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CONFINEMENT AND CHIRAL SYMMETRY BREAKING IN CP^{n-1} MODELS WITH QUARKS

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We study the low-energy physics of quarks in two dimensions, which are minimally coupled to CP^{n-1} fields. Within the $1/n$ expansion, heavy quarks are confined, the $U(1)_A$ chiral symmetry is spontaneously broken and a light isoscalar pseudoscalar boson is avoided *via* the axial anomaly.

1. Introduction

That spontaneous chiral symmetry breaking in QCD is not accompanied by an isoscalar Goldstone boson has only recently been understood as an instanton effect [1,2]. The infrared divergent dilute instanton gas approximation is the only calculation supporting this explanation and one has no means to show that its predictions are qualitatively correct at low energies. To get an idea about this and related questions, we here propose to study a two-dimensional theory, the CP^{n-1} model [3,4] with quarks [5,4], which is very similar to QCD_4: it is asymptotically free, there are instantons and the $U(1)_A$ current has an anomaly proportional to the topological density. In contrast to QCD_4, CP^{n-1} models can be $1/n$ expanded, and since this expansion is infrared stable, one is able to reliably analyze, e.g., the spontaneous breaking of chiral symmetry.

At first sight, one might object that a continuous symmetry in two dimensions cannot be spontaneously broken and consequently there is no $U(1)_A$ problem. Because of the anomaly, however, Coleman's theorem does not apply here so that it may be perfectly consistent to have $\langle\bar{\psi}\psi\rangle \neq 0$ and an η-propagator $\langle(\bar{\psi}\gamma_5\psi)(x) \times (\bar{\psi}\gamma_5\psi)(0)\rangle$ decaying with some power of $|x|$ (i.e., there would be no mass gap but no Goldstone boson either). Besides this, the $U(1)_A$ symmetry could be realized in a phase, where it is almost spontaneously broken in the sense that quarks acquire a spontaneous mass, but $\langle\bar{\psi}\psi\rangle = 0$ and $\langle(\bar{\psi}\psi)(x)(\bar{\psi}\psi)(0)\rangle \sim |x|^{-\alpha}$ $(|x| \to \infty)$ (this is the Berezinski-Kosterlitz-Thouless phase [6] familiar from the XY model). Thus, massless excitations in two dimensions are not *a priori* excluded and it is a non-trivial property of the theory if there is a mass gap in the η channel.

The glue part in CP^{n-1} models with quarks is the pure CP^{n-1} non-linear σ-model, which has already been analyzed in great detail. We shall assume here that the reader has read our first paper [4] on this subject. To find a geometrically natural way of how to couple quarks to CP^{n-1} fields, we discuss in sect. 2 the supersymmetric version of the CP^{n-1} model. From the point of view of supersymmetry this model is in itself very interesting, because it has an O(2) extended supersymmetry (sect. 3). Readers not interested in the supersymmetric model may skip sects. 2 and 3 and jump to sect. 4, where we write down an action for heavy flavoured quarks interacting in an $U(N) \times U(N)$ symmetric manner with CP^{n-1} fields. The $1/n$ expansion of this model then reveals a rich physical structure, which we discuss in sect. 5. Conclusions are drawn in sect. 6 and an appendix is included describing some properties of the Dirac equation in the presence of a background CP^{n-1} field.

2. The supersymmetric CP^{n-1} model

The supersymmetric CP^{n-1} model has been constructed by Cremmer and Scherk [5] in the context of supergravity. In this section we want to construct it in two dimensions proceeding in exactly the same way as in the case of the supersymmetric $O(n)$ σ-model [7,8].

One starts from a superfield $\phi_\alpha(x, \theta)$, $\alpha = 1, ..., n$,

$$\phi_\alpha(x, \theta) = z_\alpha(x) + i\theta\chi_\alpha(x) + \tfrac{1}{2}i\theta\gamma_5\theta F_\alpha(x) , \tag{1}$$

whose component fields z_α, χ_α and F_α are complex fields, which transform according to the fundamental representation of $SU(n)$, while θ is a real two-component spinor *. Under an Abelian gauge transformation, ϕ transforms as follows:

$$\phi'(x, \theta) = \phi(x, \theta)\, e^{i\Lambda(x,\theta)} , \tag{2}$$

where Λ is a real scalar superfield.

In order to construct a super and gauge invariant Lagrangian we need a supercovariant derivative **

$$\nabla = D - A . \tag{3}$$

* We use the following representation for the Euclidean γ matrices:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

** The formalism for a supersymmetric gauge theory in two dimensions has been developed by Ferrara [9,7].

Here, D is the usual supersymmetric covariant derivative

$$D = \frac{\partial}{\partial\theta} + i\not{\partial}\theta ,$$

and A is a fermionic real superfield, which transforms as an Abelian gauge field under a gauge transformation:

$$A' = A + iD\Lambda . \tag{4}$$

Of course, D and A also carry a two-dimensional Lorentz spinor index which is suppressed for simplicity.

A supersymmetric, gauge and SU(n) invariant action is now given by

$$S = \frac{n}{8f}\int \mathrm{d}^2x \; \mathrm{d}\theta \; \gamma_5 \; \mathrm{d}\theta \; \overline{\nabla\phi} \cdot \gamma_5 \nabla\phi , \tag{5}$$

where the bar denotes complex conjugation. The action (5) together with the constraint

$$\bar{\phi} \cdot \phi = 1 , \tag{6}$$

provides a supersymmetric extension of the CP^{n-1} models. The gauge field A can be eliminated from the action by using its equation of motion:

$$A = \bar{\phi} \cdot D\phi . \tag{7}$$

Inserting eq. (7) back into eq. (5) we obtain an action involving the field ϕ only:

$$S = \frac{n}{8f}\int \mathrm{d}^2x \; \mathrm{d}\theta \; \gamma_5 \; \mathrm{d}\theta \, \{\overline{D\phi} \cdot \gamma_5 D\phi - \bar{\phi} \cdot D\phi\gamma_5\bar{\phi} \cdot D\phi\} . \tag{8}$$

In terms of the superfield ϕ one can also write a "topological charge"

$$Q = \frac{i}{8\pi}\int \mathrm{d}^2x \; \mathrm{d}\theta\gamma_5 \, \mathrm{d}\theta \; \overline{\nabla\phi} \cdot \nabla\phi , \tag{9}$$

the supersymmetric version of the self-duality condition

$$\nabla\phi = (\pm) \, \gamma_5 \nabla\phi , \tag{10}$$

and the equation of motion (which is implied by eq. (10))

$$\nabla\gamma_5\nabla\phi_\alpha + (\overline{\nabla\phi} \cdot \gamma_5 \nabla\phi)\phi_\alpha = 0 . \tag{11}$$

Next, we rewrite the action (8) in terms of the component fields defined in eq. (1). The constraint (6) gives

$$\bar{z} \cdot z = 1, \qquad \bar{z} \cdot \chi + z \cdot \bar{\chi} = 0 ,$$
$$\bar{z} \cdot F = \bar{F} \cdot z = i\bar{\chi}\gamma_5\chi . \tag{12}$$

Performing the integration over the θ's in eq. (8) and using eqs. (12) we get

$$S = \frac{n}{2f}\int d^2x \; \{|\partial_\mu z|^2 - i\bar{\psi}\not{\partial}\psi + i\bar{\psi}\gamma_\mu\psi\bar{z}\partial_\mu z$$
$$- (\bar{G} + \rho\gamma_5\bar{\psi})(G - \rho\gamma_5\psi) + (\bar{z}\partial_\mu z)(\bar{z}\partial_\mu z)$$
$$+ \tfrac{1}{4}[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2 - (\bar{\psi}\gamma_\mu\psi)^2]\} \; . \tag{13}$$

Here, we introduced the following new fields:

$$\psi_\alpha = \chi_\alpha - z_\alpha(\bar{z}\cdot\chi) \, ,$$
$$G_\alpha = F_\alpha - z_\alpha(\bar{z}\cdot F) \, ,$$
$$\rho = \tfrac{1}{2}i(z\cdot\bar{\chi} - \bar{z}\cdot\chi) \, . \tag{14}$$

Eliminating the dummy field G_α using its equation of motion, we see that ρ disappears, an effect, which can be traced back to gauge invariance. We are then left with

$$S = \frac{n}{2f}\int d^2x \; \{\overline{D_\mu z}\cdot D_\mu z - i\bar{\psi}\not{D}\psi$$
$$+ \tfrac{1}{4}[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2 - (\bar{\psi}\gamma_\mu\psi)^2]\} \; , \tag{15}$$

where

$$D_\mu = \partial_\mu - \bar{z}\cdot\partial_\mu z \; , \tag{16}$$

and the fields are constrained by

$$\bar{z}\cdot z = 1 , \qquad \bar{z}\cdot\psi = \bar{\psi}\cdot z = 0 \; . \tag{17}$$

Obviously, the action (15) is invariant under gauge transformations

$$z'_\alpha(x) = e^{i\Lambda(x)}z_\alpha(x) \; ,$$
$$\psi'_\alpha(x) = e^{i\Lambda(x)}\psi_\alpha(x) \; , \tag{18}$$

and the supersymmetry transformations are

$$\delta z_\alpha = i\epsilon\psi_\alpha \; ,$$
$$\delta\psi_\alpha = -\tfrac{1}{2}i\epsilon z_\alpha(\bar{\psi}\psi) + \tfrac{1}{2}i\gamma_5\epsilon z_\alpha(\bar{\psi}\gamma_5\psi)$$
$$+ \gamma_\mu\epsilon[D_\mu z_\alpha - \tfrac{1}{2}iz_\alpha(\bar{\psi}\gamma_\mu\psi)] \; . \tag{19}$$

As in the case of the supersymmetric O(n) σ-model, supersymmetry requires the presence of four fermion interaction terms, which are also chirally invariant in our case.

We conclude this section with the remark that the supersymmetric CP^1 model is identical to the supersymmetric O(3) σ-model. This can be shown trivially as in the non-supersymmetric case by defining the field

$$q^i = \bar{\phi} \cdot \sigma^i \phi, \qquad i = 1, 2, 3 \ , \tag{20}$$

and checking that the action (8) and the constraint (6) written in terms of q^i become the action and the constraint of the O(3) supersymmetric σ-model.

3. Complex supersymmetry

As in the case of the O(3) σ-model [8] and the Higgs model [7], the supersymmetric CP^{n-1} models have an additional supersymmetry for any n. In other words, the action (8) is actually invariant under an O(2) extended supersymmetry, whose Noether supercurrent is given by

$$J_\mu = \overline{\partial\!\!\!/^{\perp} \phi} \cdot \gamma_\mu \nabla \phi \ , \tag{21}$$

where

$$\partial_\mu^\perp = \partial_\mu - \bar{\phi} \cdot \partial_\mu \phi, \qquad \nabla = D - \bar{\phi} \cdot D\phi \ .$$

The structure of an O(2) supersymmetry implies furthermore the existence of a conserved O(2) bosonic current

$$V_\mu = \overline{\nabla\phi} \cdot \gamma_\mu \nabla \phi \ . \tag{22}$$

The most direct way to prove invariance under complex supersymmetry is to rewrite the action (8) and the constraint (6) using only O(2) superfields. This is a function $R(x, \theta, \bar{\theta})$ of x_μ, a complex spinor θ and its conjugate $\bar{\theta}$, transforming as follows under a supersymmetry transformation:

$$\delta R = \left\{ \alpha \left[\frac{\partial}{\partial \theta} - \tfrac{1}{2} i \partial\!\!\!/ \bar{\theta} \right] + \bar{\alpha} \left[\frac{\partial}{\partial \bar{\theta}} - \tfrac{1}{2} i \partial\!\!\!/ \theta \right] \right\} R \ . \tag{23}$$

This transformation law follows directly from the transformation laws of x_μ and θ:

$$\delta x_\mu = -\tfrac{1}{2} i [\alpha \gamma_\mu \bar{\theta} + \bar{\alpha} \gamma_\mu \theta] \ ,$$

$$\delta \theta = \alpha, \qquad \delta \bar{\theta} = \bar{\alpha} \ . \tag{24}$$

The supersymmetric covariant derivatives are thus given by

$$D = \frac{\partial}{\partial \theta} + \tfrac{1}{2} i \partial\!\!\!/ \bar{\theta} \ , \qquad \bar{D} = \frac{\partial}{\partial \bar{\theta}} + \tfrac{1}{2} i \partial\!\!\!/ \theta \ . \tag{25}$$

It is well-known [10] that the superfield R can be reduced by imposing the invariant constraint

$$\bar{D} R = 0 \ , \tag{26}$$

which is equivalent to

$$R = R(x_\mu - \tfrac{1}{2} i \bar{\theta} \gamma_\mu \theta, \theta) \ . \tag{27}$$

The complex supersymmetry transformation (23) then simplifies:

$$\delta R = \left[\alpha \frac{\partial}{\partial\theta} - i\bar{\alpha}\partial\!\!\!/\theta\right] R \,. \tag{28}$$

Introducing complex component fields z, χ and F by

$$\begin{aligned} R(x_\mu - \tfrac{1}{2}i\bar{\theta}\gamma_\mu\theta, \theta) &= z(x) - \tfrac{1}{2}i\bar{\theta}\gamma_\mu\theta\, \partial_\mu z(x) \\ &- \tfrac{1}{16}(\bar{\theta}\gamma_5\bar{\theta})(\theta\gamma_5\theta)\,\square\, z(x) + i\theta\chi(x) - \tfrac{1}{4}(\theta\gamma_5\theta)(\bar{\theta}\gamma_5\partial\!\!\!/\chi(x)) \\ &+ \tfrac{1}{2}i\theta\gamma_5\theta\, F(x)\,, \end{aligned} \tag{29}$$

eq. (28) reads

$$\begin{aligned} \delta z &= i\alpha\psi\,, \\ \delta\psi &= \gamma_5 F\alpha + \partial\!\!\!/ z\bar{\alpha}\,, \\ \delta F &= i\bar{\alpha}\gamma_5\partial\!\!\!/\psi\,. \end{aligned} \tag{30}$$

The complex conjugate superfield $\bar{R}$ satisfies the opposite "chirality" condition:

$$D\bar{R} = 0\,. \tag{31}$$

In addition to R one must also introduce a O(2) real superfield $V(x, \theta, \bar{\theta})$ that is the O(2) supersymmetric generalization of the vector field λ_μ appearing in the purely bosonic theory.

Under an infinitesimal Abelian gauge transformation these fields transform as follows:

$$\delta R = i\Lambda R\,, \qquad \delta\bar{R} = -i\bar{\Lambda}\bar{R}\,, \qquad \delta V = -i(\Lambda - \bar{\Lambda})\,, \tag{32}$$

where the gauge function $\Lambda(x_\mu - \frac{1}{2}i\bar{\theta}\,\gamma_\mu\theta, \theta)$ is a "chiral" O(2) superfield.

In the Wess-Zumino gauge the vector superfield V has the following form

$$\begin{aligned} V &= \tfrac{1}{2}\bar{\theta}\gamma_\mu\theta\lambda_\mu + \tfrac{1}{2}\bar{\theta}\theta M + \tfrac{1}{2}i(\bar{\theta}\gamma_5\theta)N + (\theta\gamma_5\theta)(\bar{\theta}\bar{\varphi}) \\ &+ (\bar{\theta}\gamma_5\bar{\theta})(\theta\varphi) + (\bar{\theta}\gamma_5\bar{\theta})(\theta\gamma_5\theta)D\,. \end{aligned} \tag{33}$$

An O(2) supersymmetric and gauge invariant action is given by *:

$$S = \frac{n}{2f}\int \mathrm{d}^2x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, [-V + R\bar{R}\,\mathrm{e}^V]\,. \tag{34}$$

By varying S with respect to the vector superfield V one gets the following equation of motion:

$$R\bar{R} = \mathrm{e}^{-V}\,, \tag{35}$$

* We thank S. Ferrara for very useful discussions on complex supersymmetry.

which implies the following identities in terms of the component fields:

$$\bar{z} \cdot z = 1, \qquad \psi \cdot \bar{z} = 0, \qquad M = \bar{\psi}\psi, \qquad N = i\bar{\psi}\gamma_5\psi\ ,$$

$$\lambda_\mu = i[z \cdot \partial_\mu \bar{z} - \bar{z} \cdot \partial_\mu z - i\bar{\psi} \cdot \gamma_\mu \psi]\ ,$$

$$D = \tfrac{1}{4}[\partial_\mu \bar{z} \partial_\mu z - i\bar{\psi}\not{\partial}\psi - F\bar{F}] + \tfrac{1}{16}(-\lambda_\mu\lambda_\mu - M^2 - N^2)\,,$$

$$\bar{\varphi} = \tfrac{1}{4}\gamma_5(\not{\partial}\bar{z} \cdot \psi - \not{\partial}\psi \cdot \bar{z}) - \tfrac{1}{2}F \cdot \bar{\psi}\ . \tag{36}$$

Inserting (35) back into (34) after the integration over θ and $\bar{\theta}$ one gets the action (15), while the first two equations (36) give the constraints (17). That proves that the action (15) and the constraints (17) are invariant under a complex supersymmetry.

It is interesting to notice that in a manifestly invariant O(2) formalism one must not add any constraint with a Lagrange multiplier as in the case of real supersymmetry, because in this case the constraints are obtained from the equation of motion for the vector superfield V.

4. $1/n$ expansion of CP^{n-1} models with quarks

From the supersymmetric action (15) we see that the coupling between quarks and CP^{n-1} fields is described by the Dirac equation (neglecting the four quark interaction)

$$\not{D}\psi_\alpha = \lambda z_\alpha\ , \tag{37}$$

where the Lagrange multiplier λ is to be eliminated using the constraint

$$\bar{z}_\alpha \psi_\alpha = 0\ . \tag{17}$$

Some properties of the classical solutions of this equation such as the Atiyah-Singer index theorem are discussed in the appendix.

The supersymmetric model can now easily be generalized to incorporate flavored massive quarks with an "electric" charge e possibly different from that of the z-particle. The total action then reads:

$$S = \int d^2x \left\{ \overline{D_\mu z} \cdot D_\mu z + \bar{\psi}(\not{D} - M_B)\psi + \frac{1}{2n}(g_s + e^2 f)(\bar{\psi}\gamma_\mu\psi)^2 \right.$$

$$\left. - \frac{g_v}{2n}\left[(\bar{\psi}\tau^i\psi)^2 + (\bar{\psi}\gamma_5\tau^i\psi)^2\right] \right\}\ , \tag{38}$$

with

$$|z|^2 = \frac{n}{2f}\ , \qquad \bar{z} \cdot \psi = \bar{\psi} \cdot z = 0\ . \tag{39}$$

In addition to the color index α, the quark field ψ_α^a now carries a flavor index $a = 1, ..., N$, too. For simplicity, we assume that the bare mass M_B is flavor independent. The $N \times N$ matrices τ^i, $i = 0, 1, ..., N^2 - 1$ form a complete set of hermitian flavor matrices normalized such that

$$\tau^0 = \frac{1}{\sqrt{N}} \mathbb{1}, \qquad \mathrm{Tr}(\tau^i \tau^k) = \delta^{ik} . \tag{40}$$

The four quark interaction in eq. (38) is the most general one invariant under chiral $U(N) \times U(N)$ transformations:

$$\begin{aligned} \psi' &= \exp\{\tfrac{1}{2}i(1 \pm i\gamma_5)\tau^k \omega^k\}\psi , \\ \bar{\psi}' &= \bar{\psi} \exp\{-\tfrac{1}{2}i(1 \mp i\gamma_5)\tau^k \omega^k\} . \end{aligned} \tag{41}$$

The covariant derivatives D_μ act in a different way on z and ψ fields:

$$\begin{aligned} D_\mu z_\alpha &= \partial_\mu z_\alpha - \frac{f}{n}(\bar{z} \cdot \overleftrightarrow{\partial}_\mu z) z_\alpha , \\ D_\mu \psi_\alpha^a &= \partial_\mu \psi_\alpha^a - e\frac{f}{n}(\bar{z} \cdot \overleftrightarrow{\partial}_\mu z)\psi_\alpha^a . \end{aligned} \tag{42}$$

Correspondingly, the gauge transformations are

$$\begin{aligned} z'_\alpha(x) &= e^{i\Lambda(x)} z_\alpha(x) , \\ \psi'^a_\alpha(x) &= e^{ie\Lambda(x)} \psi_\alpha^a(x) . \end{aligned} \tag{43}$$

The supersymmetric model (15) is the special case, where $N = 1$, $e = 1$, $g_s = 0$, $g_v = f$ and $M_B = 0$. To match eq. (38) with eq. (15), the fields have to be rescaled according to *

$$z \to \left(\frac{n}{2f}\right)^{1/2} z, \qquad \psi \to \left(\frac{n}{2f}\right)^{1/2} \psi, \qquad \bar{\psi} \to (-i)\left(\frac{n}{2f}\right)^{1/2} \bar{\psi} .$$

As in the Thirring model the coupling constant g_s is not infinitely renormalized and since it does not give rise to any interesting interactions, we put it henceforth equal to zero. No other interactions than those included in the action (38) will arise through renormalization, because there are no other gauge, $U(N) \times U(N)$ and parity-invariant possibilities to couple z and ψ fields that satisfy the constraint (39) (the mass term breaks chiral symmetry only "at small momenta").

The $1/n$ expansion of the generating functional for the Euclidean Green functions of the CP^{n-1} model with quarks,

$$\begin{aligned} Z(J, \bar{J}, \eta, \bar{\eta}) &= \int \mathcal{D}z\, \mathcal{D}\bar{z}\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi} \prod_x \left[\delta\left(|z|^2 - \frac{n}{2f}\right)\delta(\bar{\psi} \cdot z)\delta(\bar{z} \cdot \psi)\right] \\ &\quad \times \exp\{-S + \int d^2x\, [\bar{J} \cdot z + \bar{z} \cdot J + \bar{\eta} \cdot \psi + \bar{\psi} \cdot \eta]\} , \end{aligned} \tag{44}$$

* The factor $(-i)$ is introduced for notational convenience.

can be done as in the case without quarks [4]. Thus, to remove the constraints and the quartic interaction terms one introduces a set of Lagrange multipliers α, c, $\bar{c}$, λ_μ, ϕ^i and ϕ_5^i:

$$\prod_x \left[\delta\left(|z|^2 - \frac{n}{2f}\right)\delta(\bar{\psi}\cdot z)\delta(\bar{z}\cdot\psi)\right] \exp\int d^2x \left\{-\frac{f}{2n}[(\bar{z}\cdot\overleftrightarrow{\partial}_\mu z) - e\bar{\psi}\gamma_\mu\psi]^2 + \frac{g_V}{2n}[(\bar{\psi}\tau^i\psi)^2 + (\bar{\psi}\gamma_5\psi)^2]\right\}$$

$$= \int \mathcal{D}\alpha\, \mathcal{D}c\, \mathcal{D}\bar{c}\, \mathcal{D}\lambda_\mu\, \mathcal{D}\phi\, \mathcal{D}\phi_5 \exp\int d^2x \left\{\frac{i}{\sqrt{n}}\alpha\left(|z|^2 - \frac{n}{2f}\right) + \frac{i}{\sqrt{n}}(\bar{c}\bar{z}\cdot\psi + \bar{\psi}\cdot zc) + \frac{i}{\sqrt{n}}\lambda_\mu((\bar{z}\cdot\overleftrightarrow{\partial}_\mu z) - \bar{\psi}\gamma_\mu\psi) - \left(m^2 + \frac{1}{n}\lambda_\mu\lambda_\mu\right)|z|^2 + \frac{1}{\sqrt{n}}(\phi^i\bar{\psi}\tau^i\psi + \phi_5^i\bar{\psi}\gamma_5\tau^i\psi) - \frac{1}{2g_V}(\phi^i\phi^i + \phi_5^i\phi_5^i)\right\}. \tag{45}$$

For later use, a parameter m^2 has been introduced here. It is completely irrelevant at this stage, because $|z|^2$ = constant. Inserting eq. (45) back into eq. (44) and performing the Gaussian integral on ψ, $\bar{\psi}$, z and $\bar{z}$ one gets

$$Z(J, \bar{J}, \eta, \bar{\eta}) = \int \mathcal{D}\alpha\, \mathcal{D}c\, \mathcal{D}\bar{c}\, \mathcal{D}\lambda_\mu\, \mathcal{D}\phi\, \mathcal{D}\phi_5 \times \exp\left\{-S_{\text{eff}} + \int d^2x \left[\bar{\eta}\cdot\Delta_F^{-1}\eta + \left(\bar{J} + \frac{i}{\sqrt{n}}\bar{\eta}\cdot\Delta_F^{-1}c\right) \cdot \left(\Delta_B + \frac{1}{n}\bar{c}\cdot\Delta_F^{-1}c\right)^{-1}\left(J + \frac{i}{\sqrt{n}}\bar{c}\cdot\Delta_F^{-1}\eta\right)\right]\right\}, \tag{46}$$

where

$$\Delta_B = -D_\mu D_\mu + m^2 - \frac{i}{\sqrt{n}}\alpha, \qquad D_\mu = \partial_\mu + \frac{i}{\sqrt{n}}\lambda_\mu, \tag{47}$$

$$\Delta_F = \not{D} - M_B - \frac{1}{\sqrt{n}}(\phi^i + \phi_5^i\gamma_5)\tau^i, \qquad D_\mu = \partial_\mu + \frac{ie}{\sqrt{n}}\lambda_\mu. \tag{48}$$

The effective action S_{eff} is given by

$$S_{\text{eff}} = n\,\text{Tr}\log\left(\Delta_B + \frac{1}{n}\bar{c}\Delta_F^{-1}c\right) - n\,\text{Tr}\log\Delta_F + \int d^2x \left\{\frac{i\sqrt{n}}{2f}\alpha + \frac{1}{2g_V}(\phi^i\phi^i + \phi_5^i\phi_5^i)\right\}. \tag{49}$$

We wish to expand S_{eff} in a power series of $1/\sqrt{n}$ around a minimum. Such a minimum occurs at a non-zero constant field ϕ, i.e., within the $1/n$ expansion chiral symmetry is spontaneously broken. The quark mass term acts like an external magnetic field on a ferromagnet, i.e., it defines the direction of spontaneous $\mathrm{U}(N) \times \mathrm{U}(N)$ symmetry breakdown:

$$\langle\phi^0\rangle = \sqrt{nN}\, M_{\mathrm{S}}, \qquad M_{\mathrm{S}} > 0\,,$$
$$\langle\phi^i\rangle = 0, \qquad (i \neq 0); \qquad \langle\phi_5^i\rangle = 0\,. \tag{50}$$

M_{S} is the spontaneously generated quark mass. Defining

$$\varphi^0 = \phi^0 - \sqrt{nN}\, M_{\mathrm{S}}\,,$$
$$\varphi^i = \phi^i\,, \qquad (i \neq 0); \qquad \varphi_5^i = \phi_5^i\,, \tag{51}$$

and expanding

$$S_{\text{eff}} = \text{constant} + \sum_{\nu=1}^{\infty} (n)^{1-\nu/2} S^{(\nu)}\,, \tag{52}$$

one finds

$$S^{(1)} = i\tilde{\alpha}(0)\left\{\frac{1}{2f} - \int \frac{\mathrm{d}^2 q}{(2\pi)^2}(q^2 + m^2)^{-1}\right\}$$
$$+ \tilde{\varphi}^0(0)\sqrt{N}\left\{\frac{M_{\mathrm{S}}}{2g_{\mathrm{v}}} - M\int \frac{\mathrm{d}^2 q}{(2\pi)^2}(q^2 + M^2)^{-1}\right\}\,, \tag{53}$$

where

$$\tilde{\alpha}(p) = \int \mathrm{d}^2 x\, \mathrm{e}^{-ipx}\alpha(x)\,, \qquad \text{etc.,} \tag{54}$$

and

$$M = M_{\mathrm{B}} + M_{\mathrm{S}} \tag{55}$$

is the total quark mass. Regularizing the divergent integrals in eq. (53) with a Pauli-Villars cutoff Λ, the saddle-point condition $S^{(1)} = 0$ requires the bare coupling constants f and g_{v} to vary with Λ according to

$$\frac{2\pi}{f} = \log\frac{\Lambda^2}{m^2}\,, \tag{56}$$

$$\frac{2\pi}{g_{\mathrm{v}}} = \frac{M}{M_{\mathrm{S}}}\log\frac{\Lambda^2}{M^2}\,. \tag{57}$$

The renormalization of f is identical with what we found in the pure CP^{n-1} models [4]. In particular, fermions do not affect asymptotic freedom no matter how many flavors are introduced.

The quadratic part of S_{eff} is given by

$$S^{(2)} = \int d^2x\, d^2y\, \{\tfrac{1}{2}\alpha(x)\Gamma^{\alpha}(x-y)\alpha(y) + \tfrac{1}{2}\lambda_\mu(x)\Gamma^{\lambda}_{\mu\nu}(x-y)\lambda_\nu(y)$$

$$+ \tfrac{1}{2}\varphi^i(x)\Gamma^{\varphi}_{ij}(x-y)\varphi^j(y) + \tfrac{1}{2}\varphi_5^i(x)\Gamma^{\varphi 5}_{ij}(x-y)\varphi_5^j(y)$$

$$+ \lambda_\mu(x)\Gamma^{\lambda\varphi}_\mu(x-y)\varphi_5^0(y) + \bar{c}(x)\Gamma^{\bar{c}c}(x-y)c(y)\} \,, \tag{58}$$

where

$$\tilde{\Gamma}^{\alpha}(p) = A(p; m^2) \,, \tag{59}$$

$$\tilde{\Gamma}^{\lambda}_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Big\{(p^2 + 4m^2)A(p;m^2)$$

$$- 4Ne^2M^2A(p;M^2) + (Ne^2 - 1)\frac{1}{\pi}\Big\} \,, \tag{60}$$

$$\tilde{\Gamma}^{\varphi}_{ij}(p) = \delta_{ij}\left\{\frac{1}{g_{\text{V}}} - \frac{1}{2\pi}\log\frac{\Lambda^2}{M^2} + (p^2 + 4M^2)A(p;M^2)\right\} \,, \tag{61}$$

$$\Gamma^{\varphi 5}_{ij}(p) = \delta_{ij}\left\{\frac{1}{g_{\text{V}}} - \frac{1}{2\pi}\log\frac{\Lambda^2}{M^2} + p^2A(p;M^2)\right\} \,, \tag{62}$$

$$\tilde{\Gamma}^{\lambda\varphi}_{\mu}(p) = -\epsilon_{\mu\nu}p_\nu 2e\sqrt{N}\,MA(p;M^2) \,, \tag{63}$$

$$\tilde{\Gamma}^{\bar{c}c}(p) = i\not{p}\left\{\frac{1}{4\pi p^2}\log\frac{M}{m} + \frac{1}{2p^2}(p^2 + m^2 - M^2)A(p;m^2;M^2)\right\}$$

$$- MA(p;m^2;M^2) \,. \tag{64}$$

Here, $A(p;m^2;M^2)$ denotes a one-loop integral

$$A(p;m^2;M^2) = \int \frac{d^2q}{(2\pi)^2}\{(q^2+m^2)((p+q)^2+M^2)\}^{-1}$$

$$= \frac{1}{4\pi}[(p^2+m^2-M^2)^2 - 4m^2M^2]^{-1/2}$$

$$\times \log\frac{p^2+m^2+M^2+\sqrt{(p^2+m^2+M^2)^2-4m^2M^2}}{p^2+m^2+M^2-\sqrt{(p^2+m^2+M^2)^2-4m^2M^2}} \,, \tag{65}$$

and $A(p;m^2) = A(p;m^2;m^2)$. The calculation of Γ^{φ} and $\Gamma^{\varphi 5}$ involves a divergent fermion loop integral, which has been regularized with a Pauli-Villars cutoff. In the limit $\Lambda \to \infty$ we require that

$$\frac{1}{g_{\text{V}}} - \frac{1}{2\pi}\log\frac{\Lambda^2}{M^2} = \epsilon \geqslant 0 \,. \tag{66}$$

To match with eq. (57), the bare mass M_B must scale as

$$M_\mathrm{B}(\Lambda) = \epsilon M_\mathrm{S} \frac{2\pi}{\log(\Lambda^2/M_\mathrm{S}^2)} \ . \tag{67}$$

Thus, M_B goes to zero as $\Lambda \to \infty$ so that the total quark mass M coincides with the spontaneously generated mass M_S in this limit. Taking eq. (66) into account, Γ^φ and $\Gamma^{\varphi 5}$ read

$$\tilde{\Gamma}^\varphi_{ij}(p) = \delta_{ij}\{\epsilon + (p^2 + 4M^2)A(p; M^2)\} \ , \tag{68}$$

$$\tilde{\Gamma}^{\varphi 5}_{ij}(p) = \delta_{ij}\{\epsilon + p^2 A(p; M^2)\} \ . \tag{69}$$

The higher-order terms in the $1/n$ expansion of S_eff are finite and need not be renormalized. From eqs. (56), (57) and (67) we see that the coupling constants f and g_v and the bare mass M_B have disappeared and are absorbed into the physical parameters m, M and ϵ. The "electric charge" e is not renormalized at all at this level.

The free parameters N, m, M, ϵ and e take special values, when some symmetry is realized: in the chiral limit $M_\mathrm{B} = 0$, we have $\epsilon = 0$, and in the supersymmetric case $N = 1$, $e = 1$, $m = M$ and $\epsilon = 0$. When supersymmetry is broken by a mass term only, then $M > m$ and the chiral symmetry breaking parameter ϵ is fixed to the value $(1/2\pi)\log(M^2/m^2)$ by the requirement that supersymmetry be recovered at high energies.

As in the pure CP^{n-1} model, the $1/n$ expansion of Euclidean vacuum expectation values of z and ψ fields are obtained by expanding the integrand of the gener-

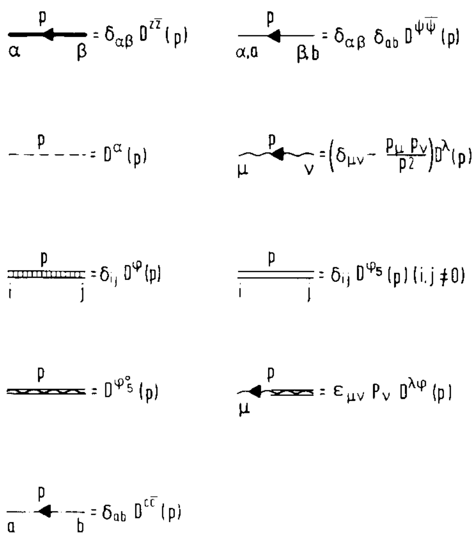

Fig. 1. Graphical representation of the propagators occurring in the $1/n$ expansion.

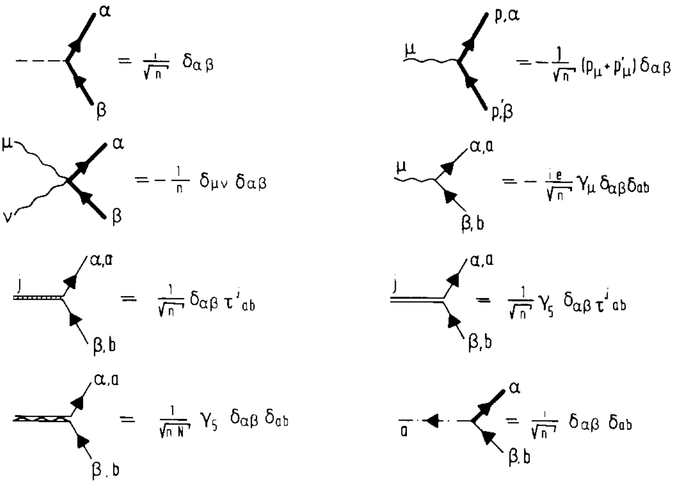

Fig. 2. Vertices for the $1/n$ expansion.

ating functional (46) in powers of $1/n$ and performing the resulting Gaussian integrals over the Lagrange multiplier fields. The corresponding Feynman diagrams are composed of the propagators and vertices displayed in figs. 1, 2, respectively. The graphs collected in fig. 3 should not be drawn, since "they have already been

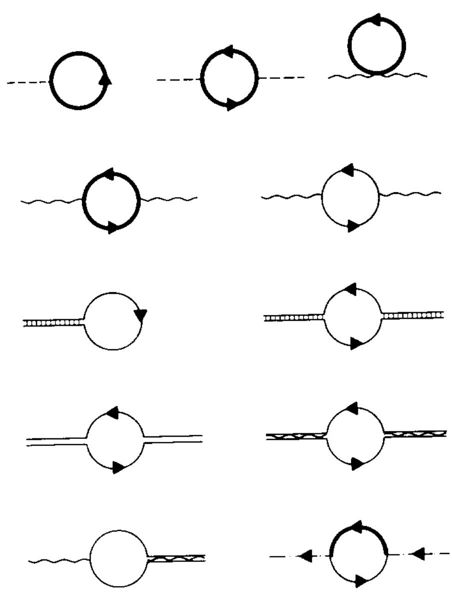

Fig. 3. Forbidden (sub-) diagrams.

included in the propagators of the Lagrange multipliers". We use the following convention for the propagators:

$$\langle A_\alpha(x) B_\beta(y)\rangle = \int \frac{d^2p}{(2\pi)^2}\, e^{ip(x-y)} D^{AB}_{\alpha\beta}(p)\,. \tag{70}$$

Explicitly, in the Lorentz gauge $\partial_\mu \lambda_\mu = 0$, one finds

$$D^{z\bar{z}}(p) = (p^2 + m^2)^{-1}\,, \tag{71}$$

$$D^{\psi\bar{\psi}}(p) = (i\not{p} - M)^{-1}\,, \tag{72}$$

$$D^{\alpha}(p) = (A(p; m^2))^{-1}\,, \tag{73}$$

$$D^{\lambda}(p) = \tilde{\Gamma}^{\varphi_5}(p)[\tilde{\Gamma}^{\lambda}(p)\tilde{\Gamma}^{\varphi_5}(p) + p^2(\tilde{\Gamma}^{\lambda\varphi}(p))^2]^{-1}\,, \tag{74}$$

$$D^{\varphi}(p) = [\epsilon + (p^2 + 4M^2)A(p; M^2)]^{-1}\,, \tag{75}$$

$$D^{\varphi_5}(p) = [\epsilon + p^2 A(p; M^2)]^{-1}\,, \tag{76}$$

$$\overset{0}{D}{}^{\varphi_5}(p) = \tilde{\Gamma}^{\lambda}(p)[\tilde{\Gamma}^{\lambda}(p)\tilde{\Gamma}^{\varphi_5}(p) + p^2(\tilde{\Gamma}^{\lambda\varphi}(p))^2]^{-1}\,, \tag{77}$$

$$D^{\lambda\varphi}(p) = \tilde{\Gamma}^{\lambda\varphi}(p)[\tilde{\Gamma}^{\lambda}(p)\tilde{\Gamma}^{\varphi_5}(p) + p^2(\tilde{\Gamma}^{\lambda\varphi}(p))^2]^{-1}\,, \tag{78}$$

$$D^{c\bar{c}}(p) = (\tilde{\Gamma}^{c\bar{c}}(p))^{-1}\,, \tag{79}$$

where (*cf.* eqs. (60), (63) and (69))

$$\tilde{\Gamma}^{\lambda}_{\mu\nu}(p) = (\delta_{\mu\nu} - (p_\mu p_\nu/p^2))\tilde{\Gamma}^{\lambda}(p)\,,$$

$$\tilde{\Gamma}^{\lambda\varphi}_{\mu}(p) = \epsilon_{\mu\nu}p_\nu\tilde{\Gamma}^{\lambda\varphi}(p)\,,$$

$$\tilde{\Gamma}^{\varphi_5}_{ij}(p) = \delta_{ij}\tilde{\Gamma}^{\varphi_5}(p)\,.$$

5. Physical interpretation of results

The $1/n$ expansion provides a description of quarks and partons (the z-particles, *cf.* ref. [4]) valid at low energies. The Feynman rules of sect. 4 reveal that both kinds of particles are massive and interact by exchanging α, λ, φ, φ_5 and c quanta. From eqs. (73), (75), (79) and

$$A(p; m^2) = \frac{1}{4\pi m^2}\left\{1 - \frac{p^2}{6m^2} + O(p^+)\right\}\,, \qquad (p \to 0)\,, \tag{80}$$

it follows that the α, φ and c exchanges are short ranged, i.e., the corresponding Yukawa forces have a range of order $1/m$ or $1/M$. The other interactions can conveniently be summarized by writing a low-energy effective Lagrangian density:

$$\mathcal{L}_{\text{eff}} = \overline{D_\mu z} \cdot D_\mu z + m^2|z|^2 + \bar{\psi}(\not{D} - M)\psi$$

$$+\frac{M^2+2Ne^2m^2}{24\pi m^2M^2}F^2+\frac{1}{8\pi M^2}(\partial_\mu\varphi_5^i)^2+\tfrac{1}{2}\epsilon(\varphi_5^i)^2$$

$$+\frac{ie\sqrt{N}}{2\pi M}F\varphi_5^0-\frac{1}{\sqrt{n}}\bar{\psi}\tau^i\gamma_5\psi\varphi_5^i\,, \tag{81}$$

where $F=\epsilon_{\mu\nu}\partial_\mu\lambda_\nu$ and $D_\mu=\partial_\mu+i\lambda_\mu/\sqrt{n}$ when acting on z, and $D_\mu=\partial_\mu+ie\lambda_\mu/\sqrt{n}$, when acting on ψ, respectively.

As in the case without quarks, the propagator of λ_μ has a pole at $p^2=0$:

$$D^\lambda(p)=c/p^2+\mathrm{O}(1)\,, \tag{82}$$

with

$$c=\epsilon\,\frac{12\pi^2m^2M^2}{3Ne^2m^2+\epsilon\pi(M^2+2Ne^2m^2)}\,. \tag{83}$$

When $\epsilon\neq 0$, i.e., when the bare quark mass M_B is not zero, the exchange of λ quanta gives rise to a linear Coulomb potential so that quarks and partons are confined: the physical Hilbert space contains only states with zero n-ality. This effect has already been discussed at length in refs. [4,11] so that we are not going into it any further.

When the bare quark mass vanishes, $\epsilon=0$ and the pole eq. (82) disappears. Consequently, quarks and partons are liberated, i.e., there are one-particle states carrying the fundamental SU(n) quantum numbers. This phenomenon is a screening effect: any externally applied "electric" field is bleached by chiral quark-antiquark dipoles. To make this explicit, we add two infinitely heavy static charges with values $\pm\theta/(2\pi)$ (relative to the z-particle) at $x_1=\pm\infty$ to our system. This amounts to a change of the effective action by

$$S_\mathrm{eff}^\theta=S_\mathrm{eff}-i\,\frac{\theta}{2\pi\sqrt{n}}\int\mathrm{d}^2x\;\epsilon_{\mu\nu}\partial_\mu\lambda_\nu\,. \tag{84}$$

The external charges produce a constant background "electric" field. Due to vacuum polarization, the actually measured field (*cf.* eq. (83)),

$$\langle F\rangle_\theta=ic\,\frac{\theta}{2\pi\sqrt{n}}+\mathrm{O}(n^{-3/2})\,, \tag{85}$$

is much reduced, when ϵ is small, and disappears for $\epsilon=0$.

That the screening of an external "electric" field is due to chiral quark-antiquark pairs is also supported by the following observation. From the effective low-energy Lagrangian, eq. (81), or directly from eq. (78) one finds that the mixed propagator $D^{\lambda\varphi}$ has a pole at $p^2=0$:

$$D^{\lambda\varphi}(p)=-\frac{d}{p^2}+\mathrm{O}(1)\,, \tag{86}$$

where

$$d = \frac{6\pi\sqrt{N}\, em^2 M}{3Ne^2m^2 + \epsilon\pi(M^2 + 2Ne^2m^2)} \ . \tag{87}$$

This implies that $\varphi_5^0 = (g_v/\sqrt{nN})\, \bar{\psi}\gamma_5\psi$ acquires a non-vanishing expectation value in the presence of external charges:

$$\langle g_v \bar{\psi}\gamma_5\psi\rangle_\theta = \frac{\theta}{2\pi}\sqrt{N}\, d + O\left(\frac{1}{n}\right) \ , \tag{88}$$

i.e., there is a constant background density of chiral quark-antiquark dipoles, which is indeed maximal for $\epsilon = 0$.

Although quarks are liberated in the chiral limit $\epsilon = 0$, it is not straightforward to construct one-quark states. For example, the quark field ψ in the Coulomb gauge $\lambda_1 = 0$ creates a state of infinite mean energy when applied to the vacuum. The problem roots in the gauge variance of $\psi(x)$, which requires a non-local tail such as

$$\exp\left\{\frac{ie}{\sqrt{n}} \int_{-\infty}^{x} dz_\mu \lambda_\mu(z)\right\}$$

to be added. Roughly speaking, this phase factor represents the *unscreened* Coulomb field due to a point charge at x. One can show, within a non-relativistic approximation, that the infinite Coulomb energy can be reduced to a finite value by multiplying with an appropriate exponential of $\bar{\psi}\gamma_5\psi$ in complete agreement with the physical picture explained above.

Chiral symmetry is spontaneously broken: from eq. (50) and $\phi^0 = (g_v/\sqrt{nN})\bar{\psi}\psi$ it follows that

$$\langle g_v \bar{\psi}\psi\rangle = nNM_S \ , \tag{89}$$

where $M_S = M$ is the spontaneously generated quark mass. From the Goldstone theorem, we then expect the existence of a multiplet of pseudoscalar bosons with a mass of the order of the chiral symmetry breaking parameter ϵ. Indeed, from eq. (81) it is obvious that the φ_5^i, $i = 1, ..., N^2 - 1$, are interpolating fields for mesons ("pions") with a mass

$$m_\pi^2 = 4\pi\epsilon M^2 \ . \tag{90}$$

Due to the mixing term between λ_μ and φ_5^0, however, there is no light particle associated to the isoscalar φ_5^0, i.e., when the propagator (77) of φ_5^0 is worked out in the low-p^2 range, it turns out to be analytic there, no matter how small ϵ is. This shows that there is no $U(1)_A$ problem in our model: the "η" (if it exists) is not a Goldstone boson! No contradiction between the absence of a low-mass isoscalar boson and the Ward identities of the axial U(1) current arises, because of the anomaly:

$$\partial_\mu j_\mu^5(x) = -2enNiq(x) + 2\epsilon M g_v(\bar{\psi}\gamma_5\psi)(x) \ . \tag{91}$$

Here, $j_\mu^5 = \bar{\psi}\gamma_5\gamma_\mu\psi$ and $q(x)$ is the topological density

$$q(x) = \frac{if}{n\pi}\,\epsilon_{\mu\nu}\partial_\mu\bar{z}\cdot\partial_\nu z = \frac{1}{2\pi\sqrt{n}}\,\epsilon_{\mu\nu}\partial_\mu\lambda_\nu\,. \tag{92}$$

Note that $g_\mathrm{v}\bar{\psi}\gamma_5\psi = \sqrt{nN}\,\varphi_5^0$ is the *renormalized* axial quark density, so that it makes sense to call ϵM the current quark mass.

If there would be no anomaly in eq. (91), the Ward identity in the chiral limit $\epsilon = 0$,

$$\partial_\mu\langle j_\mu^5(x) g_\mathrm{v}(\bar{\psi}\gamma_5\psi)(0)\rangle = -2\delta(x)\langle g_\mathrm{v}\bar{\psi}\psi\rangle\,, \tag{93}$$

would imply that there was a pole at $p^2 = 0$ in the two-point function $\langle j_\mu^5(x) g_\mathrm{v}(\bar{\psi}\gamma_5\psi)(0)\rangle$. However, from eq. (86) we see that the anomalous term $-2enNiq(x)$ contributes at $p^2 = 0$ and a short calculation reveals that it cancels the right-hand side of eq. (93) exactly there. In particular, $\langle j_\mu^5(x) g_\mathrm{v}(\bar{\psi}\gamma_5\psi)(0)\rangle$ is analytic around $p^2 = 0$. Remarkably, the way the $\mathrm{U}(1)_\mathrm{A}$ problem is resolved in our model looks much the same as in the dilute instanton gas approximation [1,2], except that due to the many zero modes of the Dirac equation in an instanton field (*cf.* appendix), this approximation predicts $\langle\bar{\psi}\psi\rangle = 0$ (instead chiral symmetry is broken by a non-zero vacuum expectation value of a product of $2n$ quark fields).

When there is more than one quark flavor and $\epsilon = 0$, the $1/n$ expansion breaks down: higher-order diagrams involving internal φ_5^i lines are infrared divergent, because of the pion poles at $p^2 = 0$. This was to be expected from Coleman's theorem [12] and the fact that the flavored axial currents are anomaly free. If the theory nevertheless exists, $\mathrm{SU}(N)\times\mathrm{SU}(N)$ chiral symmetry cannot be spontaneously broken, in particular $\langle\bar{\psi}\psi\rangle = 0$. On the other hand, the currents

$$j_\mu^i = \bar{\psi}\gamma_\mu\tau^i\psi\,,\qquad j_\mu^{i,5} = \bar{\psi}\gamma_5\gamma_\mu\tau^i\psi\,, \tag{94}$$

are both conserved for $i \neq 0$ and since $\gamma_5\gamma_\mu = -\epsilon_{\mu\nu}\gamma_\nu$ in two dimensions, they are actually free massless fields. The presence of these massless excitations suggests that the $\mathrm{SU}(N)\times\mathrm{SU}(N)$ chiral symmetry is realized in a phase similar to the Berezinski-Kosterlitz-Thouless phase of the XY model [6], but we did not analyze this question any further.

6. Conclusions

An outstanding property of CP^{n-1} models with quarks is their asymptotic freedom. Some high-energy processes could therefore be calculated in perturbation theory yielding logarithmic scaling violations as in QCD_4. On the other hand, from the $1/n$ expansion one can derive a superrenormalizable effective Lagrangian valid at low energies. In this regime, chiral symmetry is spontaneously broken and the quarks consequently acquire a large constituent mass. If they are given a bare mass M_B, they are also confined by a linear Coulomb force, which is produced by the

CP^{n-1} fields with non-trivial topology. If $M_B = 0$, a screening effect takes place and the fundamental SU(n) quantum numbers are liberated. Also due to CP^{n-1} fields with non-vanishing winding number is the absence of a light isoscalar boson in connection with chiral symmetry breaking.

We believe that the breaking of the $U(1)_A$ chiral symmetry in QCD_4 looks very much the same as in our two-dimensional model. In particular, we conclude that the mechanism resolving the $U(1)_A$ problem as suggested by the dilute-gas approximation is likely to be valid beyond this approximation. We doubt, however, that confinement breaks down in QCD_4 in the chiral limit $M_B = 0$. Topologically non-trivial glue fields are suppressed in this limit too, but it seems that confinement in four dimensions is not due to the contribution of these fields. It might be more closely related to the dynamical mass creation mechanism, which occurs independently of the existence of a topological number and is still poorly understood.

Another mismatch between the CP^{n-1} model and QCD_4 is the fact that the latter is not straightforwardly $1/n$ expandable (n is the number of colors), although some simplifications occur in the large-n limit. A topological explanation of this has recently been given by Atiyah [13]. In case of the CP^{n-1} model, the function space to be integrated over in the Feynman path integral becomes contractible in each instanton sector when $n = \infty$. On the other hand, the space of all gauge potentials modulo gauge transformations does not become topologically trivial for $n = \infty$. From this point of view one cannot expect the $n = \infty$ Yang-Mills theory to be representable in terms of free fields.

There are two very interesting questions concerning the CP^{n-1} models with quarks, which we could not answer so far. First, it is not clear to us how precisely the perturbation expansion (including instanton contributions) and the results of the $1/n$ expansion can be patched together so as to obtain a consistent physical picture for all energy ranges. Secondly, in the case of more than one flavor and $M_B = 0$, the $1/n$ expansion breaks down, because it is impossible to break the anomaly free $SU(N)_A$ chiral symmetry spontaneously. On the other hand, the flavored axial currents are free massless fields so that chiral symmetry is maybe almost spontaneously broken. We do not know whether such a non-Abelian Berezinski-Kosterlitz-Thouless phase exists and what its properties might be.

Appendix

Solutions of the Dirac equation in a CP^{n-1} background field

In the geometrically most natural case where $e = 1, N = 1$ and $M_B = 0$, the Dirac equation in an external CP^{n-1} field $z_\alpha(x)$ reads:

$$(\delta_{\alpha\beta} - z_\alpha \bar{z}_\beta) \not{D} \psi_\beta = 0, \qquad z \cdot \psi = 0 . \tag{A.1}$$

As usual, Dirac indices have been suppressed and

$$\not{D} = \gamma_\mu(\partial_\mu + iA_\mu)\,, \qquad A_\mu = \tfrac{1}{2}\bar{z}\cdot\overleftrightarrow{\partial}_\mu z\,, \qquad |z|^2 = 1\,. \tag{A.2}$$

Solutions of eq. (A.1) split into eigenstates of γ_5:

$$\psi = \begin{pmatrix}1\\ i\end{pmatrix}\psi^+ + \begin{pmatrix}1\\ -i\end{pmatrix}\psi^-\,, \tag{A.3}$$

$$(\delta_{\alpha\beta} - z_\alpha\bar{z}_\beta)D_s\psi^+_\beta = 0\,, \qquad \bar{z}\cdot\psi^+ = 0\,, \tag{A.4}$$

$$(\delta_{\alpha\beta} - z_\alpha\bar{z}_\beta)D_{\bar{s}}\psi^-_\beta = 0\,, \qquad \bar{z}\cdot\psi^- = 0\,, \tag{A.5}$$

where

$$D_s = \tfrac{1}{2}(D_0 + iD_1)\,, \qquad D_{\bar{s}} = \tfrac{1}{2}(D_0 - iD_1)\,. \tag{A.6}$$

The number of solutions of the Dirac equation, which are normalizable when projected on the sphere, i.e., for which

$$\int d^2x\,(1+x^2)^{-1}|\psi(x)|^2 < \infty\,, \tag{A.7}$$

is related to the topological charge,

$$Q = \frac{1}{2\pi}\int d^2x\ \epsilon_{\mu\nu}\partial_\mu A_\nu\,,$$

of the background field $z_\alpha(x)$. More precisely, denoting by z_+ and z_- the number of solutions of eq. (A.4), (A.5), respectively, we have

$$z_+ - z_- = -nQ\,. \tag{A.8}$$

This is a special case of the Atiyah-Singer index theorem. A physicist's proof of eq. (A.8) can be given following Schroer and Nielson's argumentation [15].

In the case when $z_\alpha(x)$ is a multi-instanton solution, the Dirac equation can actually be solved explicitly. The most general q-instanton solution can conveniently be written in the form

$$z_\alpha = \frac{p_\alpha}{(\bar{p}_\beta p_\beta)^{1/2}}\,, \qquad \alpha = 1, ..., n\,. \tag{A.9}$$

Here, p_α denotes a set of polynomials of $s = x_0 - ix_1$ with no common root and maximal degree q. The general solution of eq. (A.4) is

$$\psi^+_\alpha = (\bar{p}_\beta p_\beta)^{1/2} f_\alpha\,, \tag{A.10}$$

where f_α is an arbitrary vector of holomorphic functions of $\bar{s}$ orthogonal to p_α: $\bar{p}_\alpha f_\alpha = 0$. On the other hand, the general solution of eq. (A.5) is

$$\psi^-_\alpha = (\bar{p}_\gamma p_\gamma)^{-3/2}(\delta_{\alpha\beta}(\bar{p}_\delta p_\delta) - p_\alpha\bar{p}_\beta)g_\beta\,, \tag{A.11}$$

with g_β an arbitrary set of holomorphic functions of s.

None of the solutions (A.10) is normalizable so that we have a vanishing theorem:

$$z_+ = 0 \,, \qquad \text{for instanton solutions.} \tag{A.12}$$

From the index theorem (A.8), we then expect precisely $n \cdot q$ of the solutions (A.11) to be normalizable. These solutions are obtained by choosing g_β to be a polynomial of degree of at most q, the coefficient u_α of s^q being proportional to that one of s^q in p_α.

Note added

The CP^{n-1} models with quarks have also been discussed by E. Witten [14].

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