Nuclear Physics B136 (1978) 461–474

BIANCHI IDENTITIES FOR SUPERSYMMETRIC GAUGE THEORIES

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Received 5 December 1977
(Revised 19 January 1978)

The Bianchi identities for gauge theories in an extended flat superspace are evaluated. They permit better understanding of possible constraint equations, and can serve as a starting point for further constructions of gauge theories with extended supersymmetry.

1. Introduction

Supersymmetric gauge theories have been developed for Abelian [1] and non-Abelian [2] gauge groups, starting from the multiplet structure of the simplest supersymmetric field theories [3]. Gauge theories with N supersymmetries (extended supersymmetry) have been constructed by the authors of ref. [2] and by Fayet [4] for $N = 2$, and by Gliozzi, Scherk and Olive for $N = 4$ [5]. The latter authors used a dimensional reduction technique which allowed interpretation of the (simpler) non-extended theory in 10 dimensions as extended supersymmetry in 4 dimensions. However, they do not reach, or attempt to reach, an off-shell supersymmetric theory with all its auxiliary fields.

Wess and Zumino suggested the use of differential geometry in superspace to reach a better understanding of both supergravity and supersymmetric gauge theories [6,7]. These techniques are displayed once more in sect. 2 of this paper. They have been used to derive the fully supercovariant gauge theory for $N = 2$ [8]. In any case, the most difficult task left in the construction of new theories is to guess gauge and supercovariant constraint conditions on the basic fields which are compatible with the identities, i.e. which do not lead to equations of motion or a pure-gauge theory. Therefore the Bianchi identity in superspace is examined closely in this paper. In the appendix, we calculate which of its components are algebraically independent for the different N; the results of this calculation are presented in sect. 3. In sects. 4 and 5, we briefly indicate the connection with the known theories of refs. [1,2,8]. Sect. 6 indicates some of the problems left for $N > 2$, sect. 7 shows how the scheme connects with the results of ref. [5] for $N = 4$, without, however, featuring any factual results beyond those of refs. [5,9]. Concluding remarks are contained in sect. 8.

2. Yang-Mills superfields

Our *superspace* is spanned by the space-time variables x^μ and the $4N$ anticommuting variables θ^A, $\bar{\theta}^{\dot{A}}$. The capital indices A and $\dot{A}$ stand for double indices ${}^\alpha_i$ and ${}^{\dot{\alpha}i}$, respectively ($\alpha, \dot{\alpha} = 1, 2$ are SL(2, C) spinor indices; $i, j = 1, ..., N$ number the internal degrees of freedom of the extended supersymmetry). Lower indices ${}_A$ and ${}_{\dot{A}}$ denote index pairs ${}^i_\alpha$ and ${}_{\dot{\alpha}j}$.

On this space we represent the *algebra* of N-extended supersymmetry ($\sigma^\mu_{A\dot{A}} \equiv \sigma^\mu_{\alpha\dot{\alpha}}\delta^i_j$, $\sigma^\mu \equiv (1, \vec{\sigma})$ is a set of Pauli matrices),

$$\{Q_A, \bar{Q}_{\dot{A}}\} = 2\sigma^\mu_{A\dot{A}} P_\mu \,,$$
$$\{Q_A, Q_B\} = \{\bar{Q}_{\dot{A}}, \bar{Q}_{\dot{B}}\} = [P_\mu, P_\nu] = 0 \,,$$
$$[Q_A, P_\mu] = [\bar{Q}_{\dot{A}}, P_\mu] = 0 \,, \tag{1}$$

as differential operators ($\partial\!\!\!/_{A\dot{A}} \equiv \sigma^\mu_{A\dot{A}}\,\partial_\mu$)

$$P_\mu = i\partial_\mu \,,$$
$$Q_A = \frac{\partial}{\partial\theta^A} - i\partial\!\!\!/_{A\dot{A}}\,\bar{\theta}^{\dot{A}} \,,$$
$$\bar{Q}_{\dot{A}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{A}}} + i\theta^A\partial\!\!\!/_{A\dot{A}} \,. \tag{2}$$

A *superfield* $\phi(x, \theta, \bar{\theta})$ transforms under supertransformations as

$$\delta\phi = (\zeta^A Q_A + \bar{Q}_{\dot{A}}\bar{\zeta}^{\dot{A}})\phi \,. \tag{3}$$

Under a *gauge transformation*, a gauge-covariant superfield is supposed to transform according to

$$\phi \to e^{-i\Lambda}\phi \,, \tag{4}$$

where Λ is a generating matrix of the Lie algebra and depends on the point in superspace:

$$\Lambda = \sum_l T^l\lambda_l(x, \theta, \bar{\theta}) \,. \tag{5}$$

The generating matrices of the gauge group T^l act on the (unwritten) gauge group indices of ϕ.

While $\partial_\mu\phi$ is still a superfield, we know that we have to replace ∂_μ by the (gauge-) *covariant derivative*

$$\mathcal{D}_\mu \equiv \partial_\mu + i\mathcal{A}_\mu \,, \tag{6}$$

so that $\mathcal{D}_\mu\phi$ is again a gauge-covariant superfield. The *Yang-Mills potential* $\mathcal{A}_\mu$ is a Lie-algebra valued superfield

$$\mathcal{A}_\mu = \sum_l T^l\mathcal{A}_{\mu l}(x, \theta, \bar{\theta}) \,, \tag{7}$$

whose gauge transformation properties are

$$\mathcal{A}_\mu \to e^{-i\Lambda}(\mathcal{A}_\mu - i\partial_\mu)\, e^{i\Lambda} \,. \tag{8}$$

In the same sense, the "covariant" spinor derivatives

$$D_A \equiv \frac{\partial}{\partial\theta^A} + i\not{\partial}_{A\dot{A}}\bar{\theta}^{\dot{A}} \,,$$
$$\bar{D}_{\dot{A}} \equiv -\frac{\partial}{\partial\bar{\theta}^{\dot{A}}} - i\theta^A \not{\partial}_{A\dot{A}} \tag{9}$$

must be augmented by *Yang-Mills spinor potentials* to yield gauge-covariant quantities

$$\mathcal{D}_A \equiv D_A + i\mathcal{A}_A \,,$$
$$\bar{\mathcal{D}}_{\dot{A}} \equiv \bar{D}_{\dot{A}} + i\bar{\mathcal{A}}_{\dot{A}} \,. \tag{10}$$

The Lie-algebra valued spinor superfields $\mathcal{A}_A$, $\bar{\mathcal{A}}_{\dot{A}}$ transform under gauge transformations very similarly to $\mathcal{A}_\mu$:

$$\mathcal{A}_A \to e^{-i\Lambda}(\mathcal{A}_A - iD_A)\, e^{i\Lambda} \,,$$
$$\bar{\mathcal{A}}_{\dot{A}} \to e^{-i\Lambda}(\bar{\mathcal{A}}_{\dot{A}} - i\bar{D}_{\dot{A}})\, e^{i\Lambda} \,. \tag{11}$$

The commutators of two covariant derivatives yield the six *Yang-Mills field strengths* (or curvatures):

$$\{\mathcal{D}_A, \mathcal{D}_B\} = iF_{AB} \,,$$
$$\{\bar{\mathcal{D}}_{\dot{A}}, \bar{\mathcal{D}}_{\dot{B}}\} = i\bar{F}_{\dot{A}\dot{B}} \,,$$
$$\{\mathcal{D}_A, \bar{\mathcal{D}}_{\dot{B}}\} = iF_{A\dot{B}} - 2i\sigma^\mu_{A\dot{B}}\mathcal{D}_\mu \,,$$
$$[\mathcal{D}_\mu, \mathcal{D}_A] = iF_{\mu A} \,,$$
$$[\mathcal{D}_\mu, \bar{\mathcal{D}}_{\dot{A}}] = i\bar{F}_{\mu\dot{A}} \,,$$
$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = iF_{\mu\nu} \,, \tag{12}$$

which are again Lie-algebra valued superfields:

$$F_{AB} = D_A\mathcal{A}_B + D_B\mathcal{A}_A + i\{\mathcal{A}_A, \mathcal{A}_B\} \,,$$
$$\bar{F}_{\dot{A}\dot{B}} = \bar{D}_{\dot{A}}\bar{\mathcal{A}}_{\dot{B}} + \bar{D}_{\dot{B}}\bar{\mathcal{A}}_{\dot{A}} + i\{\bar{\mathcal{A}}_{\dot{A}}, \bar{\mathcal{A}}_{\dot{B}}\} \,,$$
$$F_{A\dot{B}} = D_A\bar{\mathcal{A}}_{\dot{B}} + \bar{D}_{\dot{B}}\mathcal{A}_A + i\{\mathcal{A}_A, \bar{\mathcal{A}}_{\dot{B}}\} + 2i\sigma^\mu_{A\dot{B}}\mathcal{A}_\mu \,,$$
$$F_{\mu A} = \partial_\mu\mathcal{A}_A - D_A\mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_A] \,,$$
$$\bar{F}_{\mu\dot{A}} = \partial_\mu\bar{\mathcal{A}}_{\dot{A}} - \bar{D}_{\dot{A}}\mathcal{A}_\mu + i[\mathcal{A}_\mu, \bar{\mathcal{A}}_{\dot{A}}] \,,$$
$$F_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu] \,, \tag{13}$$

and, of course, gauge-covariant fields in the adjoint representation:

$$F \to e^{-i\Lambda} F\, e^{i\Lambda}\ . \tag{14}$$

From the six eqs. (12), which we may call "Ricci identities", we can derive, by means of generalized Jacobi identities, the following set of ten *Bianchi identities*:

$$\mathcal{D}_A F_{BC} + \mathcal{D}_B F_{CA} + \mathcal{D}_C F_{AB} = 0\ , \tag{15.1}$$

$$\bar{\mathcal{D}}_{\dot A} \bar F_{\dot B\dot C} + \bar{\mathcal{D}}_{\dot B} \bar F_{\dot C\dot A} + \bar{\mathcal{D}}_{\dot C} \bar F_{\dot A\dot B} = 0\ , \tag{15.2}$$

$$\bar{\mathcal{D}}_{\dot A} F_{BC} + \mathcal{D}_B F_{C\dot A} + \mathcal{D}_C F_{B\dot A} + 2i\sigma^{\mu}_{B\dot A} F_{\mu C} + 2i\sigma^{\mu}_{C\dot A} F_{\mu B} = 0\ , \tag{15.3}$$

$$\mathcal{D}_A \bar F_{\dot B\dot C} + \bar{\mathcal{D}}_{\dot B} F_{A\dot C} + \bar{\mathcal{D}}_{\dot C} F_{A\dot B} + 2i\sigma^{\mu}_{A\dot B} \bar F_{\mu\dot C} + 2i\sigma^{\mu}_{A\dot C} \bar F_{\mu\dot B} = 0\ , \tag{15.4}$$

$$\mathcal{D}_\mu F_{AB} - \mathcal{D}_A F_{\mu B} - \mathcal{D}_B F_{\mu A} = 0\ , \tag{15.5}$$

$$\mathcal{D}_\mu \bar F_{\dot A\dot B} - \bar{\mathcal{D}}_{\dot A} \bar F_{\mu\dot B} - \bar{\mathcal{D}}_{\dot B} \bar F_{\mu\dot A} = 0\ , \tag{15.6}$$

$$\mathcal{D}_\mu F_{A\dot B} - \mathcal{D}_A \bar F_{\mu\dot B} - \bar{\mathcal{D}}_{\dot B} F_{\mu A} - 2i\sigma^{\nu}_{A\dot B} F_{\mu\nu} = 0\ , \tag{15.7}$$

$$\mathcal{D}_\mu F_{\nu A} - \mathcal{D}_\nu F_{\mu A} + \mathcal{D}_A F_{\mu\nu} = 0\ , \tag{15.8}$$

$$\mathcal{D}_\mu \bar F_{\nu\dot A} - \mathcal{D}_\nu \bar F_{\mu\dot A} + \bar{\mathcal{D}}_{\dot A} F_{\mu\nu} = 0\ , \tag{15.9}$$

$$\mathcal{D}_\mu F_{\rho\sigma} + \mathcal{D}_\rho F_{\sigma\mu} + \mathcal{D}_\sigma F_{\mu\rho} = 0\ , \tag{15.10}$$

which are indeed identities, given the solution (13) for the F's.

3. Independent identities

One of the main tasks in finding supersymmetric gauge theories is to get rid of superfluous fields by imposing suitable covariant conditions on the field strengths. These conditions should neither lead to equations of motion nor render the theory flat (i.e. $F_{\mu\nu} = 0$). The all-important tool to find out whether or not this is the case is the set of generalized Bianchi identities (15). These are, however, not all independent, and it seems important to know the truly independent subset.

The evaluation of (15) for the independent identities is done in the appendix, and we want to summarize the results here.

Eqs. (15.1, 2). They are independent for any N.

Eqs. (15.3, 4). We get for any N ($\bar\sigma^{\dot A A}_{\mu} \equiv \bar\sigma^{\dot\alpha\alpha}_{\mu}\delta^{j}_{i}$, $\bar\sigma^{\mu} \equiv (1, -\vec\sigma)$)

$$\begin{aligned} &4i(N+1)F_{\mu A} - 2i(\sigma_\mu\bar\sigma^\nu)_A{}^B F_{\nu B} \\ &\quad = -(\bar{\mathcal{D}}\bar\sigma_\mu)^B F_{AB} - (\bar\sigma_\mu \mathcal{D})^{\dot A} F_{A\dot A} - \mathcal{D}_A \bar\sigma^{\dot B B}_{\mu} F_{B\dot B}\ , \end{aligned} \tag{15.3a}$$

$$\begin{aligned} &4i(N+1)\bar F_{\mu\dot A} - 2i\bar F_{\nu\dot B}(\bar\sigma^\nu\sigma_\mu)^{\dot B}{}_{\dot A} \\ &\quad = -(\bar{\mathcal{D}}\bar\sigma_\mu)^B F_{B\dot A} - (\bar\sigma_\mu \mathcal{D})^{\dot B} \bar F_{\dot A\dot B} - \bar{\mathcal{D}}_{\dot A} \bar\sigma^{\dot B B}_{\mu} F_{B\dot B}\ . \end{aligned} \tag{15.4a}$$

These equations determine $F_{\mu A}$ and $\bar{F}_{\mu\dot{A}}$ completely in terms of the other F's for $N>1$, but only the spin-$\frac{3}{2}$ components in the case of $N=1$. For $N>1$, a further consequence of eqs. (15.3, 4) is *

$$(\epsilon\sigma_{\mu\nu})^{\beta\gamma}[\bar{\mathcal{D}}_{\dot{\alpha}i}F^{jk}_{\beta\gamma} + \mathcal{D}^k_\gamma F^j_{\beta\dot{\alpha}i} + \mathcal{D}^j_\gamma F^k_{\beta\dot{\alpha}i}]$$
$$= \frac{1}{N+1}\delta^j_i(\epsilon\sigma_{\mu\nu})^{\beta\gamma}[\bar{\mathcal{D}}_{\dot{\alpha}l}F^{lk}_{\beta\gamma} + \mathcal{D}^k_\gamma F^l_{\beta\dot{\alpha}l} + \mathcal{D}^l_\gamma F^k_{\beta\dot{\alpha}l}]$$
$$+ (j,k \text{ interchanged}) , \tag{15.3b}$$

$$(\bar{\sigma}_{\mu\nu}\epsilon)^{\dot{\beta}\dot{\gamma}}[\mathcal{D}^i_\alpha \bar{F}_{\dot{\beta}j\dot{\gamma}k} + \bar{\mathcal{D}}_{\dot{\beta}j}F^i_{\alpha\dot{\gamma}k} + \bar{\mathcal{D}}_{\dot{\beta}k}F^i_{\alpha\dot{\gamma}j}]$$
$$= \frac{1}{N+1}\delta^i_j(\bar{\sigma}_{\mu\nu}\epsilon)^{\dot{\beta}\dot{\gamma}}(\mathcal{D}^l_\alpha \bar{F}_{\dot{\beta}l\dot{\gamma}k} + \bar{\mathcal{D}}_{\dot{\beta}l}F^l_{\alpha\dot{\gamma}k} + \bar{\mathcal{D}}_{\dot{\beta}k}F^l_{\alpha\dot{\gamma}l})$$
$$+ (j,k \text{ interchanged}) . \tag{15.4b}$$

For $N>2$, yet another consequence of eqs. (15.3, 4) is

$$\epsilon^{\beta\gamma}[\bar{\mathcal{D}}_{\dot{\alpha}i}F^{jk}_{\beta\gamma} + \mathcal{D}^k_\gamma F^j_{\beta\dot{\alpha}i} - \mathcal{D}^j_\gamma F^k_{\beta\dot{\alpha}i}]$$
$$= \frac{1}{N-1}\delta^j_i\epsilon^{\beta\gamma}[\bar{\mathcal{D}}_{\dot{\alpha}l}F^{lk}_{\beta\gamma} + \mathcal{D}^k_\gamma F^l_{\beta\dot{\alpha}l} - \mathcal{D}^l_\gamma F^k_{\beta\dot{\alpha}l}]$$
$$- (j,k \text{ interchanged}) , \tag{15.3c}$$

$$\epsilon^{\dot{\beta}\dot{\gamma}}[\mathcal{D}^i_\alpha \bar{F}_{\dot{\beta}j\dot{\gamma}k} + \bar{\mathcal{D}}_{\dot{\beta}j}F^i_{\alpha\dot{\gamma}k} - \bar{\mathcal{D}}_{\dot{\beta}k}F^i_{\alpha\dot{\gamma}j}]$$
$$= \frac{1}{N-1}\delta^i_j\epsilon^{\dot{\beta}\dot{\gamma}}[\mathcal{D}^l_\alpha \bar{F}_{\dot{\beta}l\dot{\gamma}k} + \bar{\mathcal{D}}_{\dot{\beta}l}F^l_{\alpha\dot{\gamma}k} - \bar{\mathcal{D}}_{\dot{\beta}k}F^l_{\alpha\dot{\gamma}l}]$$
$$- (j,k \text{ interchanged}) . \tag{15.4c}$$

For any N, eqs. (15.3a–c) and (15.4a–c) are equivalent to eqs. (15.3) and (15.4), respectively.

Eqs. (15.5, 6). They are independent only for $N=1$.

Eq. (15.7). For any N, it gives $F_{\mu\nu}$ in terms of other F's:

$$8iNF_{\mu\nu} = \bar{\sigma}_\nu{}^{\dot{B}A}(\mathcal{D}_\mu F_{A\dot{B}} - \mathcal{D}_A\bar{F}_{\mu\dot{B}} - \bar{\mathcal{D}}_{\dot{B}}F_{\mu A})$$
$$- (\mu,\nu \text{ interchanged}) . \tag{15.7a}$$

* $\sigma_{\mu\nu} \equiv \frac{1}{2}i(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)$, $\bar{\sigma}_{\mu\nu} \equiv \frac{1}{2}i(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu)$,

$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} = -\epsilon_{\alpha\beta}$, $\epsilon^{12} = 1$;

same for dotted indices.

For $N = 1$, an additional consequence of eq. (15.7) is

$$0 = \bar{\sigma}^{\mu\dot{\beta}\alpha}(\mathcal{D}_\mu F_{\alpha\dot{\beta}} - \mathcal{D}_\alpha \bar{F}_{\mu\dot{\beta}} - \bar{\mathcal{D}}_{\dot{\beta}} F_{\mu\alpha}) , \tag{15.7b}$$

while for $N = 2$, we get

$$0 = \bar{\sigma}^{\mu\dot{\beta}\alpha}(\mathcal{D}_\mu F^i_{\alpha\dot{\beta}j} - \mathcal{D}^i_\alpha \bar{F}_{\mu\dot{\beta}j} - \bar{\mathcal{D}}_{\dot{\beta}j} F_{\mu\alpha}{}^i)_{\text{traceless in } i,\, j} . \tag{15.7c}$$

The eqs. (15.7a–c) are equivalent to eq. (15.7) for any N.

Eqs. (15.8–10). They are dependent for any N.

The following table indicates whether an identity is independent (+), dependent (–), trivially fulfilled (○), or not deducible (x):

N \ Eq.	(1)	(2)	(3a)	(3b)	(3c)	(4a)	(4b)	(4c)	(5)	(6)	(7a)	(7b)	(7c)	(8)	(9)	(10)
1	+	+	+	x	x	+	x	x	+	+	+	+	o	–	–	–
2	+	+	+	+	o	+	+	o	–	–	+	–	+	–	–	–
>2	+	+	+	+	+	+	+	+	–	–	+	–	–	–	–	–

4. Non-extended supersymmetry

Since for $N = 1$ we can set

$$F_{\alpha\beta} = \bar{F}_{\dot{\alpha}\dot{\beta}} = F_{\alpha\dot{\beta}} = 0 \tag{16}$$

without getting a flat theory (since $F_{\mu\alpha}$ may still contain an undetermined spin-$\frac{1}{2}$ contribution), the theory is reasonably simple: the constraint equations (16) for the potentials have simple solutions, which correspond to a theory "flat in the Grassmann directions":

$$\begin{aligned}
\mathcal{A}_\alpha &= \mathrm{e}^{-V} D_\alpha\, \mathrm{e}^{V} , \\
\bar{\mathcal{A}}_{\dot{\alpha}} &= \mathrm{e}^{-U} \bar{D}_{\dot{\alpha}}\, \mathrm{e}^{U} , \\
\mathcal{A}_\mu &= \tfrac{1}{4} i \bar{\sigma}_\mu^{\dot{\beta}\alpha} (D_\alpha \bar{\mathcal{A}}_{\dot{\beta}} + \bar{D}_{\dot{\beta}} \mathcal{A}_\alpha + i\{\mathcal{A}_\alpha, \bar{\mathcal{A}}_{\dot{\beta}}\}) ,
\end{aligned} \tag{17}$$

where V and U are arbitrary superfields. The identities (15.1, 2) are then trivial, eqs. (15.3a) and (15.4a) have the solutions

$$\begin{aligned}
F_{\mu\alpha} &= -\tfrac{1}{8} i \sigma_{\mu\alpha\dot{\beta}}\, \bar{W}^{\dot{\beta}} , \\
F_{\mu\dot{\alpha}} &= \tfrac{1}{8} i \sigma_{\mu\beta\dot{\alpha}} W^{\beta} ,
\end{aligned} \tag{18}$$

where the W's are functions of the $\mathcal{A}$'s which can be calculated from eqs. (13). W^α is particularly simple for the (supersymmetric) gauge where $\bar{\mathcal{A}}_{\dot{\alpha}} = 0$:

$$W^\alpha = \epsilon^{\alpha\beta} \bar{D}_{\dot{\alpha}}\, \epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}_{\dot{\beta}} \mathcal{A}_\beta , \tag{19}$$

while $\bar{W}^{\dot{\alpha}}$ would be simple in the gauge where $\mathcal{A}_\alpha = 0$. It differs from $(W^\alpha)^\dagger$ only by a gauge transformation. With a little algebra, the remaining identities now take the forms

$$\mathcal{D}_\alpha \bar{W}^{\dot\beta} = 0\,,$$

$$\bar{\mathcal{D}}_{\dot\alpha} W^\beta = 0\,, \tag{20}$$

from (15.5, 6), and

$$\mathcal{D}_\alpha W^\alpha - \bar{\mathcal{D}}_{\dot\alpha} \bar{W}^{\dot\alpha} = 0\,, \tag{21}$$

from eq. (15.7b). $F_{\mu\nu}$ is given through (15.7a) as

$$F_{\mu\nu} = \tfrac{1}{32} i(\mathcal{D}_\alpha \sigma_{\mu\nu\beta}{}^\alpha W^\beta + \bar{\mathcal{D}}_{\dot\alpha} \bar{\sigma}_{\mu\nu}{}^{\dot\alpha}{}_{\dot\beta} \bar{W}^{\dot\beta})\,. \tag{22}$$

With the abovementioned supersymmetric choice of gauge, this is the theory first developed by Ferrara and Zumino, and Salam and Strathdee [2] (see Wess [7] for details of the correspondence).

5. The case $N = 2$

While a constraint similar to eq. (16) would render the theory flat, we can impose the weaker conditions

$$F^{ij}_{\alpha\beta} + F^{ij}_{\beta\alpha} = 0\,, \tag{23a}$$

$$\bar{F}_{\dot\alpha i \dot\beta j} + \bar{F}_{\dot\beta i \dot\alpha j} = 0\,, \tag{23b}$$

$$F^{i}{}_{\alpha\dot\beta j} = 0\,. \tag{23c}$$

A consequence of eqs. (23a, b) is that we can express F_{AB} and $\bar{F}_{\dot A \dot B}$ through (g^{ij} stands for the ϵ symbol in the SU(2) space)

$$F^{ij}_{\alpha\beta} = \epsilon_{\alpha\beta} g^{ij} \bar{W}\,,$$

$$\bar{F}_{\dot\alpha i \dot\beta j} = \epsilon_{\dot\alpha\dot\beta} g_{ij} W\,. \tag{24}$$

All curvatures can now be expressed in terms of the W's:

$$F_{\mu\alpha}{}^i = -\tfrac{1}{4} i g^{ij} (\sigma_\mu \epsilon)_\alpha{}^{\dot\beta} \bar{\mathcal{D}}_{\dot\beta j} \bar{W}\,,$$

$$\bar{F}_{\mu\dot\alpha i} = \tfrac{1}{4} i g_{ij} (\epsilon \bar{\sigma}_\mu)_{\dot\alpha}{}^\beta \mathcal{D}^j_\beta W\,,$$

$$F_{\mu\nu} = \tfrac{1}{32} i(\mathcal{D}\,\epsilon\sigma_{\mu\nu} g \mathcal{D} W + \bar{\mathcal{D}} \bar{\sigma}_{\mu\nu} \epsilon g \bar{\mathcal{D}} \bar{W})\,, \tag{25}$$

and the remaining identities for the W's read

$$\mathcal{D}^i_\alpha \bar{W} = \bar{\mathcal{D}}_{\dot\alpha i} W = 0\,, \tag{26}$$

from (15.1, 2), and ($\boldsymbol{\tau}_i{}^j$ are the Pauli matrices in SU(2) space)

$$\mathcal{D}\,\epsilon\boldsymbol{\tau} g \mathcal{D} W - \bar{\mathcal{D}} \epsilon g \boldsymbol{\tau} \bar{\mathcal{D}} \bar{W} = 0\,, \tag{27}$$

from (15.7c), while (15.3b) and (15.4b) are trivial due to eq. (23). This leads to the theory described in detail in ref. [8].

6. The case $N > 2$

Here the constraints (23) appear to be too stringent. Assuming that they hold, we would get as the equivalent of (24)

$$F^{ij}_{\alpha\beta} = \epsilon_{\alpha\beta}\bar{W}^{ij}\,, \qquad \bar{W}^{ij} = -\bar{W}^{ji}\,,$$
$$\bar{F}_{\dot{\alpha}i\dot{\beta}j} = \epsilon_{\dot{\alpha}\dot{\beta}}W_{ij}\,, \qquad W_{ij} = -W_{ji}\,, \tag{28}$$

and (15.1) reads for $\bar{W}$

$$\mathcal{D}^i_\alpha \bar{W}^{jk} = -\mathcal{D}^j_\alpha \bar{W}^{ik}\,, \tag{29}$$

while (15.3c) becomes

$$\bar{\mathcal{D}}_{\dot{\alpha}i}\bar{W}^{jk} = \frac{1}{N-1}(\delta^j_i\,\bar{\mathcal{D}}_{\dot{\alpha}l}\bar{W}^{lk} - \delta^k_i\,\bar{\mathcal{D}}_{\dot{\alpha}l}\,\bar{W}^{lj})\,. \tag{30}$$

For an Abelian gauge group, it is possible to use (29, 30) in order to get

$$D^i_\alpha \bar{D}_{\dot{\beta}k}\bar{W}^{kj} = -2i(N-1)\,\partial\!\!\!/_{\alpha\dot{\beta}}\bar{W}^{ij}\,, \tag{31}$$

while

$$\epsilon^{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\alpha}i}\bar{D}_{\dot{\beta}j}\bar{W}^{ij} = 0 \tag{32}$$

is an obvious consequence of the algebra of the D's. From here we can derive the following chain of equations:

$$\begin{aligned}-2i(\partial\!\!\!/\epsilon\bar{D}_k)_\alpha \bar{W}^{ki} &= \{D^i_\alpha, \bar{D}_{\dot{\beta}j}\}\,(\epsilon\bar{D}_k)^{\dot{\beta}}\,\bar{W}^{kj}\\ &= (\bar{D}_j\epsilon)^{\dot{\beta}} D^i_\alpha \bar{D}_{\dot{\beta}k}\bar{W}^{kj} = -2i(N-1)(\partial\!\!\!/\epsilon\bar{D}_k)_\alpha \bar{W}^{ki}\\ &= 0 \qquad \text{for } N > 2\,.\end{aligned} \tag{33}$$

Differentiating again by D^j_β and using (31), one gets $\Box\bar{W} = 0$, i.e. an equation of motion. Thus the constraints (23) are unacceptable for $N > 2$.

7. The particle spectrum for $N = 4$

As we have seen in sect. 6, the constraints (23) with their solutions (28) lead to equations of motion for the field strengths. This indicates that the fully supersymmetric theory (with all auxiliary fields) should not obey all of the constraints (23), but probably some of them, while the others are then the supercovariant equations of motion. However, to study the physical content of the theory, i.e. the "particle" spectrum, we may as well look at the theory with fields already governed by equations of motion, and with all auxiliary fields removed. In the case of $N = 4$, we can then even impose an additional constraint beyond (23), namely

$$\overline{W}^{ij} = \tfrac{1}{2}\epsilon^{ijkl}W_{kl}\,, \tag{34}$$

as a consequence of which the two remaining identities for W_{ij} (from eqs. (15.2) and (15.4c)),

$$\overline{\mathcal{D}}_{\dot\alpha i}W_{jk} = -\overline{\mathcal{D}}_{\dot\alpha j}W_{ik}\,, \tag{35}$$

$$\mathcal{D}^i_\alpha W_{jk} = \frac{1}{N-1}(\delta^i_j\mathcal{D}^l_\alpha W_{lk} - \delta^i_k\mathcal{D}^l_\alpha W_{lj})\,, \tag{36}$$

become equivalent to eqs. (30) and (29), respectively.

Every component of a superfield can be used as the basic field ($\theta = \overline{\theta} = 0$) of another superfield, whose components are then determined by the transformation properties of the basic field. These components are functions of the components of the original superfield. The new superfield can always be expressed in terms of covariant derivatives on the old one (because those can be used to project out any component into the basic position). As a consequence of this property, we can derive the particle spectrum of the theory by asking which independent gauge-covariant superfields can be constructed from W_{ij}. This is done by consecutively using more and more covariant derivatives on W_{ij}.

We find that the only independent ones are

$$W_{ij}, \quad \mathcal{D}^j_\alpha W_{ji}; \quad \overline{\mathcal{D}}_{\dot\alpha j}\overline{W}^{ji}\,, \quad F_{\mu\nu}\,.$$

All others can be expressed through these and the space derivatives $\mathcal{D}_\mu$. The reality constraint (that $\mathcal{A}^\dagger_\mu$ be related to $\mathcal{A}_\mu$ through a gauge transformation, i.e., that $F_{\mu\nu}$ be real), relates the third of these to the complex conjugate of the second, and the spectrum becomes that of the SU(4)-invariant theory in ref. [5]: six scalars in the $\underline{6}$ of SU(4), a Weyl spinor in the $\underline{4}$, its antiparticle in the $\underline{4}^*$, and the SU(4)-scalar Yang-Mills field.

Let us check these statements. Clearly, eq. (36) indicates that $\mathcal{D}^i_\alpha W_{jk}$ has as its independent component only a $\underline{4}$ of SU(4), namely $\mathcal{D}^l_\alpha W_{li}$. Eq. (35) says that $\overline{\mathcal{D}}_{\dot\alpha i}W_{jk}$ is totally antisymmetric in ijk, and thus contains only a $\underline{4}^*$, namely $\epsilon^{ijkl}\overline{\mathcal{D}}_{\dot\alpha j}W_{kl} = -2\overline{\mathcal{D}}_{\dot\alpha j}\overline{W}^{ji}$. Using (23c) and (28), we can derive from (15.3a) and (15.4a) the relations

$$F_{\mu\alpha}^{\ \ i} = \tfrac{1}{12}i(\sigma_\mu\epsilon)_\alpha{}^{\dot\beta}\,\overline{\mathcal{D}}_{\dot\beta j}\overline{W}^{ji}\,, \tag{37a}$$

$$\overline{F}_{\mu\dot\alpha i} = -\tfrac{1}{12}i(\epsilon\overline{\sigma}_\mu)_{\dot\alpha}{}^{\beta}\mathcal{D}^j_\beta W_{ji}\,, \tag{37b}$$

respectively, and then from (15.7a)

$$F_{\mu\nu} = \frac{-i}{12\cdot 16}(\overline{\mathcal{D}}_i\overline{\sigma}_{\mu\nu}\epsilon\overline{\mathcal{D}}_j\overline{W}^{ji} + \mathcal{D}^i\epsilon\sigma_{\mu\nu}\mathcal{D}^jW_{ji})\,. \tag{37c}$$

Clearly, eqs. (37) correspond to eqs. (25) which held for $N = 2$. Reality of $F_{\mu\nu}$ implies

$$W_{ij} = -(\bar{W}^{ij})^\dagger = -(\tfrac{1}{2}\epsilon^{ijkl}W_{kl})^\dagger \ , \tag{38}$$

which says that W_{ij} has indeed only 6 independent real components. Another consequence of (38) is

$$(\mathcal{D}^j_\alpha W_{ji})^\dagger = -\bar{\mathcal{D}}_{\dot{\alpha}j}\bar{W}^{ji} \ , \tag{39}$$

which indeed relates the $\underset{-}{4}^*$ to the complex conjugate of the $\underset{-}{4}$. The last statement to be checked is that by using more derivatives on W_{ij}, we do not get any other superfields which cannot be expressed in terms of the above and their spatial derivatives. We use the technique to anticommute the $\mathcal{D}$'s, use the identities (29, 30) and (35, 36), commute the $\mathcal{D}$'s again, use the identities again, and finally collect terms to get

$$\mathcal{D}^i_\alpha \mathcal{D}^l_\beta W_{lj} = \tfrac{1}{4}\delta^i_j \mathcal{D}^l_\alpha \mathcal{D}^k_\beta W_{kl} + \tfrac{3}{2} i\epsilon_{\alpha\beta}[\bar{W}^{il}, W_{lj}] \ , \tag{40}$$

$$\bar{\mathcal{D}}_{\dot{\alpha}i}\bar{\mathcal{D}}_{\dot{\beta}l}\bar{W}^{lj} = \tfrac{1}{4}\delta^j_i \bar{\mathcal{D}}_{\dot{\alpha}l}\bar{\mathcal{D}}_{\dot{\beta}k}\bar{W}^{kl} + \tfrac{3}{2} i\epsilon_{\dot{\alpha}\dot{\beta}}[W_{il}, \bar{W}^{lj}] \ , \tag{41}$$

$$\mathcal{D}^i_\alpha \bar{\mathcal{D}}_{\dot{\beta}l}\bar{W}^{lj} = -6i\,\mathcal{D}_{\alpha\dot{\beta}}\bar{W}^{ij} \ . \tag{42}$$

The first two of these can be rewritten, using a completeness relation for σ matrices and (37c), as

$$\mathcal{D}^i_\alpha \mathcal{D}^l_\beta W_{lj} = 6i\delta^i_j(\sigma_{\mu\nu}\epsilon)_{\alpha\beta}F^{\mu\nu} + \tfrac{3}{2} i\epsilon_{\alpha\beta}[\bar{W}^{il}, W_{lj}] \ , \tag{43}$$

$$\bar{\mathcal{D}}_{\dot{\alpha}i}\bar{\mathcal{D}}_{\dot{\beta}l}\bar{W}^{lj} = 6i\delta^j_i(\epsilon\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}}F^{\mu\nu} + \tfrac{3}{2} i\epsilon_{\dot{\alpha}\dot{\beta}}[W_{il}, \bar{W}^{lj}] \ . \tag{44}$$

Thus, using any two $\mathcal{D}$'s on one of the W's, we only get functions of the W's, $\mathcal{D}_\mu W$, and $F^{\mu\nu}$. Finally, a $\mathcal{D}$ on $F^{\mu\nu}$ yields, according to identity (15.8), only spatial derivatives on $F_{\mu A}$, which in turn is given through $\bar{\mathcal{D}}_{\dot{\alpha}j}\bar{W}^{ji}$, see eq. (37a). Any higher number of $\mathcal{D}$'s on W can be reduced out in a similar manner, using the above results as a starting point. We have thus proved our statement about the spectrum of the theory.

By quite tedious calculations, using most of the above formulae for the W's and their derivatives, it can be shown that the gauge-invariant superfield

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Big(&-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{144} i\bar{\mathcal{D}}_{\dot{\alpha}l}\bar{W}^{li}\mathcal{D}^{\dot{\alpha}\alpha}\mathcal{D}^k_\alpha W_{ki} - \tfrac{1}{32}\mathcal{D}^\mu W_{ij}\mathcal{D}_\mu \bar{W}^{ij} \\ &- \frac{i}{4\cdot 144}\bar{\mathcal{D}}_{\dot{\alpha}l}\bar{W}^{li}\epsilon^{\dot{\alpha}\dot{\beta}}[\bar{\mathcal{D}}_{\dot{\beta}k}\bar{W}^{kj}, W_{ij}] - \frac{i}{4\cdot 144}\mathcal{D}^l_\alpha W_{li}\epsilon^{\alpha\beta}[\mathcal{D}^k_\beta W_{kj}, \bar{W}^{ij}] \\ &+ \frac{1}{4\cdot 16\cdot 16}[W_{ij}, W_{kl}][\bar{W}^{ij}, \bar{W}^{kl}]\Big)\end{aligned} \tag{45}$$

has the property that $D^i_\alpha\mathcal{L}$ and $\bar{D}_{\dot{\alpha}i}\mathcal{L}$ are divergences of four-vectors. Therefore, the

$(\theta, \bar{\theta})$-independent component of $\mathcal{L}$, for which the actions of the D's and of supersymmetry transformations coincide, is a supersymmetric Lagrangian. Denoting

$$G_{\mu\nu} \equiv F_{\mu\nu}|_{\theta=\bar{\theta}=0} \,,$$

$$\lambda_{\alpha i} = \tfrac{1}{12} i \mathcal{D}^j_\alpha W_{ji}|_{\theta=\bar{\theta}=0} \,, \qquad \bar{\lambda}^i_{\dot{\alpha}} \equiv \tfrac{1}{12} i \bar{\mathcal{D}}_{\dot{\alpha}j} \bar{W}^{ji}|_{\theta=\bar{\theta}=0} \,,$$

$$M_{ij} \equiv \tfrac{1}{4} i W_{ij}|_{\theta=\bar{\theta}=0} \,, \qquad M^{ij} \equiv \tfrac{1}{4} i \bar{W}^{ij}|_{\theta=\bar{\theta}=0} \,, \tag{46}$$

this Lagrangian becomes

$$L = \mathcal{L}|_{\theta=\bar{\theta}=0} = \mathrm{Tr}\big(-\tfrac{1}{4} G_{\mu\nu} G^{\mu\nu} - i\bar{\lambda}^i \bar{\mathcal{D}}\!\!\!/ \lambda_i + \tfrac{1}{2} \mathcal{D}^\mu M_{ij} \mathcal{D}_\mu M^{ij}$$
$$+ \bar{\lambda}^i \epsilon [\bar{\lambda}^j, M_{ij}] + \lambda_i \epsilon [\lambda_j, M^{ij}] + \tfrac{1}{4} [M_{ij}, M_{kl}][M^{ij}, M^{kl}]\big) \,, \tag{47}$$

which is, in slightly different notation, the Lagrangian given in ref. [9] for the SU(4)-invariant theory, and which reduces to the form given in ref. [5], if we take only the SO(4) subgroup and build four Majorana spinors from $\lambda_{\alpha i}$ and $\bar{\lambda}^i_{\dot{\alpha}}$.

8. Concluding remarks

Obviously, we have not yet presented a fully supersymmetric gauge theory for more than 2 supersymmetry charges ($N > 2$). The results indicated in sect. 7, however, suggest the existence of such a theory. Investigations in that direction are in progress.

The non-trivial internal symmetry group has never been fixed. Its only relevant property is the dimension N of those of its representations under which the supersymmetry charges transform. All results apply equally well to orthogonal or unitary groups.

Apart from the relevance which the results of this paper may have for the study of supersymmetric gauge theories, it should be pointed out that no properties of the gauge group itself have been used. Thus local Lorentz transformations are allowed as the gauge group, and some of our results may be relevant to (extended) supergravity as well.

Appendix

The purpose of this appendix is to indicate the calculational steps which lead to the results summarized in sect. 3. We will use an Abelian gauge group here, where the Yang-Mills field strengths are invariants. It is left to the reader to work out the additional commutator terms which appear at intermediate steps for non-Abelian groups.

From (15.3) we get, after multiplication with $\bar{\sigma}_\mu^{\dot{A}C}$ eq. (15.3a). If we differentiate this by D_B and symmetrize in A and B we get, after commuting D's and using (15.1–3) (on the r.h.s.),

$$0 = 2(N+2)\, X^{ij}_{\alpha\beta\mu} - (\sigma_\mu\bar{\sigma}^\nu)_\alpha{}^\gamma X^{ji}_{\beta\gamma\nu} - (\sigma_\mu\bar{\sigma}^\nu)_\beta{}^\gamma X^{ij}_{\alpha\gamma\nu}\ ,$$

with the abbreviation

$$X^{ij}_{\alpha\beta\mu} = X_{AB\mu} \equiv D_A F_{\mu B} + D_B F_{\mu A} - \partial_\mu F_{AB}\ .$$

If we contract this equation with $\bar{\sigma}^{\mu\dot{\alpha}\alpha}$ and $\bar{\sigma}^{\mu\dot{\alpha}\beta}$ we get two equations which possess a non-trivial solution only for $N = 1$. Thus the dependence of (15.5) for $N > 1$ has been shown. (15.6) is treated similarly.

For $N > 1$, it is possible to give $F_{\mu A}$ as a function of other F's, as a consequence of (15.3a):

$$\begin{aligned} 4i(N+1)F_{\mu\alpha}{}^{i} &= \frac{1}{N-1}\,[(\bar{D}\bar{\sigma}_\mu\epsilon)_{\alpha l}\epsilon^{\beta\gamma}F^{li}_{\beta\gamma} - (\epsilon D)^{\gamma l}(\bar{\sigma}_\mu\epsilon)^{\dot{\beta}}{}_\alpha F^{i}_{\gamma\dot{\beta}l} \\ &\quad + (\epsilon D)^{\gamma i}(\bar{\sigma}_\mu\epsilon)^{\dot{\beta}}{}_\alpha F^{l}_{\gamma\dot{\beta}l}\,] - (\bar{D}\bar{\sigma}_\mu)^\gamma_l F^{il}_{\alpha\gamma} - (\bar{\sigma}_\mu D)^{\dot{\alpha}l}F^{i}_{\alpha\dot{\alpha}l} - D^i_\alpha\bar{\sigma}^{\dot{\alpha}\beta}_\mu F^{l}_{\beta\dot{\alpha}l}\,. \end{aligned} \tag{A.1}$$

This can be inserted into (15.3), and after a little algebra we get

$$\begin{aligned} 0 &= \tfrac{1}{2}(\bar{D}_{\dot{\alpha}i}F^{jk}_{\beta\gamma} + D^j_\beta F^k_{\gamma\dot{\alpha}i} + D^k_\gamma F^j_{\beta\dot{\alpha}i}) \\ &\quad + \frac{1}{N^2-1}\,\epsilon_{\gamma\beta}\delta^k_i(\bar{D}_{\dot{\alpha}l}\epsilon^{\alpha\delta}F^{lj}_{\alpha\delta} - (\epsilon D)^{\delta l}F^{j}_{\delta\dot{\alpha}l} + (\epsilon D)^{\delta j}F^{l}_{\delta\dot{\alpha}l}) \\ &\quad - \frac{1}{N+1}\,\delta^k_i(\bar{D}_{\dot{\alpha}l}F^{jl}_{\beta\gamma} + D^l_\gamma F^j_{\beta\dot{\alpha}l} + D^j_\beta F^l_{\gamma\dot{\alpha}l}) + (\,{}^j_\beta \text{ and } {}^k_\gamma \text{ interchanged}). \end{aligned}$$

The symmetric and antisymmetric parts in β and γ of this give eqs. (15.3b) and (15.3c), respectively. Eq. (15.3c) is identically fulfilled for $N = 2$, as can be shown by multiplying it with g_{jk} (the antisymmetric symbol), a procedure which does not lose any information for $N = 2$. Eq. (15.4) is treated in an analogous way, which results in eqs. (15.4a–c).

Expressing ∂_μ through $\{D, \bar{D}\}$ and then using (15.8, 9), we get

$$\begin{aligned} 4iN\partial_\mu F_{\rho\sigma} &= \bar{\sigma}_\mu^{\dot{A}A}\partial_\rho(D_A\bar{F}_{\sigma\dot{A}} + \bar{D}_{\dot{A}}F_{\sigma A}) - (\rho, \sigma \text{ interchanged}) \\ &= -4iN\,\partial_\rho F_{\sigma\mu} + 4iN\,\partial_\sigma F_{\rho\mu}\ , \end{aligned}$$

i.e. eq. (15.10). For the last step we used (15.7).

Differentiating (15.7) by D_C and using (15.3) on the first resulting term, (15.7) on the second and (15.5) on the third, after some algebra we get

$$\begin{aligned} &2i\sigma^\nu_{A\dot{B}}(D_C F_{\mu\nu} + \partial_\mu F_{\nu C} - \partial_\nu F_{\mu C}) \\ &\quad = -2i\sigma^\nu_{C\dot{B}}\,(D_A F_{\mu\nu} + \partial_\mu F_{\nu A} - \partial_\nu F_{\mu A})\ . \end{aligned}$$

After contraction with $\bar{\sigma}_\rho^{\dot{B}C}$ the r.h.s. is left antisymmetric in μ and ρ. For the l.h.s., this is found to be the case only if the bracket vanishes, i.e. if (15.8) holds. Thus (15.8) is a consequence of (15.3, 5, 7). Similarly, we show that (15.9) follows from (15.4, 6, 7).

Eq. (15.7) can be split into three parts. If we define the abbreviations

$$Y_{\mu A\dot{B}} \equiv \partial_\mu F_{A\dot{B}} - D_A \bar{F}_{\mu\dot{B}} - \bar{D}_{\dot{B}} F_{\mu A} \,,$$

$$Y_{\mu\nu}{}^{i}_{j} \equiv \bar{\sigma}_\nu^{\dot{\beta}\alpha} Y_{\mu\alpha\dot{\beta}}{}^{i}_{j} \,, \qquad Y_{\mu\nu} \equiv \delta^i_j Y_{\mu\nu}{}^{j}_{i} = \bar{\sigma}_\nu^{\dot{B}A} Y_{\mu A\dot{B}} \,,$$

then they are

$$4iNF_{\mu\nu} = \tfrac{1}{2}(Y_{\mu\nu} - Y_{\nu\mu}) \,,$$

$$0 = Y_{\mu\nu} + Y_{\nu\mu} \,, \tag{A.2}$$

$$0 = Y_{\mu\nu}{}^{j}_{i}\big|_{\text{traceless in } i,\, j} \,. \tag{A.3}$$

The first one of these is (15.7a).

If, in the following expression:

$$\bar{D}_{\dot{B}}\, \bar{\sigma}_\mu^{\dot{C}C} (\bar{D}_{\dot{C}} F_{AC} + D_C F_{A\dot{C}}) + D_A\, \bar{\sigma}_\mu^{\dot{C}C} (\bar{D}_{\dot{C}} F_{C\dot{B}} + D_C \bar{F}_{\dot{B}\dot{C}}) \,,$$

we use (15.3) and (15.4) on the brackets, we get

$$\begin{aligned} &= -2i(\sigma^\nu\bar{\sigma}_\mu)_A{}^C \bar{D}_{\dot{B}} F_{\nu C} + 2i\bar{\sigma}_\mu^{\dot{C}C} \sigma^\nu_{A\dot{B}}\, \partial_\nu F_{C\dot{C}} \\ &\quad -2i(\bar{\sigma}_\mu\sigma^\nu)^{\dot{C}}{}_{\dot{B}} D_A \bar{F}_{\nu\dot{C}} - 4iN(\bar{D}_{\dot{B}} F_{\mu A} + D_A \bar{F}_{\mu\dot{B}}) \,, \end{aligned}$$

but if we first anticommute the D's and then use (15.3, 4) only on the terms resulting from the first bracket we get

$$\begin{aligned} &= 2i\bar{\sigma}_\mu^{\dot{C}C} \sigma^\nu_{A\dot{B}} \bar{D}_{\dot{C}} F_{\nu C} + 2i(\bar{\sigma}_\mu\sigma^\nu)^{\dot{C}}{}_{\dot{B}} \bar{D}_{\dot{C}} F_{\nu A} - 4iN\partial_\mu F_{A\dot{B}} \\ &\quad + 2i\bar{\sigma}_\mu^{\dot{C}C} \sigma^\nu_{A\dot{B}} \bar{F}_{\nu\dot{C}} + 2i(\sigma^\nu\bar{\sigma}_\mu)_A{}^C D_C \bar{F}_{\nu\dot{B}} \\ &\quad - 2i(\sigma^\nu\bar{\sigma}_\mu)_A{}^C \partial_\nu F_{C\dot{B}} - 2i(\bar{\sigma}_\mu\sigma^\nu)^{\dot{C}}{}_{\dot{B}} \partial_\nu F_{A\dot{C}} \,. \end{aligned}$$

Collecting terms, we find that this yields the following equation for $Y_{\mu A\dot{B}}$:

$$0 = 2(N+2) Y_{\mu A\dot{B}} - (\sigma_\mu\bar{\sigma}^\nu)_A{}^C Y_{\nu C\dot{B}} - (\bar{\sigma}^\nu\sigma_\mu)^{\dot{C}}{}_{\dot{B}} Y_{\nu A\dot{C}} + \sigma^\nu_{A\dot{B}} \bar{\sigma}_\mu^{\dot{C}C} Y_{\nu C\dot{C}} \,. \tag{A.4}$$

After multiplication with $\bar{\sigma}_\rho^{\dot{B}A}$, after only a little algebra we find

$$0 = (N+1)(Y_{\mu\nu} + Y_{\nu\mu}) - \eta_{\mu\nu} Y_\lambda{}^\lambda \,,$$

which implies (A.2) for $N \neq 1$. For $N = 1$, eq. (A.2) is reduced to $Y_\mu{}^\mu = 0$, i.e. eq. (15.7b).

The traceless part in i, j of eq. (A.4) is

$$0 = [(N+1)Y_{\mu\nu}{}^{i}{}_{j} + Y_{\nu\mu}{}^{i}{}_{j} - \eta_{\mu\nu}Y_{\lambda}{}^{\lambda i}{}_{j}] ,$$

which implies

$$0 = [(N+2)Y_{\mu\nu}{}^{i}{}_{j} - \eta_{\mu\nu}Y_{\lambda}{}^{\lambda i}{}_{j}] ,$$

i.e. eq. (A.3) for $N \neq 2$. For $N = 2$, eq. (A.3) is reduced to

$$0 = Y_{\mu}{}^{\mu i}{}_{j}|_{\text{traceless in } i,j} ,$$

i.e. eq. (15.7c).

Helpful discussions with P. Breitenlohner, R. Grimm and J. Wess are gratefully acknowledged.

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