ON OBSERVABLES OF QCD IN HIGH ORDER CALCULATIONS

M. MOSHE
DESY, Hamburg, Germany
and Tel-Aviv University, Tel-Aviv, Israel [1]

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The observables at short distance in quantum chromodynamics defined through scaling violations in the moments of the deep inelastic structure functions are determined here in terms of the single parameter $\Lambda = \Lambda(g^2, \mu)$ to all orders in the loop expansion in the form $M_i^n = [\log(Q^2/\Lambda^2)]^{a_n} \Sigma_{k=0} \Sigma_{j=0}^{k} C_{k,j}^{n,i} [\log(Q^2/\Lambda^2)]^{-k} [\log(\log(Q^2/\Lambda^2))]^j$. The constants $C_{k,j}^{n,i}$ in terms of the renormalization group functions and the coefficient functions define a set of observables and thus are invariant under such changes of renormalization conditions that induce, for example, the recently discussed mappings $g \to g'$ of the coupling constant plane. The general procedure for deriving these coefficients to all orders is presented and their implications on the study of the g-plane mappings and on practical high order calculations of scaling violations at high Q^2 are briefly discussed.

Though the short-distance behavior of quantum chromodynamics (QCD) is well understood [1], perfectly manageable and in agreement with existing experimental data, the study of its large-distance behavior has known only a limited success. Disentangling the large-distance properties of the theory calls for an understanding of its nonperturbative properties. Thus, the analytic structure of the Green's functions in the coupling constant plane are of major interest and recently were extensively studied. As a preliminary step 't Hooft had suggested [2] using mappings $g \to g'$ so that the description of the singularity structure of the Green's function can be easily concluded already from the known low-order expansion of the renormalization group functions (e.g. in the case of zero-mass fermions). Though presently much more work is needed along these lines of formulating convergent resummation for perturbation expansion in various field theories [2], it is interesting to study the implications of such studies on the well-understood regime of short distances. Obviously, the transformations $g \to g'$ should not change the physics and thus observables are invariant under such mappings which reflect a change in the renormalization conditions.

[1] Permanent address.

Calculations in QCD, performed while using a certain set of renormalization conditions, result in quantities whose numerical values are dependent on these conditions (see e.g. ref. [4]). Only certain combinations of these quantities are independent of the calculational scheme and form the set of observables of the theory. The explicit formulation of these observables in QCD at short distances, their dependence on the renormalization group functions and a single [3] parameter Λ at high orders are presented here. It is shown that scaling violations in the moments of deep inelastic structure functions are uniquely given by series of the form

$$\sum_{k=0} \sum_{j=0}^{k} C_{k,j} [\log(Q^2/\Lambda^2)]^{-k} \, [\log(\log(Q^2/\Lambda^2))]^j ,$$

where Λ is an explicit function of g and μ (the coupling constant and its renormalization point). Transformations of the type $g \to g'$ relate different coefficients in the β, γ and the coefficient function to form the invariant combinations $C_{k,j}(\gamma, \beta, c)$ which are constructed below. Our low-order results reproduce known relations derived in the past [1,4]. Higher-order results present new invariant combinations and the scheme for deriving them at any order is given below. Their implications on high-order calculations of scaling violations at high Q^2 and some practical approximations in such calculations are briefly discussed.

The coefficient functions in Wilson's operator product expansion [5] satisfy a renormalization group equation whose solution is

$$c_i^n(q^2/\mu^2, g) = c_i^n(1, \bar{g}(t,g)) \exp\left\{ -\int^{\bar{g}(t,g)} \frac{\gamma_n(g')}{\beta(g')} \mathrm{d}g' \right\}, \quad (1)$$

where $t = \frac{1}{2}\log(-q^2/\mu^2)$, μ is the renormalization point, $\bar{g}(t,g)$ is the effective coupling constant and $g = \bar{g}(t=0)$. $c_i^n(q^2/\mu^2, g)$ are measured experimentally through the moments of deep inelastic structure functions

$$M_i^n(-q^2) = \int_0^1 \mathrm{d}\xi\, \xi^{n-1} F_i(\xi, q^2) \underset{-q^2\to\infty}{\approx} c_i^n(q^2/\mu^2, g)\, \langle \mathrm{h}|O^{(n)}|\mathrm{h}\rangle. \quad (2)$$

Only the leading twist-two operators were usually [1, 3,4] kept in eq. (2) and only a single such operator contributes to the flavor non-singlet part of the structure functions.

Eq. (2) offers the perfect clean test of QCD at short distances once $c_i^n(q^2/\mu^2, g)$ are calculated. They have been calculated up to two-loop contributions in the renormalization group functions [4] and compared with the existing data [6]. To that order

$$M_i^n(-q^2 \equiv Q^2) \approx A_i^n [\log(Q^2/\Lambda^2)]^{-\gamma_n^{(0)}/2\beta_0} \times [1 + \log^{-1}(Q^2/\Lambda^2)(a_i^n + b_i^n \log(\log(Q^2/\Lambda^2)))]. \quad (3)$$

The parameter Λ can be defined in terms of μ^2 and g and thus, as it should, QCD is determined in terms of a single parameter [3]. The constants a_i^n and b_i^n associated with two different Q^2 dependences are certain known combinations [4] of the coefficients of the γ and β functions and the first non-trivial coefficient of $c_i^n(1, \bar{g})$. These combinations were shown, explicitly, to be separately independent of the renormalization conditions [4] [‡1].

[‡1] This demonstration in ref. [4] uses a particular example of a theorem by Stueckelberg and Peterman [7].

The general pattern started in eq. (3) can be extended to all orders giving the Q^2 dependence of $M_i(Q^2)$ and thus determining an infinite set of such invariants as a_i and b_i. For this to be done consistently and uniquely the single parameter $\Lambda = \Lambda(g, \mu^2)$ has to be explicitly determined to all orders in terms of the coupling constant g and the renormalization point μ (see e.g. ref. [3] for the first loop calculations). Denote

$$\gamma_n(g) = \sum_{k=0} \gamma_n^{(k)} g^{2k+2}, \quad \beta(g) = \sum_{k=0} \beta_k g^{2k+3}, \quad (4a)$$

$$c_i^n(1, \bar{g}) = \tilde{c}_i^n \sum_{k=0} c_i^{n,k} \bar{g}^{2k} \quad (c_i^{n,0} = 1). \quad (4b)$$

As emphasized by 't Hooft [2] our only knowledge and the definition of quantities like those in eqs. (4a, 4b) come from perturbation theory and its renormalization scheme. This, however, leaves us with much freedom (e.g. to choose an appropriate set of renormalization conditions). Thus, only directly measurable quantities, like poles in gauge invariant Green's functions and other observables, are independent of our renormalization scheme whereas most of the numerical values for $\gamma_n^{(k)}, \beta_k, c_i^{n,k}$ are entirely dependent on a particular definition of g. The certain combinations of $\gamma^{(k)}, \beta_k, c_i^{n,k}$ that form the observables-invariants of the theory are derived as follows: From eq. (1) one finds

$$c_i^n(Q^2/\mu^2, g) = A_i^n (\bar{g}^2)^{\gamma_n^{(0)}/2\beta_0} \sum_{k=0} B_i^{n,k} \bar{g}^{2k}. \quad (5)$$

The $B_i^{n,k}$ are given by

$$B_i^{n,k} = \sum_{l=0}^{k} c_i^{n,l} D_n^{k-l}, \quad (6)$$

where $c_i^{n,l}$ are defined in eq. (4b) and D_n^k are given below in terms of the coefficients in eq. (4a) for $k = 0$, 1, 2, 3, 4 (listed here for up to 5-loop contributions):

$$D_n^0 = 1, \; D_n^1 = a_1/2\beta_0, \quad (7)$$

$$D_n^2 = (1/4\beta_0)(a_2 - b_1 a_1) + a_1^2/8\beta_0^2,$$

$$D_n^3 = (1/6\beta_0)(a_1 b_1^2 - a_1 b_2 - b_1 a_2 + a_3)$$
$$+ (1/8\beta_0^2)(a_1 a_2 - a_1^2 b_1) + \tfrac{1}{48} a_1^3/\beta_0^3,$$
$$D_n^4 = (1/8\beta_0)(a_2 b_1^2 + 2a_1 b_1 b_2$$
$$- a_1 b_3 - a_1 b_1^3 - a_2 b_2 - a_3 b_1 + a_4)$$
$$+ (1/24\beta_0^2)(\tfrac{3}{4} a_2^2 - \tfrac{7}{2} a_1 a_2 b_1$$
$$+ \tfrac{11}{4} a_1^2 b_1^2 - 2a_1^2 b_2 + 2a_1 a_3)$$
$$+ (1/32\beta_0^3)(a_1^2 a_2 - a_1^3 b_1) + \tfrac{1}{384} a_1^4/\beta_0^4),$$

where $b_k = \beta_k/\beta_0$ and $a_k = \gamma_n^{(k)} + b_k \gamma_n^{(0)}$. The solution of $\mathrm{d}\bar{g}/\mathrm{d}t = \beta(\bar{g})$ with the initial condition $g(-q^2 = \mu^2) = g$ results in an implicit equation for $\bar{g}(t)$:

$$-\beta_0 \log(-q^2/\mu^2) + c(g)$$
$$= -\tfrac{1}{2}\bar{g}^{-2} - \tfrac{1}{2} b_1 \log \bar{g}^2 + F(\bar{g}), \qquad (8)$$

where $F(\bar{g}) = \Sigma_{k=1}(f_k/2k)\,\bar{g}^{2k}$ is regular at $\bar{g} = 0$ and $c(g)$ equals to the r.h.s. of eq. (8) at $\bar{g} = g$. Denote

$$g_0^2 \equiv [\beta_0 \log(-q^2/\Lambda^2)]^{-1}, \qquad (9)$$

then eq. (8) can be written in the form

$$g_0^{-2} = \bar{g}^{-2} + b_1 \log \bar{g}^2 - \sum_{k=1} \frac{f_k}{k}\bar{g}^{2k}, \qquad (10)$$

where the f_k are given by (up to 5-loop contributions)

$$f_1 = b_1^2 - b_2, \quad f_2 = 2b_1 b_2 - b_3 - b_1^3,$$
$$f_3 = 2b_1 b_3 - b_4 - 3b_1^2 b_2 + b_2^2 + b_1^4,$$
$$f_4 = 2b_1 b_4 - 3b_1^2 b_3 + 4b_1^3 b_2$$
$$- 3b_1 b_2^2 + 2b_2 b_3 - b_1^5 - b_5, \qquad (11)$$

and Λ is then given by

$$\Lambda^2 = \mu^2 (g^2)^{-\beta_1/\beta_0^2} \exp\left\{\frac{1}{\beta_0}\left[-\frac{1}{g^2} + \sum_{k=1}\frac{f_k}{k} g^{2k}\right]\right\}. \qquad (12)$$

Eq. (12) is the generalization [‡2] to all orders of eq. (2.4) in ref. [3]. The solution of eq. (10) for $\bar{g}(g_0)$ is:

$$\bar{g}^2(Q^2/\Lambda^2) = g_0^2 + g_0^4 b_1 \log g_0^2$$
$$+ g_0^6 b_1^2 [\log^2 g_0^2 + \log g_0^2 - f_1/b_1^2]$$
$$+ g_0^8 b_1^3 [\log^3 g_0^2 + \tfrac{5}{2}\log^2 g_0^2$$
$$+ (1 - 3f_1/b_1^2)\log g_0^2 - f_1/b_1^2 - f_2/2b_1^3] \qquad (13)$$
$$+ g_0^{10} b_1^4 [\log^4 g_0^2 + \tfrac{13}{3}\log^3 g_0^2 + (\tfrac{9}{2} - 6f_1/b_1^2)\log^2 g_0^2$$
$$+ (1 - 7f_1/b_1^2 - 2f_2/b_1^3)\log g_0^2$$
$$- f_1/b_1^2 + 2f_1^2/b_1^4 - f_2/2b_1^3 - f_3/3b_1^4] + O(g_0^{12}).$$

When $\bar{g}(Q^2/\Lambda^2)$ is inserted in eq. (5) one has

$$c_i^n(Q^2/\mu^2, g) = c_i^n(Q^2/\Lambda^2)$$
$$= A_i^n (g_0^2)^{\gamma_n^{(0)}/2\beta_0} \sum_{k=0} \sum_{j=0}^{k} C_{k,j}^{n,i}\, g_0^{2k} [\log g_0^2]^j. \qquad (14)$$

Since $c_i^n(Q^2/\Lambda^2)$ is measured through the moments in eq. (2) and each $C_{k,j}^{n,i}$ are associated with different Q^2 behavior they form a set of observables that are each independent of the specific scheme in which they had been calculated [‡3].

The first two orders in g_0^2 give the coefficients

$$C_{0,0}^{n,i} = 1, \quad C_{1,1}^{n,i} = (\gamma_n^{(0)}/2\beta_0^2)\beta_1, \qquad (15)$$
$$C_{1,0}^{n,i} = B_i^{n,1} = \frac{1}{2\beta_0}\left(\gamma_n^{(1)} - \frac{\beta_1}{\beta_0}\gamma_n^{(0)}\right) + c_i^{n,1},$$

which only reproduce the known results [8] that β_0, β_1, $\gamma_n^{(0)}$ and [4] (eq. (3)) $I_2 = (1/2\beta_0)\gamma_n^{(1)} + c_i^{n,1}$ are invariants. Higher orders in eq. (14) produce new results; but let us first note the structure of eq. (14) implied by eqs. (5) and (13). All $C_{k,k}^{n,i}$ and $C_{k,k-1}^{n,i}$ namely

‡2 Note that the definition of Λ^2 in eq. (12) differs from the Λ^2 defined in ref. [6] by $\Lambda_{[6]}^2 = \Lambda^2(\beta_0)^{-\beta_1/\beta_0^2}$.

‡3 Including β_0, which is by itself invariant, in the definition of $g_0(Q^2/\Lambda^2)$ does not invalidate this statement as will be shown below.

the "leading" and "next to leading" logarithms of g_0^2 at a given order of g_0^{2k}, are exactly determined by the calculations up to two loops in the renormalization group functions and are therefore known for all k. In general $C^{n,i}_{k,k-m}$ for $k \geqslant m \geqslant 1$ is fully determined by contributions from up to $m+1$ loops in $\beta(g)$, $\gamma(g)$ and $c_i^n(1,\bar{g})$. Thus, going from an m-loop to an $(m+1)$-loop calculation there is only one new $C^{n,i}_{m,j}$ that has to be calculated in the coefficient of g_0^{2m} in eq. (14). This is $C^{n,i}_{m,0}$, whereas all $C^{n,i}_{m,j}$ with $m \geqslant j > 0$ are known already from the m-loop calculations.

Thus, at each order in the loop expansion we find only one new invariant in $C^{n,i}_{m,0}$ and all $C^{n,i}_{m,j}$ with $m \geqslant j > 0$ are functions only of the invariants of the lower-loop calculations [‡4]. For example, at the three-loop level we have

$$C^{n,i}_{2,0} = B_i^{n,2} - f_1\,\gamma_n^{(0)}/2\beta_0, \tag{16a}$$

which determines a new invariant of the theory [‡5]. But

$$C^{n,i}_{2,2} = \frac{\gamma_n^{(0)}}{2\beta_0^5}\,\beta_1^4, \quad C^{n,i}_{2,1} = \frac{\gamma_n^{(0)}}{2\beta_0^2}\,\beta_1^2 + \frac{\beta_1}{\beta_0}\left(1 + \frac{\gamma_n^{(0)}}{2\beta_0}\right)\left[c_i^{n,1} + \frac{1}{2\beta_0}\left(\gamma_n^{(1)} - \frac{\beta_1}{\beta_0}\gamma_n^{(0)}\right)\right], \tag{16b}$$

define no new invariant since $\beta_0, \beta_1, \gamma_n^{(0)}$ and $I_2 = c_i^{n,1} + \gamma_n^{(1)}/2\beta_0$ (eq. 15) have been known already to be invariants from the two-loop calculations.

Similarly at the 4-loop and 5-loop level the new invariants are contained in $C^{n,i}_{3,0}$ and $C^{n,i}_{4,0}$ only. They are [‡6]

$$I_4 = B_i^{n,3} - \frac{\gamma_n^{(0)}}{4\beta_0}\,f_2 - \left[B_i^{n,1}\left(1 + \frac{\gamma_n^{(0)}}{2\beta_0}\right) + \frac{\gamma_n^{(0)}}{2\beta_0^2}\beta_1\right] f_1, \tag{17a}$$

$$I_5 = B_i^{n,4} - B_i^{n,2} f_1\left(2 + \frac{\gamma_n^{(0)}}{2\beta_0}\right) - B_i^{n,1}\left(1 + \frac{\gamma_n^{(0)}}{2\beta_0}\right)\left(f_1\frac{\beta_1}{\beta_0} + \tfrac{1}{2} f_2\right) - f_1\left(\frac{\beta_1}{\beta_0}\right)^2 \frac{\gamma_n^{(0)}}{2\beta_0} - \tfrac{1}{4} f_2 \frac{\beta_1}{\beta_0^2}\,\gamma_n^{(0)} - \frac{\gamma_n^{(0)}}{6\beta_0}\,f_3 + \tfrac{1}{4} f_1^2\,\frac{\gamma_n^{(0)}}{\beta_0}\left(\frac{\gamma_n^{(0)}}{2\beta_0} + 3\right). \tag{17b}$$

It is easy to see now how to derive higher such combinations but the expressions become pretty long and we will stop here at the 5-loop level. Indeed also here one finds that $C_{3,3}$, $C_{3,2}$ and $C_{3,1}$ are functions only of $\beta_0, \beta_1, \gamma_n^{(0)}, I_2$ and I_3. Similarly all $C_{4,j}$ with $4 \geqslant j > 0$ are functions of these invariants and of I_4.

In detailed m-loop calculations, certain renormalization conditions can be shown to ease one piece of the calculation or other, but the existence of the invariants $C^{n,i}_{k,j}$, which are independent of the scheme used, restrict one's ability to simplify the exact calculation.

The two-loop calculations [4] are sufficient for accommodating [6] the presently available data while more accurate future experiments may test higher-loop corrections. The structure of eq. (14) enables one, as discussed above, to extract certain approximations for the m-loop results from the known $m-1$ calculations. For example, from the already available two-loop calculations [4] one has the exact values for the "leading" pieces $g_0^4\,[C_{2,2}\log^2 g_0^2 + C_{2,1}\log g_0^2]$ (eq. (16b)) while only in $g_0^4 C_{2,0}$ there is a missing contribution that has to be evaluated by detailed three-loop calculations. An estimate of the 4-loop contribution can be obtained once the 3-loop calculations are done and so on. In view of the complexity of such calculations an approximate result is certainly of some use for a comparison with future data.

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Note added. The fact, discussed above, that the observables $C^{n,i}_{k,j}$ are indeed independent of the gauge and the renormalization scheme had been also emphasized in a recent study, received after the completion of the

[‡4] This confirms the remark in footnote 3.

[‡5] As in the case of $C^{n,i}_{1,0}$ also $C^{n,i}_{2,0}$ contains pieces that are known to be invariant from the lower loop (2-loops) result. When these pieces are subtracted off one finds the new invariant

$$I_3 = \frac{\gamma_n^{(2)}}{4\beta_0} + \frac{\beta_2\gamma_n^{(0)}}{4\beta_0^2} - \frac{\beta_1\gamma_n^{(1)}}{4\beta_0^2} + \frac{\gamma_n^{(1)2}}{8\beta_0^2} + c_i^{n,1}\frac{\gamma_n^{(1)}}{2\beta_0} + c_i^{n,2}.$$

[‡6] One can further subtract from I_4 (and I_5) pieces that depend only on $\beta_0, \beta_1, \gamma_n^{(0)}, I_2, I_3$ (and I_4).

present work, by W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta (FNAL-78/42, May 78) who had recalculated $C_{1,0}^{n,i}$ and discussed its phenomenological implications.

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