IN QUEST OF A RELATIVISTIC CONSTITUENT QUARK MODEL – SOME CONSTRUCTIVE REMARKS

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The set-up of a relativistic constituent quark model in four dimensions is one of the outstanding problems in particle physics. For the time being this involves a great deal of model building which, very probably, will not change in the near future. In this paper we shall offer some general remarks which might help putting such models into shape. Most of the earlier attempts are found controversial. In particular, a convential quark constituent interpretation could not be recovered.

The most orthodox tool for discussing the relativistic bound state problem in quantum field theory is the Bethe–Salpeter (BS) equation. The BS equation is the canonical relativistic, field theoretical counterpart of the Schrödinger equation. This is to say that it provides a unique laboratory for extending and incorporating the nonrelativistic dynamical models, such as the charmonium picture, into a fully relativistic particle dynamics.

Nowadays, in the age of QCD, the BS equation has, however, become a highly complex vehicle, though it still bears some great advantages over other approaches. It has been emphasized that the relativistic bound state problem being faced in QCD is afflicted with two coupled integral equations [‡1] [1,2]:

$$\Delta_F^{-1}(q) = Z(q^2 - m^2) - i\int d^4k K(q,k;P=0)\Delta_F(k)\,, \quad (1)$$

$$\Delta_F^{-1}(\tfrac{1}{2}P+q)\,\Delta_F^{-1}(\tfrac{1}{2}P-q)\phi_P(q) = i\int d^4k K(q,k;P)\phi_P(k)\,, \quad (2)$$

rather than simply the BS equation in the ladder approximation. It is to be expected that Δ_F undergoes a drastic change from the free propagator in the presence of long-range (confining) forces. Eq. (1) is the so called Schwinger–Dyson (SD) equation which, in practice, may raise some difficulties as it is nonlinear. A possible confinement scheme is to demand that Δ_F has no discontinuity in the mass squared variable, i.e., is an entire function. [3,4].

Eq. (1) is derived from the BS equation for the vertex (see e.g. ref. [5])

$$\Gamma_\mu(q,P) = Z\Gamma_\mu^{(0)}(q,P) - i\int d^4k K(q,k;P)\Delta_F(\tfrac{1}{2}P+q)\Delta_F(\tfrac{1}{2}P-q)\Gamma_\mu(k,P)\,, \quad (3)$$

and the Ward identity

$$\Gamma_\mu(q,0) = (\partial/\partial q_\mu)\Delta_F^{-1}(q)\,. \quad (4)$$

As it stands eq. (1) holds for convolution-type BS kernels [‡2]. It can easily be generalized as to include any given BS kernel.

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[‡1] For simplicity we shall assume spin-zero quarks and restrict to quark–antiquark bound states. We have omitted the inhomogeneous term from the BS equation as demanded by confinement.

[‡2] Including those of ref. [2].

The main advantage of the BS scheme over other approaches, such as the bag models [6–8], lies in its predictivity. Once the solutions to eqs. (1) and (2) are known it is straightforward to calculate currents, structure functions, etc., similar to two-dimensional QCD. In order to appreciate this it should be recalled that, e.g., the MIT bag meets with great difficulties already in the derivation of the form factor [9].

The BS approach has been realized in two-dimensional QCD [10,11][‡3]. So far it has not been seriously attempted to build a constituent quark model along the lines of eqs. (1) and (2) which has some phenomenological relevance. There exist, however, some promising steps in that direction [2] which, together with two-dimensional QCD, oppose to the widespread prejudice that a fully relativistic quark model cannot be accomodated within conventional field theory.

In four dimensions it is generally believed that confinement comes about because the quark–gluon coupling constant $g(t)$ goes to infinity at large distances. This is equivalent to an assumption on the strong coupling behavior of the Callan–Symanzik function $\beta(g)$. In this spirit the BS kernel may eventually be approximated by a few effective low-order gluon exchange diagrams. For the time being this leaves, of course, a great deal of freedom for model building since we do not know $\beta(g)$ explicitly but only have some gross idea of its functional behaviour.

But it will soon become clear that not any BS kernel one might be led to is physically sensible. The SD equation proves here to be the real clue to the confinement problem. It not only confines the class of possible BS kernels but brings in a new dimension into the search for a relativistic quark dynamics. Most of the "relativistic models" being proposed miss this point completely, and it seems to us that there is a lot of confusion in the literature of what really is a *relativistic* constituent quark model.

For the purpose of illustration and for historical reasons [14,15] let us take the BS kernel to be of the form

$$K(q,k;P) \sim g^2(t)/t\,,\qquad g(t) \underset{t\to 0}{\simeq} \sqrt{\beta}/t\,,\quad t=(g-k)^2\,, \tag{5}$$

which we will regularize in the usual way:

$$K(q,k;P) = \operatorname*{Res}_{\lambda=3} \beta/t^{\lambda} = i\beta\,\Box\,\delta^4(q-k)\,. \tag{6}$$

This is the relativistic analogue of the harmonic oscillator potential. The SD equation has been solved for this interaction kernel in ref. [1].

Let us now assume, in close analogy to the nonrelativistic Schrödinger picture and following the main stream of relativistic approaches (such as the Dirac equation, the covariant parton model, bag models, etc.) which treat the quark as a real particle (independent coordinate), that the quark propagator be of the form

$$\Delta_F(q) = \frac{1}{Z}\,\frac{1}{q^2-m^2} + \text{something}\,. \tag{7}$$

In ref. [1] this type of solution has not been found possible. But, for the moment, we shall adopt a purely phenomenological point of view rather than going in for field theoretical consistency. After all, eqs. (6), (7) and (2) reflect that many authors understand by a relativistic quark model.

The question is, if such a heuristic picture is (in fact) good for a (relativistic) constituent quark model. It has been argued that the quark propagator should not have any singularities in the mass squared variable [1–3]. But in some sense this is also true for the Schrödinger equation which we know does lead to confinement for the oscillator potential [16]. The quark propagator may also not be of primary significance because of its gauge dependence.

Therefore, let us investigate the bound state equation. For the BS kernel (6) and propagator (7) this simplifies to

$$\left[\frac{\partial^2}{\partial q_0^2} - \frac{\partial^2}{\partial|\boldsymbol{q}|^2} - \frac{2(l+1)}{|\boldsymbol{q}|}\frac{\partial}{\partial|\boldsymbol{q}|} - \frac{Z^2}{\beta}(q_+^2-m^2)(q_-^2-m^2)\right]u(q_0,|\boldsymbol{q}|)=0\,, \tag{8}$$

$$\phi_P(q_0,\boldsymbol{q}) = |\boldsymbol{q}|^l\,Y_{lm}(\theta,\phi)\,u(q_0,|\boldsymbol{q}|)\,,$$

where $q_\pm^2 = (q_0 \pm \frac{1}{2}M)^2 - \boldsymbol{q}^2$, $M^2=P^2$. We have only kept the pole term of the quark propagator which, however, will do the job. The normalization condition reads

[‡3] Recently, this has been criticized in refs. [12,13].

$$\int \mathrm{d}q_0 \mathrm{d}|\boldsymbol{q}||\boldsymbol{q}|^2 \bar{u}(q_0,|\boldsymbol{q}|)(q_\pm^2 - m^2) u(q_0,|\boldsymbol{q}|) < \infty , \quad (9)$$

which can be rewritten in the form

$$\int \mathrm{d}q_0 \mathrm{d}|\boldsymbol{q}||\boldsymbol{q}|^2 \bar{u}(q_0,|\boldsymbol{q}|) \frac{1}{q_\mp^2 - m^2} \times \left[\frac{\partial^2}{\partial q_0^2} - \frac{\partial^2}{\partial |\boldsymbol{q}|^2} - \frac{2(l+1)}{|\boldsymbol{q}|} \frac{\partial}{\partial |\boldsymbol{q}|} \right] u(q_0,|\boldsymbol{q}|) < \infty . \quad (10)$$

Under no further assumption than that $u(q_0, |\boldsymbol{q}|)$ be normalizable, eq. (10) can be partially integrated to give

$$\int \mathrm{d}q_0 \mathrm{d}|\boldsymbol{q}| \left\{ \left(\frac{\partial^2}{\partial q_0^2} - \frac{\partial^2}{\partial |q|^2} + \frac{\partial}{\partial |\boldsymbol{q}|} \frac{2(l+1)}{|\boldsymbol{q}|} \right) \times \frac{|\boldsymbol{q}|^2}{q_\mp^2 - m^2} \bar{u}(q_0, |\boldsymbol{q}|) \right\} u(q_0, |\boldsymbol{q}|) < \infty . \quad (11)$$

This tells us that $u(q_0, |\boldsymbol{q}|)$ must vanish at $q_+^2 = m^2$ and $q_-^2 = m^2$.

Let us now examine the solutions of eq. (8) in the vicinity of ‡4 $q_+^2 = m^2$ and $q_-^2 = m^2$. Since we are looking for a regular solution we may write

$$u(q_0, |\boldsymbol{q}|) = \sum_{m,n=0}^{\infty} a_{mn} (q_+^2 - m^2)^m (q_-^2 - m^2)^n , \quad (12)$$

where $a_{mn} = a_{nm}$. By comparison of coefficients it is then easy to show (but what we do not have space to carry out explicitly) that any regular solution must be nonvanishing for $q_+^2 \to m^2$ and/or $q_-^2 \to m^2$. If eq. (12) should vanish all coefficients must be zero. This is to say that any vanishing solution, and hence any normalizable solution, must have branch points at $q_+^2 = m^2$ and $q_-^2 = m^2$ in complete accordance with conventional field theory.

Consequently, eq. (8) will not admit any bound states above $M = 2m$. Such states would be unstable as in the case of conventional BS kernel, despite the long-range forces and in contrast to confinement. In other words, it is not true relativistically that the harmonic oscillator has a linear Regge spectrum.

This result is not an artifact of our somewhat special BS kernel but holds true quite generally, at least we could not find any counterexample among BS kernels which appeared to us realistic. The true BS kernel will probably be less singular such that the infrared singularities can be controlled uniquely ($K \sim 1/t^2$). In this case our result will be all the more valid. Had we assumed that $\Delta_F(q)$ has a cut starting at $q^2 = m^2$, and if we think conventional there should be a cut associated with the quark propagator (see also ref. [13] and our following discussion), this result would be immediate.

It is to be noted that the situation is completely different in two-dimensional QCD [11]. Here the interaction is instantaneous and the BS equation reduces to a Schrödinger-like equation which, as in the nonrelativistic case, may be consistent with confinement. But it is not clear if this has any relevance for the real world.

Our conclusion is that there is really no basis for a relativistic constituent quark model with *quasi-free constituent motion,* ‡5,6 not mentioning its field theoretical inconsistency. This also questions the use of all the semi-relativistic derivatives of the BS equation such as, e.g., the Blankenbecler–Sugar equation. The Schrödinger equation, which gives a rather successful account of the spectroscopy of the new particles, will no sooner have a simple interpretation in terms of on-mass-shell quarks but must have a rather sophisticated dynamical origin. In general, we can say that the meaning of the "quark mass" entering in most of the approaches is rather obscure.

Let us now drop eq. (7) and consider the true quark propagator as being given by the SD equation. This leads to ‡7

$$\Delta_F(q) = (1/\sqrt{3}\beta)\sqrt{-q^2} , \quad (13)$$

which is to be interpreted as a coherent quark–gluon state with zero quark mass. It is a simple exercise to show that the inhomogeneous term in the BS equation (cf. eq. (4.2) of ref. [2]), which is connected with the continuum of free quark states, vanishes for the prop-

‡4 In order that both quarks can be "on-mass-shell" we must demand $M^2 \geqslant 4m^2$.

‡5 It is to be stressed that it is the pole of the quark propagator which led to out result.

‡6 Including the covariant parton model [17].

‡7 The constant propagator does not lead to an eigenvalue equation. Note that there is a misprint in eq. (3.15) of ref. [1] which is corrected here.

agator (13), similar to the constant propagator and in contrast to eq. (7). So free quark states are truly absent from the spectrum. But this does not mean yet confinement.

The BS equation for this, so to speak, "true" relativistic harmonic oscillator has been discussed in great detail in ref. [18]. It is found that there exist no bound states at all. So the BS wave function will describe a continuum of coherent quark–gluon states.

The result is not surprising anymore since the propagator (13) can be represented as a superposition of free propagators with mass (m) extending down to zero. In this light the result appears again not to be restricted to our special example so that, quite generally, there will be no confinement for quark propagators having a Lehmann representation.

So we have to demand that Δ_F be an entire function. This has been suggested a long time before, but we have seen that this is not only *sufficient* but also *necessary* for confinement. The most simple entire function we can imagine is $\Delta_F(q)$ = const. which also has a nice physical interpretation:

$$\Delta_F(x-z) \sim \delta^4(x-z). \tag{14}$$

Another choice would be a polynomial (in q^2). This has, however, to be discarded because it brings in an untolerable singularity in the electromagnetic vertex ‡8 via the Ward identity (4). For the same reason we cannot allow an essential singularity at infinity so that only the constant propagator remains ‡9. In the more realistic case of spin-1/2 quarks this is not necessarily true. Here [20] a first-order polynomial (in $\not q$) is possibly allowed because of the traces to be taken.

This reduces the number of possible BS kernels in two respects. First of all, the BS kernel has, of course, to be consistent with (in case of spinless quarks) the constant propagator and, secondly the BS equation must give rise to an eigenvalue equation. The latter is not automatically the case. The BS kernel (6), e.g., admits a constant propagator but does not lead to an eigenvalue equation. In this respect strict convolution-type BS kernels must be discarded. If the BS kernel is phrased in terms of an effective one-gluon exchange the "gluon" must have spin one. But even this P^2 dependence is not sufficient for a reasonable spectrum as can be inferred from ref. [2]. What else can come up for any further P^2 dependence? It seems to us that *hadronic* intermediate states may be the answer. These also might simulate the quark mass singularities appearing in the nonrelativistic Schrödinger picture.

‡8 And, hence, in the vacuum polarization, etc.

‡9 That this is compatible with nonzero charges has been demonstrated in ref. [19]. Cf. also for its internal consistency.

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