} U^{(\gamma)} = U^{(\gamma)} U^{(\gamma^*)+} = \mathbb{1} . \qquad (20)$$

As in the O(3) case, these equations constitute a family of linear eigenvalue problems closely associated with the equations of motion. Imposing the condition (11) we may require that $U^{(\gamma)}(t, -\infty) = \mathbb{1}$, the $2^{[n/2]} \times 2^{[n/2]}$ unit matrix ($[\lambda]$ denotes the largest integer less than or equal to λ).

With the appropriate changes, in particular

$$\not{q} = q^a \Gamma^a , \qquad (21)$$

eqs. (12)–(17) remain valid. Thus, the solution vectors q, $q^{(\gamma)}$, $q^{(\gamma)'}$ can be constructed from the matrices $U^{(\pm\gamma)}$, $U^{(\pm i\gamma)}$ (and $q(-\infty)$, $q^{(\gamma)}(-\infty)$ and $q^{(\gamma)'}(-\infty)$, respectively).

4. Derivation of an infinite set of constants of motion

For simplicity we again concentrate on the O(3) case. Also, $q(t, x)$ will always be assumed to fulfill condition (11).

At any given fixed time t, the "scattering problem", eq. (10b) and the boundary condition $U(t, -\infty) = \mathbb{1}$ define $U(t, x)$ uniquely for any given Cauchy data $q^a(t, x)$, $\dot{q}^a(t, x)$. Especially, for any value of the "spectral" parameter w, $w \neq \pm 1$,

$$Q(w) = U(t, \infty) \qquad (22)$$

is a well defined functional of q^a, $\dot{q}^a$, which, by eq. (11), is time-independent. Furthermore, it can be expanded in a power series in w, thus leading to an infinite set of conserved charges:

$$Q(w) = \sum_{n=0}^{\infty} w^n Q_n \,, \qquad \frac{\mathrm{d}}{\mathrm{d}t} Q_n = 0 \,. \tag{23}$$

Recall that $Q(w)$ is a 2×2 matrix, i.e.

$$Q_n = Q_n^0 + Q_n^a i\sigma^a \,. \tag{24}$$

Under O(3) rotations Q_n^0 behaves as a scalar and Q_n^a as a vector.

A more explicit representation of the charges is obtained by rewriting the defining differential equation as an integral equation:

$$U(t,x) = \mathbb{1} + \frac{w}{1-w^2} \int_{-\infty}^{x} \mathrm{d}y \,\{(\dot{q} \times q)^a(t,y) - w(q_y \times q)^a(t,y)\}\, i\sigma^a U(t,y). \tag{25}$$

Inserting the expansion

$$U(t,x) = \sum_{n=0}^{\infty} w^n U_n(t,x) \,, \tag{26}$$

we get the recurrence relation

$$\begin{aligned} U_n(t,x) = \int_{-\infty}^{x} \mathrm{d}y \,\{&(\dot{q} \times q)^a(t,y)\, i\sigma^a \sum_{0 \leqslant k \leqslant (n-1)/2} U_{n-2k-1}(t,y) \\ &- (q_y \times q)^a(t,y)\, i\sigma^a \sum_{1 \leqslant l \leqslant n/2} U_{n-2l}(t,y)\} \qquad (n \geqslant 1) \,, \\ U_0(t,x) = \mathbb{1} \,. & \end{aligned} \tag{27}$$

Thus, we can calculate the coefficients $U_n(t,x)$, and hence the charges Q_n, recursively. They are a sum of k-fold ($k \leqslant n$) ordered integrals. For example, we find

$$Q_1^0 = 0 \,, \qquad Q_1^a = \int \mathrm{d}y \,(\dot{q} \times q)^a(t,y) \,, \tag{28}$$

$$Q_2^0 = -\tfrac{1}{2} Q_1^a Q_1^a \,, \tag{29}$$

$$\begin{aligned} Q_2^a = &-\int \mathrm{d}y_1 \,\mathrm{d}y_2 \,\Theta(y_1 - y_2)\,\epsilon^{abc} (\dot{q} \times q)^b(t,y_1)(\dot{q} \times q)^c(t,y_2) \\ &- \int \mathrm{d}y \,(q_y \times q)^a(t,y) \,. \end{aligned} \tag{30}$$

Q_1^a is just the generator of O(3) rotations, whereas Q_2^a generates a non-local symmetry.

The fact that Q_2^0 can be expressed through the vector charges is not accidental. Indeed, by eq. (12), $\det U(t,x) = 1$, and therefore

$$\det Q(w) = (Q^0(w))^2 + Q^a(w)\,Q^a(w) = 1 \ .$$

In other words, we have

$$Q^0(w) = (1 - Q^a(w)\,Q^a(w))^{1/2} \ , \tag{31}$$

which, upon expansion in w, yields Q_n^0 as a polynomial of Q_m^a, $m \leqslant n-1$.

Our charges Q_n $(n \geqslant 2)$ cannot be written as an integral over a *local* charge density. To illustrate this fact, let us consider a spin string $q^a(t,x)$, which, at some time t_0, is composed of two separated substrings $l^a(t,x)$ and $r^a(t,x)$, i.e.

$$q^a(t_0,x) = \begin{cases} l^a(t_0,x) & (x \leqslant \alpha) \\ r^a(t_0,x) & (x \geqslant \alpha) \ , \end{cases}$$

$$\dot{q}^a(t_0,x) = \begin{cases} \dot{l}^a(t_0,x) & (x \leqslant \alpha) \\ \dot{r}^a(t_0,x) & (x \geqslant \alpha) \ , \end{cases}$$

where $(\partial_\mu l^a)(t_0,x) = 0$ $(x > \alpha)$ and $(\partial_\mu r^a)(t_0,x) = 0$ $(x < \alpha)$. Due to the non-locality of Q_n we then find that, in general,

$$Q_n[q] \neq Q_n[l] + Q_n[r] \ . \tag{32}$$

Here, $Q_n[s]$ denotes the nth charge evaluated for the spin string $s^a(t,x)$.

Nevertheless, we still have an addition law; for, as is easily seen from eq. (10b), the respective generating functionals $Q[q](w)$, $Q[l](w)$ and $Q[r](w)$ satisfy:

$$Q[q](w) = Q[r](w) \cdot Q[l](w) \tag{33}$$

Expanding in powers of w we get, for example,

$$Q_1^a[q] = Q_1^a[r] + Q_1^a[l] \ ,$$

$$Q_2^a[q] = Q_2^a[r] + Q_2^a[l] - \epsilon^{abc} Q_1^b[r]\,Q_1^c[l] \ . \tag{34}$$

In view of the non-Abelian composition law (33) it is sensible to look more closely at the charges carried by special spin strings $q^a(t,x)$. Let us for example consider a massless lump moving to the right (sect. 2). Eq. (10b) then reduces to

$$\frac{\partial}{\partial x} U(t,x) = -\frac{w}{1-w}\,(q_x \times q)^a(t,x)\, i\sigma^a U(t,x) . \tag{35}$$

For lumps, $Q(w)$ has a simple geometric interpretation. Namely, eq. (35) tells us that $Q(w)$ is the product of all the infinitesimal rotations (t is fixed)

$$R(x) = \mathbb{1} - \frac{w}{1-w}\,\omega^a(x)\, i\sigma^a \ ,$$

$$\omega^a(x) = (q_x \times q)^a(t,x)\,\mathrm{d}x = (\mathrm{d}q \times q)^a(t,x)\,.$$

From this picture it is obvious that there exist non-trivial lumps with $Q(w) = \mathbb{1}$ for all w. Indeed, this happens always if $q^a(t,x)$ runs back the same curve when x goes from x_0 to $+\infty$, as it moved along when x increased from $-\infty$ to x_0. We therefore conclude that we cannot construct a complete set of integrals of motion in involution (i.e. the invariant manifold in the phase space for q^a) from the charges Q_n^a and the energy-momentum tensor $\Theta_{\mu\nu}$ alone.

The statement above suggests that there are more constants of motion. One might speculate that these can be obtained by forming Poisson brackets among the old charges. Unfortunately, due to non-vanishing boundary terms, the Poisson bracket between say Q_n^a and Q_m^b is not unambiguously defined through the fundamental brackets.

In order to get a safe definition of a Poisson bracket $\{Q_n^a, Q_m^b\}$ let us introduce volume cutoff charges $Q_n^{L,a}$. These are the same as Q_n^a where, however, the multiple integrals involved range only from $-L$ to $+L$. Then the limit

$$\{Q_n^a, Q_m^b\} = \lim_{L_2\to\infty} \big(\lim_{L_1\to\infty} \{Q_n^{L_1,a}, Q_m^{L_2,b}\}\big) \tag{36}$$

is well-defined. Had we first taken the $L_2 \to \infty$ limit, the outcome would differ from the above by a polynomial in Q_l^c, $l \leqslant n+m-2$, and $q(\pm\infty)$.

By a rather lenghty argument (it will be omitted here) one can show that $\{Q_n^a, Q_m^b\}$ is a polynomial in Q_l^c, $l \leqslant n+m-1$ and $q(\pm\infty)$. For example, we found that

$$\{Q_2^a, Q_m^b\} = -\epsilon^{abc} Q_{m+1}^c + P_m^{a,b}\,, \tag{37}$$

where $P_m^{a,b}$ is a combination of Q_l^c, $l \leqslant m$, and $q(\pm\infty)$. Thus, we cannot produce new constants of motion this way.

We remark finally that a simple interpretation (such as particle-number conservation etc.) of our charges is lacking so far. Of course, this is due to the fact that classical spinwaves do not decay into a superposition of Abelian waves for large times.

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