Nuclear Physics B151 (1979) 353–356

SELF-DUAL Z(N) GAUGE THEORIES ‡

Aharon CASHER
Deutsches Elektronen-Synchrotron DESY, Hamburg, Germany
and
Dept. of Physics, Tel-Aviv University, Ramat-Aviv, Israel *

Received 1 September 1978

It is shown that the Villain Z(N) gauge theory in four dimensions is self-dual for all N. The critical temperature is computed and shown to vanish when $N \to \infty$.

1. Introduction

It has been suggested [1,2] that the field configurations which control the behaviour of the Wilson loop in a pure gauge theory are distributions of quantized fluxons. The fluxons induce a change of phase in units which belong to the centre of the gauge group (Z(N) for SU(N)), and if the distribution is sufficiently random [2], the average of the loop will decrease according to an area law. Due to the quantization condition the fluxon system may be approximately described by an effective Z(N) gauge theory, and the central issue is then to determine whether the effective coupling is above the critical point. It should be stressed that according to this view, the relevant effective coupling is not subject to variation but is completely fixed by the interaction between the Z(N) system and the background of SU(N)/Z(N) fields. It thus seems useful to investigate the properties of lattice Z(N) gauge theories. Such investigations have recently been carried out [3,4] and self-duality was proved for the Z(2,3,4) theories in four dimensions. This note is concerned with a simple generalization of these results to the Z(N) case. More specifically, by using a Villain-type action the duality transformation may be exhibited explicitly for all N and an extremely simple expression for the critical temperature ensues which vanishes as $N \to \infty$. The Wilson loop is easily evaluated in the high-temperature limit and as expected behaves according to the area law. No attempt is made to analyse the transition and identify the excitations whose condensation controls the critical behaviour.

‡ Work supported in part by the Israel Commission for basic research and the US-Israel Bi-National Foundation.
* Permanent address.

2. The Z(N) Villain gauge theory

The concept of duality involves a transformation between order and disorder variables. More concretely, the idea is that in a phase where the relevant variables are ordered, their canonical conjugates should be disordered and *vice versa.* Hence, the basic idea is to perform a Fourier transform in the configuration space of the local variables. It may thus be expected that gaussian interactions should have particularly transparent duality properties. These considerations have led to a fruitful use of the Villain form of lattice spin systems in analysing critical behaviour [5–7]. In what follows, we shall use the technique employed by BKM [6] to investigate the Villain Z(N) gauge theory in four dimensions.

As usual in lattice gauge theory, the basic variables are defined on links, and will be the Z(N) phases:

$$A_\mu(x) = \frac{2\pi}{N} a_\mu(x) , \quad a_\mu = 0, 1, ..., N-1 . \tag{1}$$

The field strengths are defined on plaquettes by forming the discrete curl of A:

$$F_{\mu\nu}(x) = A_\mu(x) - A_\mu(x+\nu) - A_\nu(x) + A_\nu(x+\mu) , \tag{2}$$

where $\mu = 1, ..., 4$ designates a unit vector on the lattice.

The Villain action is defined by:

$$S = \sum_{x,\mu\nu} S_{\mu\nu}(x) , \tag{3}$$

$$e^{-\beta S_{\mu\nu}(x)} = \sum_{n_{\mu\nu}(x)=-\infty}^{\infty} \exp\{-\tfrac{1}{2}\beta[F_{\mu\nu}(x) - 2\pi n_{\mu\nu}(x)]^2\} . \tag{4}$$

Clearly the low-temperature ($\beta \to \infty$) behaviour is equivalent to that of the usual ($S_{\mu\nu} = 1 - \cos F_{\mu\nu}$) action. We now follow BKM and use the Poisson summation formula to Fourier-transform eq. (4) (equivalently, a well-known theta function identity may be used):

$$e^{-\beta S_{\mu\nu}} = (2\pi\beta)^{-1/2} \sum_{l_{\mu\nu}=-\infty}^{\infty} \exp\left\{il_{\mu\nu}F_{\mu\nu} - \frac{1}{2\beta} l_{\mu\nu}^2\right\} . \tag{5}$$

The partition function may now be evaluated by summing over the configurations $\{A_\mu\}$:

$$Z(\beta) = \sum_{\{A_\mu\}} e^{-\beta S} = (2\pi\beta)^{-3V} \sum_{\{l_{\mu\nu}(x)\}} \exp\left\{-\frac{1}{2\beta} \sum_{x,\mu\nu} l_{\mu\nu}^2(x)\right\} \sum_{\{A_\mu(x)\}} \times \exp\{i \sum_{x,\mu\nu} A_\mu \partial_\nu l_{\mu\nu}\} . \tag{6}$$

In eq. (6) V is the total number of links, while ∂ is the discrete derivative. The sum over A_μ may now be performed, resulting in a δ-function constraint on the variables $l_{\mu\nu}$ (using eq. (1))

$$N^{-1} \sum_{a_\mu=0}^{N-1} \exp\, i \frac{2\pi}{N} a_\mu \partial_\nu l_{\mu\nu} = \delta\,[\partial_\nu l_{\mu\nu}, 0\ (\mathrm{mod}\ N)]\ . \tag{7}$$

In order to satisfy the constraint identically we now represent $l_{\mu\nu}(x)$ in the form:

$$l_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}(\partial_\alpha b_\beta - \partial_\beta b_\alpha - N m_{\alpha\beta})\ , \tag{8}$$

where (b, m) are integers. Substituting eq. (8) in the partition function (6) and pulling out a factor $2\pi/N$ out of the brackets of eq. (8) now leads back to the original expression with a new temperature:

$$Z(\beta) = \left(\frac{2\pi\beta}{N^2}\right)^{-3V} Z\left(\frac{N^2}{4\pi^2\beta}\right) . \tag{9}$$

In other words, the Fourier variables $(2\pi/N)\, l_{\mu\nu}$ are dual to the original fields $F_{\mu\nu} - 2\pi n_{\alpha\beta}$ and in terms of them a low-temperature system is converted into the same high-temperature system. Assuming there is one critical point, we readily find from eq. (9):

$$\beta_c = \frac{N}{2\pi}\ . \tag{10}$$

The high-temperature limit of the Wilson loop easily be calculated by using the dual form of the action. The loop is defined by:

$$W(\mathrm{C}) = Z^{-1} Z(J) \tag{11}$$

where J_μ is the conserved "current" of external charges directed along the loop C, and $Z(J)$ is:

$$Z(J) = \sum_{\{A\}} \exp\{-\beta S + i \sum A_\mu J_\mu\}\ . \tag{12}$$

Applying the transformation (5) to eq. (12) leads to eqs. (6), (7) with the change:

$$\partial_\mu l_{\mu\nu} = 0(\mathrm{mod}\ N) \to \partial_\nu l_{\mu\nu} + J_\mu = 0(\mathrm{mod}\ N)\ . \tag{13}$$

Now when $\beta \to 0$, only the lowest allowed values of $l_{\mu\nu}$ will contribute due to the Boltzman factor $e^{-l^2/2\beta}$. The lowest configuration which satisfies the constraint eq. (13) is:

$$l_{\mu\nu}(x) = \begin{cases} J(\mathrm{mod}\ N)\,, & x \in \mathcal{A}(\mathrm{C})\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{14}$$

where $\mathscr{A}(\mathrm{C})$ is the smallest area enclosed by C. Hence, we find for $W(\mathrm{C})$:

$$W(\mathrm{C}) \underset{\beta\to\infty}{\sim} \exp\left\{-\frac{1}{2\beta}\,\mathscr{A}(\mathrm{C})\,J(\mathrm{mod}\,N)\right\}. \qquad (15)$$

Note that the dominant high-temperature configuration which contributes is one in which the "canonical conjugate" of A_μ has a definite value fixed by a generalized Gauss theorem. This is obviously a realization of the flux-tube confinement mechanism. We further note that eq. (15) is periodic with respect to the magnitude of the external charges which flow along the loop C.

We finally note that the limit $N \to \infty$ may easily be taken in eq. (4). In fact, when $N \to \infty$, the sum over A becomes an integral in the interval $(0,2\pi)$ and the Z(N) theory becomes Abelian compact QED in the version studied by BKM. The dual form makes it clear that in this limit the (mod N) may be dropped from eq. (7) since values of $l_{\mu\nu}$ of order $N \to \infty$ will cost an infinite action (provided β is fixed). We thus recover the BKM duality transformation. Note, however, that the critical temperature, eq. (10), goes to zero when $N \to \infty$. Thus, the phase transition of compact QED studied by BKM is a phenomenon associated with the limit $N \to \infty$ which disappears for finite N. A partial formal explanation is that the transition to a monopole gas may be performed explicitly only when the range of the variables b_μ (eq. (8)) is infinite; otherwise the integrations performed by BKM are not gaussian. Presumably the effective non-linear interactions represented by the restricted integration range destroy the BKM phase transition. In view of the above it would seem unlikely that the monopole-pair dissociation mechanism is responsible for confinement in the non-Abelian theory.

I thank K. Symanzik for several profitable discussions. I would also like to express my gratitude to the DESY theory group for their hospitality and generosity.

References

[1] G. 't Hooft, Utrecht preprint (1978).
[2] Y. Aharonov, A. Casher and S. Yankielowitz, Nucl. Phys. B146 (1978) 256.
[3] T. Yoneya, CCNY preprint (1978).
[4] C.P. Korthals Altes, Nucl. Phys. B142 (1978) 315.
[5] J. Jose, L. Kadanoff, S. Kirkpatrick and D. Nelson, Phys. Rev. B16 (1977) 1217.
[6] T. Banks, J. Kogut and R. Myerson, Nucl. Phys. B129 (1977) 493.
[7] M. Peskin, Harvard preprint 77/A083.