OPERATOR SOLUTION TO BROKEN QCD(N)$_2$ FOR MASSLESS QUARKS

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By explicit construction of solutions in gauges satisfying the Lorentz condition we give an operator realization of the dynamically broken Patrascioiu phase in QCD(N)$_2$ for massless quarks. The SU(N) colour gauge group is reduced to its maximal abelian subgroup. The $N-1$ diagonal gluons become massive and there is a massless free U(1) current. The contrast with the 't Hooft phase, established by the $1/N$ method, supports the idea of two distinct phases.

Are there two distinct phases [1] in QCD(N)$_2$ (i.e., SU(N) quantum chromodynamics in two-dimensional spacetime [2]) depending on the value of the bare gauge coupling g? The $1/N$ expansion method [3], pioneered by 't Hooft [4], has established a nontrivial dynamics in the limit when $g \to 0$ proportional to $N^{-1/2}$ as $N \to \infty$. This scheme in which gluons remain massless (and hence are not independent degrees of freedom [‡1]) while colour is completely confined – we shall call the 't Hooft phase. It is fairly [‡2] well-defined in the weak coupling domain $g(N/\pi)^{1/2} < M$ – the mechanical quark mass in the QCD lagrangian. For $g(N/\pi)^{1/2} > M$, the validity of the 't Hooft phase is dubious. Gauge invariance difficulties with the 't Hooft solution for M tending to zero were found some time ago [5]. It now appears [6] that in a general axial gauge there is trouble in solving the quark self-energy equation as soon as M is put lower than $g(N/\pi)^{1/2}$. It is quite likely that a different physical picture holds in the strong coupling regime $g(N/\pi)^{1/2} > M$. This intriguing possibility has already been considered by Patrascioiu [7]. He proposed a new phase by working out a different solution to QCD(N)$_2$ for $M = 0$ and conjectured the validity of its properties for nonzero M up to $g(N/\pi)^{1/2}$.

In Patrascioiu's phase the colour SU(N) group spontaneously breaks down to its maximum abelian subgroup S(U(1) × U(1) × ..., N factors). The dynamics is given by an underlying free field theory in terms of $N-1$ col-

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[‡1] Since there are no transverse directions in one spatial dimension, massless gluons cannot represent independent degrees of freedom. This may also be seen by counting the dynamical degrees of freedom from the time-evolution parts and the constraint parts of the QCD equations of motion [7].

[‡2] There is some controversy with the proponents of the alternative infrared regularization procedure advocated by Wu [12]. We merely note that even in this procedure the branch-cuts in the poleless quark propagator are along the imaginary axis for $(g^2N/\pi)^{1/2} < M$ but are spacelike for $(g^2N/\pi)^{1/2} > M$ again suggesting two distinct phases. For spacelike branch cuts there cannot be an indefinitely rising 't Hooft bound-state spectrum since in a general axial gauge the Bethe–Salpeter equation cannot be once integrated to give a Schrödinger-like equation as done in [2] and the singularities of the quark propagator show up in the B–S amplitude [13]. In fact, for the $M = 0$ case the bound-state equation has been considered in Wu's approach [14] and, as in the Patriascioiu phase, only massless colour-singlet bound states have been found.

our diagonal gluons, all massive and expressible as curls of free massive pseudoscalar fields. However, as admitted by Patrascioiu himself, his deduction of these results was not very reliable since he did not fully take into account the operator nature of various fields. His procedure involved naive manipulations of operator products as well as the treatment of quark fields as c-number functions. Thus his argument for the existence of the new phase has not seemed very compelling so far. It is clear nevertheless that a more rigorous construction of Patrascioiu's solution will support the idea of two distinct phases in $\mathrm{QCD}(N)_2$.

We have found an operator realization of Patrascioiu's picture (for massless quarks and any positive integral $N > 1$) as described here in essence. Our results are not a priori surprising since many massless fermionic field theories in (1 + 1) dimensions are known to be explicitly solvable. The method used by us is a generalization of that of Lowenstein and Swieca [8], employed for Schwinger electrodynamics, to which the reader is referred for procedural introduction. We first give an operator definition of $M = 0$ $\mathrm{QCD}(N)_2$ as an SU(N)-invariant theory in terms of equations of motion involving ordered products and regularized fermionic source currents introduced as immediate nonabelian generalizations of that in ref. [8]. Then, in gauges obeying the Lorentz-condition, we construct quark and gluon field solutions in terms of free massless fermions and free massive and massless scalar bosons. (This last set of objects are defined with indefinite metric.) We believe that our construction makes Patrascioiu's solution much more acceptable.

It is in the constructed solutions (and *not in the operator definition* of the theory) that SU(N) is dynamically broken down to $\mathrm{S}(\{\mathrm{U}(1)\}^N)$. Diagonal gluons become realizable physical states as free massive vector fields and represent longitudinal degrees of freedom. Nondiagonal ones remain massless, do not [‡1] represent independent degrees of freedom, are not canonical quantum operators and can be chosen to vanish. One may compare with abelian Schwinger electrodynamics which possesses only one independent longitudinal degree of freedom manifested through the photon acquiring a mass. In the present case, despite one U(1) and $N^2 - 1$ SU(N) currents, they have between them only N independent degrees of freedom coming in fact from N quarks (see also the construction of nondiagonal SU(2) currents in terms of diagonal U(2) currents in ref. [9]). One of these is the U(1) current and the $N - 1$ others are mutually commuting SU(N) currents which may be chosen to be diagonal. When the SU(N) interaction is switched on, a dynamical Higgs mechanism operates so that the $N - 1$ massless fields existing in the form of SU(N) currents get devoured by $N - 1$ gauge gluons and provide mass as well as physical existence [‡1] to the latter. This mechanism is essentially abelian and breaks down the nonabelian group to its maximum abelian subgroup (cf. [10]).

From our construction, the quark equation of motion is satisfied by straight differentiation. Further, from the regularization of quark source currents, it is automatic that finiteness for colour-diagonal members must mean vanishing for colour nondiagonal components of the current multiplet. Gluon equations of motion are then satisfied modulo certain zero-norm vector operators. In analogy with the Gupta–Bleuler formalism, an explicit construction of a physical subspace is given where these zero-norm vectors vanish between any two physical states. The theory then is well-defined, and may be seen to manifestly contain the features that Patrascioiu had. Consider first our operator definition of $\mathrm{QCD}(N)_2$. The equations of motion in a transparent notation are

$$-\mathrm{i}\partial\!\!\!/ q(x) = g\,\tfrac{1}{2}\lambda^i\!:\! A\!\!\!/^i q\!:\!(x), \qquad (\nabla^\nu F_{\mu\nu})^i(x) = gJ^i_\mu(x). \tag{1, 2}$$

The double dots mean some well-defined normal ordered product. The nonlinear term in the gauge-covariant derivative ∇, if present, must also be defined with appropriate normal-ordering. As generalized from ref. [8], the quark source currents are stipulated to have the regularized form [‡3]

$$J^i_\mu(x) = \lim_{\epsilon\to 0} f^{-1}(\epsilon)[\bar{q}(x+\epsilon)\gamma_\mu\,\tfrac{1}{2}\lambda^i q(x) - \text{v.e.v.} + \mathrm{i}g\epsilon^\nu A^j_\nu(x)\langle 0|\bar{q}(x+\epsilon)\gamma_\mu\,\tfrac{1}{2}\lambda^j\,\tfrac{1}{2}\lambda^i q(x)|0\rangle]\,. \tag{3}$$

In eq. (3) $f(\epsilon)$ is a Lorentz and SU(N)-invariant c-number function which makes the square-bracketed quantity

‡3 This is the regularised infinitesimal form of the gauge-covariant path-ordered bilinear $[\bar{q}(x+\epsilon)\gamma_\mu P\exp[\mathrm{i}g\,\tfrac{1}{2}\lambda^k\int_x^{x+\epsilon}\mathrm{d}z^\mu A^k_\mu(z)] \times \tfrac{1}{2}\lambda^i q(x)]$.

regular as $\epsilon \to 0$. This being a superrenormalizable theory, in some special gauges the latter object may be expected to be regular as $\epsilon \to 0$ in which case $f^{-1}(0)$ would be a finite wave-function renormalization constant. But this is untrue in general gauges and specifically in those we are interested in. There the quark-bilinear in question has a divergence in the limit of vanishing ϵ that is multiplicatively removed from the source current by the right choice of $f(\epsilon)$ which is also singular in the same limit.

To construct our solution, we start with the massless free fermionic field $\psi^a(x)$ obeying $-\mathrm{i}\not\partial\psi^a(x) = 0$. These belong to the fundamental SU(N) representation, the superscript a being usually not displayed. Let $\tilde{\phi}^d$ be $N-1$ free pseudoscalar fields of mass m (a parameter to be specified later) and $\tilde{\eta}^d$ be $N-1$ massless (quantized with indefinite metric) [8] pseudoscalar fields. Throughout, we use the superscripts i, d and n respectively to denote N^2-1 operators in the adjoint representation typified by the matrices λ^i, $N-1$ operators corresponding to the element set D comprising the diagonal λ's and N^2-N operators corresponding to the nondiagonal λ's respectively. Double dots will indicate Klaiber ordering, i.e. $:AB: A^{(+)}B + BA^{(-)}$ for a field A canonically decomposable into positive and negative frequency parts, $A = A^{(+)} + A^{(-)}$, $A^{(-)}|0\rangle = 0$. The interacting quark field is constructed as

$$q(x) = :\exp[\mathrm{i}\sqrt{2\pi}\,\gamma^5\,\tfrac{1}{2}\lambda^d\{\xi\tilde{\phi}^d(x) + \tilde{\eta}^d(x)\}]\,\psi(x):, \qquad \xi^2 > 1, \tag{4a, b}$$

where ξ is a real parameter obeying eq. (4b). The gluons are chosen as

$$A^d_\mu(x) = -g^{-1}\sqrt{2\pi}\,\epsilon_{\mu\nu}\partial^\nu\{\xi\tilde{\phi}^d(x) + \tilde{\eta}^d(x)\}, \qquad A^n_\mu(x) = 0, \tag{5a, b}$$

i.e. the diagonal ones obey the equation $(\Box + m^2)A^d_\mu = 0$, nondiagonal ones are absent and $\partial^\mu A^i_\mu = 0$. In such gauges the field tensors $F^i_{\mu\nu}$ are

$$F^d_{\mu\nu}(x) = -g^{-1}m^2\sqrt{2\pi}\,\xi\epsilon_{\mu\nu}\tilde{\phi}^d(x), \qquad F^n_{\mu\nu}(x) = 0. \tag{6a, b}$$

Among the equations of motion, eq. (1) is automatically satisfied. The lhs of eq. (2) is $\partial^\nu F^d_{\mu\nu}$ and zero respectively for $i = d$ and n. To obtain the quark source current, consider the quark bilinear $\not{\Sigma}_r q^{\dagger a}_r(x+\epsilon)q^b_r(x)$ where r is a spinor index and $\not{\Sigma}$ implies no summation despite repetition. The bilinear is factorizable as a c-number times an operator if all exponentials involving (+, –) frequency parts are moved to the (left, right) à la Klaiber. Thus [8]

$$\begin{aligned}\not{\sum_r} q^{\dagger a}_r(x+\epsilon)q^b_r(x) = {}&:\exp[-\mathrm{i}(-1)^r\sqrt{2\pi}\,\{(\lambda^d/2)^{aa}(\xi\tilde{\phi}^d(x+\epsilon) + \tilde{\eta}^d(x+\epsilon))\\ &-(\lambda^d/2)^{bb}(\xi\tilde{\phi}^d(x) + \tilde{\eta}^d(x))\}]\,\psi^{\dagger a}_r(x+\epsilon)\psi^b_r(x):\,\exp[\pi(\delta^{ab} - N^{-1})^{-1}\mathrm{i}\{\xi^2\Delta^{(-)}(\epsilon) - D^{(-)}(\epsilon)\}]\,.\end{aligned} \tag{7}$$

Here Δ and D are the two-point functions for massive and massless pseudoscalar bosons respectively and we have used the relation $(\lambda^d)^{aa}(\lambda^d)^{bb} = 2(\delta^{ab} - N^{-1})$. Since $\xi^2 > 1$ by construction, $\mathrm{i}^{-1}\{\xi^2\Delta^{(-)}(\epsilon) - D^{(-)}(\epsilon)\}$ blows up as $\epsilon \to 0$. Hence the only divergence in the rhs of eq. (7) in that limit is for the colour diagonal case $a = b$ and is multiplicatively removed as in eq. (3) only by the choice

$$f(\epsilon) = \exp[-\mathrm{i}\pi(1 - N^{-1})\{\xi^2\Delta^{(-)}(\epsilon) - D^{(-)}(\epsilon)\}] \tag{8}$$

at the expense of making J^n_μ vanish. Eq. (7) now reduces via the vacuum expectation value

$$\langle 0|\psi^{\dagger a}_r(x+\epsilon)\psi^b_s(x)|0\rangle = -\mathrm{i}(2\pi)^{-1}[\epsilon^0 - (-1)^r\epsilon^1]^{-1}\delta_{rs}\delta^{ab}$$

to

$$J^d_\mu(x) = j^d_\mu(x) - (2\pi)^{-1/2}\partial_\mu\eta^d(x) - (2\pi)^{-1/2}\xi\epsilon_{\mu\nu}\partial^\nu\tilde{\phi}^d(x), \qquad J^n_\mu(x) = 0. \tag{9a, b}$$

Here η^d are massless scalar fields associated with $\tilde{\eta}^d$, $\partial_\mu\eta^d = \epsilon_{\mu\nu}\partial^\nu\tilde{\eta}^d$ and $j^d_\mu(x) = \mathrm{n}[\bar{\psi}\gamma_\mu\frac{1}{2}\lambda^d\psi](x)$ are the Wick-ordered free fermionic SU(n) currents.

The choice [‡4]

[‡4] The nonabelian legacy lingers in the form of the factor 2π related to $[J^d_0(x^0,x^1), J^{d'}_0(x^0,y^1)] = \mathrm{i}(2\pi)^{-1}\delta^{dd'}\partial_1\delta(x^1 - y^1)$. The corresponding factor is π in the abelian case [8].

$$m^2 = g^2(2\pi)^{-1} \tag{10}$$

and the substitution of eqs. (5) reduce the nontrivial gluon equations of motion to

$$(\nabla^\nu F_{\mu\nu})^d - gJ^d_\mu = g[j^d_\mu - (2\pi)^{-1/2}\partial_\mu \eta^d]\,. \tag{11}$$

Thus Maxwell's equations are not satisfied in strict operator form, but the culprit operators in the rhs of eq. (11) are of zero norm

$$\langle 0|\,[j^d_\mu(x) - (2\pi)^{-1/2}\partial_\mu \eta^d(x)]\,[j^d_\nu(y) - (2\pi)^{-1/2}\partial_\nu \eta^d(y)]\,|0\rangle = 0. \tag{12}$$

In fact, this is a posteriori the rationale for the choice of $\sqrt{2\pi}$ as the factor to the left of γ^5 in eq. (4a). The zero norm operator can be rendered physically irrelevant by choosing a subspace $\mathcal{H}_{\rm ph}$ of physical states obeying the condition

$$[j^d_\mu(x) - (2\pi)^{-1/2}\partial_\mu \eta^d(x)]^{(-)}|\Psi\rangle = 0, \qquad |\Psi\rangle \in \mathcal{H}_{\rm ph}. \tag{13}$$

$\mathcal{H}_{\rm ph}$ can be generated by polynomials of the operators $F^d_{\nu\lambda}, j^d_\lambda - (2\pi)^{-1/2}\partial_\lambda \eta^d$ and $:\exp[\mathrm{i}(\pi/2)^{1/2}\lambda^d \eta^d]\,q:$ (all of which commute with $j^d_\mu - (2\pi)^{-1/2}\partial_\mu \eta^d$) operating on the vacuum.

We can make the following additional comments. 1) The U(1) source current is obtainable as a generalization of eqs. (3) and (9):

$$J_\mu(x) = \lim_{\epsilon\to 0} f^{-1}(\epsilon)[\bar q(x+\epsilon)\gamma_\mu q(x) - \text{v.e.v.} + \mathrm{i}g\epsilon^\nu A^i_\nu \langle 0|\bar q(x+\epsilon)\gamma_\mu \tfrac{1}{2}\lambda^i\, q(x)|0\rangle] = j_\mu(x), \tag{14}$$

where $j_\mu(x) = \mathcal{N}\,[\bar\psi\gamma_\mu\psi](x)$ is the Wick-ordered massless [‡5] free fermionic U(1) current. This is in agreement with the conclusion of refs. [7] and [10]. 2) All the quark Wightman functions can be evaluated readily as generalizations of the abelian expression given by Lowenstein and Swieca [8] and need not be reproduced here. 3) The solution is a covariant one in that q has the same Lorentz spin (i.e. 1/2) and the same statistics (i.e. Fermi) as the free field. 4) The vanishing of A^n_μ is symptomatic of the dynamical breakdown of SU(N). Nondiagonal SU(N) charges do not exist, otherwise they would have to satisfy the impossible relation $[Q^i, A^j_\mu] = \mathrm{i} f^{ijk}A^k_\mu$. Diagonal SU($N$) charges do exist and are $Q^d = \int_{-\infty}^{\infty} \mathrm{d}x^1\, J^d_0(x)$, the gluon-contribution vanishing in the absence of nondiagonal gluons. 5) The quark source currents J^i_μ do not obey the SU(N) current-algebra contrary to the expectation from a naive extension of the classical Noether argument. This is because in this operator field theory with dynamical symmetry breakdown, the naive argument is invalidated [11]. The conserved free fermionic currents j^i_μ do obey SU(N) current algebra and can be rewritten as appropriately gluon-regulated SU(N) currents of the interacting quark field. However, these currents are a priori gauge-noncovariant and hence uninteresting. 6) Our solution may be compared with the previous work [15] of Bhattacharya and one of us (P.R.). There the SU(N) Thirring model without abelian coupling was given a gauge interpretation as a massless QCD in (1 + 1) dimensions. However, that required identifying the fermionic SU(N) Thirring currents as gluons. The latter then are Lie fields and invariance under c-number gauge transformations is necessarily lost; only that under a specific operator colour transformation is preserved. Such is not the case in the present approach where the $N-1$ diagonal gluons are ordinary massive vector fields and not Lie fields.

We have not touched upon the question of a fully positive metric solution in this brief communication. In fact, as in [8], a new set of such solutions can be obtained by the operator gauge transformation $q(x) \to q'(x) = :\exp \times [\mathrm{i}(\pi/2)^{1/2}\lambda^d\eta^d(x)]\,q(x):$ and $A^d_\mu \to A^{d\prime}_\mu(x) = A^d_\mu(x) + g^{-1}(2\pi)^{1/2}\epsilon_{\mu\nu}\partial^\nu\tilde\eta^d(x)$. However, this is at the expense of cluster decomposition and covariance; the Lorentz spin of q' is $(2N)^{-1}$ and it obeys a mixed statistics. The structure of the vacuum is correspondingly complicated but admits of a classification in terms of sectors labelled by a set of θ-phases. Another interesting point is the evaluation of gauge-invariant quantities and the comparison to the lowest nontrivial order in g with perturbative results. Also, the effects of the insertion of a mass term in the quark

‡5 Presumably, if there is a small quark mass, this becomes a solitonic degree of freedom.

equation of motion can be studied by the methods employed recently in analyzing the Schwinger model. We shall address ourselves to these issues in a future lengthier publication.

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